Moments of light-cone distribution amplitudes from lattice QCD

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RQCD DA calculations

- **Introduction**
- **Moments of pseudoscalar twist-2 DAs**[1,2]
- **Moments of leading twist octet baryon DAs**[3]
- DAs from X-space and δ_2^π[4]
- **Normalization of the photon DA**[5]
- **Outlook**

1) RQCD: GB, VM Braun, M Göckeler, M Gruber, F Hutzler, P Korcyl, B Lang, A Schäfer, 1705.10236; 1811.06050;
RQCD: GB, VM Braun, S Bürger, M Göckeler, M Gruber, F Hutzler, P Korcyl, A Schäfer, A Sternbeck, P Wein, 1903.08038 (with "addendum").

2) GB, V Braun, S Collins, A Schäfer, J Simeth, 2106.05398.

3) RQCD: GB, VM Braun, S Bürger, S Collins, M Göckeler, M Gruber, F Hutzler, P Korcyl, A Schäfer, W Söldner, A Sternbeck, P Wein, 1903.12590;
RQCD: GB, V Braun, S Bürger, M Göckeler, M Gruber, F Kaiser, B Kniehl, O Veretin, P Wein, 2411.19091.

4) RQCD: GB, VM Braun, B Gläßle, M Göckeler, M. Gruber, F Hutzler, P Korcyl, B Lang, A Schäfer, P Wein, J-H Zhang, 1709.04325; 1807.06671.

5) GB, G Endrődi, S Piemonte, 2004.08778.

Distribution amplitudes

Light front wavefunction (distribution amplitude, DA) of a hadron parameterizes the distribution of the longitudinal momentum among its partons.

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Wavefunction of a hadron (here pion) near the infinite momentum frame, written as a superposition of different Fock states:

$$|\pi\rangle = c_1|\bar{q}q\rangle + c_2|\bar{q}gq\rangle + c_3|\bar{q}q\bar{q}q\rangle + \ldots$$

At leading twist (twist-2) only the valence quarks contribute (unlike in PDFs):

$$x = x_q = 1 - x_{\bar{q}}, \quad \xi = x_q - x_{\bar{q}} = 2x - 1 \in [-1, 1].$$

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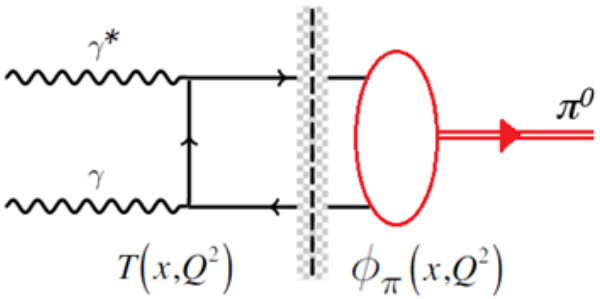
PDFs are (within the parton model) single particle probability densities and can **directly be extracted from fits to DIS and SIDIS data**.

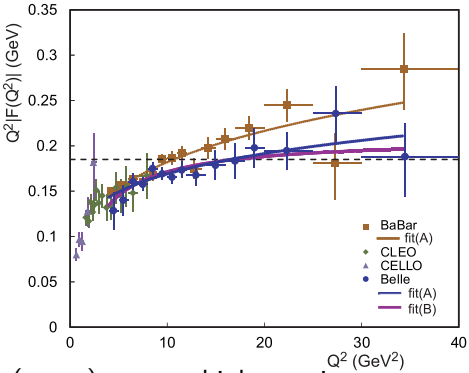
DAs are wavefunctions and only appear within convolutions in hard exclusive processes. Due to other hadronic uncertainties (and experimental techniques), it is much **harder to extract these reliably from experimental data**.

Distribution amplitudes 2

DAs appear in theoretical description of hard exclusive processes.

Example: collinear factorization of the $\gamma\gamma^* \to \pi^0$ photoproduction formfactor ($Q \gtrsim \mu \gg \Lambda$).





$$F_{\pi\gamma}(Q^2) = \frac{2F_\pi}{3}\int_0^1 \mathrm{d}x \underbrace{T_{\bar{q}q\gamma}(x,\mu_F,Q^2)}_{\text{hard coefficient function}} \cdot \underbrace{\phi_\pi(x,\mu_F)}_{\text{soft factor (DA)}} + \underbrace{\text{higher twist}}_{F_\pi\cdot\mathcal{O}(1/Q^4)}.$$

$F_\pi \approx 92\,\text{MeV}$. Factorization scale $\mu_F \sim Q/2$. $\exists\,\mathcal{O}(1/Q^4)$ non-factorizable soft contribution.

Definition of meson DAs

Non-local light front matrix element at a separation n ($n^2=0$):

$$\left\langle 0\left|\bar d\left(\frac{n}{2}\right)\not n\gamma_5\left[\frac{n}{2},-\frac{n}{2}\right]u\left(-\frac{n}{2}\right)\right|\pi^+(p)\right\rangle$$
$$= i\sqrt{2}\,F_\pi\,n\cdot p\int_0^1\mathrm{d}x\,\exp\left\{i\underbrace{[x-(1-x)]}_{=\xi}\frac{(n\cdot p)}{2}\right\}\phi_\pi(x,\mu)$$

$[n/2,-n/2]$ above denotes a gauge covariant connection.

The DA cannot directly be computed in Euclidean spacetime but its moments can:

$$\langle\xi^n\rangle=\int_0^1\mathrm{d}x\,(2x-1)^n\phi_\pi(x,\mu)\,,\quad\langle\xi^0\rangle=1\,,\quad\langle\xi^1\rangle=0.$$

$\langle\xi^{0,2}\rangle$ can be extracted from local matrix elements $\langle 0|O^\pm_{\mu\nu\rho}|\pi^+(p)\rangle$ with

$$O^\pm_{\mu\nu\rho}=\bar d\left\{\left[\overleftarrow{D}_{(\mu}\overleftarrow{D}_\nu\pm 2\overleftarrow{D}_{(\mu}\overrightarrow{D}_\nu+\overrightarrow{D}_{(\mu}\overrightarrow{D}_\nu\right]\gamma_{\rho)}\gamma_5\right\}u,$$

where $(\cdots)$ gives a traceless, symmetrized expression.

Mesons: Gegenbauer moments

Gegenbauer expansion (only even moments for the pion):

$$\phi_\pi(x,\mu) = 6x(1-x)\left[1 + \sum_{n\in\mathbb{N}} a_{2n}^\pi(\mu)\, C_{2n}^{3/2}(2x-1)\right]$$

Collinear conformal symmetry: $C_n^{3/2}(\xi)$ in $\mathrm{SL}(2,\mathbb{R})$ analogous to $Y_{\ell m}(\theta,\phi)$ in $\mathrm{SO}(3)\cong\mathrm{SU}(2)$.

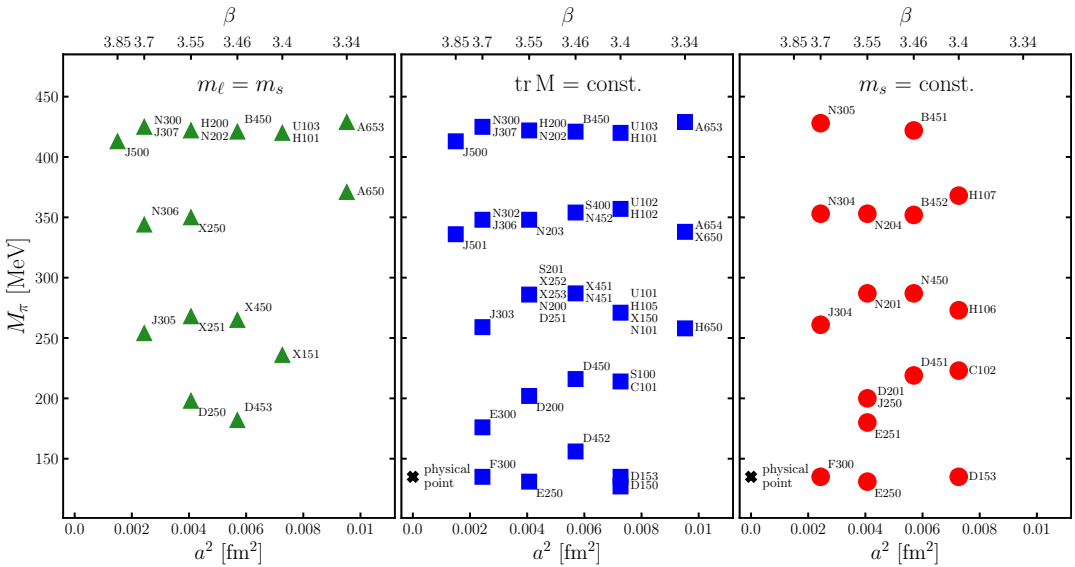
$\langle\xi^{2n}\rangle$ and a_{2n}^π are related by simple algebraic expressions ($n=1$ example):

$$a_2^\pi(\mu) = \frac{7}{12}\left(5\langle\xi^2\rangle - 1\right) = \frac{7}{12}\left(5\langle\xi^2\rangle - \langle\xi^0\rangle\right)$$

$a_{2n}^\pi(\mu) \to 0$ as $\mu\to\infty$: At large scales the lower moments will dominate.

Note the difference in the counting: 2nd DA-moment $\sim$ 3rd PDF-moment.

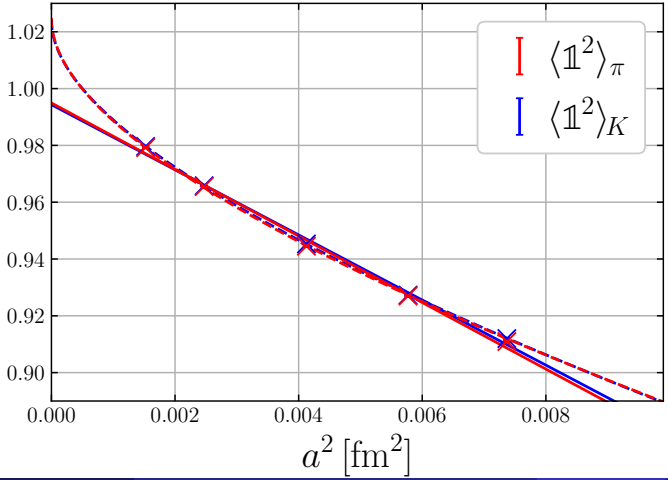
CLS ensemble overview



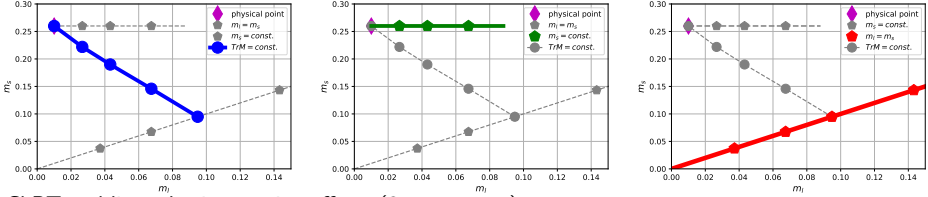
Results 1: test of $\langle \xi^0 \rangle \to 1 + \mathcal{O}(\alpha_s^4) \quad (a \to 0)$

Lattice $\to$ RI'-SMOM non-perturbatively, RI'-SMOM $\to \overline{\rm MS}$ at 3-loop (N^{3}LO).

Test of renormalization: $\langle \xi^0 \rangle = 1 + \mathcal{O}(a) \sim (\zeta_{22} - 5\zeta_{12}) O^+$



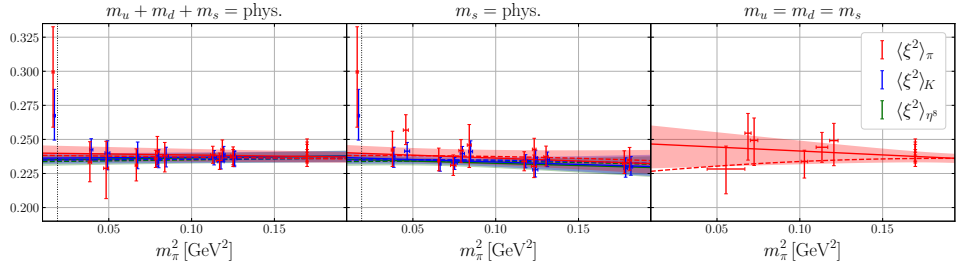
Results 2: quark masss extrapolation



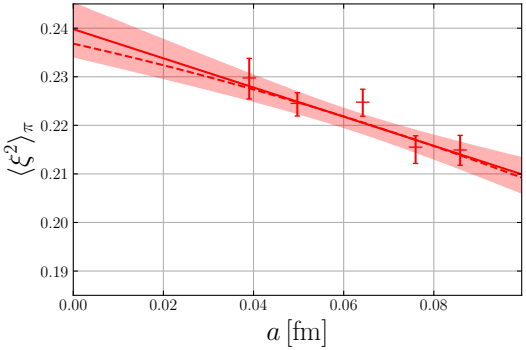
SU(3) NLO ChPT and linear lattice spacing effects (8 parameters):

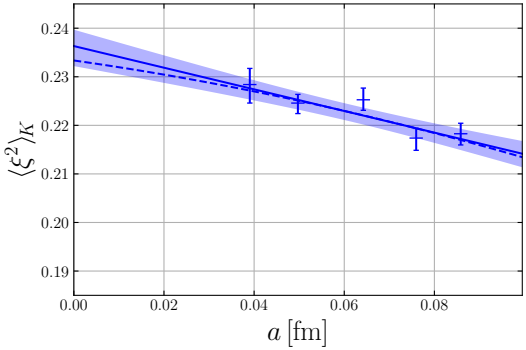
$$\xi_X^2(M_\pi, M_K, a) = \left[\xi_0^2 + \overline{A}\overline{M}^2 + A_X \delta M^2\right]\left[1 + a\left(c_0 + \overline{c}\overline{M}^2 + c_X \delta M^2\right)\right],\ X \in \{\pi, K, \eta_8\},$$

$$A_\pi = -2A_K = -A_{\eta_8},\ \ \delta M^2 = 2M_K^2 - 2M_\pi^2,\ \ \overline{M}^2 = \frac{1}{3}\left(2M_K^2 + M_\pi^2\right).$$



Results 3: the continuum limit





Continuum limit, at the physical point: $(a_2 \propto 5\langle\xi^2\rangle - 1)$

$$a_2^{\pi,\overline{\rm MS}}(2\,{\rm GeV}) = 0.116^{+16}_{-17}(4)_r(9)_a(5)_m, \qquad \langle\xi^2\rangle_\pi = 0.240^{+6}_{-6}(2)_r(3)_a(2)_m,$$

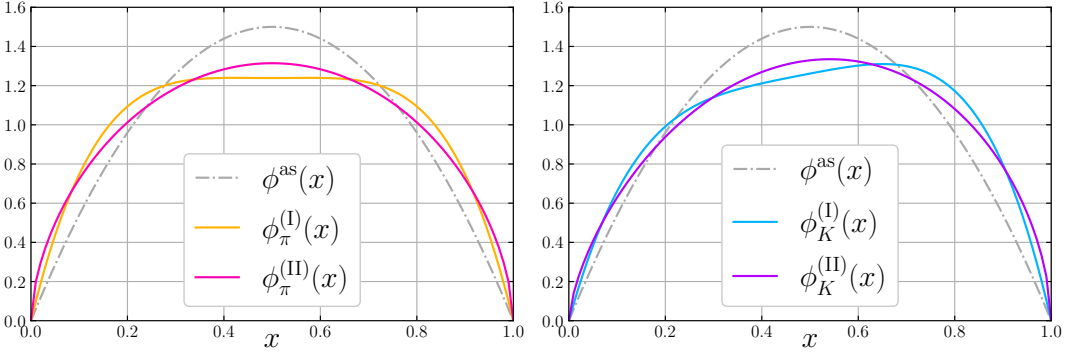
$$a_2^{K,\overline{\rm MS}}(2\,{\rm GeV}) = 0.106^{+10}_{-12}(4)_r(9)_a(4)_m, \qquad \langle\xi^2\rangle_K = 0.236^{+3}_{-4}(1)_r(3)_a(1)_m,$$

$$a_1^{K,\overline{\rm MS}}(2\,{\rm GeV}) = 0.0525^{+17}_{-19}(3)_r(20)_a(17)_m = \tfrac{5}{3}\langle\xi^1\rangle_K.$$

Connecting the moments to DA models

$\phi^{(\mathrm{I})}(x) = 6x(1-x)\left[1 + a_1 C_1^{3/2}(\xi) + a_2 C_2^{3/2}(\xi)\right]$, $\phi^{(\mathrm{II})}(x) = \frac{\Gamma(2+\alpha^+ +\alpha^-)}{\Gamma(1+\alpha^+)\Gamma(1+\alpha^-)}\, x^{\alpha^+}(1-x)^{\alpha^-}$

with a_1, a_2 fixed from our analysis ($\overline{\mathrm{MS}}$ at 2 GeV). π: $\alpha^+ = \alpha^-$, $a_1 = 0$.

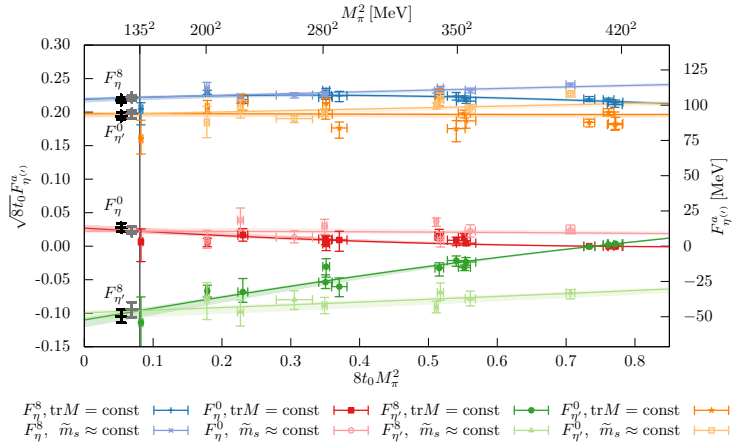


Model I: $\langle\xi^4\rangle_\pi = 0.109^{+5}_{-5}$. Model II: $\langle\xi^4\rangle_\pi = 0.112^{+7}_{-6}$. Discrimination requires very high precision of $\langle\xi^4\rangle_\pi$! Is a 1% error on $\langle\xi^4\rangle_\pi$ achievable? The present error on $\langle\xi^2\rangle_\pi$ is 3.0%.

At low Q^2 higher twist may be more relevant?

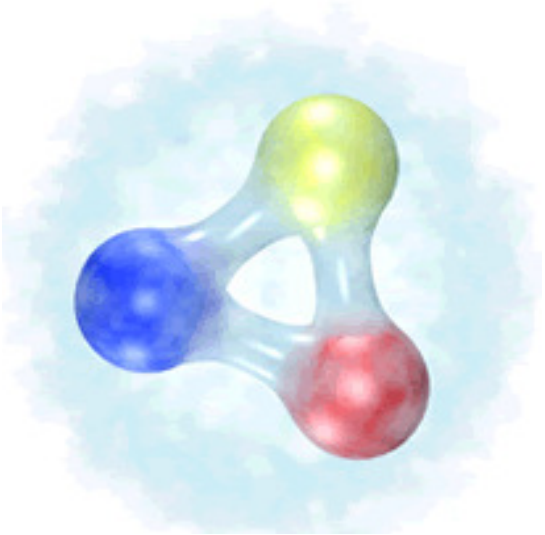
Outlook: towards moments of η and η' DAs

η/η' decay constants (normalization of twist-2 DAs) [RQCD: 2106.04398]



Data shifted to the continuum limit. $F_\pi \approx 92\,\text{MeV}$. Singlet decay constants given in $\overline{\text{MS}}$ scheme at the scale $\mu = \infty$.

Baryons



Baryon distribution amplitudes

f, g, h: quark flavours. p: $fgh = uud$, Σ^0, Λ: $fgh = sud$, Ξ^0: $fgh = ssu$.

$$\langle 0|f_\alpha(a_1 n)g_\beta(a_2 n)h_\gamma(a_3 n)|B_{p,\lambda}\rangle = \frac{1}{4}\int[\mathrm{d}x]\; e^{-ip\cdot n\sum_i a_i x_i}\left[v^B_{\alpha\beta;\gamma} V^B(x_1,x_2,x_3)\right.$$
$$\left. + a^B_{\alpha\beta;\gamma} A^B(x_1,x_2,x_3) + t^B_{\alpha\beta;\gamma} T^B(x_1,x_2,x_3) + \ldots\right]$$

$$\int[\mathrm{d}x] = \int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\delta(1-x_1-x_2-x_3)$$

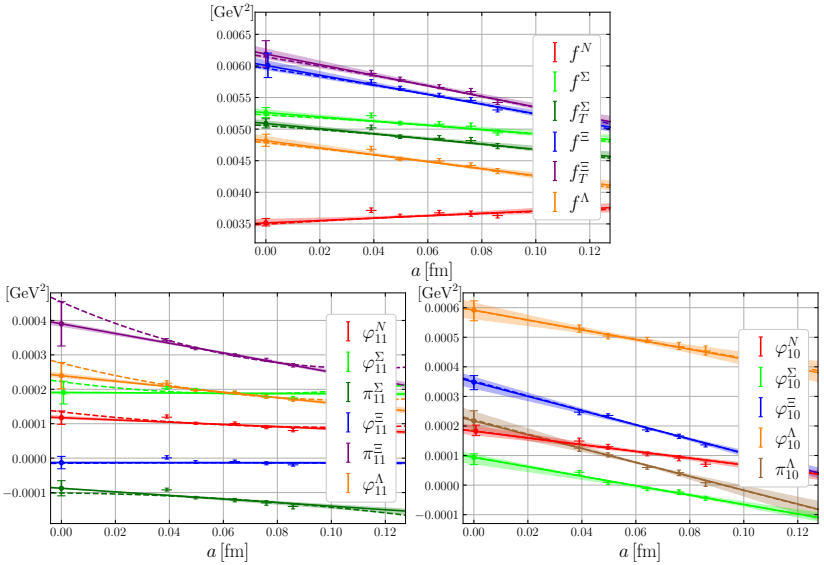
$$\Phi^B_+(x_1,x_2,x_3) = 120x_1x_2x_3[\varphi^B_{00} + \varphi^B_{11}7(x_1-2x_2+x_3)+\ldots] \propto (V-A)^B_{123} + (V-A)^B_{132}$$
$$\Phi^B_-(x_1,x_2,x_3) = 120x_1x_2x_3[\varphi^B_{10}21(x_1-x_3)+\ldots] \propto (V-A)^B_{123} - (V-A)^B_{132}$$
$$\Pi^{B\neq\Lambda}(x_1,x_2,x_3) = 120x_1x_2x_3[\pi^B_{00} + \pi^B_{11}7(x_1-2x_2+x_3)+\ldots] \propto T^B_{132}$$
$$\Pi^{\Lambda}(x_1,x_2,x_3) = 120x_1x_2x_3[\pi^\Lambda_{10}21(x_1-x_3)+\ldots] \propto T^B_{132}$$

Normalizations $f^B = \varphi^B_{00}$, $f^{B\neq\Lambda}_T = \pi^B_{00}$ similar to decay constants (no derivatives).

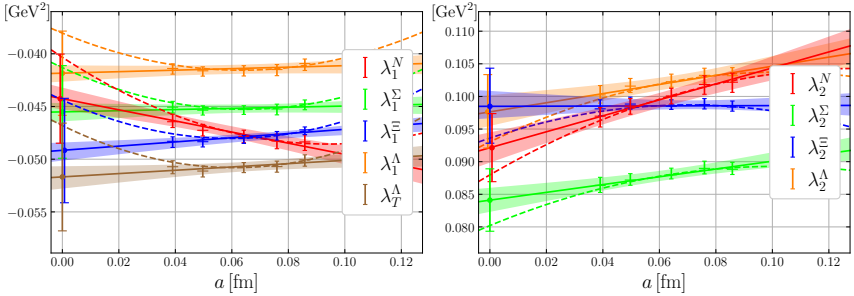
φ^B_{10}, π^Λ_{10}, φ^B_{11}, $\pi^{B\neq\Lambda}_{11}$ are first moments of twist-3 BDAs (operators contain a derivative).

Additional twist-4 BDA normalizations λ^B_1, λ^B_2, λ^Λ_T.

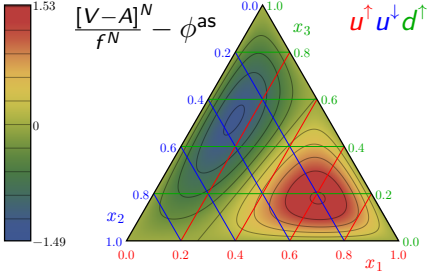
BDA results 1: the continuum limit

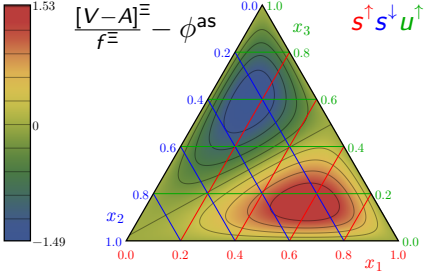


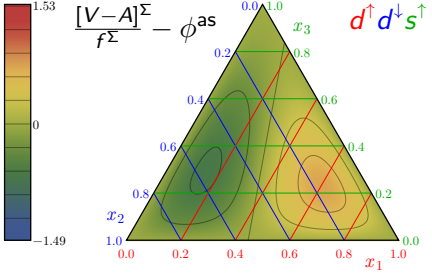
BDA results 2: twist-4 normalizations

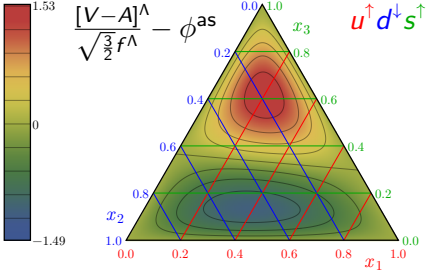


BDA results 3: "barycentric plots"









Photon distribution amplitude: the tensor coefficient

Constant background magnetic (not chromomagnetic) field F_{xy}.
A quark of flavour f carries the charge $q_f \in \{-e/3, 2e/3\}$.

$$\left\langle \bar{f}\sigma_{xy} f \right\rangle = -q_f\,\tau_f F_{xy} + \mathcal{O}(F^3).$$

Tensor coefficient τ_f undergoes additive and multiplicative renormalization:

$$\tau_{fr} = Z_T \tau_f - \tau_f^{\rm div}, \quad \tau_f^{\rm div} = m_f \frac{3}{4\pi^2} \log(\mu_{\rm QED}^2 a^2) + \mathcal{O}(\alpha_s).$$

It turns out that $f^\perp_{\gamma,f} = \tau_{fr}$ is the normalization of the distribution amplitude of finding $\bar{f}f$ inside a photon.

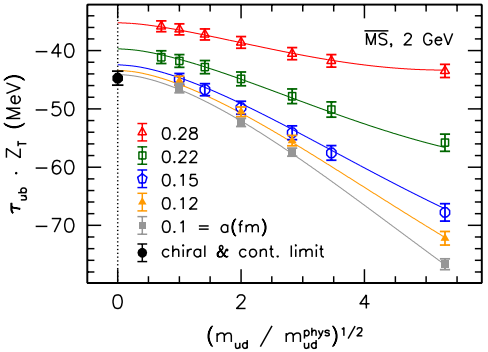
Using the background field method, it is also possible to compute higher moments, see [GB, G Endrődi, S Piemonte, 2004.08778] for details.

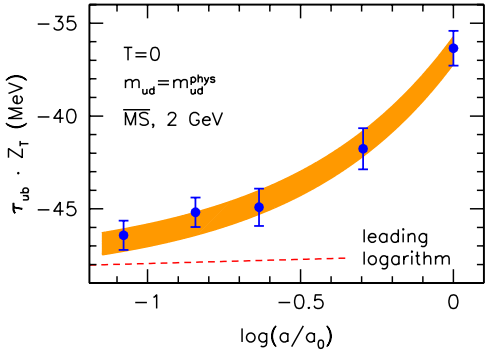
"Susceptibility of the quark condensate":

$X_u = \tau_{ur}/\langle \bar{u}u \rangle$, $X_u(1\,\mathrm{GeV}) = [542(11)\,\mathrm{MeV}]^{-2}$ (see the next slide for τ_{ur}).

Normalization of the photon distribution amplitude

τ_f diverges for $m_u > 0$ as discussed above.





$f_\gamma^\perp = \tau^{\overline{\rm MS}}(\mu = 2\,{\rm GeV}) = \lim_{a\to 0} \lim_{m_u \to 0} Z_T \tau_u = -44.7(1.2)\,{\rm MeV}$.

One can also obtain $f_\gamma^\perp$ for massive quarks. Difference between the lattice and $\overline{\rm MS}$ schemes: $({\rm finite}\,\#) \cdot m_f$. Applications: short distance hadronic light-by-light correction to $g_\mu - 2$ [J Bijnens et al, 1908.03331], $B \to \gamma\,\ell\bar{\nu}_\ell$ decay [T Janowski et al, 2106.13616] etc.

Outlook

Gegenbauer/Mellin moments: controlled continuum and physical quark mass limit, with renormalization to $\overline{\rm MS}$ at N^{3}LO (BDA at N^{2}LO, N^{3}LO now available). What next?

- $\exists$ [RQCD: 2106.04398] η and η' decay constants as well as $\langle 0|F\widetilde{F}|\eta^{(\prime)}\rangle$. What about higher twist-2 and twist-4 DA moments?
- Higher photon DA moments?
- Combine with X-space methods to explore higher twist contributions?

  $\delta_2^\pi(2\,\mathrm{GeV}) = 0.22(3)\,\mathrm{GeV}^2$ [RQCD:1807.06671]

  Can also be computed directly: $\epsilon^{\mu\nu\rho\sigma}\langle 0|\bar{d}\gamma_\nu \overleftrightarrow{D}_\rho \overleftrightarrow{D}_\sigma u|\pi^+\rangle = 4p^\mu\sqrt{2}F_\pi\,\delta_2^\pi$.
- Higher Gegenbauer moments are almost impossible to obtain within meaningful errors.

  Often these are not needed, but $\left\langle \frac{1}{\bar{x}} \right\rangle_M = 2\int_{-1}^{1}\frac{\mathrm{d}\xi}{1-\xi}\phi_M(\xi) = 3\sum_{n=0}^{\infty} a_n^M.$
- Renormalization at an order consistent with continuum calculations requires pQCD input.
- The future: combine different approaches within global fits.