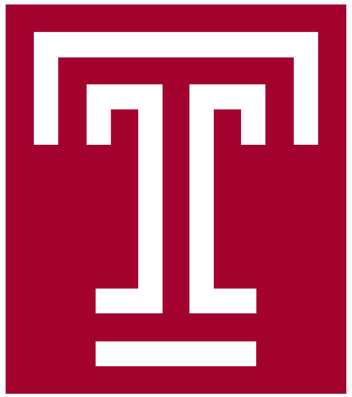
Pion and Kaon Mellin Moments From Lattice QCD

Joshua Miller

Temple University

Preliminary work with:
J. Torsiello (Temple), **K. Cichy** (Adam Mickiewicz University),
M. Constantinou (Temple), **J. Delmar** (ANL)



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Outline

- Theoretical Formulation
- Lattice Formulation (focus of this talk)
  - Background
  - Lattice Methodology
  - Results
    - Matrix Elements
    - Double Ratio
    - Fixed z Analysis and combined Analysis
    - DGLAP Evolution
    - Results Comparison
    - SU(3) Symmetry Breaking
  - Summary and Future Work

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Pion and Kaon PDFs from Lattice QCD via Large Momentum Effective Theory and Short-Distance Factorization

Joshua Miller,[1,*] Joseph Torsiello,[1,†] Isaac Anderson,[2,1] Krzysztof Cichy,[3] Martha Constantinou,[1,‡] Joseph Delmar,[1] and Sarah Lampreich[1]

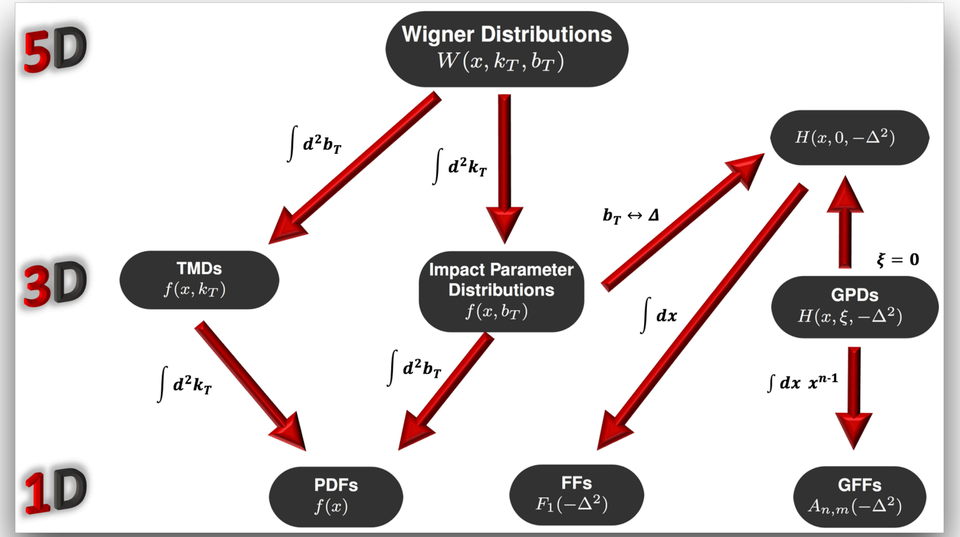
Mellin Moments of Pion and Kaon Unpolarized PDFs from Nonlocal Operators in Lattice QCD

Joshua Miller,[1,*] Joseph Torsiello,[1] Krzysztof Cichy,[2] Martha Constantinou,[1,†] and Joseph Delmar[1,3]

(To be submitted)

Parton Distribution Functions

- **PDFs are 1D:**
  - Describe the probability a quark or gluon carry a fraction x of the hadron's longitudinal momentum



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  - … but meson PDFs are relatively unknown
    - Pion information is extracted from pion-induced Drell-Yan experiments
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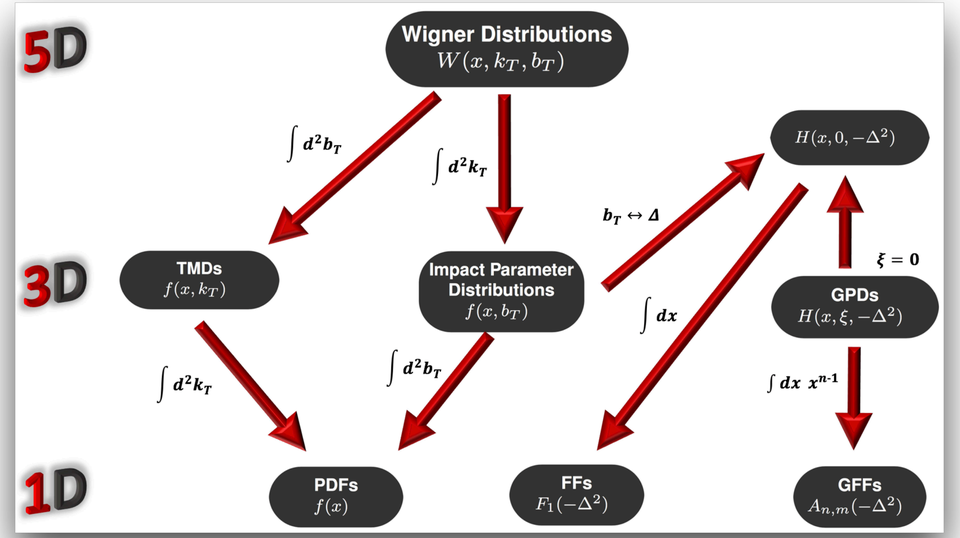

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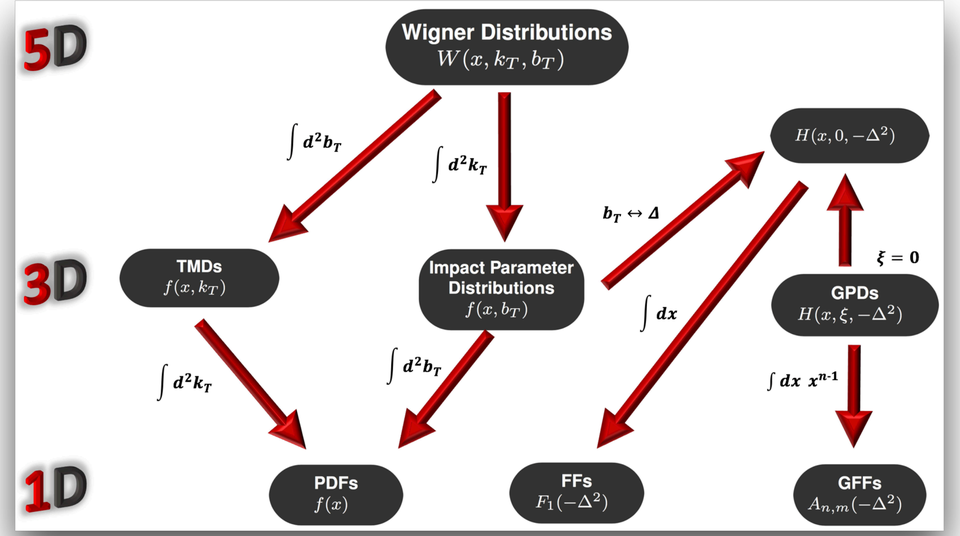
- **Mellin Moments**
  - Embedded information in the PDF
  - As n increases, the larger x region increases in weight



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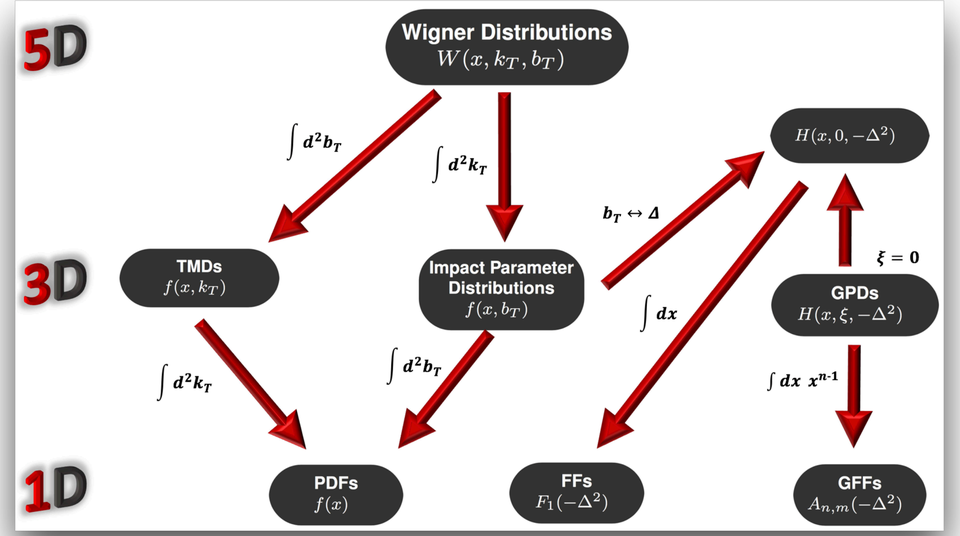

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Multiple methods to calculate moments in Lattice QCD

Mellin Moments from Lattice QCD

- Traditionally, moments are calculated from local operators

$$\partial_\mu \psi = \frac{\psi(x+\hat{\mu}) - \psi(x-\hat{\mu})}{2a}$$

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- More derivatives leads to higher order Mellin moments
  - Operator mixing becomes larger at higher moments
  - Moments decrease as n increases
  - $\langle x^3 \rangle$ **[Alexandrou, et al. Phys. Rev. D 104, 054504 (2021)] [Alexandrou, et al. arXiv 2605.29998 [hep-lar] (2026)]***
  - $\langle x^5 \rangle$ **[Francis, et al. Phys. Rev. D 113, 074520 (2026)] [Francis, et al. Phys. Rev. Lett. 136, 171903 (2026)]**

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Difficult to extract higher moments

Mellin Moments from Lattice QCD

- Recent developments have allowed full $x-$dependence of PDFs to be calculated from non-local matrix elements

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  - Large Momentum Effective Theory (LaMET)
    - Light-cone distribution extracted in $P_3 \to \infty$ limit
      - Matching kernel $\mathcal{O}\left(\frac{\Lambda_{QCD}^2}{x^2 P_3^2}, \frac{\Lambda_{QCD}^2}{(1-x)^2 P_3^2}\right)$
    - $x < 0$ region not well constrained
    - Not reliable to extract Mellin moments

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    - $x < 0$ region not well constrained
    - Not reliable to extract Mellin moments
  - Short Distance Expansion $\to$ inferred from OPE
    - Reliable for $z \lesssim 0.3$ fm
      - Matching kernel $\mathcal{O}(z^2 \Lambda_{QCD}^2)$
    - Utilize multiple momentum boosts to extract Mellin moments

Lattice Setup

- $N_f = 2 + 1 + 1$ Twisted mass fermions with a clover term

Parameters							
Ensemble	β	a [fm]	volume $L^3 \times T$	N_f	m_π [MeV]	Lm_π	L [fm]
cA211.32	1.726	0.093	$32^3 \times 64$	u, d, s, c	260	4	3.0

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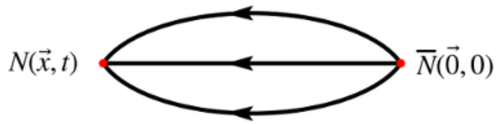
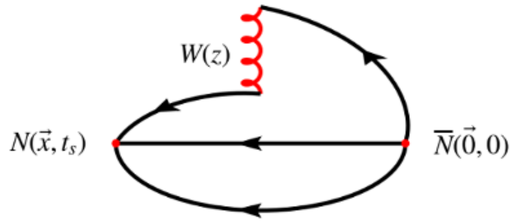


$$\langle N(P) \,|\, N(P) \rangle$$

- Calculate 2pt and 3pt Correlation functions at both $+P_3$ and $-P_3$
  - As P_3 increases, higher statistics needed



$$\langle N(P_f) \,|\, \bar{\Psi}(z)\, \Gamma\, \mathscr{W}(z,0) \Psi(0) \,|\, N(P_i) \rangle$$

P_3 [GeV]	0	± 0.41	± 0.83	± 1.25	± 1.66	± 2.07
t_s/a	12	12	12	$10^\star$, $12^\dagger$	10	10
N_{confs}	1,198	1,198	1,198	1,198	1,198	1,198
N^π_{src}	1	8	8	56	84	400
N^K_{src}	1	8	8	8	48	300
N^π_{tot}	1,198	9,584	9,584	67,088	100,632	479,200
N^K_{tot}	1,198	9,584	9,584	9,584	57,504	359,400

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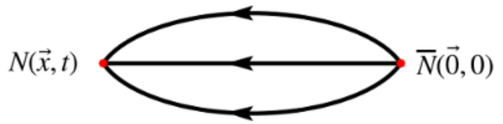

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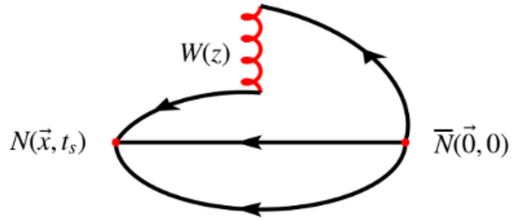

$\langle N(P_f) \,|\, \bar{\Psi}(z)\, \Gamma\, \mathscr{W}(z,0) \Psi(0) \,|\, N(P_i) \rangle$

- Large t_s needed to ensure ground state is extracted
  - t_s decreased for larger P_3 to ensure good signal to noise ratio
- Kaon requires less statistics since signal is $\sim e^{-m}$

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Lattice Methodology

- Calculate 2pt and 3pt correlation functions
- Formulate a ratio

$$R_\mu(z, p; t_s, \tau) = \frac{C_\mu^{3pt}(z, p; t_s, \tau)}{C^{2pt}(p; t_s)}$$

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- Renormalize

$$\nu = P \cdot z$$

$$\mathcal{M}(\nu, z^2) = \frac{\Pi_0^R(\nu, z^2)/\Pi_0^R(\nu,0)}{\Pi_0^R(0,z^2)/\Pi_0^R(0,0)} = \frac{\Pi_0(\nu, z^2)/\Pi_0(\nu,0)}{\Pi_0(0,z^2)/\Pi_0(0,0)}$$

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- Relate renormalized Matrix Elements to Mellin moments by Wilson Coefficients

$$\mathscr{M}(\nu,z^2)=\sum_{n=0}^{\infty}\frac{(i\nu)^n}{n!}c_n(\mu^2z^2)\langle x^n\rangle$$

$$H_n^{(j)}=\sum_{i=1}^{n}\frac{1}{i^n}$$

At NLO

$$c_n(\mu^2z^2)=1+\frac{\alpha_sC_F}{2\pi}\left[\ln\left(z^2\mu^2\frac{e^{2\gamma_E+1}}{4}\right)\left(H_n+H_{n+2}-\frac{3}{2}\right)+2\left(\frac{1}{n+1}-\frac{1}{n+2}-H_n^2-H_n^{(2)}\right)\right]$$

$$H_n^{(1)}\equiv H_n$$

Lattice Methodology

Fixed z Analysis

$$\chi^2 = \sum_{P_3} \frac{(\mathrm{Re/Im}[\mathscr{M}^{SDF}(\nu, z^2)] - \mathrm{Re/Im}[\mathscr{M}^{data}(\nu, z^2)])^2}{\sigma^2_{\mathrm{Re/Im}}}$$

- Fit even moments (real part) and odd moments (imaginary part) separately
- Test the proper truncation for the real and imaginary part separately
- Allows one to see residual z– dependence

Lattice Methodology

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Combined P_3 and z Analysis

$$\chi^2 = \sum_{P_3} \sum_{z=z_{min}}^{z_{max}} \left(\frac{(\mathrm{Re}[\mathscr{M}^{SDF}(\nu, z^2)] - \mathrm{Re}[\mathscr{M}^{data}(\nu, z^2)])^2}{\sigma^2_{\mathrm{Re}}} + \frac{(\mathrm{Im}[\mathscr{M}^{SDF}(\nu, z^2)] - \mathrm{Im}[\mathscr{M}^{data}(\nu, z^2)])^2}{\sigma^2_{\mathrm{Im}}} \right)$$

- Fit all moments at once
  - P_3 and z separated since combinations that lead to the same Ioffe value are not statistically equivalent
- Test different ranges for z_{min} and z_{max}
- Allows one to see systematic effects of higher twist contamination

Lattice Methodology

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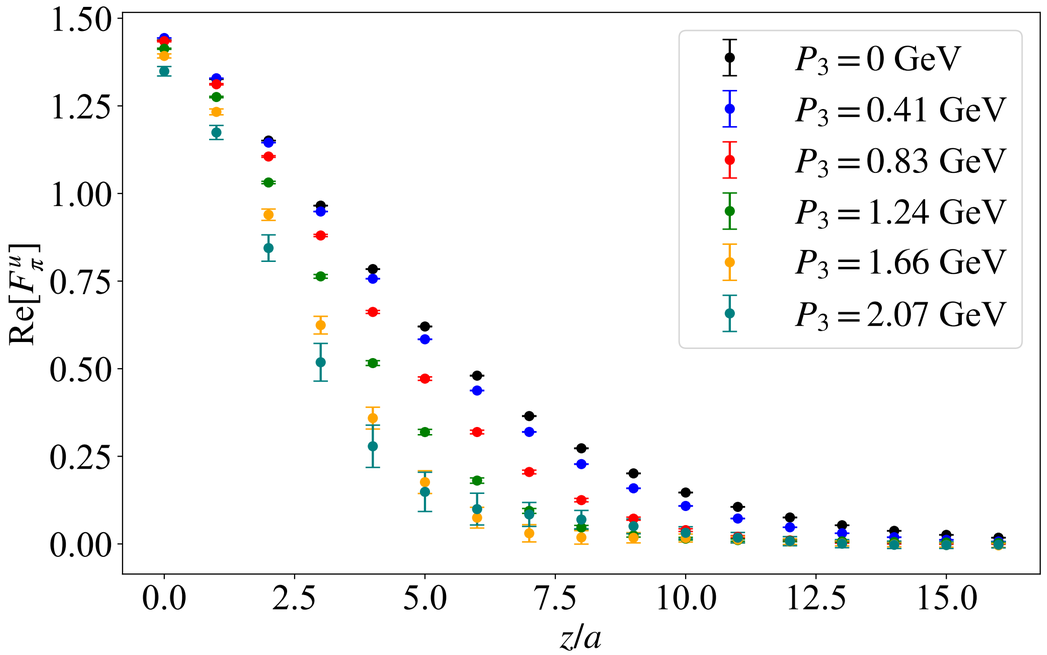
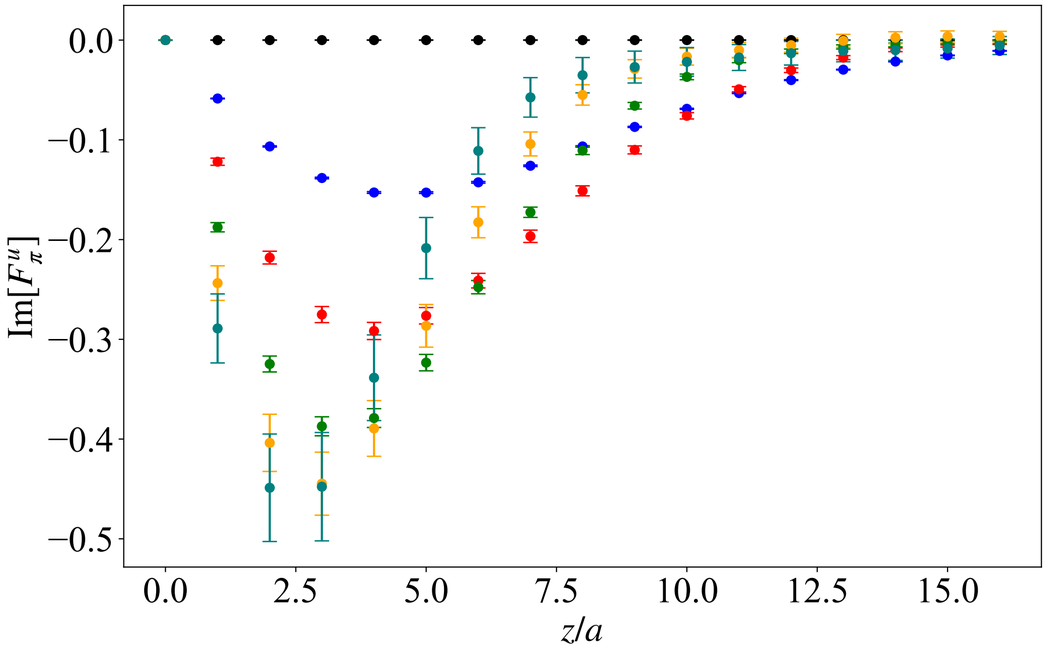
Majority of results are from using fixed $\mu = 2$ GeV

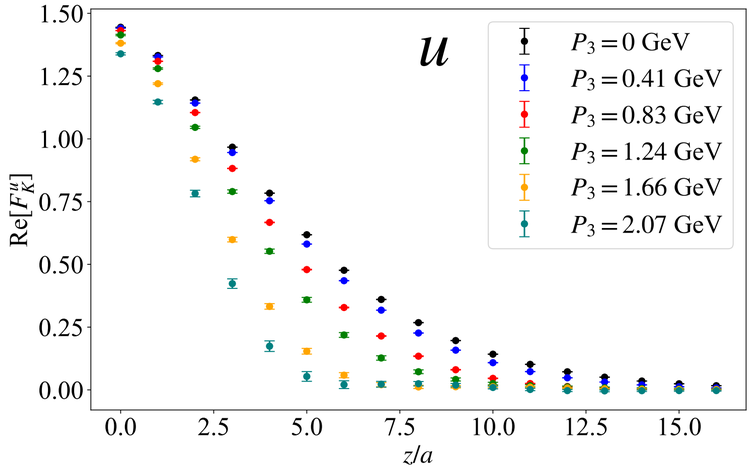
Matrix Elements: Pion

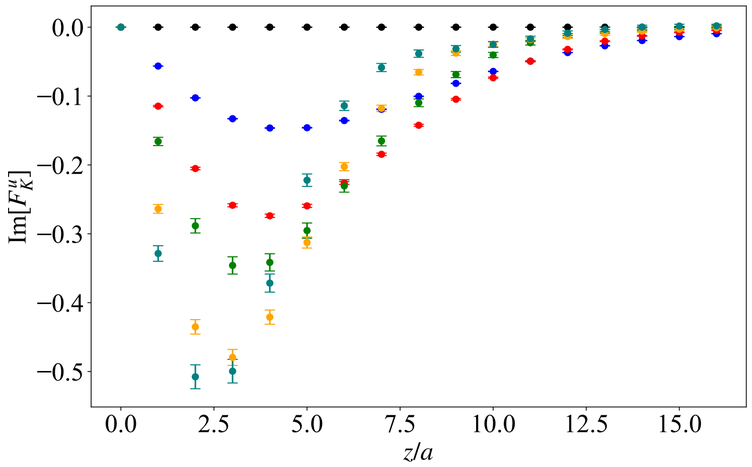


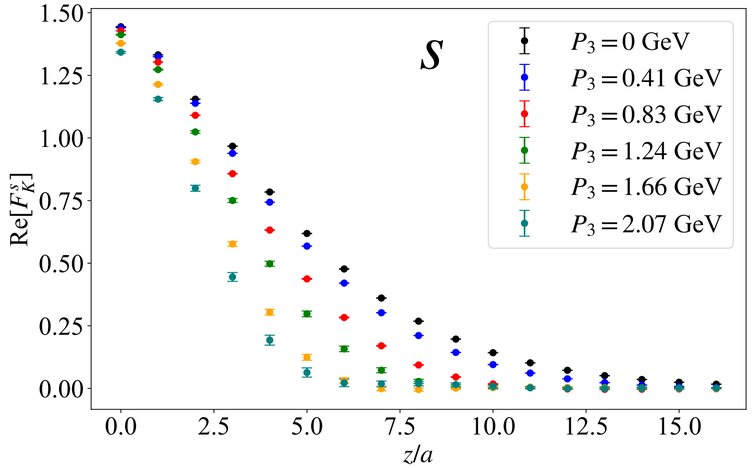


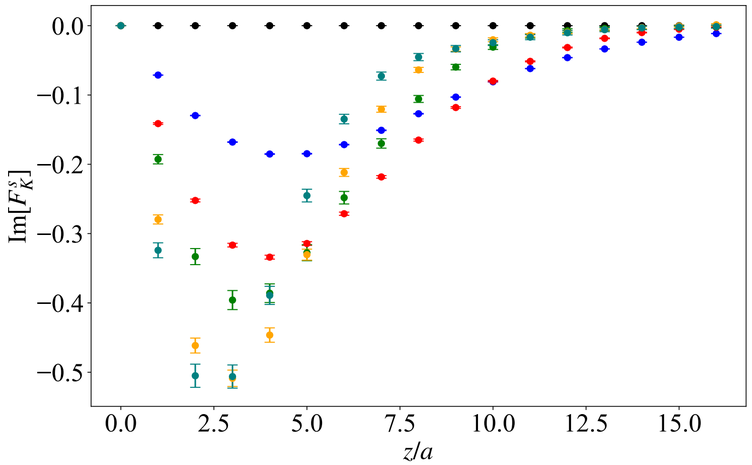
- Good signal to noise ratio
  - Increased statistics for larger P_3 still present larger error
- Real part becomes more narrow as momentum increases
- Imaginary part has a larger peak as momentum increases
- Symmetrized by $F_M^f(-z \cdot P) = F_M^{f*}(z \cdot P)$

Matrix Elements: Kaon



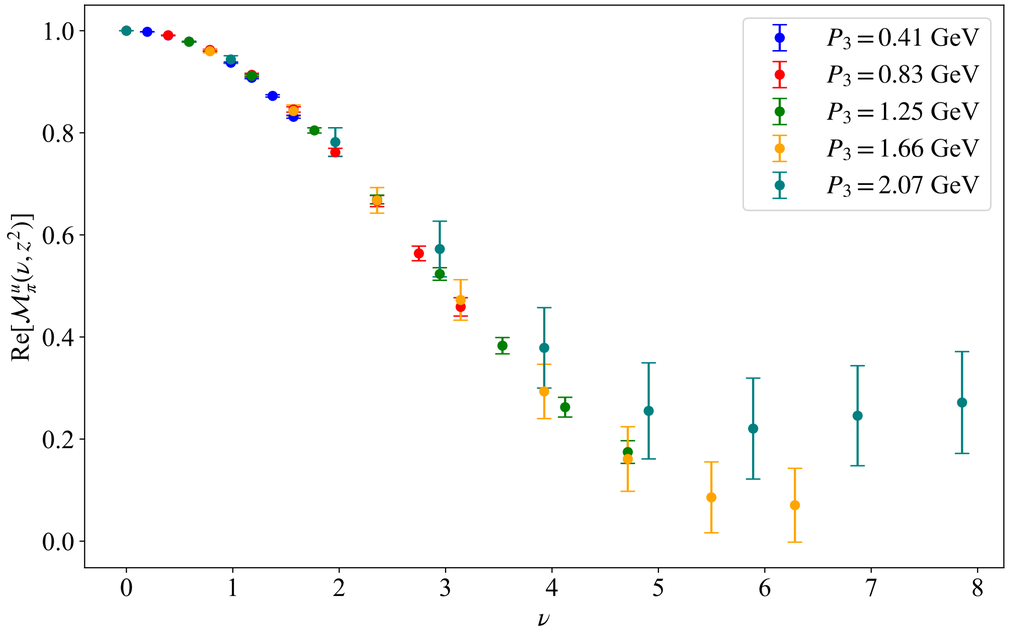


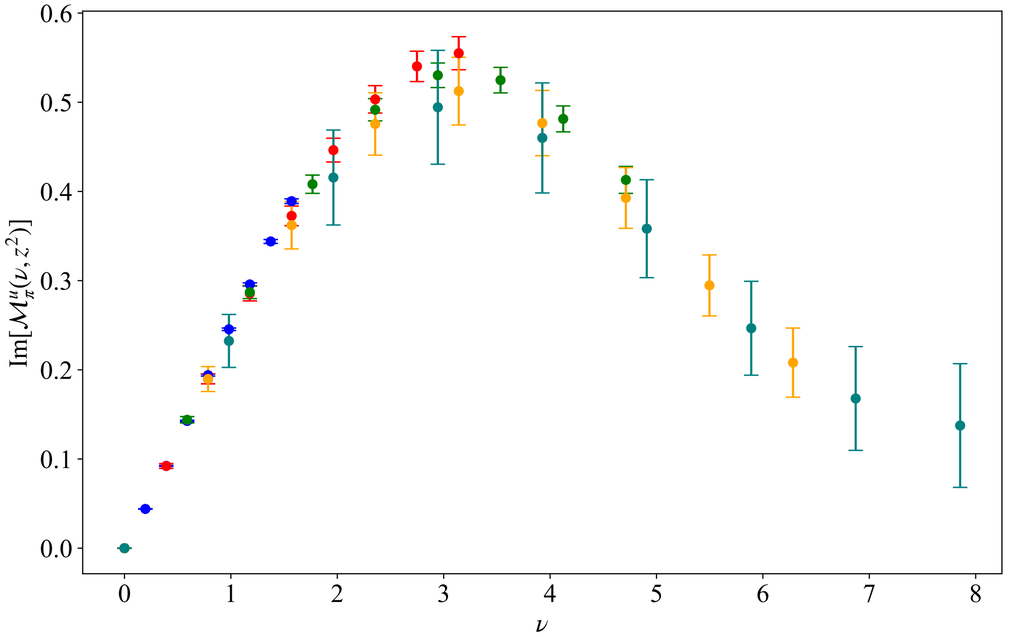




- Calculation of the up-quark and strange-quark separately
- Good signal to noise ratio
- Increased statistics for larger P_3 still present larger error
- Similar trends as the pion
- More accurate than the pion
- Symmetrized by $F_M^f(-z \cdot P) = F_M^{f*}(z \cdot P)$

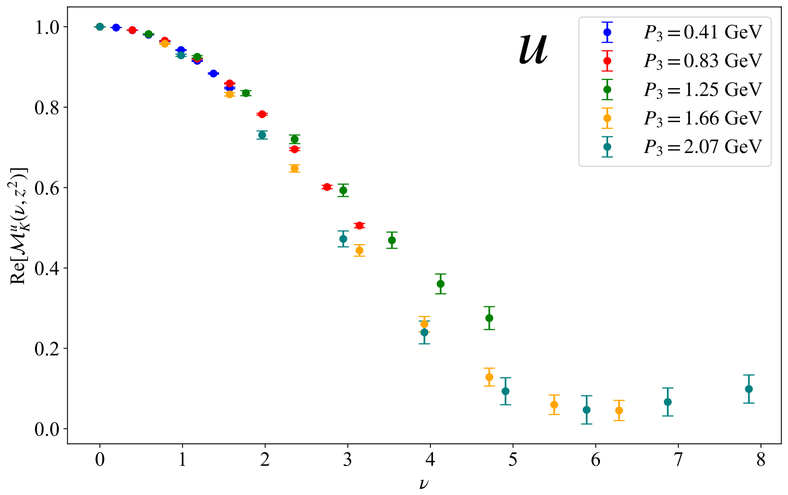
Double Ratio: Pion

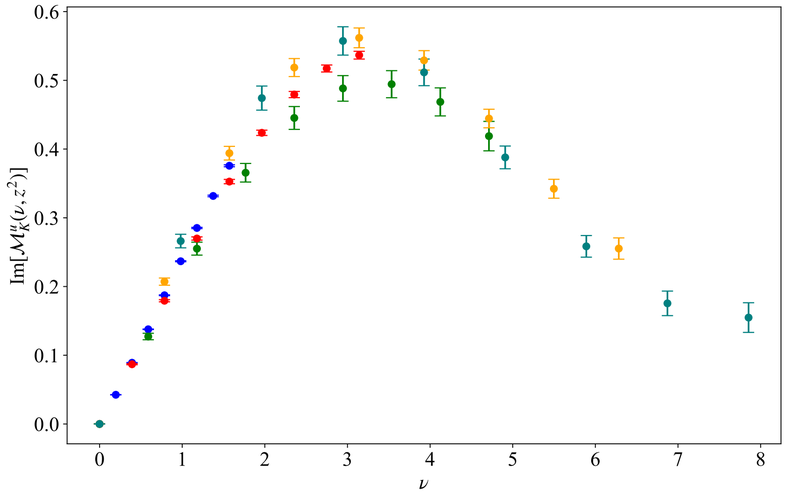


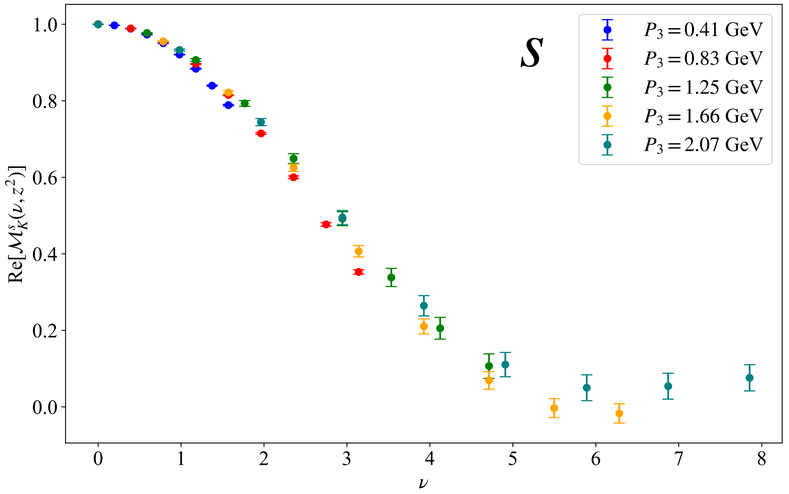


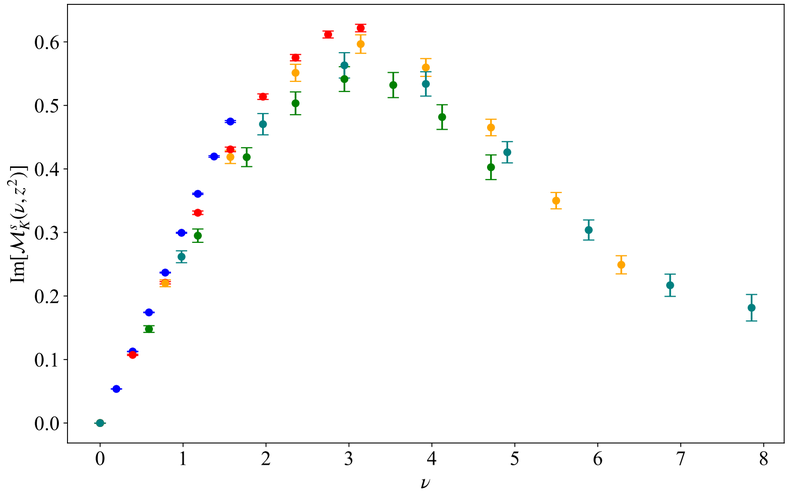
- Good signal to noise ratio
  - Large ν noisier and deviates between $P_3 = 1.66$ and 2.07 GeV
    - However this out outside the acceptable z region
- Real part decays towards zero
- Real part at $\nu = 0$ expected to be 1 since $\langle x^0 \rangle = \int_{-1}^{1} dx\ q(x) = 1$

Double Ratio: Kaon



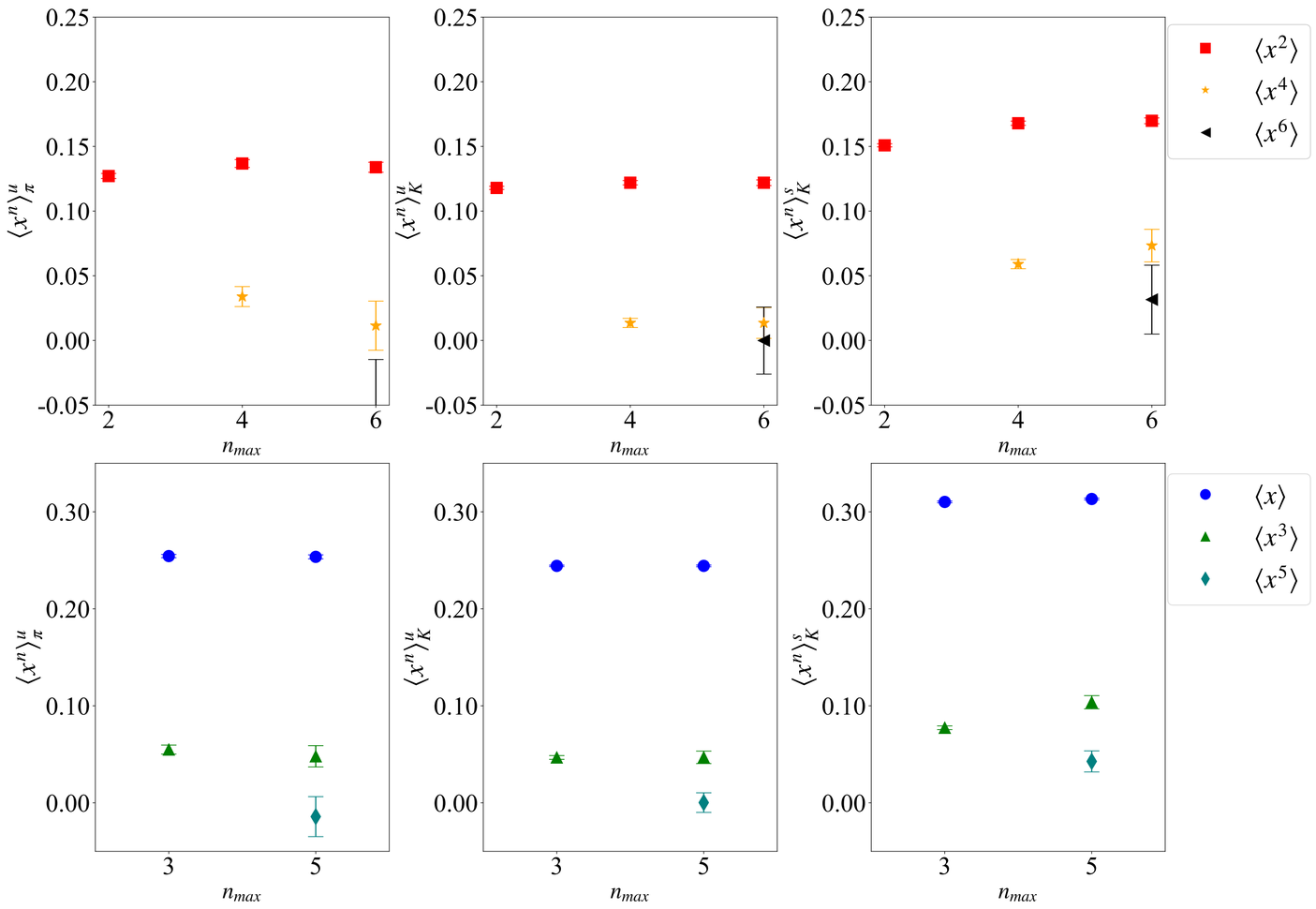






- Better signal to noise ratio than the pion
- Similar trends as the pion

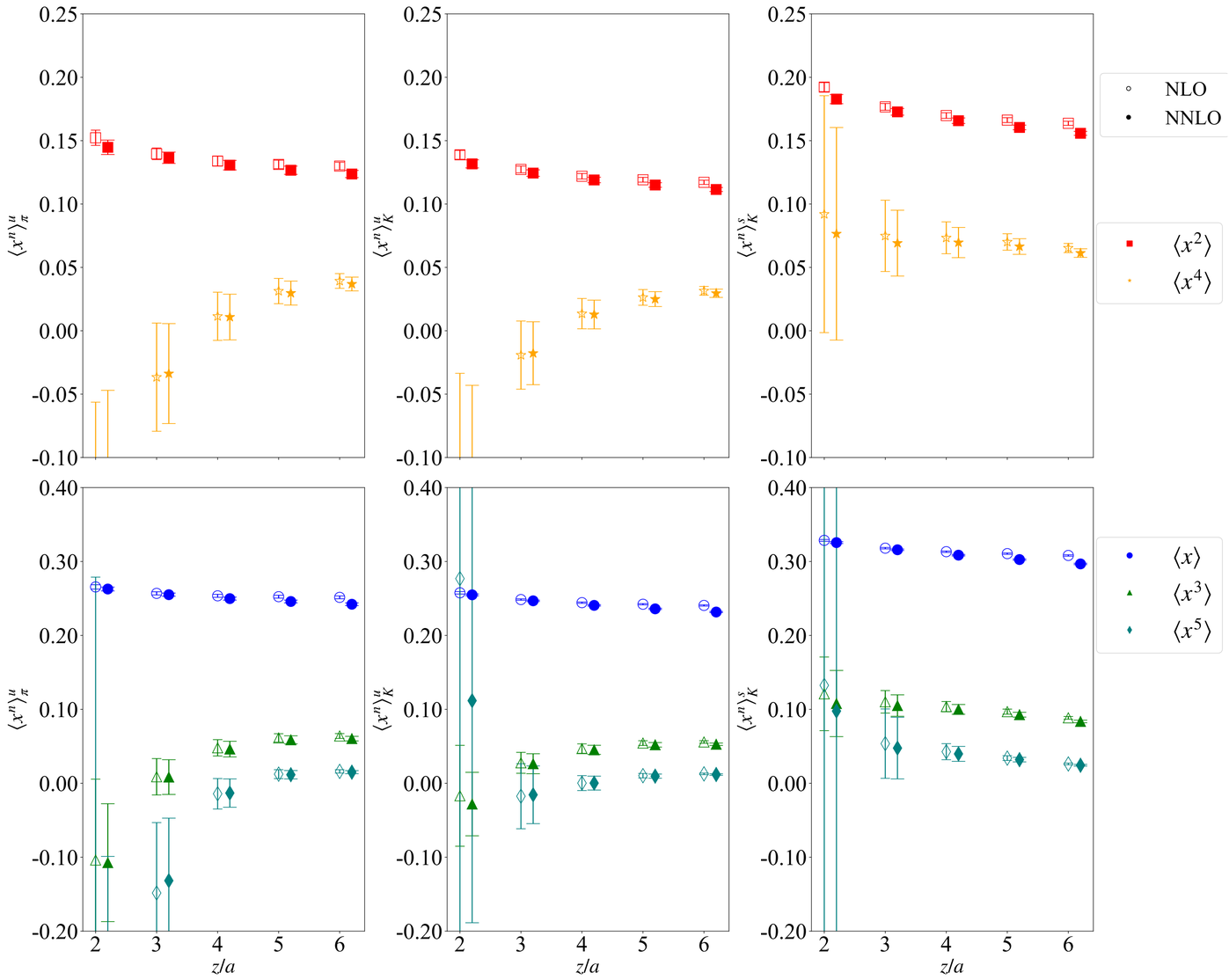
Fixed z Analysis: n_{max} Test



$$\mathcal{M}(\nu, z^2) = \sum_{n=0}^{\infty} \frac{(i\nu)^n}{n!} C_n(\mu^2 z^2) \langle x^n \rangle$$

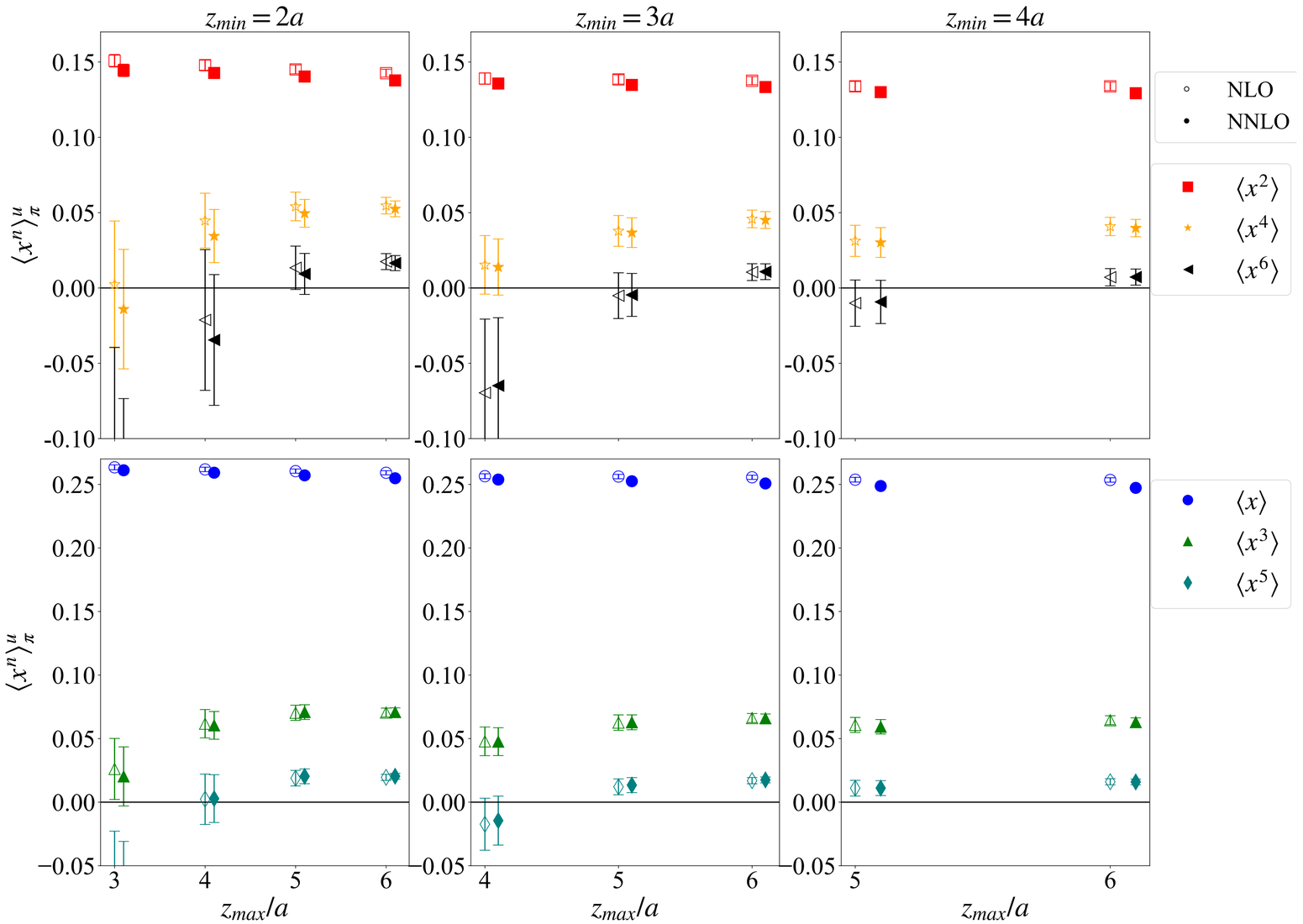
- Real and imaginary parts fit separately
  - Real part $\rightarrow \langle x^{2m} \rangle$
  - Imaginary part $\rightarrow \langle x^{2m+1} \rangle$
- Stability as n_{max} increases
  - $n_{max} = 6$ for the real part
  - $n_{max} = 5$ for the imaginary part

Fixed z Analysis: NLO vs NNLO C_n



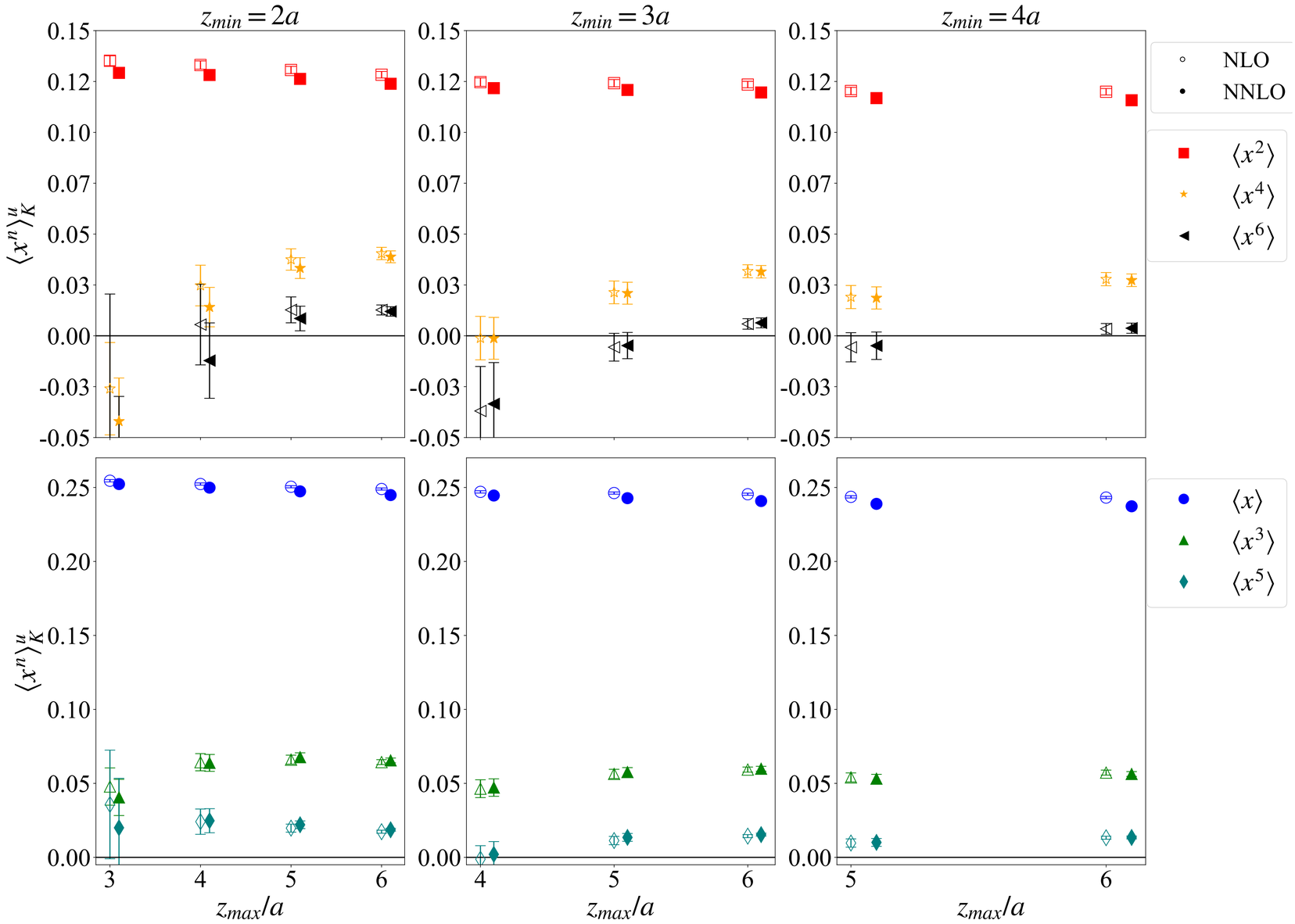
- Moments extracted from NLO and NNLO Wilson coefficients have a minimal shift
- Nontrivial $z-$dependence of the moments
- Higher moments stabilize with larger z
- No plateau until large z values
  - Indication we need to implement combined P_3 and z fits

Combined P_3 and z Analysis: $z-$ranges

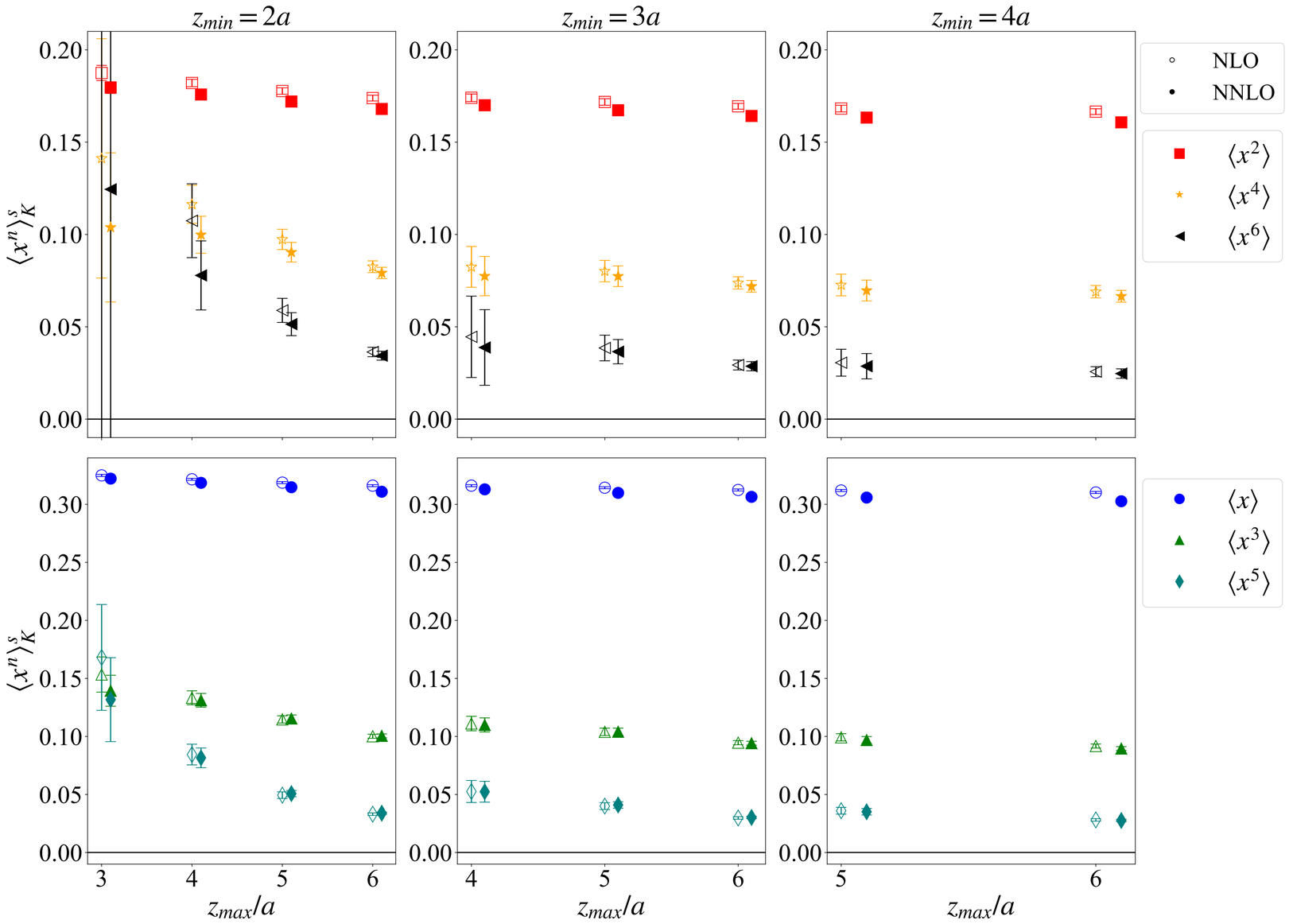


- Moments extracted from NLO and NNLO Wilson coefficients have a minimal shift
- Avoid $z_{min} = a$ due to discretization effects
- Inclusion of higher z allows for higher moments
- Systematics in different choices of z_{min} and z_{max}
- Final results are a weighted average of $z_{\min} \in [3a, 4a]$ and $z_{max} \in [4a, 6a]$ with $z_{max} - z_{min} \geq a$

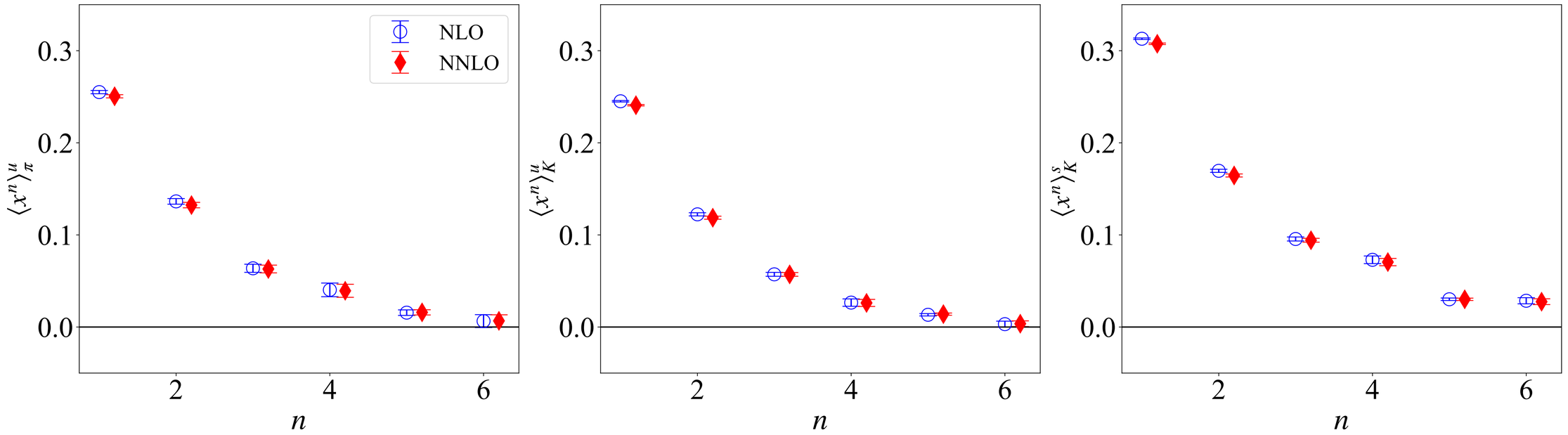
Combined P_3 and z Analysis: $z-$ ranges



Combined P_3 and z Analysis: $z-$ranges



Combined P_3 and z Analysis: Final Results



- Moments at both NLO and NNLO accuracy have a small difference between results
- PDFs decay to zero as $x \to 1$
  - As n increases, larger weight is given to higher x, causing the moment to be close to zero
- The strange-quark in the kaon has larger moments at each n
  - This indicates the strange quark carries more of the hadron momentum than the up-quark

DGLAP Evolution

- Unlike local operators, we have perturbative dependent results
  - C_n calculated at NLO, NNLO, …

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- Choose $\mu_0 = \frac{2\kappa}{ze^{\gamma_E}}$ such that the z-dependence in $\ln\left(z^2\mu_0^2\frac{e^{2\gamma_E+1}}{4}\right)$ cancels
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$$C_n^{NLOevo}(\mu^2, z^2) = C_n^{NLO}(\mu_0^2, z^2)\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{\gamma_n^{(0)}/\beta_0}$$

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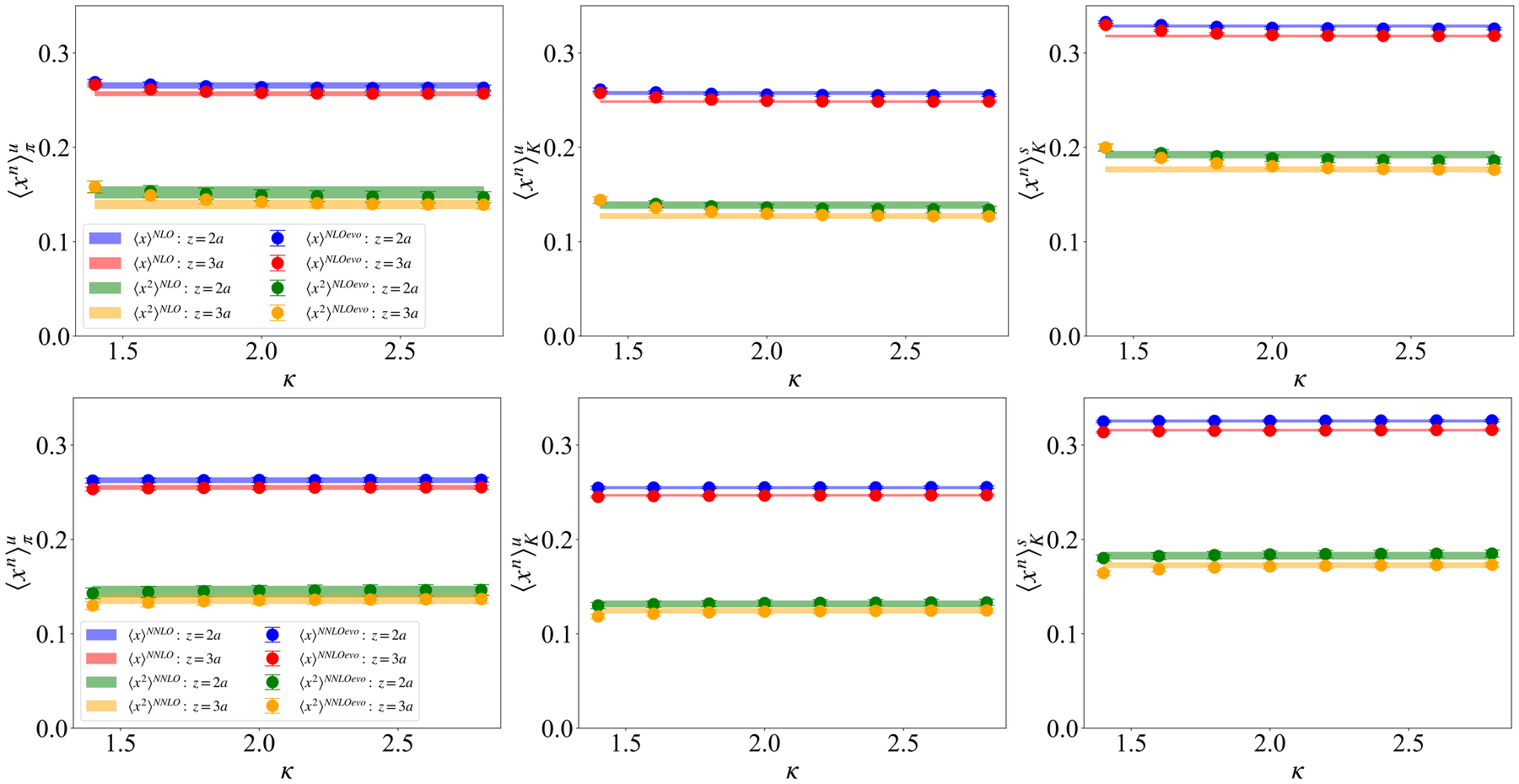
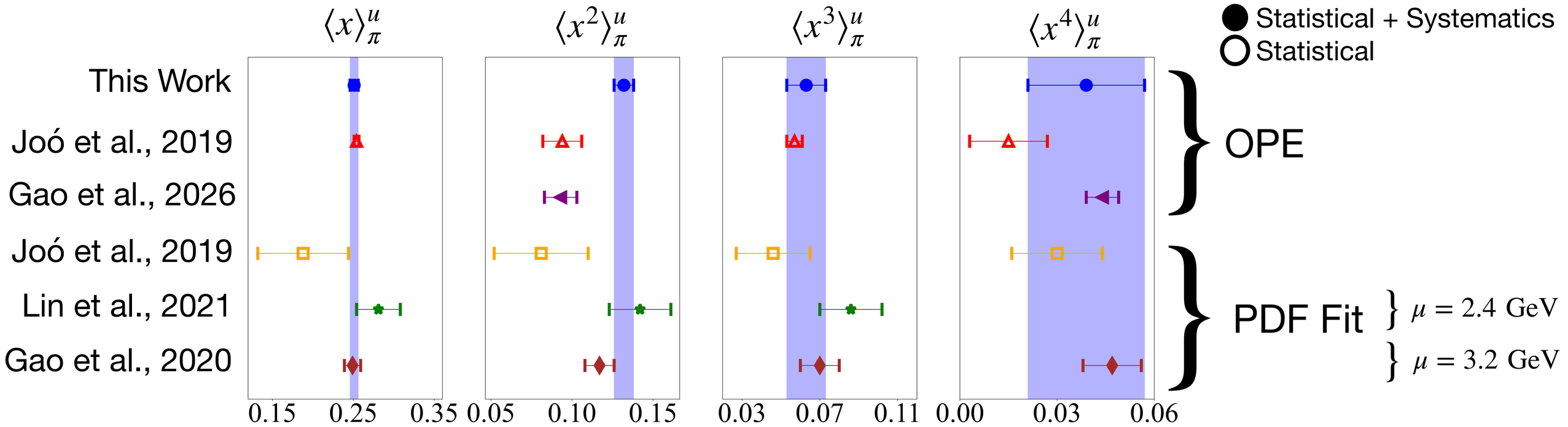
DGLAP Evolution



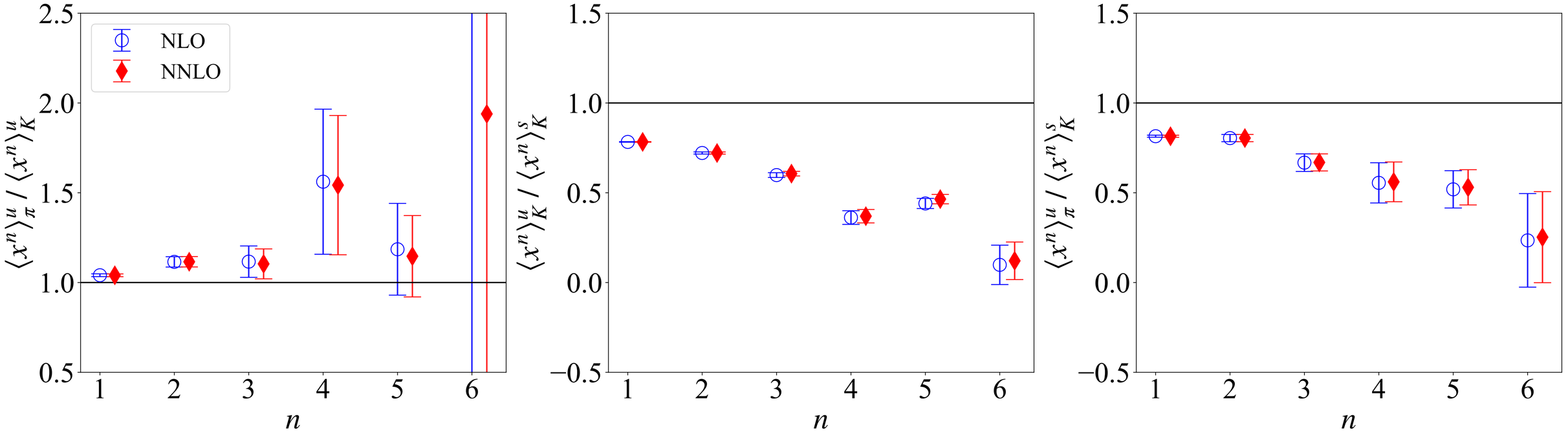
- Less dependence on κ for NNLOevo
  - $\langle x^2 \rangle$ requires larger κ to agree with the fixed μ results at NLOevo
- NNLOevo consistent with fixed μ results

Results Comparison



- Good agreement with other calculation
  - Not a 1:1 comparison due to different pion masses, lattice spacing, etc.

SU(3) Symmetry Breaking



- Pion and kaon up-quark are similar with statistical fluctuations
  - Up to 4% difference from unity
- Flavor dependence much stronger in comparison of up-quark and strange-quark
  - Deviation from unity becomes stronger as n increases, but lower moments give a more robust conclusion
- Consistent with previous studies

Summary and Future Work

- Calculate Mellin moments at NLO and NNLO accuracy
  - Utilize combined P_3 and z fits
  - Extract up to $\langle x^6 \rangle$
- Test perturbative accuracy through DGLAP evolution at NLO and NNLO accuracy
- Test SU(3) symmetry breaking

Summary and Future Work

- Calculate Mellin moments at NLO and NNLO accuracy
  - Utilize combined P_3 and z fits
  - Extract up to $\langle x^6 \rangle$
- Test perturbative accuracy through DGLAP evolution at NLO and NNLO accuracy
- Test SU(3) symmetry breaking
- Continue work to Generalized Form Factors
- Take continuum limit
- Implement Neural Networks

Thank You!!!

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- Interdisciplinary Centre for Mathematical and Computational Modeling of the Warsaw University by Okeanos
- Academic Computer Center in Gdańsk by Tryton

Award Number:
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Backup Slides

Wilson Coefficient Derivation

❖ Focusing only on NLO accuracy, start with known matching relation

$$\mathcal{M}(\nu, z^2) = \mathcal{Q}(\nu, \mu^2) + \frac{\alpha_s C_F}{2\pi} \int_0^1 du \left[\ln\left(z^2\mu^2 \frac{e^{2\gamma_E+1}}{4} \right) \left(\frac{1+u^2}{u-1} \right)_+ + \left(4\frac{\ln(1-u)}{u-1} - 2(u-1) \right) + \right] \mathcal{Q}(u\nu, \mu^2)$$

$$\int dx\ [f(x)]_+ g(x) = \int dx\ f(x)(g(x) - g(1))$$

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- Note that $\mathcal{Q}$ is the Fourier transform on the light-cone PDF

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- Substitute in and apply plus prescription

$$\mathcal{M}(\nu, z^2) = \sum_{n=0}^{\infty} \frac{(i\nu)^n}{n!} \langle x^n \rangle \left[1 + \frac{\alpha_s C_F}{2\pi} \int_0^1 du \left[\ln\left(z^2\mu^2 \frac{e^{2\gamma_E+1}}{4} \right) \left(\frac{1+u^2}{u-1} \right) + \left(4\frac{\ln(1-u)}{u-1} - 2(u-1) \right) \right] (u^n - 1) \right]$$

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- Solve the integrals

$$\mathcal{M}(\nu, z^2) = \sum_{n=0}^{\infty} \frac{(i\nu)^n}{n!} \langle x^n \rangle \left[1 + \frac{\alpha_s C_F}{2\pi} \left[\ln\left(z^2\mu^2 \frac{e^{2\gamma_E+1}}{4} \right) \left(H_n + H_{n+2} - \frac{3}{2} \right) + 2\left(\frac{1}{n+1} - \frac{1}{n+2} - H_n^2 - H_n^{(2)} \right) \right] \right]$$

DGLAP Evolution Derivation

- Start with the evolution equation for Wilson coefficients

$$\ln\left(\frac{C_n(\mu)}{C_n(\mu_0)}\right) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d(\alpha_s(\mu')) \, \frac{\gamma_n(\alpha_s(\mu'))}{\beta(\alpha_s(\mu'))}$$

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- Substitute the order you want into the integral and solve
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- NNLOevo

$$\ln\left(\frac{C_n(\mu)}{C_n(\mu_0)}\right) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d(\alpha_s(\mu')) \, \frac{\frac{\alpha_s}{4\pi}\gamma_n^{(0)} + \frac{\alpha_s^2}{(4\pi)^2}\gamma_n^{(1)}}{-\frac{\alpha_s^2}{4\pi}\beta_0 - \frac{\alpha_s^3}{(4\pi)^2}\beta_1} = -\frac{\gamma_n^{(0)}}{\beta_0} \ln\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right) - \left(\frac{\gamma_n^{(1)}}{\beta_1} - \frac{\gamma_n^{(0)}}{\beta_0}\right) \ln\left(\frac{4\pi\beta_0 + \beta_1\alpha_s(\mu)}{4\pi\beta_0 + \beta_1\alpha_s(\mu_0)}\right)$$