The leading nuclear-structure electrostatic correction in arbitrary β decays

Model-independent finite-size Coulomb correction for arbitrary multipole rank *J*

Daniel Benatar

Racah Institute of Physics, The Hebrew University

ECT Workshop. From nuclear structure to new physics

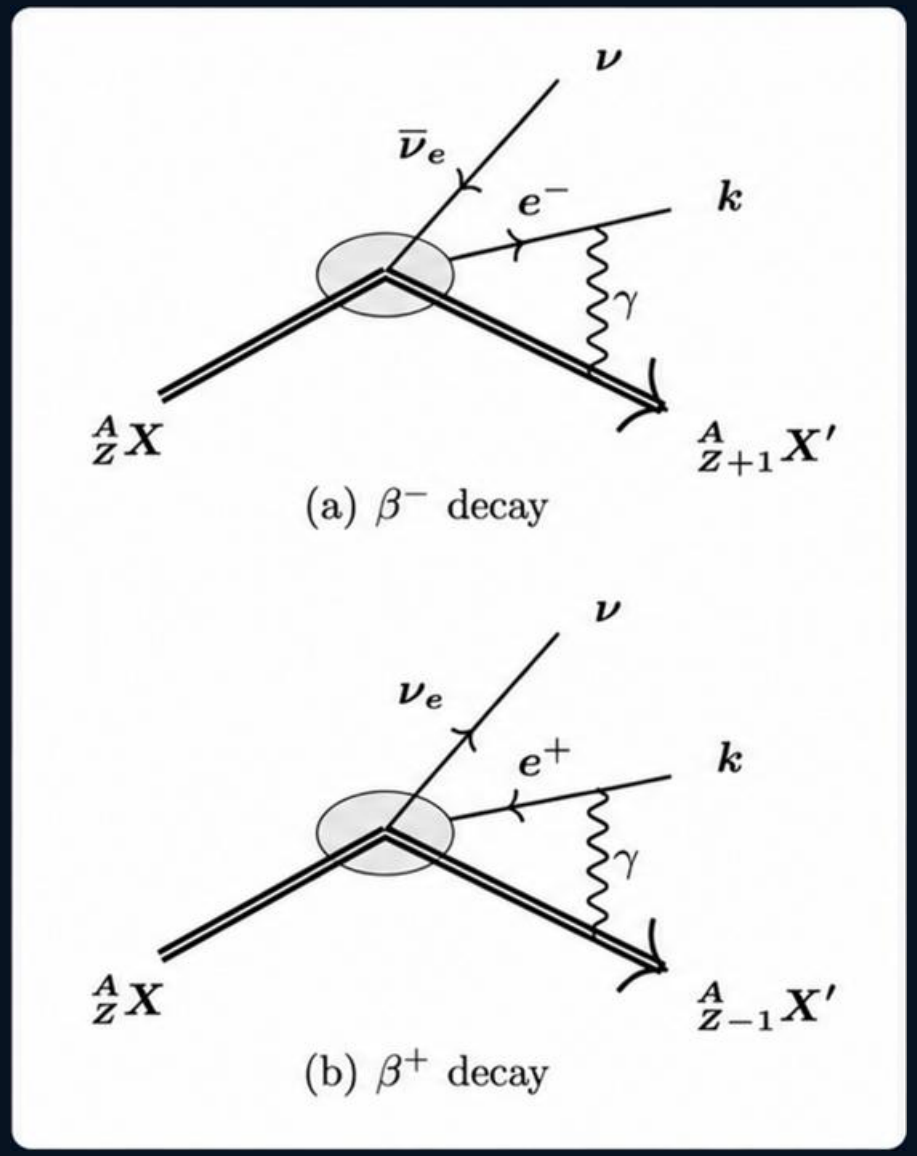
Submitted to Physical Review C — currently under review
arXiv:2606.13073 [nucl-th]: Daniel Benatar, Ayala Glick-Magid, Doron Gazit

Purpose of this presentation

Arbitrary J finite-size Coulomb corrections for precision β decay

- **Develop** a framework for *β* decay of arbitrary angular momentum transfer *J*
- **Derive** the first-order Coulomb correction from one-photon exchange
- **Work** at leading order in the finite-size parameter expanded in $\epsilon_{EC} = \alpha Z E_e R$
  - **Point-Coulomb effects:** Fermi-function expansion parameter $\alpha Z E_e / k$, independent of nuclear structure It can be significantly larger than ϵ_{EC}, particularly for heavier nuclei.
- **Identify** three leading contributions:
  1. Charge form factor
  2. Shift in lepton traces
  3. Shift in nuclear multipole operators
- **Present** results for allowed GT and unique first-forbidden transitions, relevant to precision searches for physics beyond the Standard Model

Physical picture: one Coulomb photon



The emitted charged lepton propagates in the nuclear electrostatic field via one-photon exchange.

- Momentum transfer is shifted:

$$q = k - \nu \qquad \text{and} \qquad q' = q - k_\gamma$$

Coulomb-Corrected $\beta^{\mp}$ Decay:
One-photon exchange between the emitted lepton and the nuclear charge distribution.

- Finite-size R and lepton energy E_e enter via:

$$\epsilon_{EC} = \alpha Z E_e R/(\hbar c)$$

ϵ_{EC} provides the electrostatic correction scale

X and X’ denote the initial and final nuclear states, with mass number A and charge Z

Starting point: distorted wave and Coulomb kernel

Distorted electron wave function including the nuclear potential

$$\bar{\phi}_{e^-}(\mathbf{r}) = \bar{u}(k)e^{-i\mathbf{k}\cdot\mathbf{r}} - \gamma_0(\not{k} + m_e)\bar{u}(k)\int_0^\infty dk_\gamma\, k_\gamma^2\, \frac{1}{(2\pi)^3}\int d\Omega_\gamma\, e^{-i(\mathbf{k}-\mathbf{k}_\gamma)\cdot\mathbf{r}}\,\Theta(k,k_\gamma)$$

Coulomb kernel

$$\Theta(k,k_\gamma) \equiv 8\pi\alpha Z\,\frac{\mathcal{F}(k_\gamma^2)}{(k_\gamma^2+i\epsilon)(k_\gamma^2-2kk_\gamma\cos\theta+i\epsilon)}$$

$$\mathcal{F}(k_\gamma^2) = \frac{1}{1+a^2k_\gamma^2}$$

Soft exchange

k_γ is the Coulomb photon momentum

Finite size

$\mathcal{F}(k_\gamma^2)$, Is the nuclear charge form factor and contains the nuclear charge radius. Our chosen form is exact to first order in nuclear structure, enabling a universal correction

The form factor induces finite-size suppression, resulting in $k_\gamma/k \sim 1/(MR) \ll 1$

Angular dependence

θ couples k_γ and k

Coulomb-corrected leptonic current and matrix-element split

Hamiltonian and Coulomb-corrected current

$$H_w^c(\mathbf{r}) = -\frac{G_F V_{ud}}{\sqrt{2}}\left[J^\mu(\mathbf{r})\, j_\mu^c(\mathbf{r}) + \text{h.c.}\right], \qquad j_\mu^c(\mathbf{r}) = \phi_e(\mathbf{r})\gamma_\mu(1+\gamma_5)\psi_{\nu_e}(\mathbf{r})$$

Replacing the free electron wave function with the distorted wave function $\phi_e(\mathbf{r})$ yields the corrected matrix element.

Matrix-element split

$$\langle f|\hat{H}_w^c|i\rangle = \underbrace{\langle f|\hat{H}_w|i\rangle}_{\text{Weak matrix element}} + \underbrace{\langle f|\hat{H}^c|i\rangle}_{\text{Coulomb matrix element}}$$

(no Coulomb) (first order in $Z\alpha$)

Weak matrix element

Standard Walecka multipole matrix element

Coulomb matrix element

Coulomb kernel $\Theta(k, k_\gamma)$, corrected leptonic current j_μ^c , and shifted momentum q' in the same multipole basis

Coulomb-corrected matrix element

The exchanged photon shifts the momentum transfer and induces Coulomb-corrected leptonic currents currents

Total Coulomb correction matrix element element

$\langle f|\tilde{H}_w|i\rangle$ **Weak matrix element** (no Coulomb)

$$\langle f|\hat{\mathcal{H}}_w^c(\mathbf{r})|i\rangle = -\frac{G_F V_{ud}}{\sqrt{2}} \int d^3r \, \langle f|\hat{J}_\mu(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} \, l_\mu^{\mp}|i\rangle$$
$$\pm \frac{G_F V_{ud}}{\sqrt{2}} \int_0^\infty dk_\gamma \, k_\gamma^2 \, \frac{1}{(2\pi)^3} \int d\Omega_\gamma \int d^3r$$
$$\times \langle f|\hat{J}_\mu(\mathbf{r}) e^{-i\mathbf{q}'\cdot\mathbf{r}} \, \Theta(k, k_\gamma) \, L_\mu^{c\mp}|i\rangle$$

$\langle f|\tilde{H}^c|i\rangle$ **Coulomb matrix element** (first order in $Z\alpha$)

Leptonic currents

Free currents for β∓ decay

$$l_\mu^- = \bar{u}_e(k)\gamma_\mu(1+\gamma_5)v_{\nu_e}(\nu)$$
$$l_\mu^+ = \bar{u}_{\nu_e}(\nu)\gamma_\mu(1+\gamma_5)v_e(k)$$

Coulomb-corrected currents for β∓ decay

$$L_\mu^{c-} = \bar{u}_e(k)\gamma_0(\not{k}+m_e)\gamma_\mu(1+\gamma_5)v_{\nu_e}(\nu)$$
$$L_\mu^{c+} = \bar{u}_{\nu_e}(\nu)\gamma_\mu(1+\gamma_5)(\not{k}+m_e)\gamma_0 v_e(k)$$

Coulomb-corrected weak Hamiltonian

Multipole form of the weak and Coulomb matrix elements and the nuclear multipole-operator basis

1 Coulomb-corrected weak Hamiltonian

Multipole expansions of the weak and Coulomb plane waves and spherical decomposition of the exchanged-photon momentum yield the weak matrix element plus the one-photon Coulomb correction.

$$\langle f|\hat{\mathcal{H}}_w^c(\boldsymbol{r})|i\rangle$$

$$= -\frac{G_F V_{ud}}{\sqrt{2}} \left\langle f \left| \left\{ -\sum_{J\geq 1} \sqrt{2\pi(2J+1)}(-i)^J \right.\right.\right.$$

$$\times \sum_{\lambda=\pm 1} l_\lambda^{\mp} \left[\lambda \hat{T}^{mag}_{J-\lambda}(q) + \hat{T}^{el}_{J-\lambda}(q)\right]$$

$$+ \sum_{J\geq 0} \sqrt{4\pi(2J+1)}(-i)^J \left[l_3^{\mp}\hat{\mathcal{L}}_{J0}(q) - l_0^{\mp}\hat{\mathcal{M}}_{J0}(q)\right]$$

$$\mp \int_0^\infty dk_\gamma\, k_\gamma^2 \left\{ -\sum_{J\geq 1} \sqrt{2\pi(2J+1)}(-i)^J \right.$$

$$\times \sum_{\lambda=\pm 1} \Theta(k,k_\gamma) L_\lambda^{c\mp} \left[\lambda \hat{T}^{mag}_{J-\lambda}(q') + \hat{T}^{el}_{J-\lambda}(q')\right]$$

$$+ \sum_{J\geq 0} \sqrt{4\pi(2J+1)}(-i)^J \left[\Theta(k,k_\gamma) L_3^{c\mp} \hat{\mathcal{L}}_{J0}(q')\right.$$

$$\left.\left.\left.\left. - \Theta(k,k_\gamma) L_0^{c\mp} \hat{\mathcal{M}}_{J0}(q')\right]\right\}\right\}\left|i\right\rangle$$

$\langle f|\hat{\mathcal{H}}_w(\boldsymbol{r})|i\rangle$

$\langle f|\hat{\mathcal{H}}^c(\boldsymbol{r})|i\rangle$

2 Nuclear multipole operators

Definitions of the Coulomb, longitudinal, electric, and magnetic operators entering the matrix element

Notation: M: magnetic projection; $j_J(x)$: spherical Bessel function of order J; $\mathcal{Y}_{JM}$ and $\boldsymbol{\mathcal{Y}}^M_{JJ1}$: scalar and vector spherical harmonics

$$\hat{\mathcal{M}}_{JM}(q) = \int d^3r \left[j_J(qr)\mathcal{Y}_{JM}(\Omega_r)\right]\hat{\mathcal{J}}_0(\boldsymbol{r})$$

$$\hat{\mathcal{L}}_{JM}(q) = \frac{i}{q}\int d^3r \left\{\nabla\left[j_J(qr)\mathcal{Y}_{JM}(\Omega_r)\right]\right\}\cdot\hat{\boldsymbol{\mathcal{J}}}(\boldsymbol{r})$$

$$\hat{\mathcal{T}}^{el}_{JM}(q) = \frac{1}{q}\int d^3r \left[\nabla\times j_J(qr)\boldsymbol{\mathcal{Y}}^M_{JJ1}(\Omega_r)\right]\cdot\hat{\boldsymbol{\mathcal{J}}}(\boldsymbol{r})$$

$$\hat{\mathcal{T}}^{mag}_{JM}(q) = \int d^3r \left[j_J(qr)\boldsymbol{\mathcal{Y}}^M_{JJ1}(\Omega_r)\right]\cdot\hat{\boldsymbol{\mathcal{J}}}(\boldsymbol{r})$$

Key point The Coulomb term shifts the momentum transfer to q' and introduces the kernel $\Theta(k, k_\gamma)$, while preserving the same multipole-operator structure as the weak matrix element.

Weak and Coulomb amplitudes in multipole form

Building the corrected β-decay rate from the weak amplitude, the one-photon Coulomb amplitude, and their interference

1 Standard Weak-only matrix element

Multipole expansion of the weak transition operator before Coulomb exchange

$$\langle f|\hat{\mathcal{H}}_w(\boldsymbol{r})|i\rangle = -\frac{G_F V_{ud}}{\sqrt{2}}\Bigg[-\sum_{J\geq 1}\sqrt{2\pi(2J+1)}(-i)^J \times \sum_{\lambda=\pm 1} l_\lambda^{\mp}\Big\langle f\Big|\lambda\,\hat{\mathcal{T}}^{\mathrm{mag}}_{J-\lambda}(q)+\hat{\mathcal{T}}^{\mathrm{el}}_{J-\lambda}(q)\Big|i\Big\rangle + \sum_{J\geq 0}\sqrt{4\pi(2J+1)}(-i)^J \times \Big[l_3^{\mp}\,\langle f|\hat{\mathcal{L}}_{J0}(q)|i\rangle - l_0^{\mp}\,\langle f|\hat{\mathcal{M}}_{J0}(q)\Big|i\Big\rangle\Big]\Bigg]$$

2 Coulomb matrix element

One-photon exchange contribution with shifted momentum q’ and Coulomb kernel

$$\langle f|\hat{\mathcal{H}}^c(\boldsymbol{r})|i\rangle = \mp\frac{G_F V_{ud}}{\sqrt{2}}\Bigg\{\int_0^\infty dk_\gamma\, k_\gamma^2\,\frac{1}{(2\pi)^3}\int d\Omega_\gamma \times\Bigg[-\sum_{J\geq 1}\sqrt{2\pi(2J+1)}(-i)^J \times \sum_{\lambda=\pm 1}\Theta(k,k_\gamma)\,L_\lambda^{c^{\mp}}\Big\langle f\Big|\lambda\hat{\mathcal{T}}^{\mathrm{mag}}_{J-\lambda}(q')+\hat{\mathcal{T}}^{\mathrm{el}}_{J-\lambda}(q')\Big|i\Big\rangle + \sum_{J\geq 0}\sqrt{4\pi(2J+1)}(-i)^J\Big[\Theta(k,k_\gamma)\,L_3^{c^{\mp}}\langle f|\hat{\mathcal{L}}_{J0}(q')|i\rangle - \Theta(k,k_\gamma)\,L_0^{c^{\mp}}\Big\langle f\Big|\hat{\mathcal{M}}_{J0}(q')|i\Big\rangle\Big]\Bigg]\Bigg\}$$

3 Rate through order αZ

Weak-only piece plus the Coulomb interference term; higher orders are omitted

$$\Big|\langle f|\hat{\mathcal{H}}^c_w(\boldsymbol{r})|i\rangle\Big|^2 = \frac{G_F^2\,|V_{ud}|^2}{2}\Bigg[\Big|\langle f|\hat{\mathcal{H}}_w(\boldsymbol{r})|i\rangle\Big|^2 \pm 2\mathcal{R}e\langle f|\hat{\mathcal{H}}_w(\boldsymbol{r})|i\rangle\;\langle f|\hat{\mathcal{H}}^c(\boldsymbol{r})|i\rangle^*\Bigg] + \mathcal{O}\big((\alpha Z)^2\big)$$

General β-decay rate with Coulomb corrections

Weak-only multipole response and the Coulomb contribution

1 Standard β-decay rate

$$dw^{c}_{\beta\mp} = \frac{1}{(2\pi)^5}(E_0 - E_e)\, kE_e\, dE_e\, d\Omega_k\, d\Omega_\nu \times \sum_{\text{lepton spins}} \frac{1}{2J_i+1} \sum_{M_i}\sum_{M_f} \left|\langle f|\hat{\mathcal{H}}^{c}_{w}(\boldsymbol{r})|i\rangle\right|^2$$

2 Coulomb-corrected β-decay rate

- Apply the Wigner–Eckart theorem to express the nuclear matrix elements in reduced form.
- The Coulomb traces retain the weak-trace structure, with an overall factor of $2E_e$ and the momentum shift $\boldsymbol{q} \to \boldsymbol{q}'$.

$$dw^{c}_{\beta\mp} = \frac{G_F^2|V_{ud}|^2}{2\pi^3}(E_0 - E_e)^2 kE_e\, dE_e\, \frac{d\Omega_k}{4\pi}\frac{d\Omega_\nu}{4\pi}\frac{4\pi}{2J_i+1}\left\{\mathcal{M}^0 + \mathcal{M}^c\right\}$$

3 Standard β-decay multipole expansion

$$\begin{aligned}\mathcal{M}^0 = \sum_{J\geq 1}\Bigg[&\Big(1-(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})\Big)\Big(\left|\langle J_f\|\hat{\mathcal{T}}^{\text{mag}}_J(q)\|J_i\rangle\right|^2 + \left|\langle J_f\|\hat{\mathcal{T}}^{\text{el}}_J(q)\|J_i\rangle\right|^2\Big) \pm \hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{\nu}}-\boldsymbol{\beta}) \\ &\times 2\,\text{Re}\Big(\langle J_f\|\hat{\mathcal{T}}^{\text{mag}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{T}}^{\text{el}}_J(q)\|J_i\rangle^*\Big)\Bigg] \\ +\sum_{J\geq 0}\Bigg[&\Big(1-\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta}+2(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})\Big)\left|\langle J_f\|\hat{\mathcal{L}}_J(q)\|J_i\rangle\right|^2 \\ &+\Big(1+\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta}\Big)\left|\langle J_f\|\hat{\mathcal{M}}_J(q)\|J_i\rangle\right|^2 - \hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{\nu}}+\boldsymbol{\beta}) \\ &\times 2\,\text{Re}\Big(\langle J_f\|\hat{\mathcal{L}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{M}}_J(q)\|J_i\rangle^*\Big)\Bigg]\end{aligned}$$

Standard momentum inside the leptonic traces

Standard momentum transfer inside nuclear multipole operators

4 The electrostatically corrected multipole expansion contribution

$$\begin{aligned}\mathcal{M}^c = \pm 4E_e \int_0^\infty dk_\gamma\, k_\gamma^2 \frac{1}{(2\pi)^3}\int d\Omega_{k_\gamma}\ \text{Re}\Big\{&\Theta(k,k_\gamma) \\ \times\Bigg[\sum_{J\geq 1}\Bigg[&\Big(1-(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}}')(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}}')\Big) \\ &\times\Big(\langle J_f\|\hat{\mathcal{T}}^{\text{mag}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{T}}^{\text{mag}}_J(q')\|J_i\rangle^* \\ &\quad+\langle J_f\|\hat{\mathcal{T}}^{\text{el}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{T}}^{\text{el}}_J(q')\|J_i\rangle^*\Big) \\ &\pm\Big(\hat{\boldsymbol{q}}'\cdot(\hat{\boldsymbol{\nu}}-\boldsymbol{\beta})\Big)\Big(\langle J_f\|\hat{\mathcal{T}}^{\text{mag}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{T}}^{\text{el}}_J(q')\|J_i\rangle^* \\ &\quad+\langle J_f\|\hat{\mathcal{T}}^{\text{el}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{T}}^{\text{mag}}_J(q')\|J_i\rangle^*\Big)\Bigg] \\ +\sum_{J\geq 0}\Bigg[&\Big(1-\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta}+2(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}}')(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}}')\Big) \\ &\times\langle J_f\|\hat{\mathcal{L}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{L}}_J(q')\|J_i\rangle^* \\ &+\Big(1+\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta}\Big)\langle J_f\|\hat{\mathcal{M}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{M}}_J(q')\|J_i\rangle^* \\ &-\Big(\hat{\boldsymbol{q}}'\cdot(\hat{\boldsymbol{\nu}}+\boldsymbol{\beta})\Big)\Big(\langle J_f\|\hat{\mathcal{L}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{M}}_J(q')\|J_i\rangle^* \\ &\quad+\langle J_f\|\hat{\mathcal{M}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{L}}_J(q')\|J_i\rangle^*\Big)\Bigg]\Bigg]\Big\}\end{aligned}$$

Finite nuclear charge form factor

Shifted momentum transfer inside nuclear multipole operators

Shifted momentum inside the leptonic traces

Three sources of Coulomb corrections

In the β-decay rate, the Coulomb contribution separates into three physically distinct terms

1 Correction to the rate

Using the shifted trace structures, the correction can be organized as

$$\mathcal{M}^{C} = \mathcal{M}^{C}_{\mathrm{FF}} + \mathcal{M}^{C}_{\mathrm{Tr}} + \mathcal{M}^{C}_{\mathrm{Op}}$$

2 Physical meaning of each contribution

$\mathcal{M}^{C}_{\mathrm{FF}}$

Finite nuclear charge form factor

$\mathcal{M}^{C}_{\mathrm{Tr}}$

Shifted momentum inside the leptonic traces

$\mathcal{M}^{C}_{\mathrm{Op}}$

Shifted momentum transfer inside nuclear multipole operators

This split examines the contributions from the charge form factor, the trace expansion, and the nuclear operator.

Leptonic traces with Coulomb corrections

First expand the Coulomb-shifted momentum direction, then insert it into the traces.

1 Coulomb-shifted direction

To first order in k_γ/q , the exchanged Coulomb photon changes the direction entering the angular traces.

$$\hat{\boldsymbol{q}}' \simeq \hat{\boldsymbol{q}} - \frac{\boldsymbol{k}_\gamma}{q} + \frac{\boldsymbol{k}_\gamma \cdot \hat{\boldsymbol{q}}}{q}\,\hat{\boldsymbol{q}}$$

2 Resulting leptonic traces

Substituting this expansion into the traces gives the angular structures summarized below.

TABLE I. Leptonic traces with Coulomb-interaction corrections for $\beta^{\mp}$ decay to $\mathcal{O}(k_\gamma)$, where $\beta = \frac{k}{E_e}$, $\hat{\nu} = \frac{\nu}{E_\nu}$, and $\hat{q} = \frac{q}{q}$.

Summand	Coulomb interaction correction to $\mathcal{O}(k_\gamma)$
$\frac{1}{2}\left(l \cdot L^{C*} - l_3 L_3^{C*}\right)$	$2E_e\left[1 - (\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}}) + \frac{1}{q}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\boldsymbol{k}_\gamma) + (\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\hat{\boldsymbol{\nu}}\cdot\boldsymbol{k}_\gamma) - 2(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}})\right]$
$l_0 L_0^{C*}$	$2E_e[1 + \hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta}]$
$l_3 L_3^{C*}$	$2E_e\left[1 - \hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta} + 2(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}}) - \frac{2}{q}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\boldsymbol{k}_\gamma) + (\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\hat{\boldsymbol{\nu}}\cdot\boldsymbol{k}_\gamma) - 2(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}})\right]$
$-\left(l_3 L_0^{C*}\right)$	$2E_e\left[-\hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{\nu}}+\boldsymbol{\beta}) + \frac{1}{q}(\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{\nu}} + \boldsymbol{k}_\gamma\cdot\boldsymbol{\beta}) - \frac{\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}}}{q}(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{\nu}} + \hat{\boldsymbol{q}}\cdot\boldsymbol{\beta})\right]$
$-\frac{i}{2}(l \times L^{C*})_3$	$\mp 2E_e\left[-\hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{\nu}}-\boldsymbol{\beta}) + \frac{1}{q}(\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{\nu}} - \boldsymbol{k}_\gamma\cdot\boldsymbol{\beta}) - \frac{\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}}}{q}(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{q}}\cdot\boldsymbol{\beta})\right]$
$-\left(l_0 L_3^{C*}\right)$	$2E_e\left[-\hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{\nu}}+\boldsymbol{\beta}) + \frac{1}{q}(\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{\nu}} + \boldsymbol{k}_\gamma\cdot\boldsymbol{\beta}) - \frac{\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}}}{q}(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{\nu}} + \hat{\boldsymbol{q}}\cdot\boldsymbol{\beta})\right]$

Takeaway: The Coulomb-trace results have the same structure as the non-Coulomb trace, with a factor $2E_e$, the momentum shift q → q′, and an expansion to $\mathcal{O}(k_\gamma)$.

Form-factor contribution

Nuclear charge form factor: point-Coulomb term + leading finite-size correction

1 Form factor multipole expression

$$
\begin{aligned}
\mathcal{M}^{c}_{\mathrm{FF}} = & \pm 4E_e \int_0^\infty dk_\gamma k_\gamma^2 \frac{1}{(2\pi)^3} \int d\Omega_\gamma \mathcal{R}e\Theta(k,k_\gamma) \\
& \times \Bigg\{ \sum_{J\geq 1} \Bigg[\Big((1-(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})) \Big) \\
& \times \Big(\big|\langle J_f \| \hat{\mathcal{T}}^{mag}_J(q) \| J_i \rangle\big|^2 + \big|\langle J_f \| \hat{\mathcal{T}}^{\mathrm{el}}_J(q) \| J_i \rangle\big|^2 \Big) \\
& \pm \hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{\nu}}-\boldsymbol{\beta}) 2\mathcal{R}e\langle J_f \| \hat{\mathcal{T}}^{\mathrm{mag}}_J(q) \| J_i \rangle \langle J_f \| \hat{\mathcal{T}}^{\mathrm{el}}_J(q) \| J_i \rangle^* \Bigg] \\
& + \sum_{J\geq 0} \Bigg[\Big(1-\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta} + 2(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}}) \Big) \big|\langle J_f \| \hat{\mathcal{L}}_J(q) \| J_i \rangle\big|^2 \\
& + (1+\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta}) \big|\langle J_f \| \hat{\mathcal{M}}_J(q) \| J_i \rangle\big|^2 \\
& - \hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{\nu}}+\boldsymbol{\beta}) 2\mathcal{R}e\langle J_f \| \hat{\mathcal{L}}_J(q) \| J_i \rangle \langle J_f \| \hat{\mathcal{M}}_J(q) \| J_i \rangle^* \Bigg] \Bigg\} \\
& + \mathcal{O}(k_\gamma^2)
\end{aligned}
$$

$\mathcal{M}^0$

2 Integrate over the photon momentum and solid angle, including nuclear form factor, and isolate the point-Coulomb contribution.

Fermi function to leading order in $\alpha Z/k$ and isolate the point-Coulomb term

$$
\pm 4E_e \int_0^\infty dk_\gamma k_\gamma^2 \frac{1}{(2\pi)^3} \int d\Omega_\gamma \mathcal{R}e\Theta(k,k_\gamma) = \pm\pi\alpha Z \frac{E_e}{k} \mp 4\alpha Z a E_e
$$

$$
\pm\pi\alpha Z \frac{E_e}{k} = F^{\mp}(Z,E_e) - 1 + \mathcal{O}\big((\alpha Z/k)^2\big)
$$

3 Final form-factor correction

$$
\mathcal{M}^{c}_{\mathrm{FF}} = \Big[F^{\mp}(Z,E_e) - 1 \mp 4\alpha Z a E_e \Big] \mathcal{M}^0
$$

Takeaway: The Fermi function is recovered, together with a finite-size Coulomb correction $4\alpha Z a E_e$ multiplying the standard amplitude.

Trace-shift contribution

Momentum-transfer shift in the leptonic trace: angular structures → finite-size correction

1 Full trace-shift expression

$$
\begin{aligned}
\mathcal{M}^{\tau}_{\mathrm{Tr}} &= \pm 4E_e \int_0^\infty dk_\gamma k_\gamma^2 \frac{1}{(2\pi)^3} \int d\Omega_\gamma \mathcal{R}e\Theta(k,k_\gamma) \\
&\times \frac{1}{q}\Bigg\{ \sum_{J\geq 1} \Bigg[\Big((\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\boldsymbol{k}_\gamma) + (\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\hat{\boldsymbol{\nu}}\cdot\boldsymbol{k}_\gamma) \\
&- 2(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}}) \Big) \Big(|\langle J_f \| \hat{T}^{mag}_J(q) \| J_i \rangle|^2 \\
&+ |\langle J_f \| \hat{T}^{el}_J(q) \| J_i \rangle|^2 \Big) \\
&\pm \Big(-\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{\nu}} + \boldsymbol{k}_\gamma\cdot\boldsymbol{\beta} + (\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}})(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{q}}\cdot\boldsymbol{\beta}) \Big) \\
&\times 2\mathcal{R}e\langle J_f \| \hat{T}^{mag}_J(q) \| J_i \rangle \langle J_f \| \hat{T}^{el}_J(q) \| J_i \rangle^* \Bigg] \\
&+ \sum_{J\geq 0} \Bigg[-2\Big((\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\boldsymbol{k}_\gamma) + (\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\hat{\boldsymbol{\nu}}\cdot\boldsymbol{k}_\gamma) \\
&- 2(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}}) \Big) |\langle J_f \| \hat{\mathcal{L}}_J(q) \| J_i \rangle|^2 \\
&+ \Big(\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{\nu}} + \boldsymbol{k}_\gamma\cdot\boldsymbol{\beta} - (\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}})(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{\nu}} + \hat{\boldsymbol{q}}\cdot\boldsymbol{\beta}) \Big) \\
&\times 2\mathcal{R}e\langle J_f \| \hat{\mathcal{L}}_J(q) \| J_i \rangle \langle J_f \| \hat{\mathcal{M}}_J(q) \| J_i \rangle^* \Bigg] \Bigg\} + \mathcal{O}(k_\gamma^2)
\end{aligned}
$$

2 Angular integrals

The momentum-shift expansion generates the angular structures

$$(\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}})\,, \quad (\boldsymbol{\beta}\cdot\boldsymbol{k}_\gamma)\,, \quad \text{and} \quad (\hat{\boldsymbol{\nu}}\cdot\boldsymbol{k}_\gamma)\,.$$

$$
\begin{aligned}
&\pm 4E_e \int_0^\infty dk_\gamma\, k_\gamma^2 \frac{1}{(2\pi)^3} \int d\Omega_\gamma\, \mathcal{R}e\Theta(k,k_\gamma) \\
&\quad \times \Big\{ (\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}}),\, (\boldsymbol{\beta}\cdot\boldsymbol{k}_\gamma),\, (\hat{\boldsymbol{\nu}}\cdot\boldsymbol{k}_\gamma) \Big\} \\
&\quad = \mp \alpha Z a k E_e \frac{16}{3} \Big\{ (\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{q}}),\, (\boldsymbol{\beta}\cdot\hat{\boldsymbol{k}}),\, (\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{k}}) \Big\}
\end{aligned}
$$

Then use $\hat{\boldsymbol{k}} = (E_e/k)\,\boldsymbol{\beta}$ to express the correction in β-dependent form.

3 Final trace-shift correction

Second Coulomb correction associated with the leptonic trace shift.

$$
\begin{aligned}
\mathcal{M}^{\tau}_{\mathrm{Tr}} &= \mp \alpha Z a \frac{E_e^2}{q} \frac{16}{3} \Bigg\{ \sum_{J\geq 1} \Bigg[\Big((\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})\beta^2 + (\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta}) \\
&- 2(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})^2 \Big) \Big(|\langle J_f \| \hat{T}^{mag}_J(q) \| J_i \rangle|^2 \\
&+ |\langle J_f \| \hat{T}^{el}_J(q) \| J_i \rangle|^2 \Big) \\
&\pm \Big(\beta^2 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{\nu}} + (\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{q}}\cdot\boldsymbol{\beta}) \Big) \\
&\times 2\mathcal{R}e\Big(\langle J_f \| \hat{T}^{mag}_J(q) \| J_i \rangle \langle J_f \| \hat{T}^{el}_J(q) \| J_i \rangle^* \Big) \Bigg] \\
&+ \sum_{J\geq 0} \Bigg[-2\Big((\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})\beta^2 + (\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta}) - 2(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})^2 \Big) \\
&\times |\langle J_f \| \hat{\mathcal{L}}_J(q) \| J_i \rangle|^2 + \Big(\beta^2 + \boldsymbol{\beta}\cdot\hat{\boldsymbol{\nu}} - (\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{q}}\cdot\boldsymbol{\beta}) \Big) \\
&\times 2\mathcal{R}e\Big(\langle J_f \| \hat{\mathcal{L}}_J(q) \| J_i \rangle \langle J_f \| \hat{\mathcal{M}}_J(q) \| J_i \rangle^* \Big) \Bigg] \Bigg\}
\end{aligned}
$$

Takeaway: The shift in the leptonic trace converts the momentum-shift structures into an $\alpha Z a E_e^2/q$ correction multiplying the multipole responses.

Operator-shift contribution

Coulomb-induced shift of the momentum transfer inside the nuclear multipole operators

1 Multipole-operator expansion

Expand the multipole operators under the momentum-transfer shift and use the spherical-Bessel derivative relation.

$$\langle J_f\|\{\mathcal{M}_J,\mathcal{L}_J,\mathcal{T}^{el}_J,\mathcal{T}^{mag}_J\}(q')\|J_i\rangle = \langle J_f\|\{\mathcal{M}_J,\mathcal{L}_J,\mathcal{T}^{el}_J,\mathcal{T}^{mag}_J\}(q)\|J_i\rangle - (\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}})\frac{d}{dq}\langle J_f\|\{\mathcal{M}_J,\mathcal{L}_J,\mathcal{T}^{el}_J,\mathcal{T}^{mag}_J\}(q)\|J_i\rangle + \mathcal{O}(k_\gamma^2)$$

$$\frac{d}{dq}\left\langle J_f\middle\|\{\mathcal{M}_J,\mathcal{L}_J,\mathcal{T}^{el}_J,\mathcal{T}^{mag}_J\}(q)\middle\|J_i\right\rangle = \begin{cases}\frac{J}{q}\left\langle J_f\middle\|\{\mathcal{M}_J,\mathcal{T}^{mag}_J\}(q)\middle\|J_i\right\rangle \\ \frac{J-1}{q}\left\langle J_f\middle\|\{\mathcal{L}_J,\mathcal{T}^{el}_J\}(q)\middle\|J_i\right\rangle\end{cases} \times\left[1+\mathcal{O}((qR)^2)\right]$$

2 Full operator-shift expression

Insert the derivative relations into the shifted multipole responses.

$$\begin{aligned}\mathcal{M}^{e}_{\mathrm{Op}} = {}& \mp 4E_e\int_0^\infty dk_\gamma\,k_\gamma^2\frac{1}{(2\pi)^3}\int d\Omega_\gamma \mathcal{R}e\,\Theta(k,k_\gamma)(\boldsymbol{k}_\gamma\cdot\hat{\boldsymbol{q}}) \\ &\times\Bigg\{\sum_{J\geq1}\Big[\big(1-(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})\big) \\ &\times\left(\frac{J}{q}|\langle J_f\|\hat{\mathcal{T}}^{\mathrm{mag}}_J(q)\|J_i\rangle|^2+\frac{J-1}{q}|\langle J_f\|\hat{\mathcal{T}}^{\mathrm{el}}_J(q)\|J_i\rangle|^2\right) \\ &\pm\hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{v}}-\boldsymbol{\beta}) \\ &\times\left(\frac{J-1}{q}\langle J_f\|\hat{\mathcal{T}}^{\mathrm{mag}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{T}}^{\mathrm{el}}_J(q)\|J_i\rangle^*\right. \\ &\left.+\frac{J}{q}\langle J_f\|\hat{\mathcal{T}}^{\mathrm{el}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{T}}^{\mathrm{mag}}_J(q)\|J_i\rangle^*\right)\Big] \\ &+\sum_{J\geq0}\Big[\big(1-\hat{\boldsymbol{v}}\cdot\boldsymbol{\beta}+2(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})\big) \\ &\times\frac{J-1}{q}|\langle J_f\|\hat{\mathcal{L}}_J(q)\|J_i\rangle|^2 \\ &+(1+\hat{\boldsymbol{v}}\cdot\boldsymbol{\beta})\frac{J}{q}|\langle J_f\|\hat{\mathcal{M}}_J(q)\|J_i\rangle|^2-\hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{v}}+\boldsymbol{\beta}) \\ &\times\left(\frac{J}{q}\langle J_f\|\hat{\mathcal{L}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{M}}_J(q)\|J_i\rangle^*\right. \\ &\left.+\frac{J-1}{q}\langle J_f\|\hat{\mathcal{M}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{L}}_J(q)\|J_i\rangle^*\right)\Big]\Bigg\} \\ &+\mathcal{O}(k_\gamma^2)+\mathcal{O}((qR)^2)\end{aligned}$$

3 Photon integral

Angular and exchanged-photon momentum integrations

$$\begin{aligned}\mp 4E_e\int_0^\infty dk_\gamma\,k_\gamma^2\frac{1}{(2\pi)^3}\int d\Omega_\gamma\,\Theta(k,k_\gamma)\,(k_\gamma\cdot\hat{q}) &= \pm\alpha Z a\frac{E_e^2}{q}\frac{16}{3}\,(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})\end{aligned}$$

4 Final operator-shift correction

$$\begin{aligned}\mathcal{M}^{e}_{\mathrm{Op}} = {}& \pm\alpha Z a E_e^2\frac{1}{q}\frac{16}{3}(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}}) \\ &\times\Bigg\{\sum_{J\geq1}\Big[\big(1-(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})\big) \\ &\times\left(J|\langle J_f\|\hat{\mathcal{T}}^{\mathrm{mag}}_J(q)\|J_i\rangle|^2+(J-1)|\langle J_f\|\hat{\mathcal{T}}^{\mathrm{el}}_J(q)\|J_i\rangle|^2\right) \\ &\pm\hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{v}}-\boldsymbol{\beta}) \\ &\times(2J-1)\mathcal{R}e\langle J_f\|\hat{\mathcal{T}}^{\mathrm{mag}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{T}}^{\mathrm{el}}_J(q)\|J_i\rangle^*\Big] \\ &+\sum_{J\geq0}\Big[\big(1-\hat{\boldsymbol{v}}\cdot\boldsymbol{\beta}+2(\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{q}})(\boldsymbol{\beta}\cdot\hat{\boldsymbol{q}})\big) \\ &\times(J-1)|\langle J_f\|\hat{\mathcal{L}}_J(q)\|J_i\rangle|^2 \\ &+(1+\hat{\boldsymbol{v}}\cdot\boldsymbol{\beta})J|\langle J_f\|\hat{\mathcal{M}}_J(q)\|J_i\rangle|^2-\hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{v}}+\boldsymbol{\beta}) \\ &\times(2J-1)\mathcal{R}e\langle J_f\|\hat{\mathcal{L}}_J(q)\|J_i\rangle\langle J_f\|\hat{\mathcal{M}}_J(q)\|J_i\rangle^*\Big]\Bigg\} \\ &+\mathcal{O}(k_\gamma^2)+\mathcal{O}((qR)^2)\end{aligned}$$

Standard angular and multipole dependence

Standard multipole operator

Dependence on arbitrary multipole rank J

Takeaway: **The operator shift converts the multipole momentum dependence into an $\alpha Z a E_e^2/q$ correction that multiplies the standard angular dependence and nuclear multipole responses for arbitrary multipole rank J.**

Final β decay for arbitrary transition

Separate point-Coulomb physics from the finite-size nuclear-structure Coulomb correction.

Central result:

$$dw^{c}_{\beta\mp} = \frac{G_F^2 |V_{ud}|^2}{2\pi^3} (E_0 - E_e)^2 \; k \; E_e \; dE_e \; \frac{d\Omega_k}{4\pi} \frac{d\Omega_\nu}{4\pi} \frac{4\pi}{2J_i + 1}$$
$$\times \left\{ \left[F^{\mp}(Z, E_e) \mp 4\alpha Z \, a E_e \right] \mathcal{M}^0 + \mathcal{M}^{c}_{\mathrm{Tr}} + \mathcal{M}^{c}_{\mathrm{Op}} \right\}$$

Universal scale

$$\epsilon_{EC} = \alpha Z E_e R$$

Arbitrary J

All terms live in the multipole basis

Application Examples

GT and unique 2^- as limits

Core message: **finite-size electrostatic effects are perturbative, model-independent, and compatible with modern nuclear calculations.**

Allowed Gamow–Teller transition

Long-Wavelength Limit: $J = 1$ Response with the Finite-Size Coulomb Correction

1 Allowed GT multipoles

Only the leading longitudinal operator contributes at lowest order

$$\hat{\mathcal{T}}^{\mathrm{el}}_{1(\mathrm{LW})}(q) = \sqrt{2}\,\hat{\mathcal{L}}_1(q)$$

2 Weak-only response

The angular factor gives the familiar GT correlation structure

$$\mathcal{M}^{0}_{(\mathrm{LW,GT})} = 3\left(1 - \tfrac{1}{3}\hat{\nu}\cdot\beta\right)\left|\langle J_f\|\hat{\mathcal{L}}_1(q)\|J_i\rangle\right|^2$$

3 Shift terms vanish in LW GT

$$\mathcal{M}^{\mathcal{C}}_{\mathrm{Tr}(\mathrm{LW,GT})} = \mathcal{M}^{\mathcal{C}}_{\mathrm{Op}(\mathrm{LW,GT})} = 0$$

4 Finite-size corrected GT rate

Substitute into the arbitrary-transition rate with

$$a = R/\sqrt{6} \quad \text{and} \quad \epsilon_{EC} = \alpha Z E_e R$$

$$dw^{\sigma}_{\beta^{\mp}(\mathrm{LW,GT})} = \frac{G_F^2|V_{ud}|^2}{2\pi^3}(E_0 - E_e)^2\, k\, E_e\, dE_e\, \frac{d\Omega_k}{4\pi}\,\frac{d\Omega_\nu}{4\pi} \times \frac{4\pi}{2J_i+1}\left|\langle J_f\|\hat{\mathcal{L}}_1(q)\|J_i\rangle\right|^2 3\left(1 - \frac{1}{3}\hat{\nu}\cdot\beta\right) \times \left\{F^{\mp}(Z,E_e) + \frac{4}{\sqrt{6}}\epsilon_{EC}\right\}$$

Standard β-decay rate

For the first time, the finite-size electrostatic correction to the β-decay rate is obtained independently of the nuclear model.

Takeaway: in the Long-Wavelength allowed Gamow–Teller limit, $M^{\mathcal{C}}_{\mathrm{Tr}} = M^{\mathcal{C}}_{\mathrm{Op}} = \mathbf{0}$, the standard Fermi function, angular dependence, and multipole structure are preserved, while the new finite-size correction enters through the form factor. A model-independent finite-size electrostatic correction to the Gamow–Teller $\beta^{\mp}$-decay rate, derived for the first time.

Unique first-forbidden transition

Long-wavelength limit: $J = 2^-$ response with finite-size Coulomb correction

1 Operator identity

In the LW expansion, the electric multipole is written in terms of the longitudinal operator.

$$\hat{\mathcal{T}}^{\mathrm{el}}_{2(\mathrm{LW})}(q) = \sqrt{\frac{3}{2}}\,\hat{\mathcal{L}}_2(q)$$

2 LW response and finite-size shift terms

$$\mathcal{M}^0_{(\mathrm{LW},\,2^-)} = \frac{5}{2}\left[1 - \frac{2}{5}\,\hat{\nu}\cdot\boldsymbol{\beta} + \frac{1}{5}(\hat{\nu}\cdot\hat{q})(\boldsymbol{\beta}\cdot\hat{q})\right] \times \left|\langle J_f\|\hat{\mathcal{L}}_2(q)\|J_i\rangle\right|^2$$

$$\mathcal{M}^c_{\mathrm{Tr}(\mathrm{LW},\,2^-)} = \pm\alpha Z a\,\frac{E_e^2}{q}\,\frac{16}{3}\left[(\hat{\nu}\cdot\hat{q})\boldsymbol{\beta}^2 + (\boldsymbol{\beta}\cdot\hat{q})(\hat{\nu}\cdot\boldsymbol{\beta}) - 2(\hat{\nu}\cdot\hat{q})(\boldsymbol{\beta}\cdot\hat{q})^2\right]\frac{1}{2}\left|\langle J_f\|\hat{\mathcal{L}}_2(q)\|J_i\rangle\right|^2$$

$$\mathcal{M}^c_{\mathrm{Op}(\mathrm{LW},\,2^-)} = \pm\alpha Z a\,\frac{E_e^2}{q}\,\frac{16}{3}(\boldsymbol{\beta}\cdot\hat{q})\left[1 - \frac{2}{5}\,\hat{\nu}\cdot\boldsymbol{\beta} + \frac{1}{5}(\hat{\nu}\cdot\hat{q})(\boldsymbol{\beta}\cdot\hat{q})\right] \times \frac{5}{2}\left|\langle J_f\|\hat{\mathcal{L}}_2(q)\|J_i\rangle\right|^2$$

3 Finite-size corrected unique 2^- rate

Combining the weak-only response with the Tr and Op Coulomb terms gives the final rate.

$$dw^c_{\beta^\mp(\mathrm{LW},\,2^-)} = \frac{G_F^2|V_{ud}|^2}{2\pi^3}(E_0 - E_e)^2 k E_e\,dE_e\,\frac{d\Omega_k}{4\pi}\,\frac{d\Omega_\nu}{4\pi}\,\frac{4\pi}{2J_i+1} \times \left|\langle J_f\|\hat{\mathcal{L}}_2(q)\|J_i\rangle\right|^2 \left(F^\mp(Z,E_e) \mp \frac{4}{\sqrt{6}}\,\varepsilon_{EC}\right)$$

Standard β-decay rate

$$\times \frac{5}{2}\left\{\left[1 - \frac{2}{5}\hat{\nu}\cdot\boldsymbol{\beta} + \frac{1}{5}(\hat{\nu}\cdot\hat{q})(\boldsymbol{\beta}\cdot\hat{q})\right] \pm \frac{16}{3\sqrt{6}}\,\varepsilon_{EC}\,\frac{E_e}{q}\left[\frac{1}{5}(\hat{\nu}\cdot\hat{q})\boldsymbol{\beta}^2 - \frac{1}{5}(\boldsymbol{\beta}\cdot\hat{q})(\hat{\nu}\cdot\boldsymbol{\beta}) + (\boldsymbol{\beta}\cdot\hat{q}) - \frac{1}{5}(\hat{\nu}\cdot\hat{q})(\boldsymbol{\beta}\cdot\hat{q})^2\right]\right\}$$

New finite-size electrostatic correction to the β-decay rate

Takeaway: The unique 2^- transition retains the LW response together with the nonzero M^c_{Tr} and M^c_{Op} finite-size Coulomb shift terms. The final $\beta^\mp$-decay rate includes these corrections while preserving the standard Fermi function, angular dependence, and multipole structure.

Finite-size Coulomb correction estimates

1 Leading finite-size scale

Use the endpoint electron energy to estimate the maximal finite-size Coulomb correction:

$$\epsilon_{EC}^{\max} = \alpha Z E_e^{\max} R_{\rm ch}/(\hbar c)$$

2 Fermi-function estimate

Using the leading point-Coulomb expansion at the endpoint:

$$F^{\mp}(Z, E_e^{\max}) \simeq 1 \pm \pi\alpha Z \frac{E_e^{\max}}{k_{\max}} \qquad k_{\max} = \sqrt{(E_e^{\max})^2 - m_e^2}$$

TABLE II. Maximal Coulomb-correction, suppression, and Fermi-function estimates.

Nucleus	$\epsilon_{EC}^{\max}$	$\epsilon_{EC}^{\max}\,\alpha Z$	$\epsilon_{EC}^{\max} E_e^{\max} R_{\rm ch}/(\hbar c)$	$F^{\mp}(Z, E_e^{\max}) - 1$
$^{3}_{1}$H	3.45×10^{-5}	2.51×10^{-7}	1.63×10^{-7}	$+8.73 \times 10^{-2}$
$^{6}_{2}$He	6.14×10^{-4}	8.97×10^{-6}	2.59×10^{-5}	$+4.62 \times 10^{-2}$
$^{8}_{3}$Li	4.29×10^{-3}	9.38×10^{-5}	8.39×10^{-4}	$+6.88 \times 10^{-2}$
$^{11}_{6}$C	7.57×10^{-4}	3.31×10^{-5}	1.31×10^{-5}	-1.47×10^{-1}
$^{15}_{8}$O	1.78×10^{-3}	1.04×10^{-4}	5.44×10^{-5}	-1.88×10^{-1}
$^{23}_{10}$Ne	5.26×10^{-3}	3.84×10^{-4}	3.80×10^{-4}	$+2.31 \times 10^{-1}$
$^{32}_{18}$Ar	2.37×10^{-2}	3.11×10^{-3}	4.26×10^{-3}	-4.13×10^{-1}
$^{38}_{19}$K	1.30×10^{-2}	1.80×10^{-3}	1.22×10^{-3}	-4.38×10^{-1}
$^{187}_{75}$Re	7.64×10^{-3}	4.18×10^{-3}	1.07×10^{-4}	$+1.95 \times 10^{1}$

Conclusion: **Point-Coulomb effects are incorporated through the Fermi function $F(Z, E_e)$, while the additional finite-size correction is governed by $\epsilon_{EC}^{\max}$ and remains at or below the few-percent level. For the nuclei considered, $\epsilon_{EC}^{\max}$ ranges from 10^{-5} to 10^{-2} ,with higher-order contributions remaining smaller. For ^{187}Re, the leading-order estimate $F - 1 \simeq 19.5$ becomes very large because $k_{\max}$ is very small, indicating that the exact Fermi function should be used. Nevertheless, the finite-size correction remains small.**

Discussion and Conclusions

Developed a model-independent electrostatic finite-size correction for Arbitrary β decay transitions.

- Point-Coulomb effects are isolated in the Fermi function.
- The leading nuclear-structure-dependent electrostatic correction scales as $\epsilon_{EC} = \alpha Z E_e R$
- The Coulomb correction separates into Form-Factor, Trace-shift, and Operator-shift terms, each containing one exchanged photon.
- The result is formulated directly in the standard multipole basis.
- The leading finite-size correction can be experimentally relevant, while higher-order terms remain parametrically smaller.
- Allowed GT and unique first-forbidden transitions are calculated in the long-wavelength (LW) limit.
- Higher-order terms in ϵ_{EC} will be evaluated in the near future.
- Applying this framework to Tritium (allowed decay) and $^{187}\mathrm{Re}$, for which ϵ_{EC} is very small, work in progress.

Thank you:
Ayala Glick-Magid and Leendert Hayen

Corrections of order ϵ_{EC} can imitate or obscure new physics in precision β decay.