The nucleons generalized form factors and Mellin moments up to fourth order

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with: C. Alexandrou, S. Bacchio, M. Constantinou, Y. Li, G. Spanoudes

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University of Cyprus

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AQTIVATE

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Motivation

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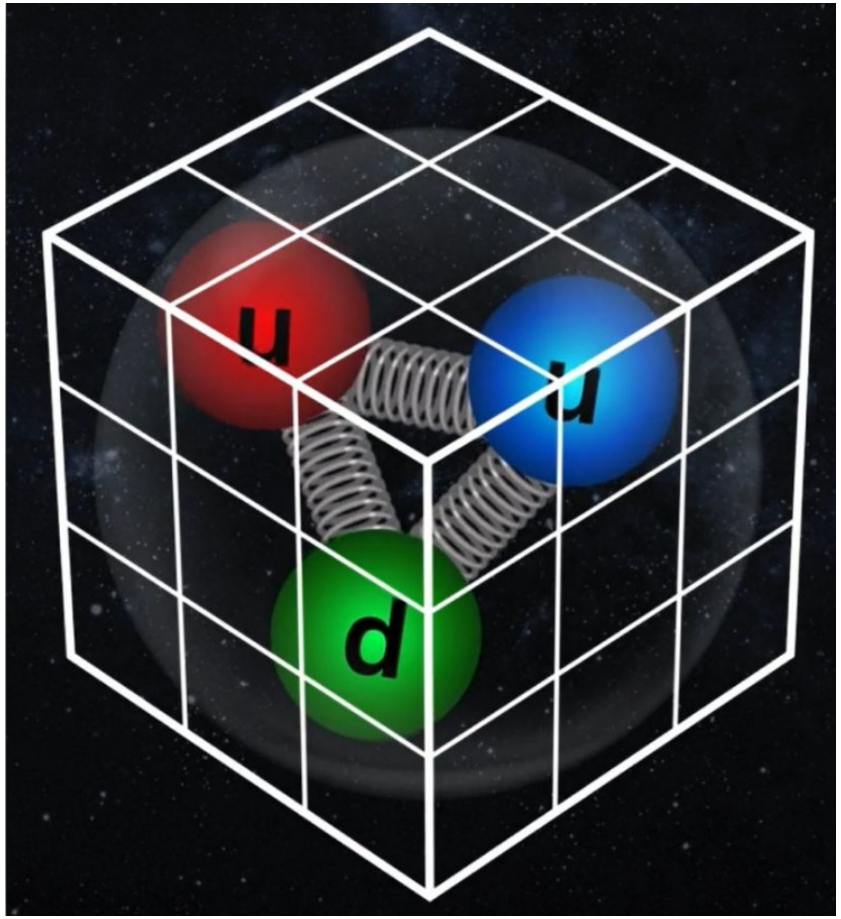
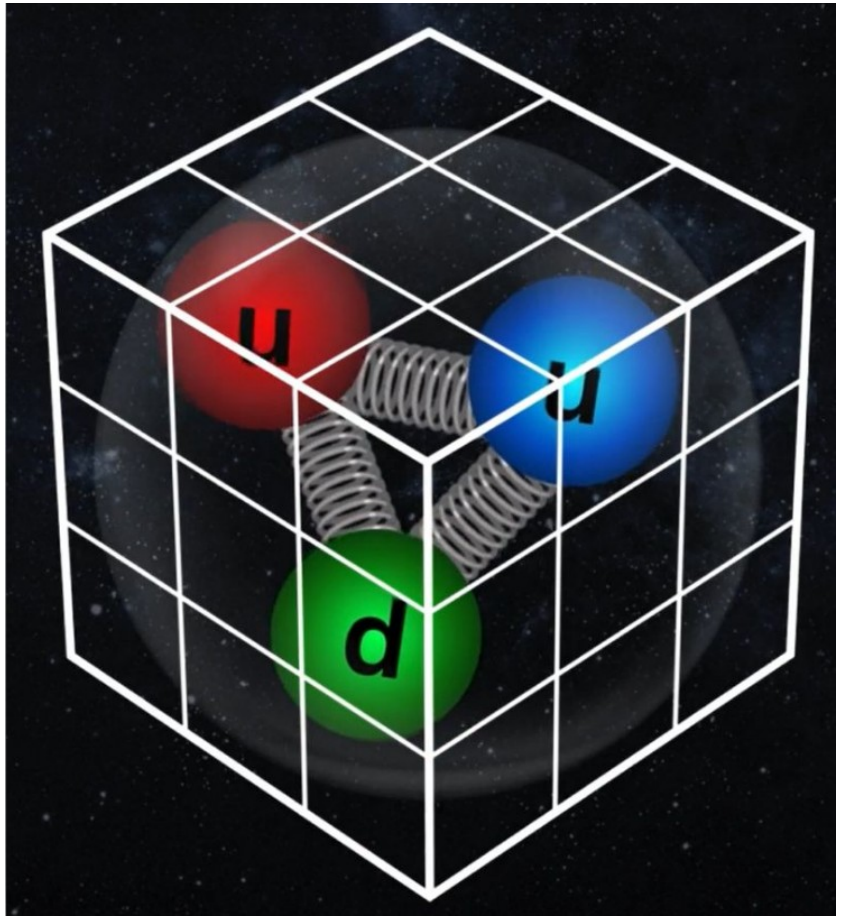


Motivation

- Proton is an excellent probe for strong processes
- Internal structure highly non-perturbative ➡ Lattice QCD
- Interest in generalized parton distributions (GPDs)

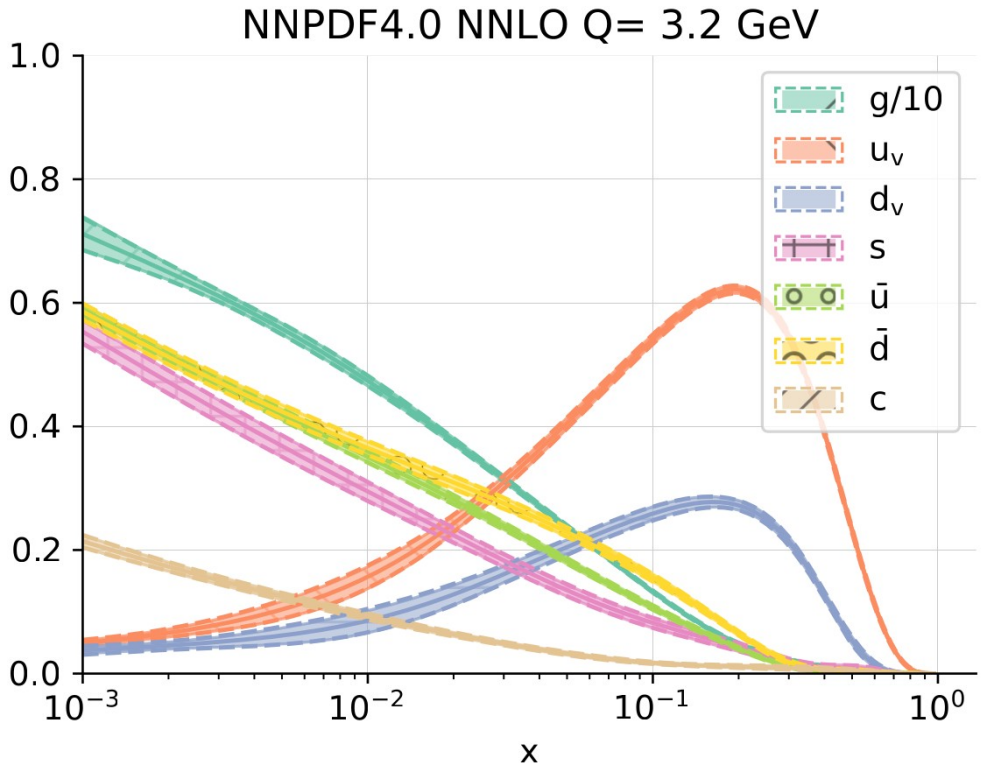
$$\langle N(p')|\overline{\psi}\gamma^{\mu}\psi|N(p)\rangle$$

$$=\overline{u}(p')\left[\gamma^{\mu}H(x,\xi,q^2)+\frac{i\sigma^{\mu\nu}q_{\nu}}{2m_N}E(x,\xi,q^2)\right]u(p)$$



Motivation

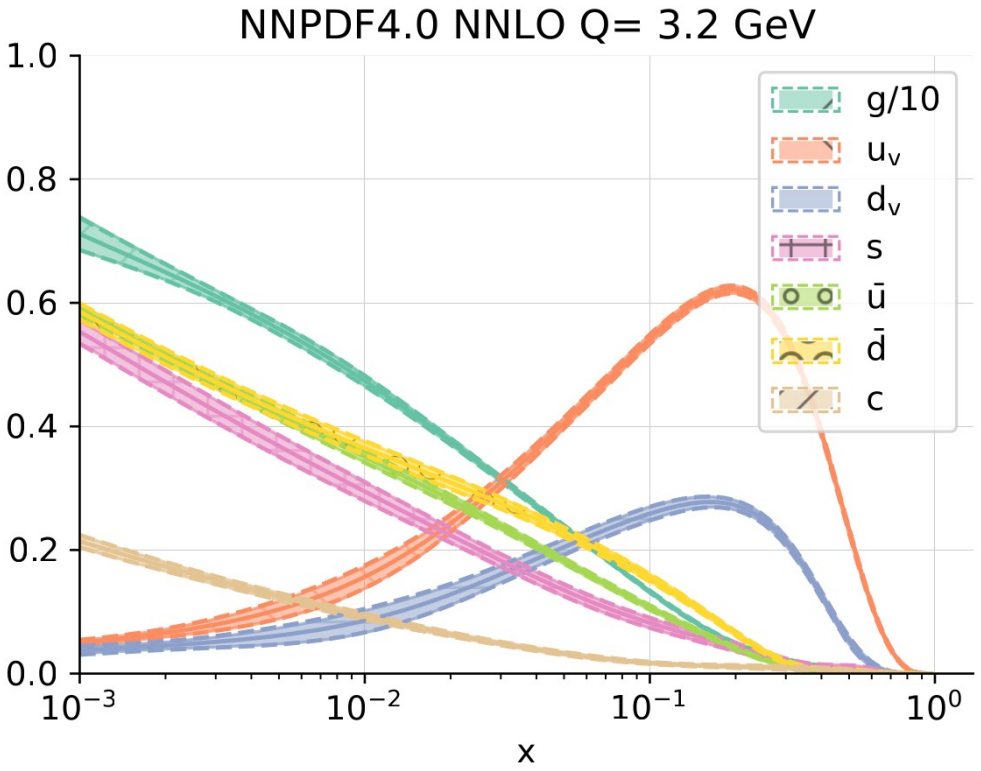
- Large Interest in GPDs (EIC)



R.D. Ball (NNPDF), EPJC 82, 428 (2022)

Motivation

- Large Interest in GPDs (EIC)
- GPDs can NOT be directly calculated on the lattice



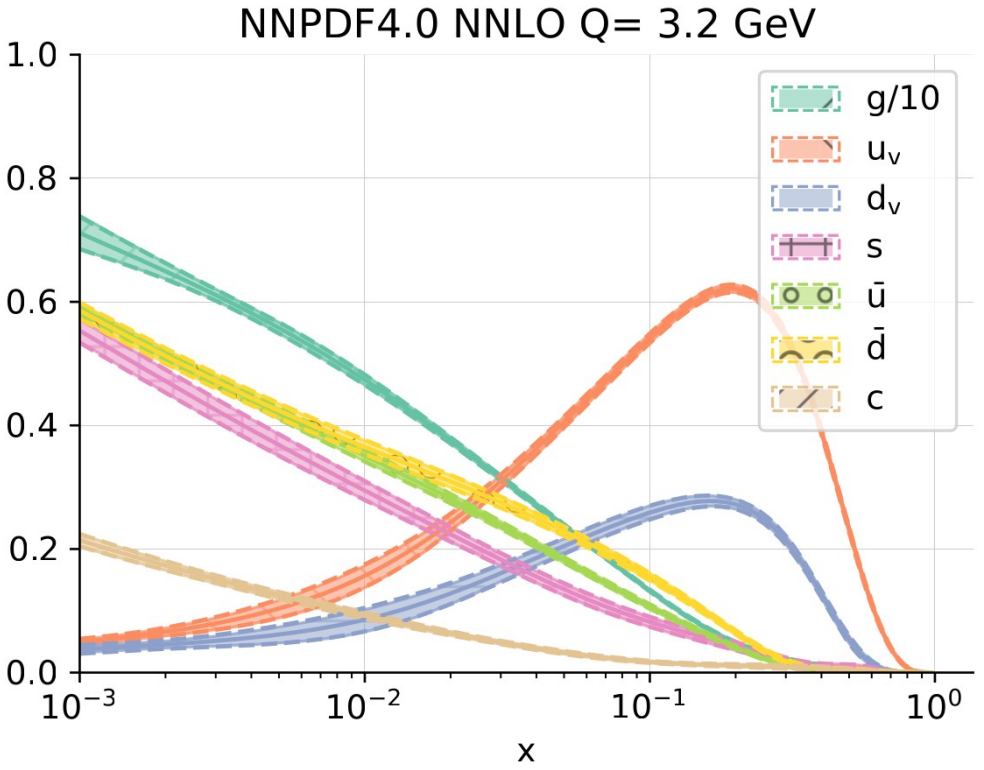
R.D. Ball (NNPDF), EPJC 82, 428 (2022)

Motivation

- Large Interest in GPDs (EIC)
- GPDs can NOT be directly calculated on the lattice
- Direct calculation of Mellin moments (and GFFs) IS possible

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} H(x,0,0)$$

$$= \int_{-1}^{1} \mathrm{d}x\, x^{n-1} q(x) = \langle x^{n-1} \rangle$$



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Definitions

- Leading twist operators

$$\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}} = \overline{\psi}\,\gamma^{\{\mu_1} i\overleftrightarrow{D}^{\mu_2}\ldots i\overleftrightarrow{D}^{\mu_n\}}\,\psi$$

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- Matrix elements

  Reminder:

$$\langle N(p')|\overline{\psi}\gamma^{\mu}\psi|N(p)\rangle$$

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$$\langle N(p')|\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}}|N(p)\rangle =$$

$$\overline{u}(p')\Bigg[\sum_{\substack{i=0\\ \text{even}}}^{n-1}\Big\{q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\gamma^{\mu_n\}}A_{n,i}(q^2)$$

$$+\,q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\frac{i\sigma^{\mu_n\}\alpha}q_\alpha}{2m_N}B_{n,i}(q^2)\Big\}$$

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X.-D. Ji, JPG 24, 1181 (1998)

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GFFs

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- Matrix elements

- Connection to GPDs

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} H(x,\xi,q^2) = \sum_{\substack{i=0 \\ \text{even}}}^{n-1} A_{n,i}(q^2)(-2\xi)^i + C_{n0}(q^2)(-2\xi)^n \Big|_{n\ \text{even}}$$

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} E(x,\xi,q^2) = \sum_{\substack{i=0 \\ \text{even}}}^{n-1} B_{n,i}(q^2)(-2\xi)^i - C_{n0}(q^2)(-2\xi)^n \Big|_{n\ \text{even}}$$

Operator Mixing

- Continuum: Lorentz group $SO(1,3)$
- Lattice: Hypercubic group $H(4)$

G. Beccarini, NPB 456, 271 (1995)

M. Göckeler, PRD 54, 5705 (1996)

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Operator Mixing

- Continuum: Lorentz group *SO*(1,3)
- Lattice: Hypercubic group *H*(4)
- Mixing!
- In general: Avoid repeated indices
- Special case: Forward limit
- Leads to boosted frame

$$\langle N(p')|\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}}|N(p)\rangle =$$

$$\overline{u}(p')\Bigg[\sum_{\substack{i=0\\ \text{even}}}^{n-1}\Big\{q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\gamma^{\mu_n\}}A_{n,i}(q^2)$$

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Methods

- Ensemble at physical point

Ensemble	a [fm]	$(L/a)^3 \times T/a$	m_π [GeV]	L [fm]
cB211.072.64	0.0801(4)	$64^3 \times 128$	0.1393(7)	5.12(3)

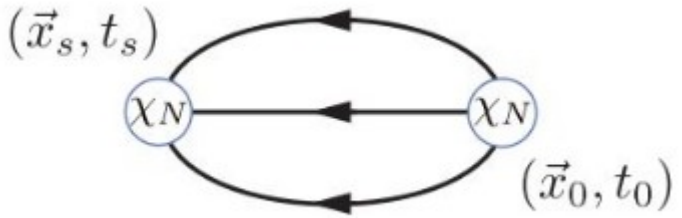
C. Alexandrou, PRD 98, 054518 (2018)

Methods

- Ensemble at physical point
- Two- and three-point-functions

$$C(\Gamma_0, \vec{p}, t_s, t_0) = \sum_{\vec{x}_s} e^{-\mathrm{i}(\vec{x}_s - \vec{x}_0)\cdot\vec{p}} \times$$

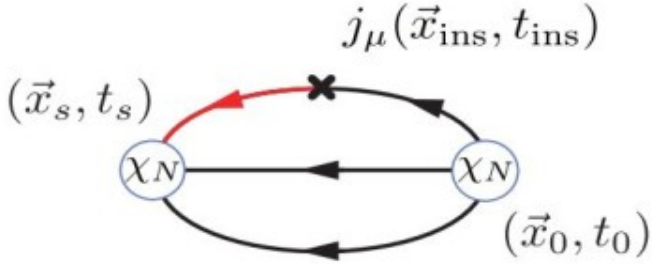
$$\mathrm{tr}[\Gamma_0 \langle J_N(t_s, \vec{x}_s) \bar{J}_N(t_s, \vec{x}_0) \rangle]$$



$$C^{\{\mu_1\mu_2\ldots\mu_n\}}(\Gamma, \vec{q}, \vec{p}', t_s, t_{\mathrm{ins}}, t_0) =$$

$$\sum_{\vec{x}_{\mathrm{ins}}, \vec{x}_s} e^{-\mathrm{i}(\vec{x}_{\mathrm{ins}} - \vec{x}_0)\cdot\vec{q}} e^{-\mathrm{i}(\vec{x}_s - \vec{x}_0)\cdot\vec{p}'} \times$$

$$\mathrm{tr}[\Gamma \langle J_N(t_s, \vec{x}_s) \mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}}(t_{\mathrm{ins}}, \vec{x}_{\mathrm{ins}}) \bar{J}_N(t_s, \vec{x}_0) \rangle]$$



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$\vec{n}'$	N_{mom}	t_s/a	N_{conf}	N_{srcs}	N_{meas}
two derivatives					
$(1,1,0)^T$	12	8	735	1	8820
$(1,1,0)^T$	12	10	735	3	26460
$(1,1,0)^T$	12	12	735	9	79380
$(1,1,0)^T$	12	14	735	18	158760
three derivatives					
$(1,1,1)^T$	8	8	735	1	5880
$(1,1,1)^T$	8	10	735	3	17640
$(1,1,0)^T$	12	14	735	18	158760

Methods

- Ensemble at physical point
- Two- and three-point-functions
- Construct Ratio to extract matrix element

$$R^{\mu\nu}(\Gamma;\vec{p}\,',\vec{p};t_s,t_{\text{ins}}) = \frac{C^{\mu\nu}(\Gamma;\vec{p}\,',\vec{p};t_s,t_{\text{ins}})}{C(\Gamma_0;\vec{p}\,';t_s)} \times$$

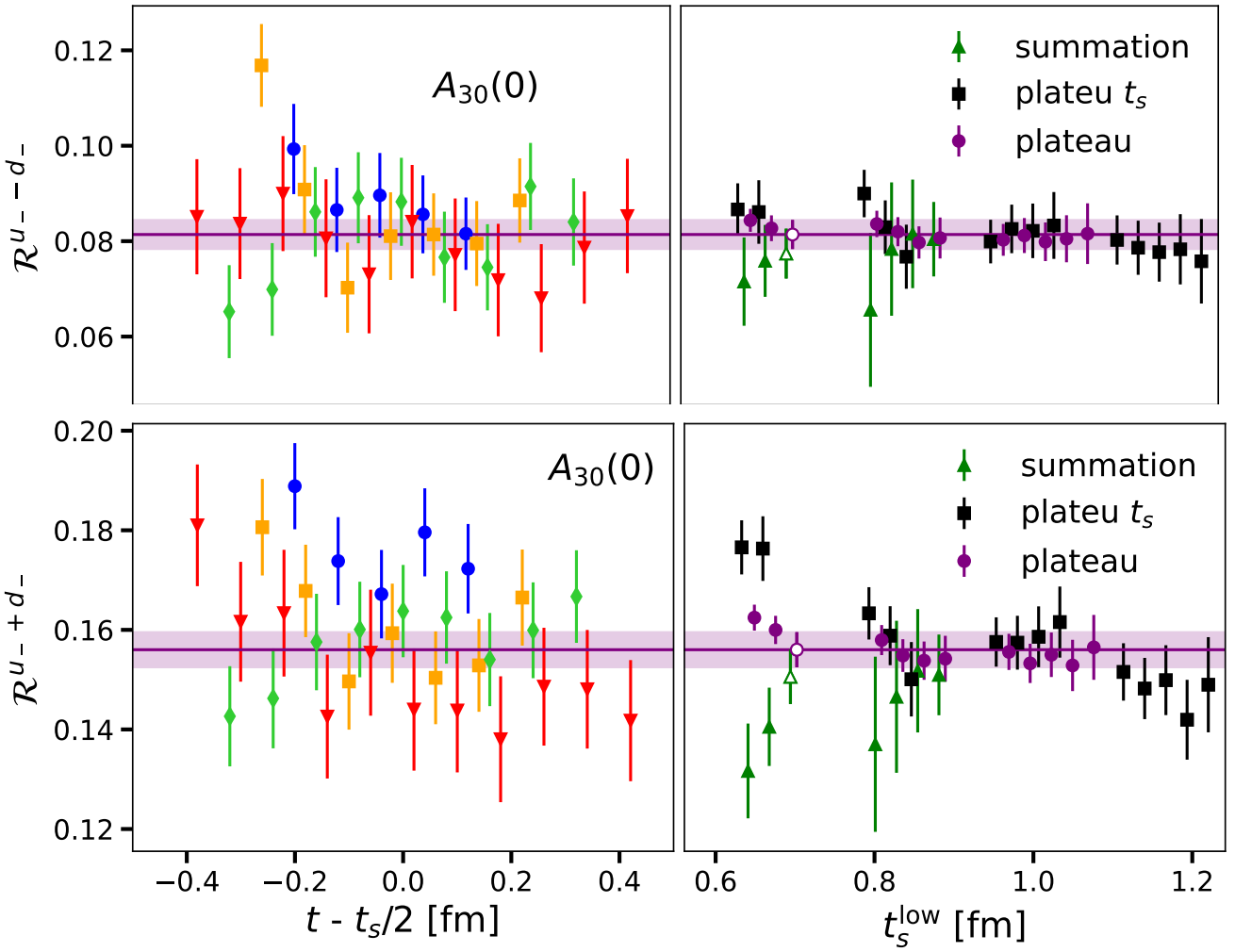
$$\sqrt{\frac{C(\Gamma_0,\vec{p};t_s-t_{\text{ins}})C(\Gamma_0,\vec{p}\,';t_{\text{ins}})C(\Gamma_0,\vec{p}\,';t_s)}{C(\Gamma_0,\vec{p}\,';t_s-t_{\text{ins}})C(\Gamma_0,\vec{p};t_{\text{ins}})C(\Gamma_0,\vec{p};t_s)}}$$

Methods

- Ensemble at physical point
- Two- and three-point-functions
- Construct Ratio to extract matrix element
- Multiplicative renormalization
- Solve LSE for GFFs
- Fit using plateau-fit and summation method

Results

- Forward limit
- Two-derivative operators



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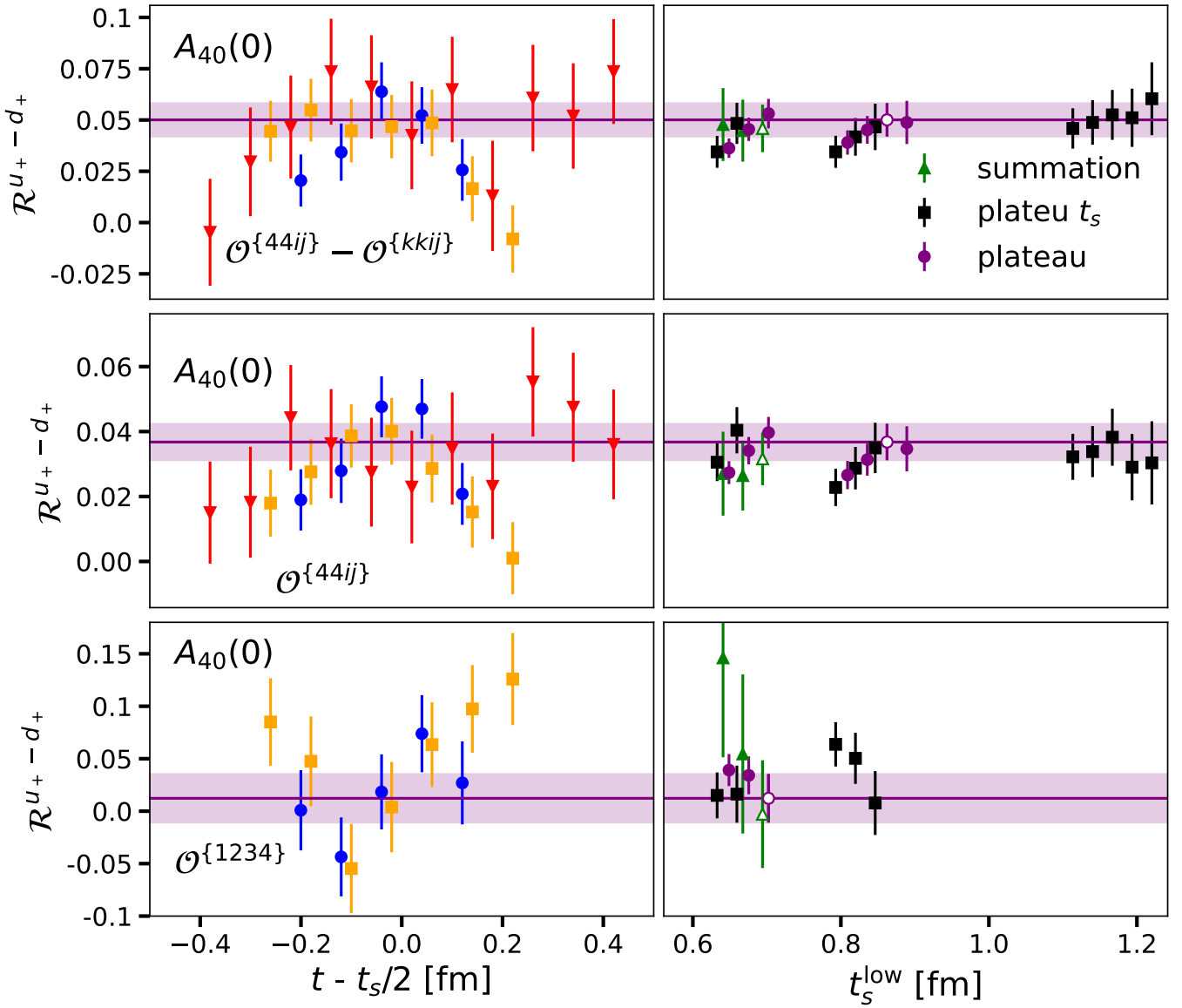
Results

- Forward limit
- Three derivative operators
- isovector



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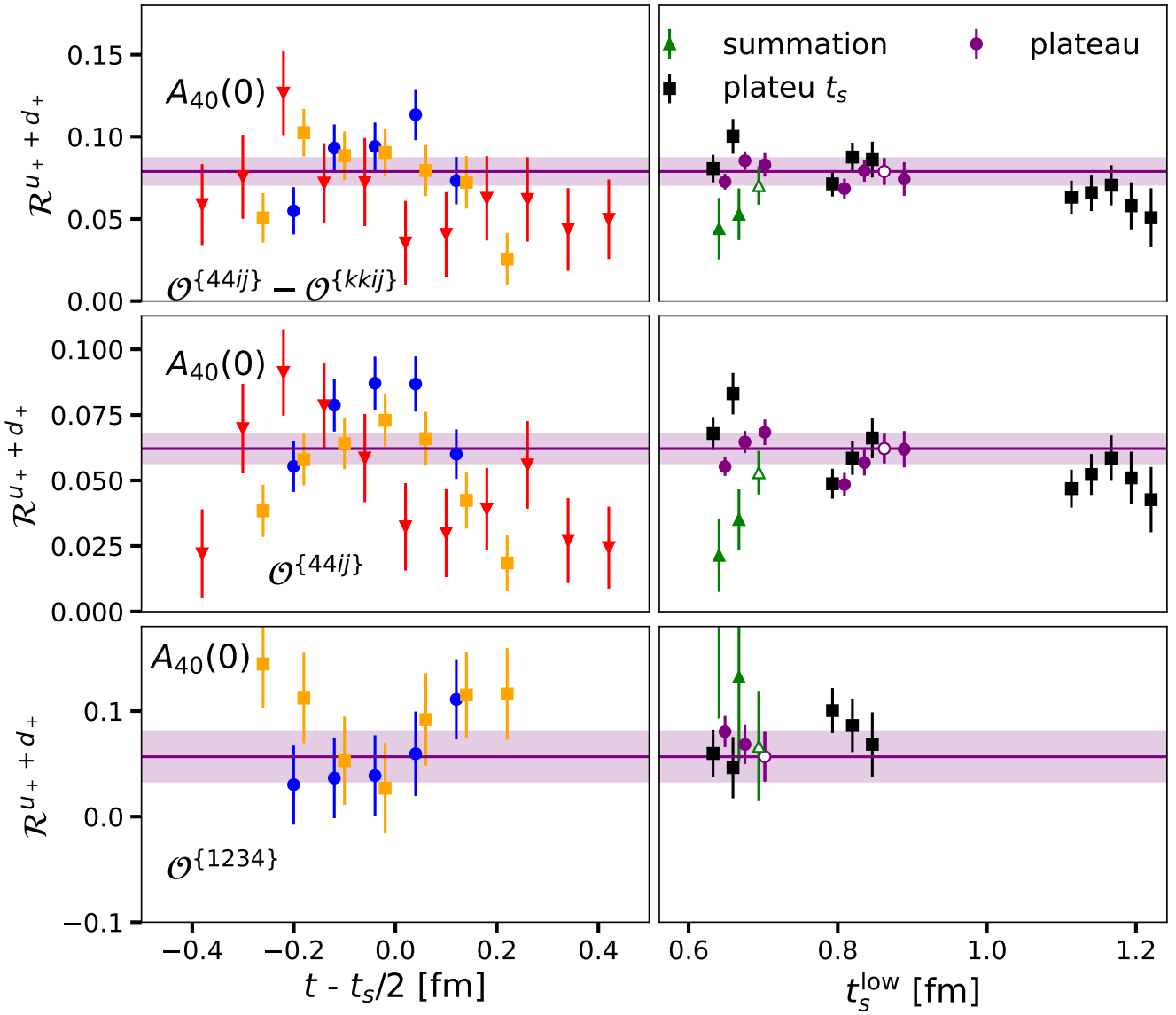
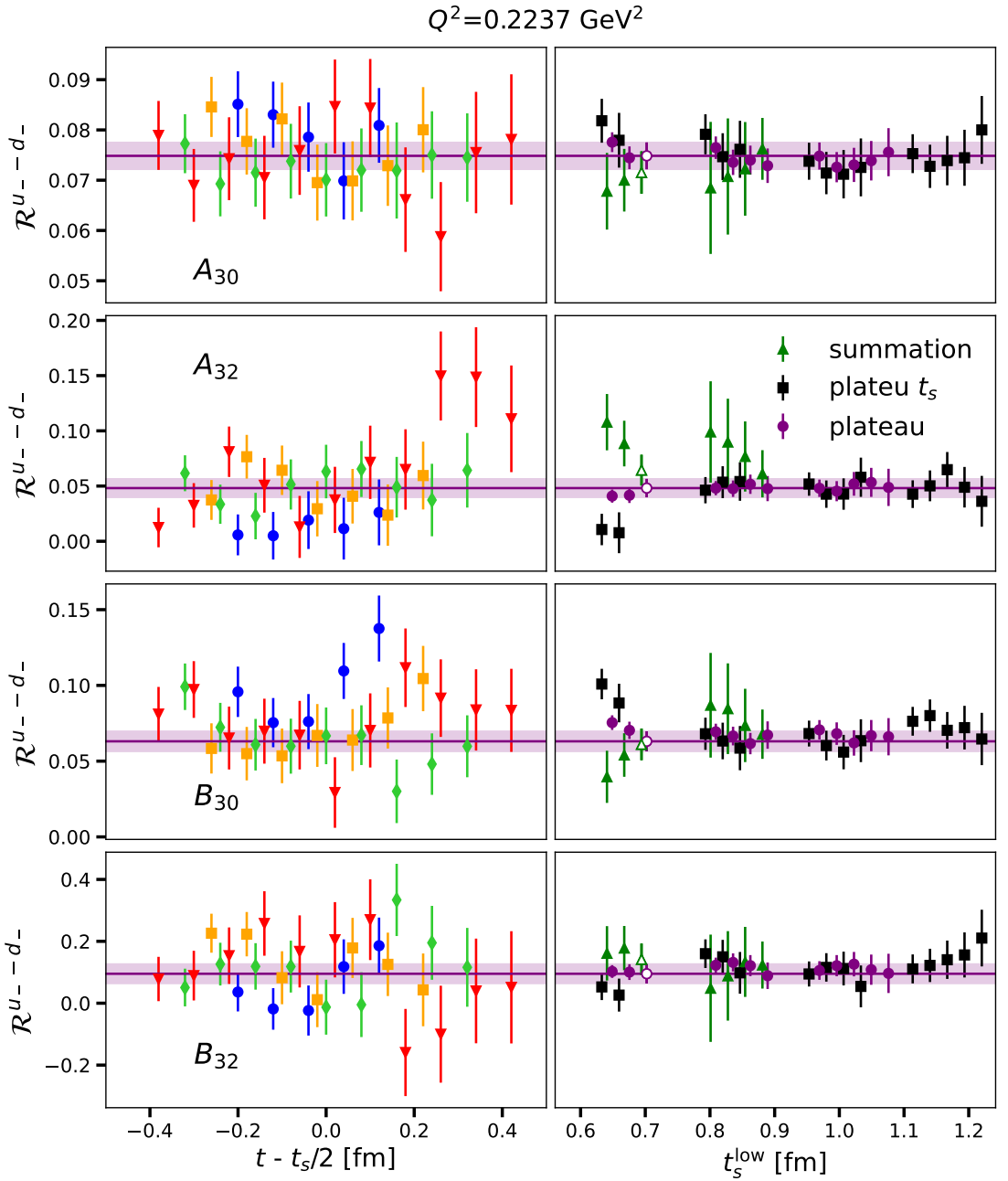
Results

- Forward limit
- Three derivative operators
- isoscalar



Results

- Momentum transfer
- Two-derivative operators
- isovector



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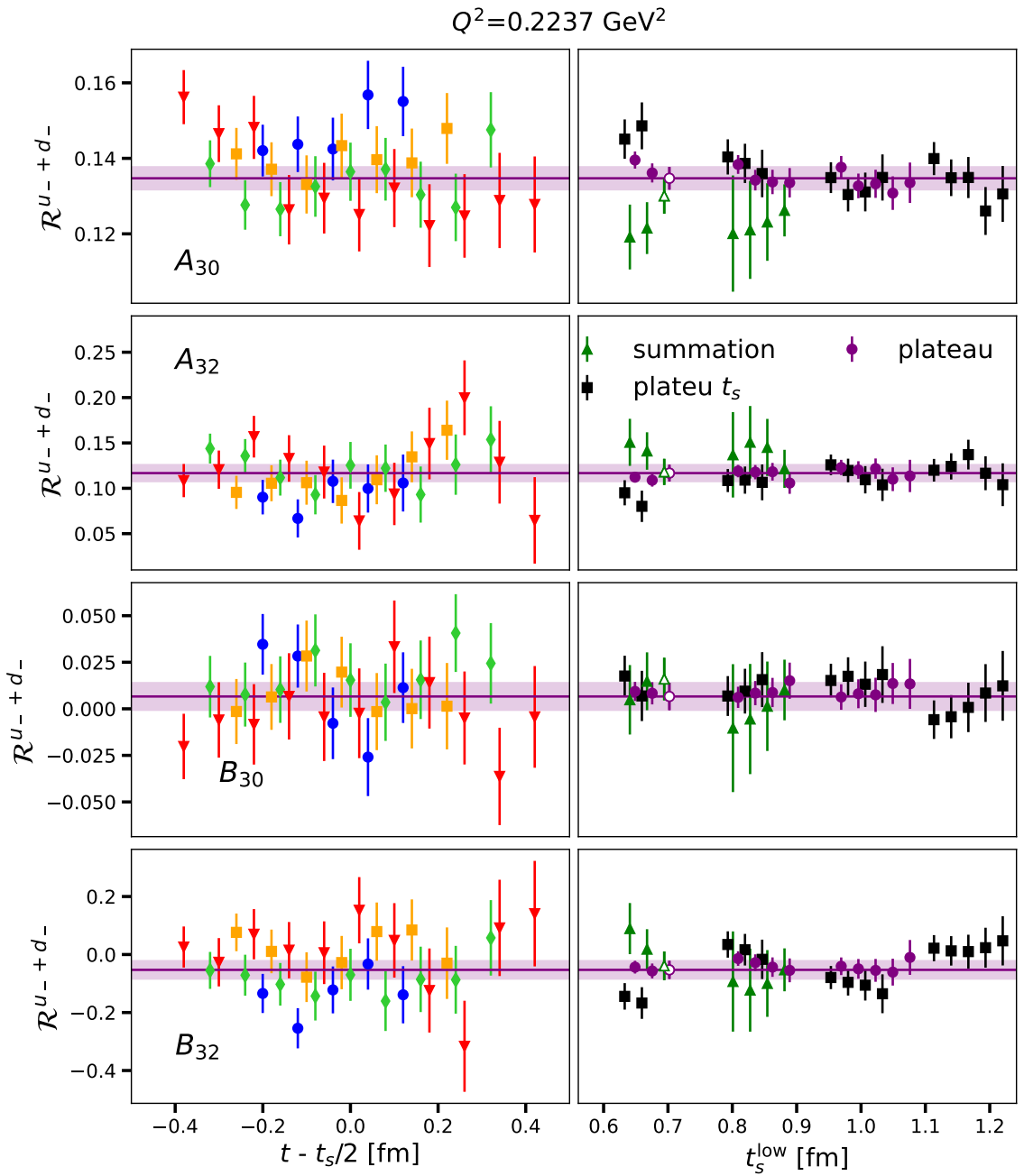

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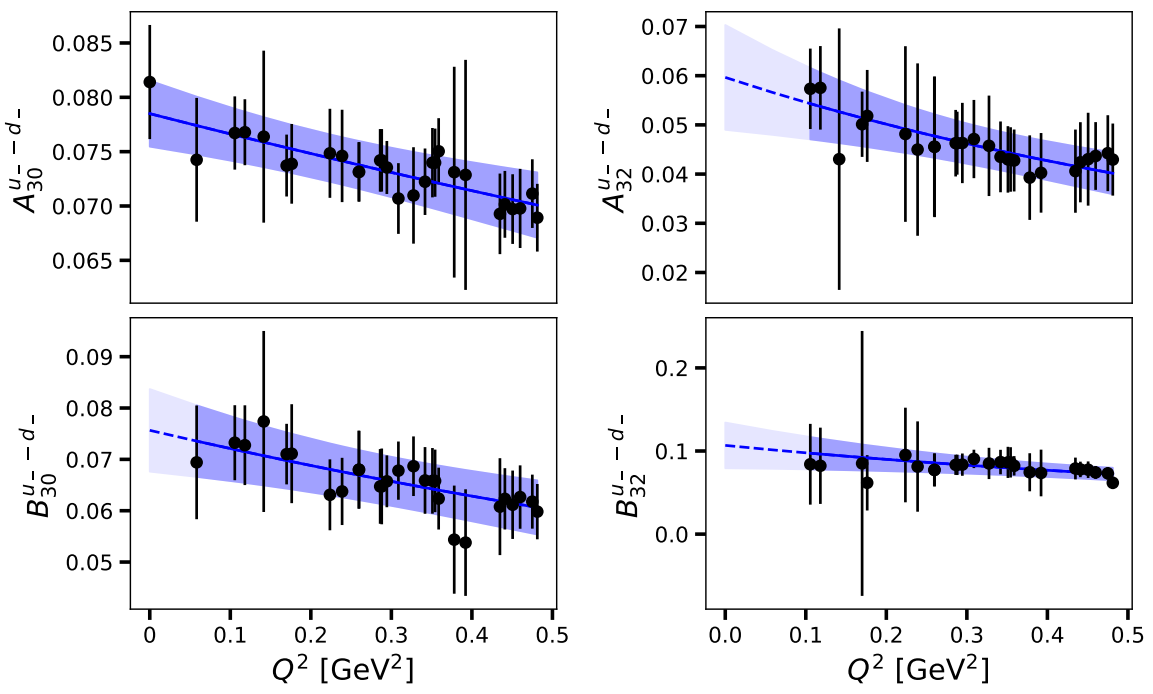
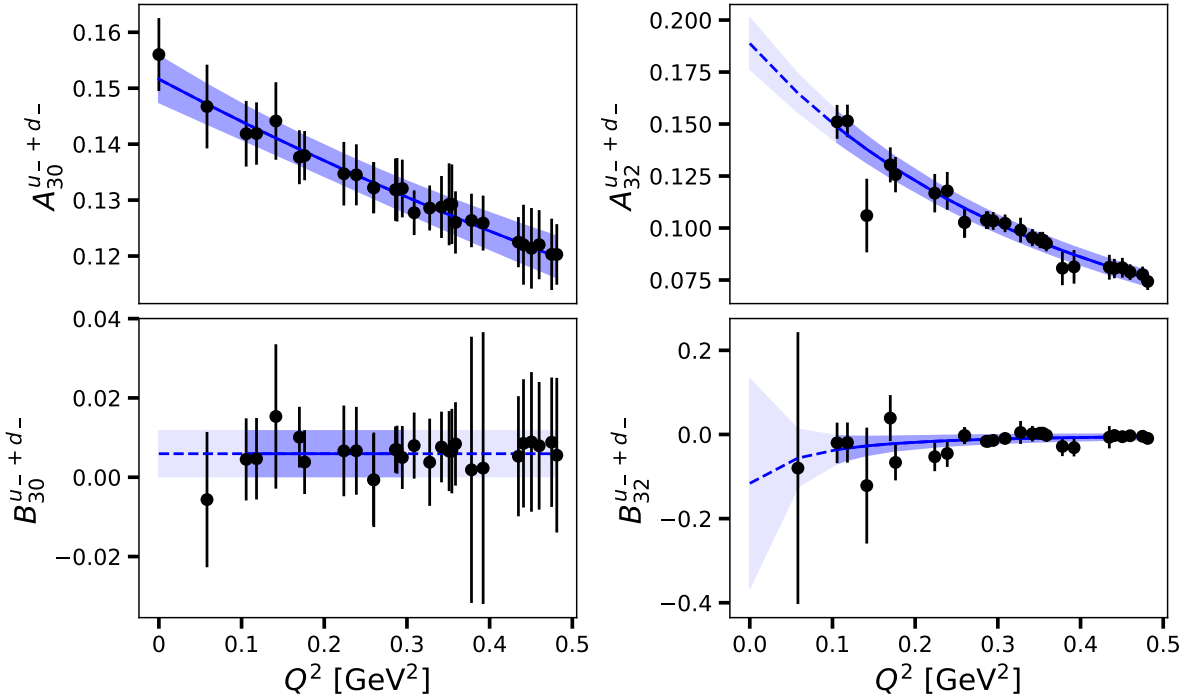
Results

- Momentum transfer
- Two-derivative operators
- Dipole fit

$$G(Q^2) = \frac{G(0)}{(1 + \frac{Q^2}{M^2})^2}$$





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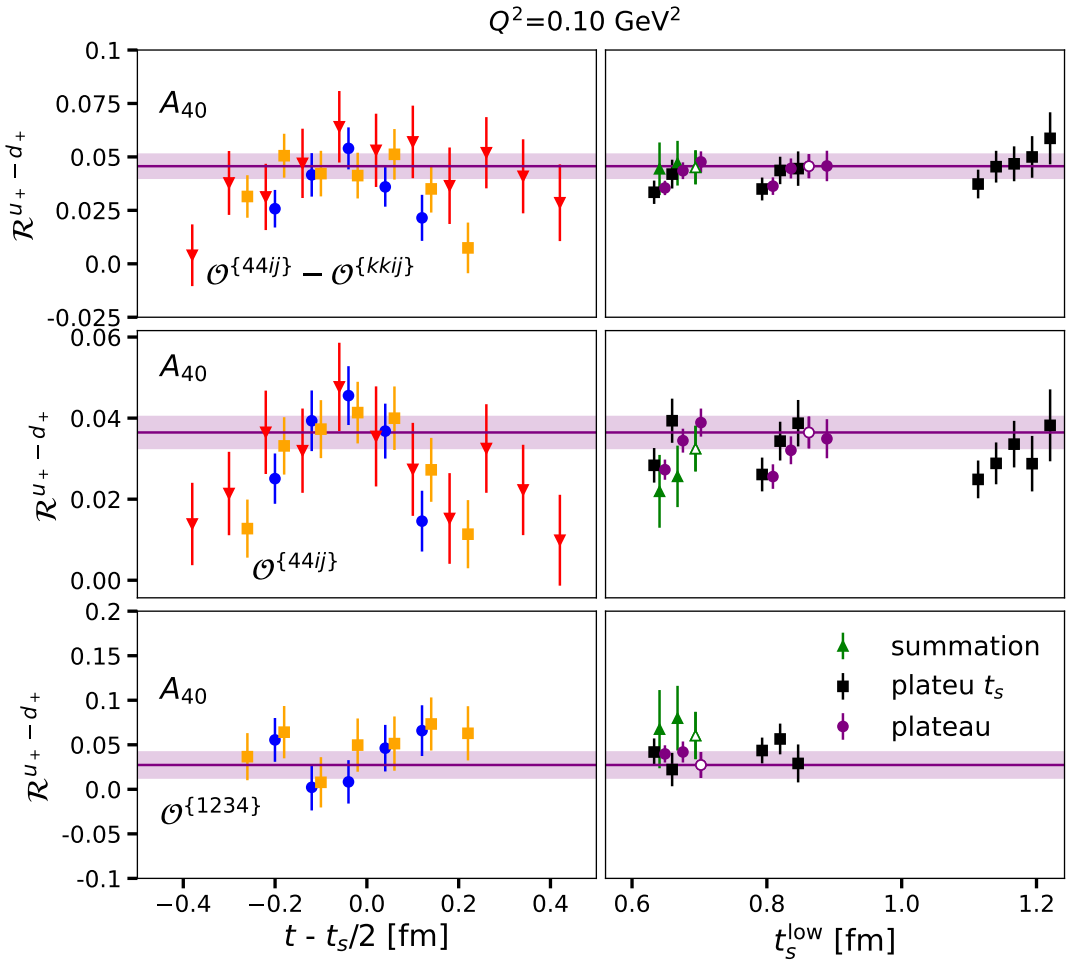
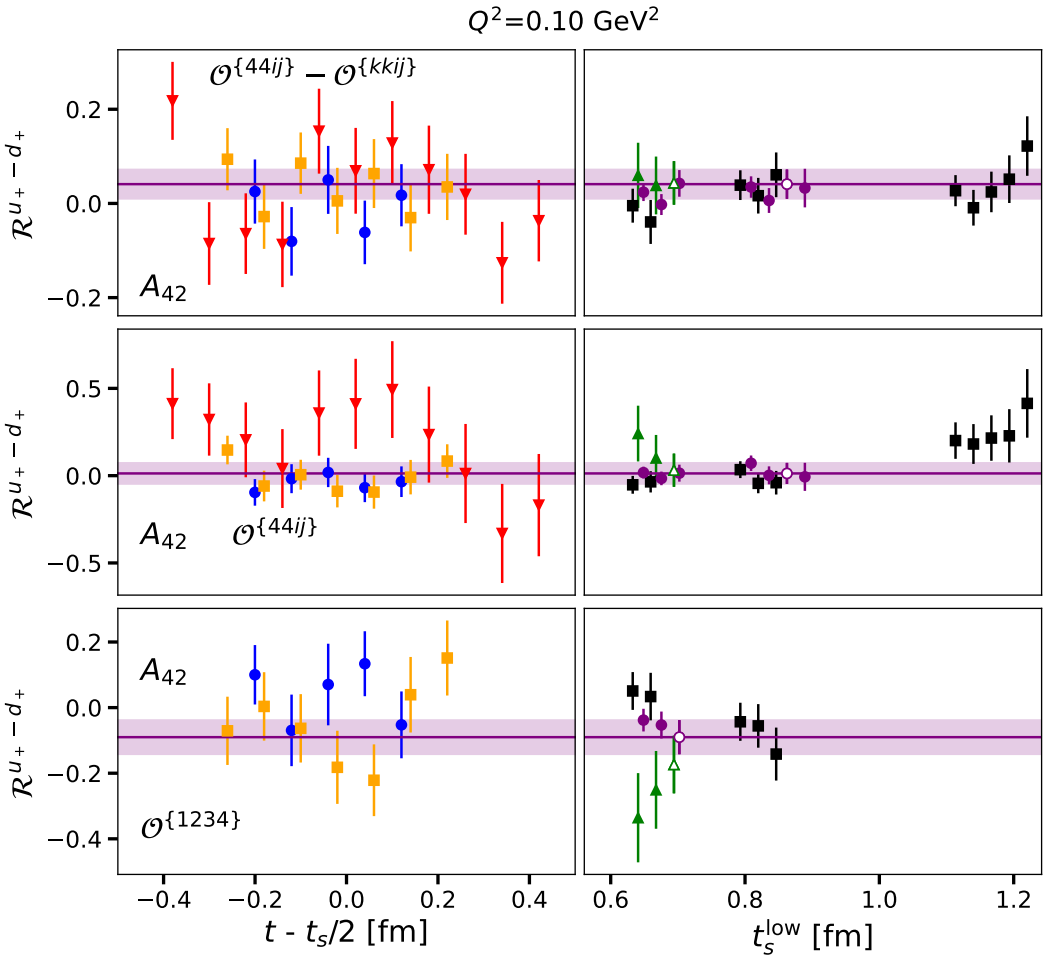
Results

- Three-derivative operators, isovector





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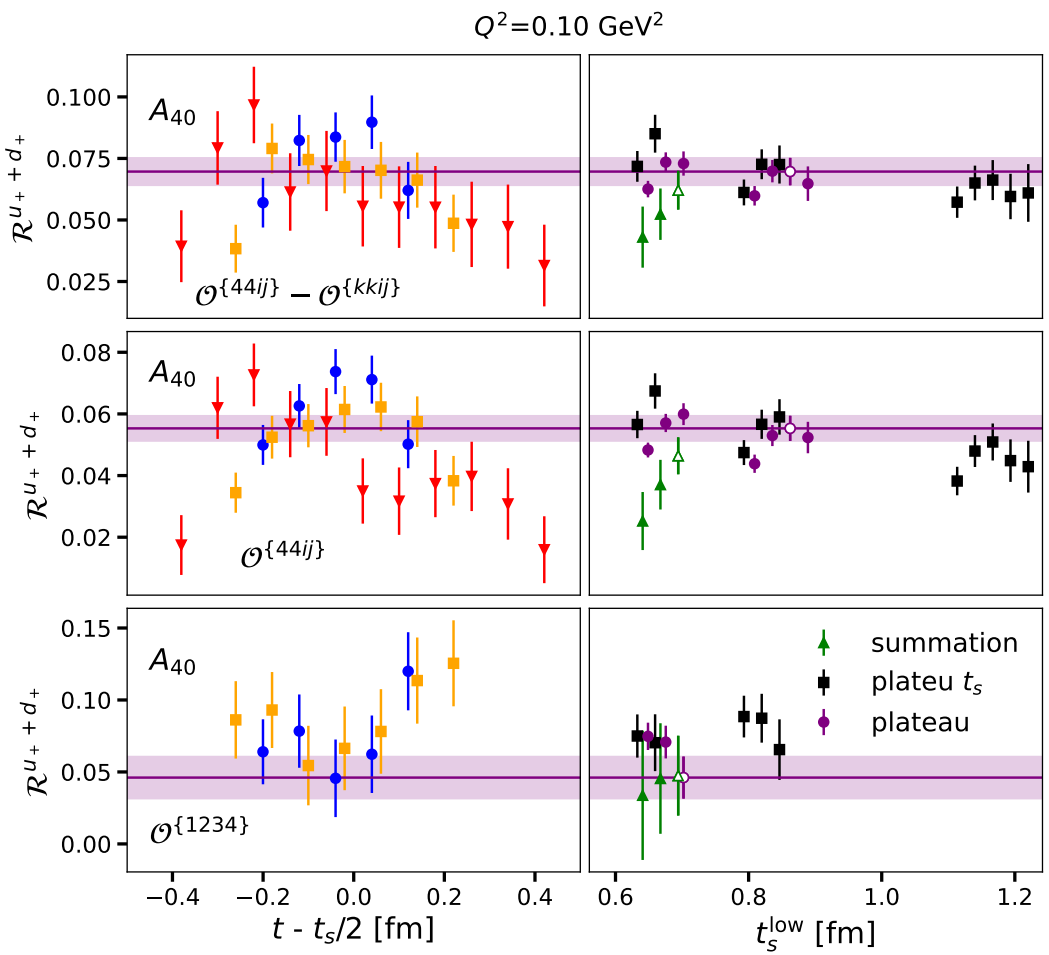
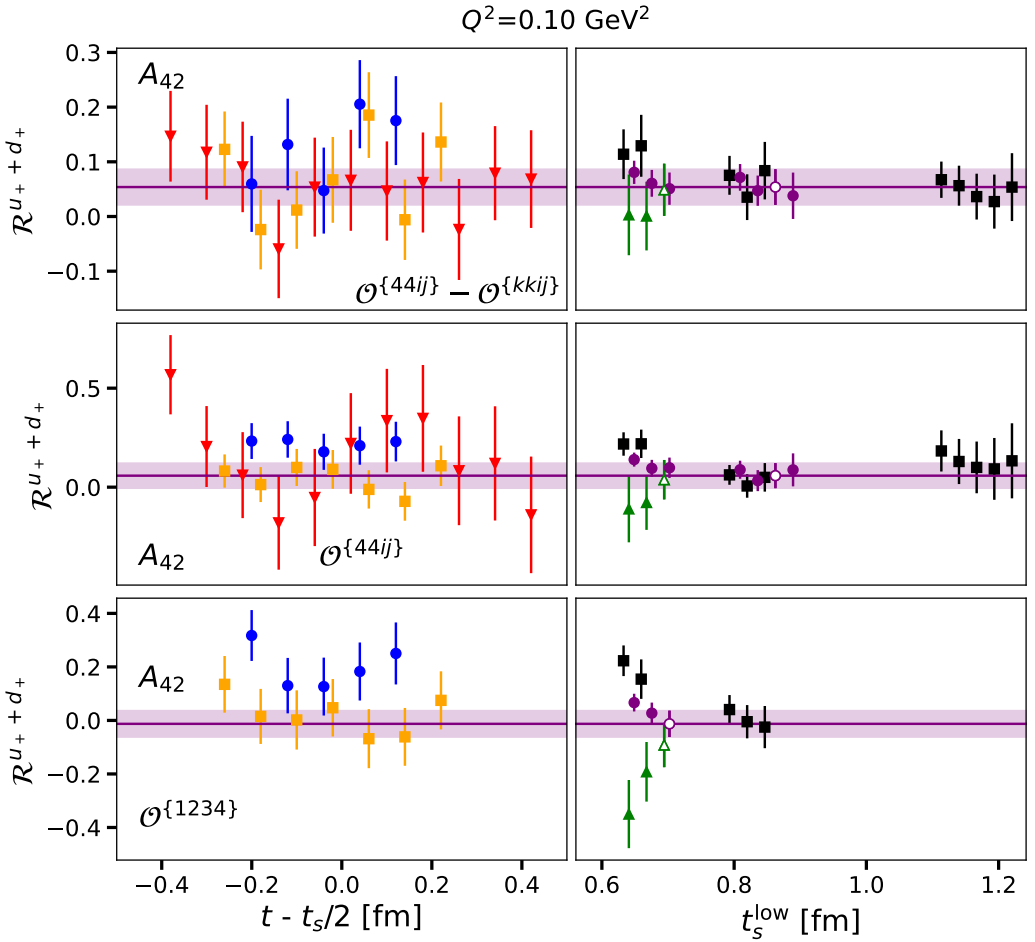
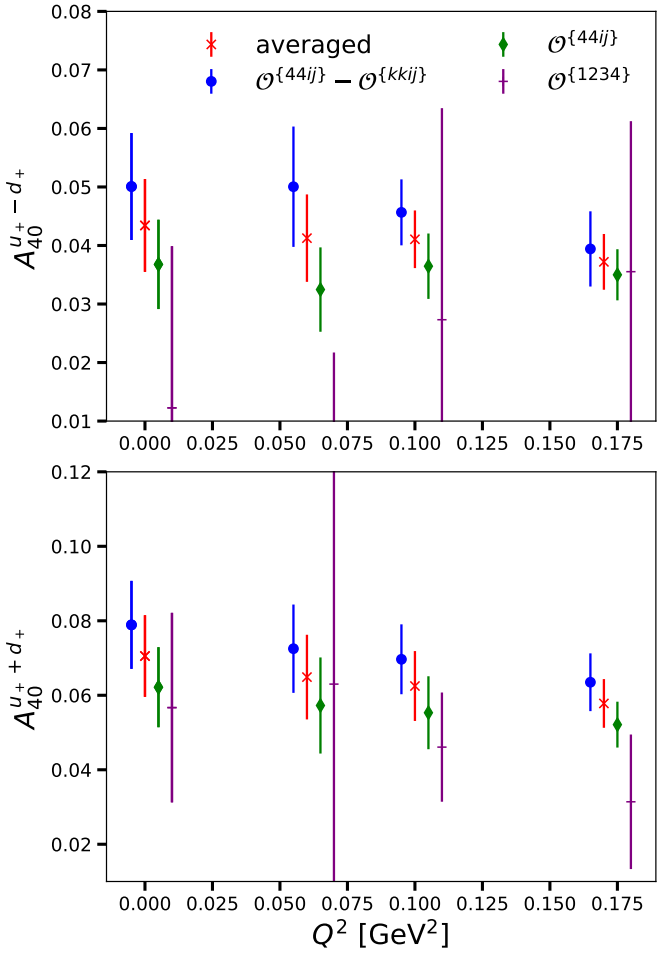
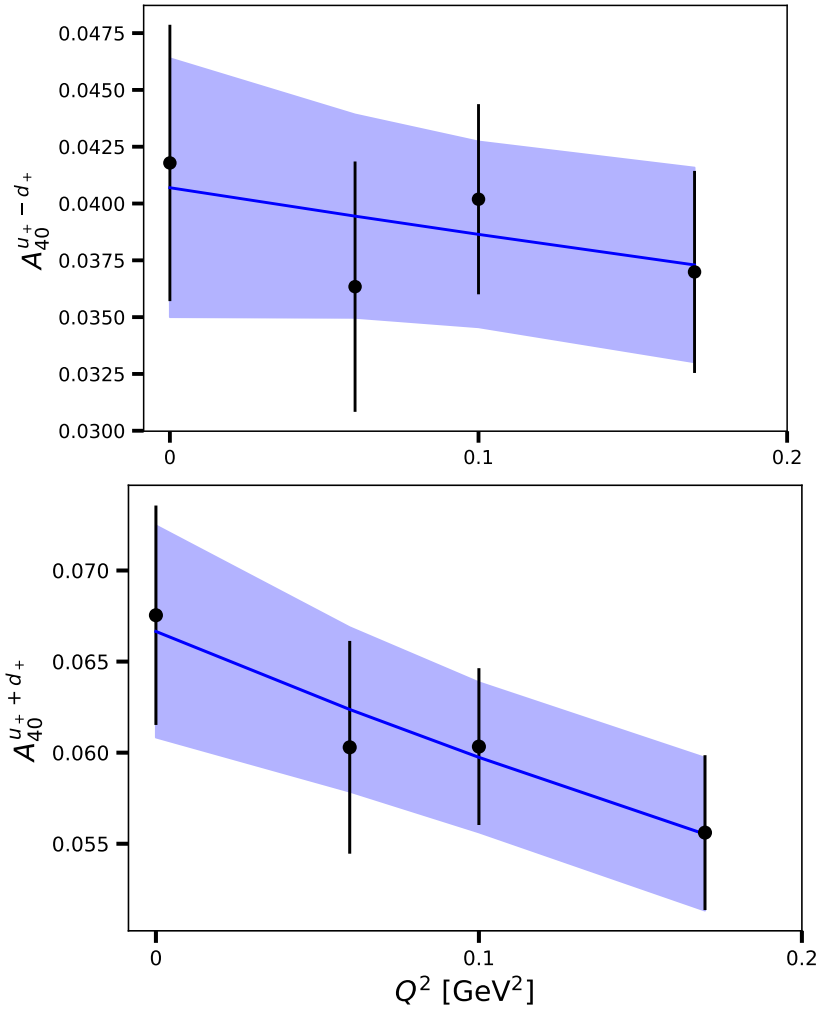




Results

- Only A_{40} nonzero





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Comparison with literature

- JAM and NNPDF are phenomenological studies

C. Cocuzza (JAM), PRD 106, L051502 (2022)

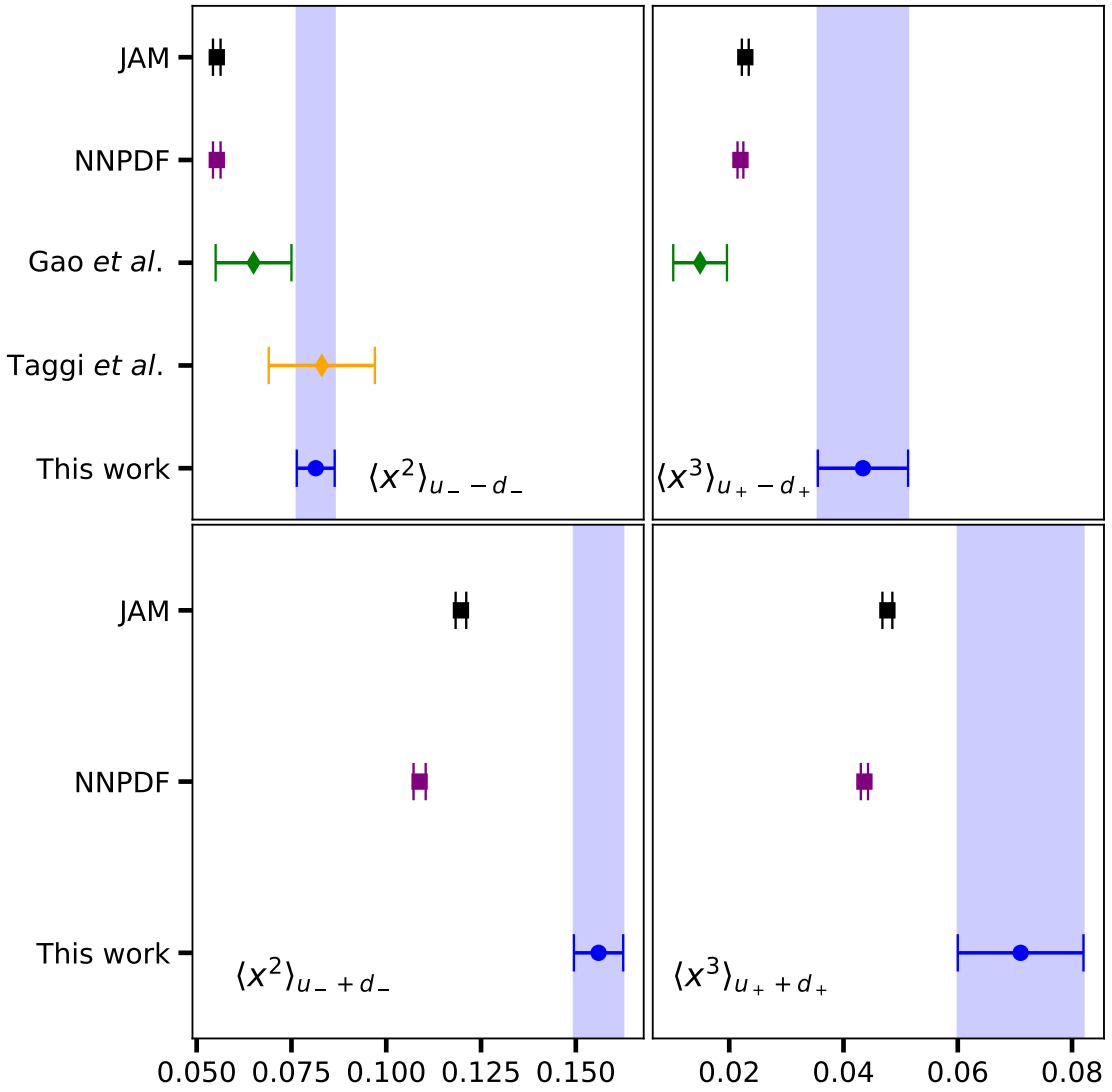
R.D. Ball (NNPDF), EPJC 82, 428 (2022)

- Gao uses non-local operators

X. Gao, PRD 107, 074509 (2023)

- Taggi is the first direct computation third moment

E. Taggi, (2026), 2605.02808



PDF reconstruction

- Ansatz

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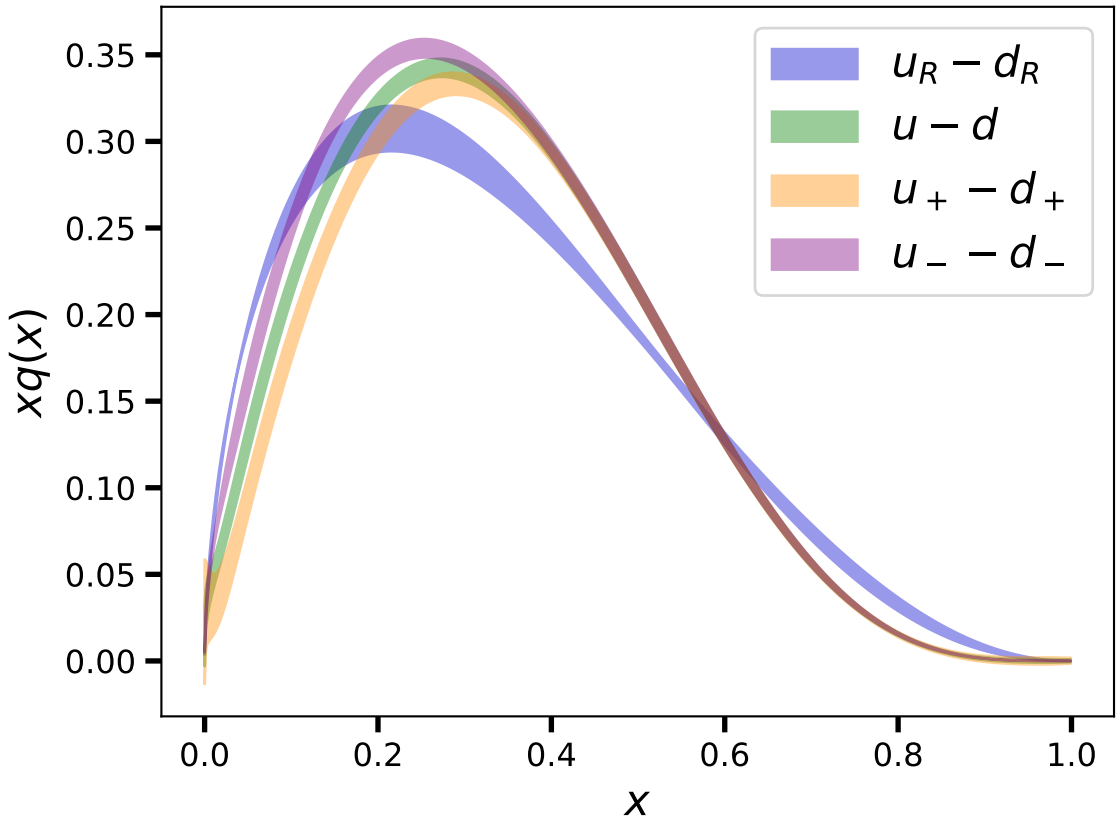
$$\langle x^{n-1} \rangle = \int_0^1 \mathrm{d}x\, x^{n-1}[q(x) - (-1)^{n-1}\bar{q}(x)]$$



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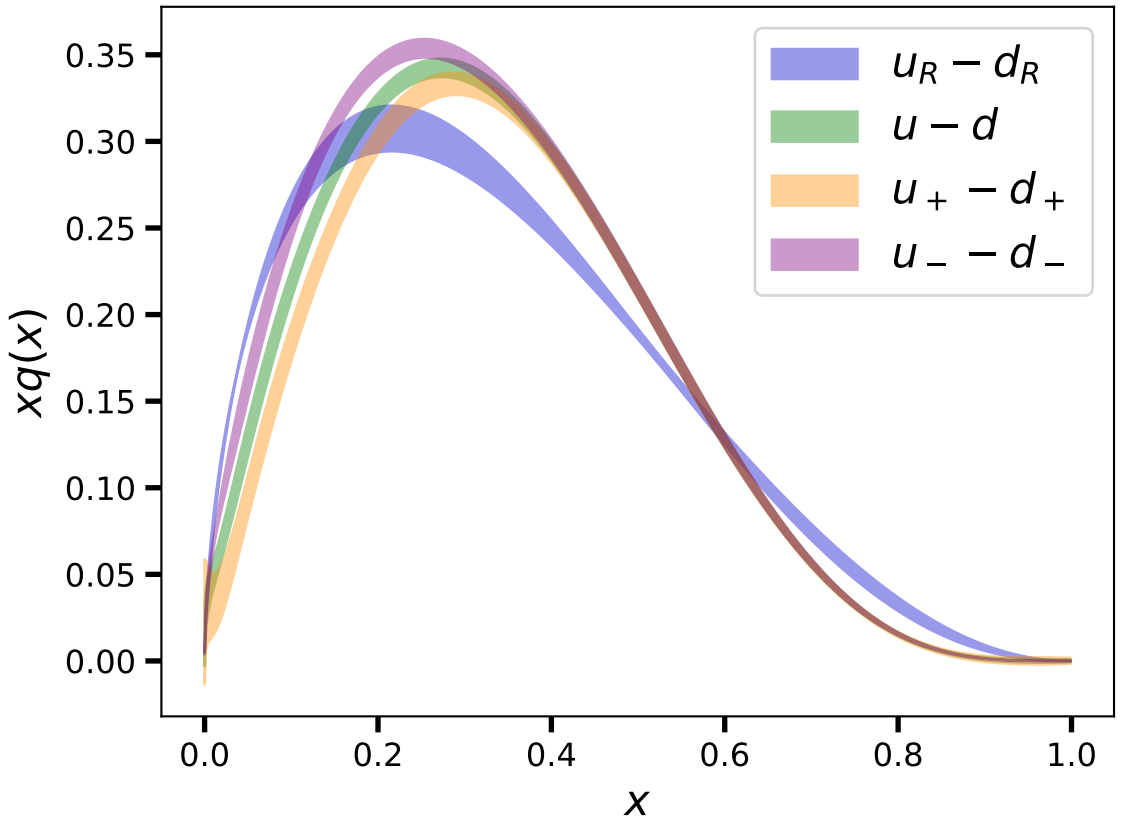
- Compare with reconstructed pdf



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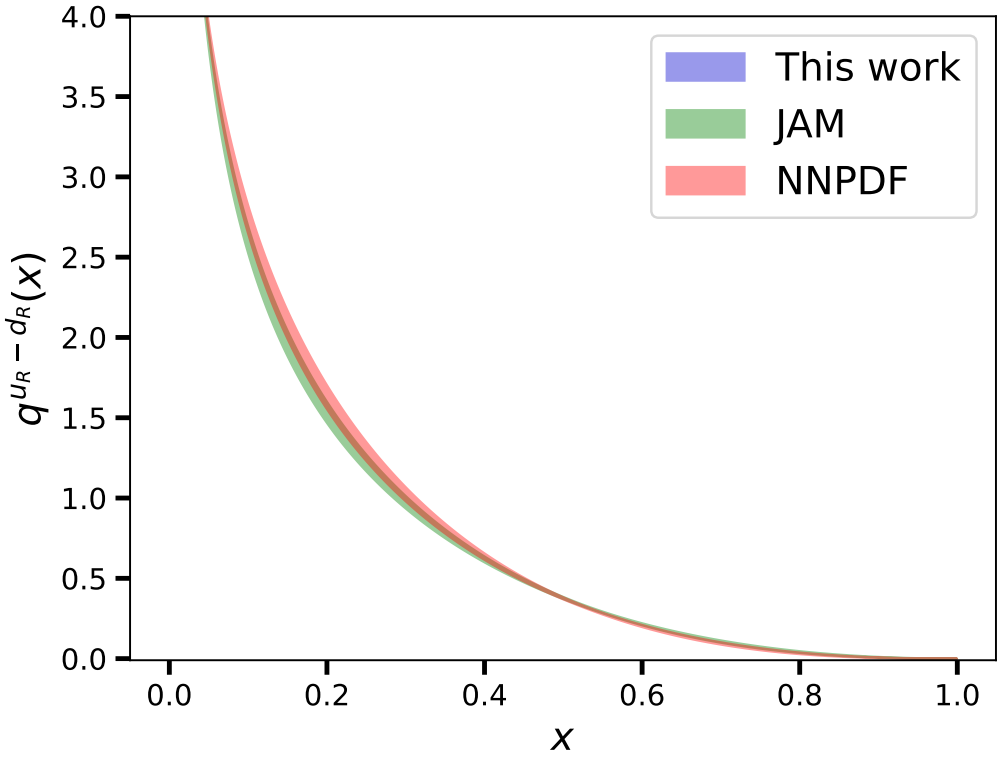

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Conclusions and Outlook

- Direct calculation of third nucleon moment
- First direct calculation of fourth moment
- First calculation of GFFs
- Reconstruction of isovector PDF

Conclusions and Outlook

- Direct calculation of third nucleon moment
- First direct calculation of fourth moment
- First calculation of GFFs
- Reconstruction of isovector PDF
- Take the continuum limit
- Investigate Axial and Tensor GFFs

Backup