Isovector moments of parton distribution functions from the lattice.

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Overview

Aim: precision determination of the lower moments of GPDs (generalised form factors, GFFs) for use as constraints in global phenomenological analyses.

★ [arXiv:2305.04717] and in preparation: S. Weishäupl, G. Bali, S. Collins, W. Söldner.

(Twist 2) **isovector moments for the baryon octet**: study $SU(3)$ flavour symmetry breaking

- Unpolarised: $\langle 1 \rangle_{u^- - d^-} \equiv g_V$, $\langle x \rangle_{u^+ - d^+}$.
- Helicity: $\langle 1 \rangle_{\Delta u^+ - \Delta d^+} \equiv g_A$, $\langle x \rangle_{\Delta u^- - \Delta d^-}$.
- Transversity: $\langle 1 \rangle_{\delta u^+ - \delta d^+} \equiv g_T$, $\langle x \rangle_{\delta u^- - \delta d^-}$.

★ [arXiv:2111.08306]: S. Bürger, T. Wurm, M. Löffler, M. Göckeler, A. Schäfer, ...

- Nucleon: twist-2 helicity third Mellin moment and twist-3 contribution.
- Some approximations made: results quoted for proton and neutron but disconnected omitted, ...

High precision with all systematics under control: $N_f = 2 + 1$ CLS ensembles.

Moments of PDFs

Mellin moment n, example of unpolarised PDF:

$$\langle x^{n-1}(\mu)\rangle_{q^{(-)^n}} = \int_0^1 \mathrm{d}x\, x^{n-1}\left[q(x,\mu) + (-)^n \bar{q}(x,\mu)\right]$$

Notation [PDFLattice,1711.07916]: $q^- := q - \bar{q}\,(= q_v)$, $q^+ := q + \bar{q}\,(= q_v + 2\bar{q})$.

OPE connects the moments to matrix elements of local operators

$$\left\langle N(p)\left|\left[\mathcal{O}^n_{\mu_1\ldots\mu_n}\right]\right|N(p)\right\rangle = \langle N(p)\,|[\mathcal{O}^n]|\,N(p)\rangle\,((p_{\mu_1}\cdots p_{\mu_n}) - \text{traces})$$

$$\langle x^{n-1}\rangle_{q^{(-)^n}} \sim \langle N(p)\,|[\mathcal{O}^n]|\,N(p)\rangle$$

$$\mathcal{O}^n_{\mu_1\ldots\mu_n} = \bar{q}\gamma_{(\mu_1} i\overleftrightarrow{D}_{\mu_2}\ldots i\overleftrightarrow{D}_{\mu_n)} q$$

where $(\mu_1\cdots)$: symmetrized, traceless.

First moments, in the isospin limit, can be related to the charges $g_{V,A,T}$.

Second moments ($n = 2$, e.g., $\langle x\rangle_{q^+}$) have one derivative (dimension 4 operators).

Moments of PDFs

Advantages of extracting information on the PDFs (GPDs) through moments (GFFs):

- ★ Determination of different flavour combinations are possible, also gluonic operators.
- ★ There are no higher twist corrections.

In principle, with knowledge of moments $\langle x^n \rangle$ to high enough n, PDF can be reconstructed. However,

- ★ Signal to noise deteriorates as n increases.
- ★ Renormalisation: reduced symmetry on the lattice.
  If mixing with lower $n' = n - \ell$ is not prohibited by symmetry $\Rightarrow a^{-\ell}$ power divergencies.

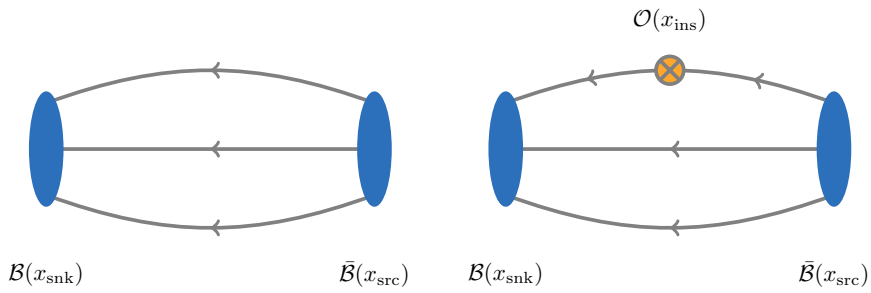
With standard methods, up to third or in some cases fourth moment possible.

Lower moments can be used as constraints in phenomenological analyses, e.g. [JAM 18,1710.09858] (averaged g_T), [JAM 22,2205.00999] (ETMC 19, g_T) and [JAM 23,2306.12998] ($\langle 1 \rangle_{\delta u^+}$ and $\langle 1 \rangle_{\delta d^+}$ from PNDME 18 and ETMC 21).

New developments: gradient flow [Shindler,2311.18704], [Francis et al.,2509.02472], ...

- ★ Improved signal and operator mixing like in the continuum.

Lattice details:



Steps in the analysis:

- **Fit** $C_{2pt}^{(B)}(t)$ (spectrum) and $C_{3pt}^{(B)}(t,\tau)$ ($\rightarrow \langle B|\mathcal{O}|B\rangle$). For $C_{3pt}^{(B)}(t,\tau)$, isovector flavour combinations only lead to connected diagrams.
- **Renormalisation**: Isovector combination: $\mathcal{O}^{\overline{\rm MS}}(\mu) = Z_{\mathcal{O}}^{\overline{\rm MS},latt}(a\mu)\mathcal{O}^{latt}(a)$
  Non-perturbative matching: lattice $\rightarrow$ RI′-(S)MOM scheme,
  Perturbative: RI′-(S)MOM $\rightarrow$ $\overline{\rm MS}$ (3-loop used, 4-loop is desirable).
  Not a significant source of systematic uncertainty anymore.

Repeat analysis on several ensembles to explore

- **Finite volume effects**: exponentially suppressed $\sim M_\pi^2 e^{-LM_\pi}/(LM_\pi)^{3/2}$, $LM_\pi > 4$.
- **Discretisation effects**: $\langle 1\rangle$ Symanzik-improved to $\mathcal{O}(a^2)$, $\langle x\rangle$ leading $\mathcal{O}(a)$.
- **Quark mass dependence**: chiral pert. theory (ChPT) $M_\pi \rightarrow M_\pi^{phys}$.

Challenges

Lattice provides (very) precise results for (see [FLAG21,2111.09849])

- α_s, m_q, $q \in \{u/d, s, c, b\}$, $K \to \pi\ell\nu$, form factor at $q^2 = 0$, $f_+(0) = 0.9698(17)$, $f_K/f_\pi = 1.1932(21)$, ...

Difficulties in the baryon sector: **Statistical noise**: signal vs noise decays with $e^{-(E-3M_\pi/2)\tau}$.

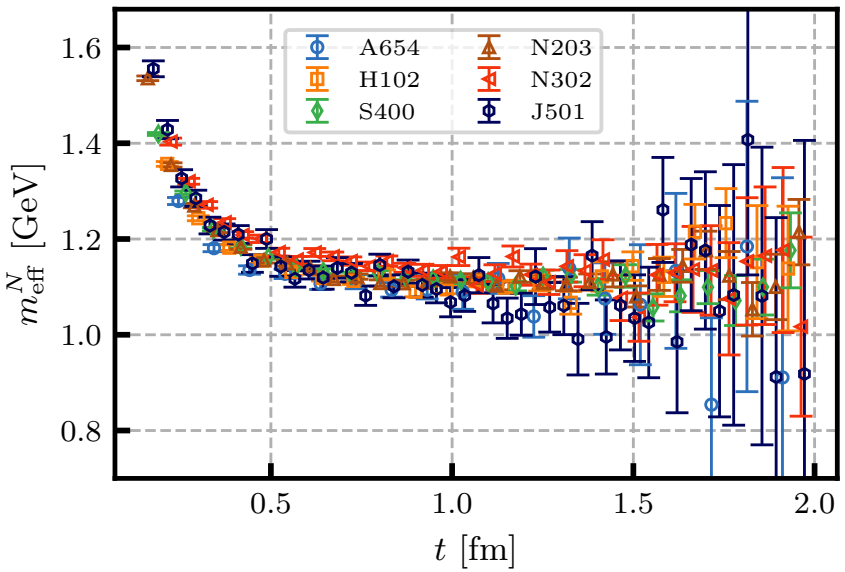
$$C^B_{2\text{pt}}(t) = \mathcal{P}^{\alpha\beta}_+ \sum_{\mathbf{x}'} \langle \mathcal{B}_\alpha(\mathbf{x}', t)\, \bar{\mathcal{B}}_\beta(\mathbf{0}, 0)\rangle,$$

$$= \sum_n |Z^B_n|^2 e^{-E^B_n t}$$

$n = 0, 1, \ldots$ and $Z^B_n \propto \langle n|\bar{\mathcal{B}}|\Omega\rangle$

For $t \to \infty$,

$$E^B_0 \approx m^B_{eff} = \ln \frac{C_{2pt}(t)}{C_{2pt}(t+a)}$$

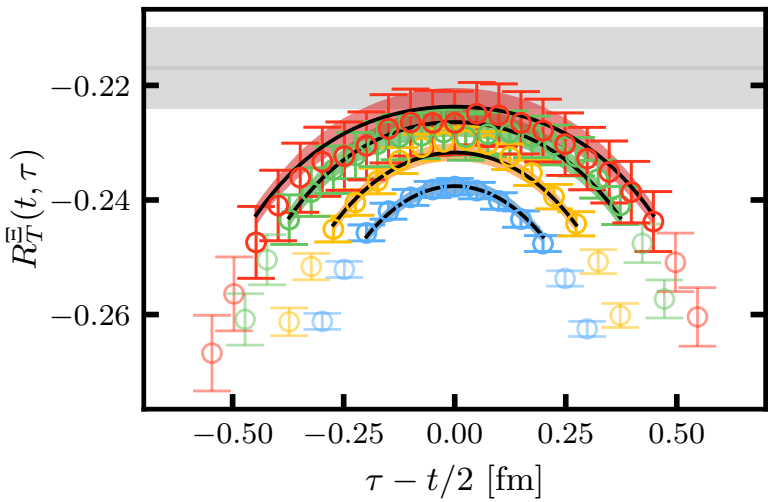


$M_\pi \sim 340$ MeV, $M_K \sim 450$ MeV,
$a = 0.098$ fm (A654) to $a = 0.039$ fm (J501)

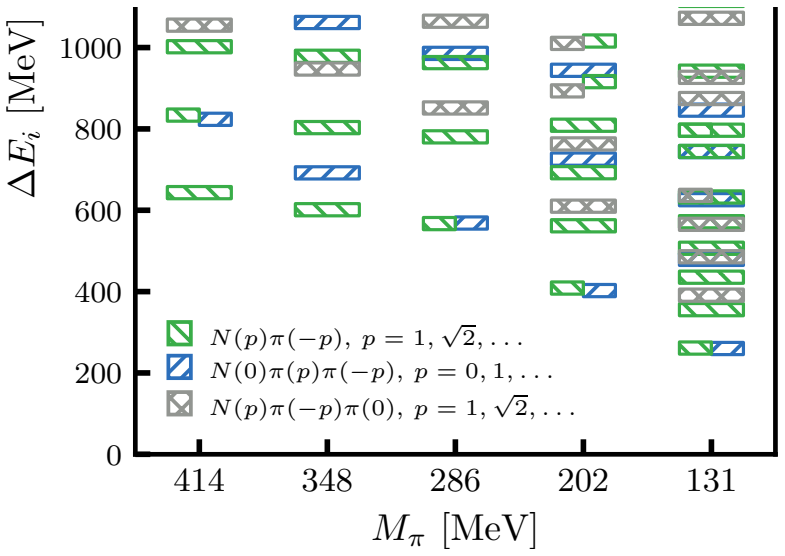
Challenges

Excited state pollution: significant since τ in $C_{3pt}^{(B)}(t,\tau)$ cannot be too large.

$$C_{3\text{pt}}^{B}(t,\tau) = \sum_{n,m} Z_n^B Z_m^B \langle n|\mathcal{O}_J|m\rangle e^{-E_n^B(t-\tau)} e^{E_m^B \tau}$$



Dense spectrum of multiparticle states as $M_\pi \to M_\pi^{phys}$.

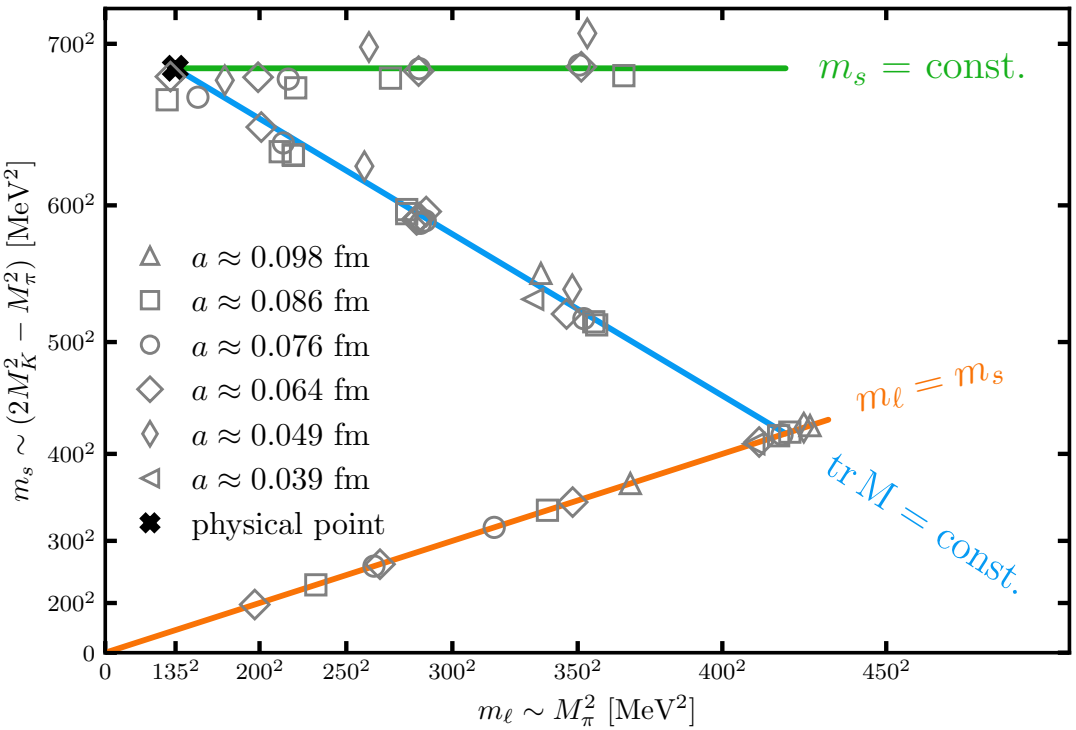


Source-sink separations: $t \sim \{0.7, 0.8, 1.0, 1.2\}$ fm. More measurements per configuration for larger t. Spatially extended interpolating operators, ($r_\ell = 0.64 - 0.82$ fm) to improve the overlap with the ground state. Up to 2 excited states included in the fit.

Quark mass dependence: not clear how well ChPT describes the quark mass dependence in the range $M_\pi \sim (M_\pi^{phys} - 300$ MeV). $\to$ Need $M_\pi \approx M_\pi^{phys}$.

Parameter space of the CLS simulations: quark mass plane

Coordinated Lattice Simulations: Berlin, Bern, Bochum, Darmstadt, Mainz, UA Madrid, Milano Bicocca, Münster, Odense, Regensburg, Rome I and II, Wuppertal, DESY-Zeuthen.



Three trajectories: good control over the quark mass dependence. Observables sensitive to m_s are tightly constrained.

$2m_\ell + m_s = \text{const.}$: investigate SU(3) flavour breaking (flavour average quantities roughly constant), approach to physical point involves $M_\pi \downarrow$ and $M_K \uparrow$.

$m_\ell = m_s$: important for determination of renormalisation factors.

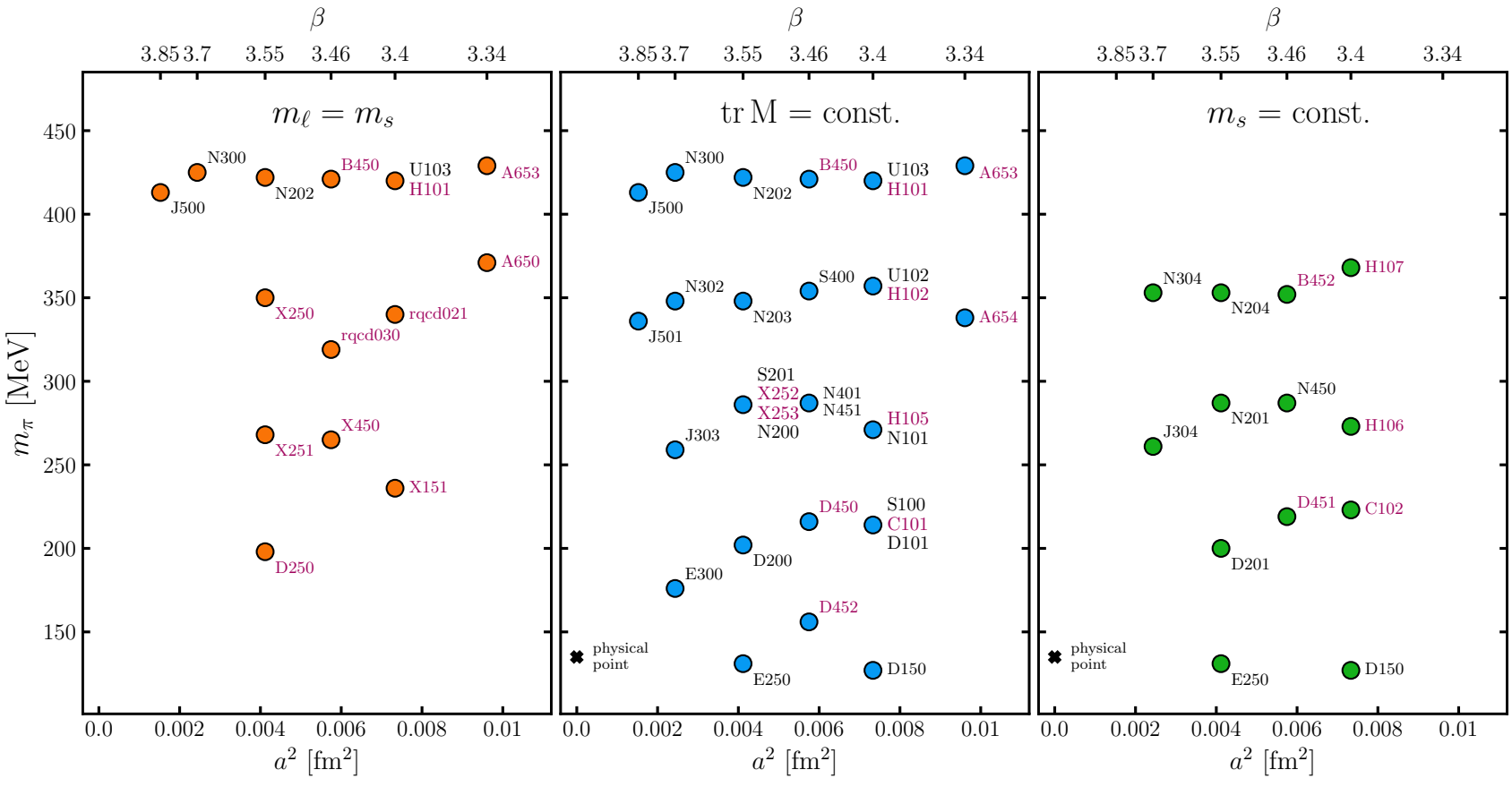
CLS ensembles

High statistics. **Control all sources of systematics** (a, m_q and V).

⋆ **Discretisation**: Six lattice spacings: $a = 0.1 - 0.04$ fm.

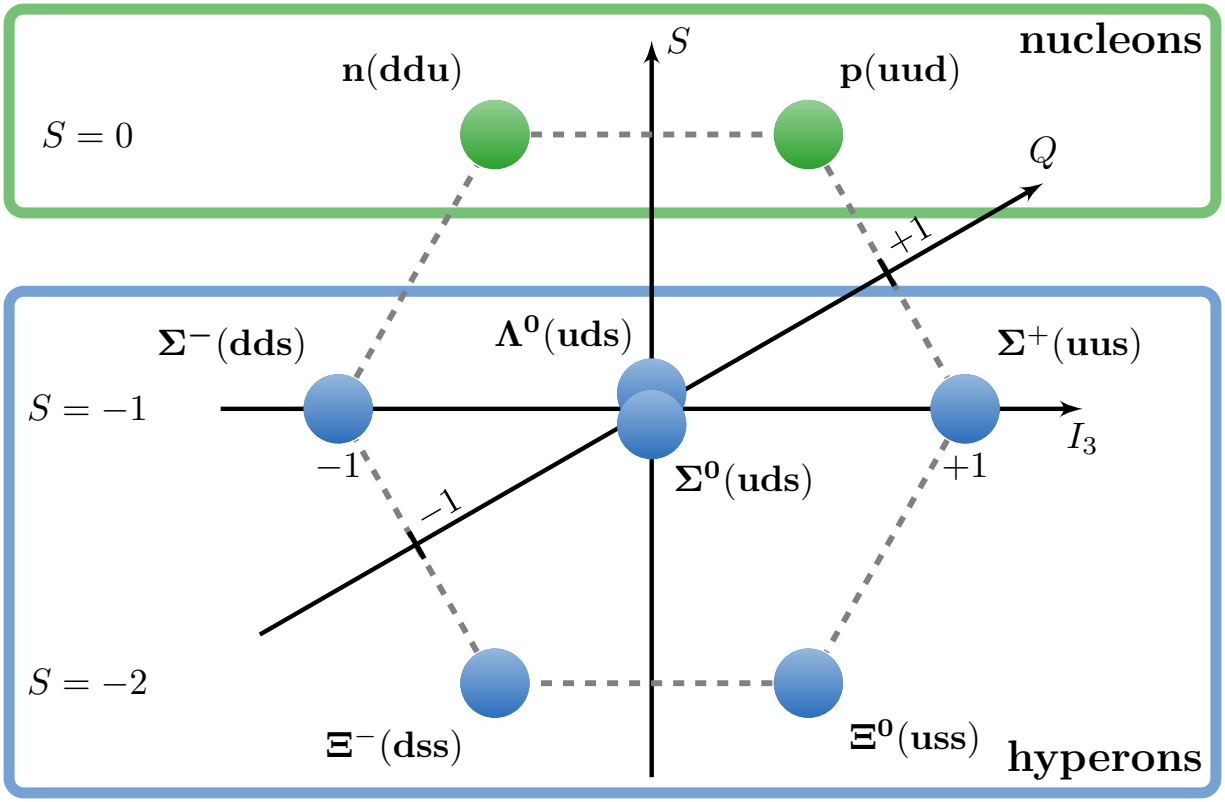
⋆ **Finite volume**: $LM_\pi \gtrsim 4$ with additional smaller volumes.

⋆ **Quark mass**: $M_\pi = 410$ MeV down to M_π^{phys}.



$m_\ell = m_s$ $\qquad$ $2m_\ell + m_s = \text{const.}$ $\qquad$ $m_s = \text{const.}$

Baryon octet



Simulating QCD (isospin-symmetric, electrically neutral): N, Σ, Λ, Ξ

The isovector moments for the Λ baryon vanish.

In addition to investigating $SU(3)$ flavour symmetry breaking, better signal for Σ and Ξ correlation functions allows further tests of systematics.

SU(3) flavour symmetry

The moments of the N, Σ and Ξ baryons are related.

Due to group theoretical constraints, for each moment ($n = 0, 1$)

$$\langle x^n \rangle^B = c_B^n + \bar{c}_B^n \overline{M}^2 + \delta c_B^n \delta M^2 + \textbf{higher orders}$$

$$\overline{M}^2 = (2M_K^2 + M_\pi^2)/3 \propto \overline{m} \qquad \delta M^2 = 2(M_K^2 - M_\pi^2) \propto m_s - m_\ell$$

Leading order term (c_B^n) depends on two constants F^n and D^n

$$c_N^n = F_n + D_n, \qquad c_\Sigma^n = 2F_n \qquad c_\Xi^n = F_n - D_n$$

$\bar{c}_B^n$ has the same structure. Seven coefficients in total (cf. $3 \times 3 = 9$).

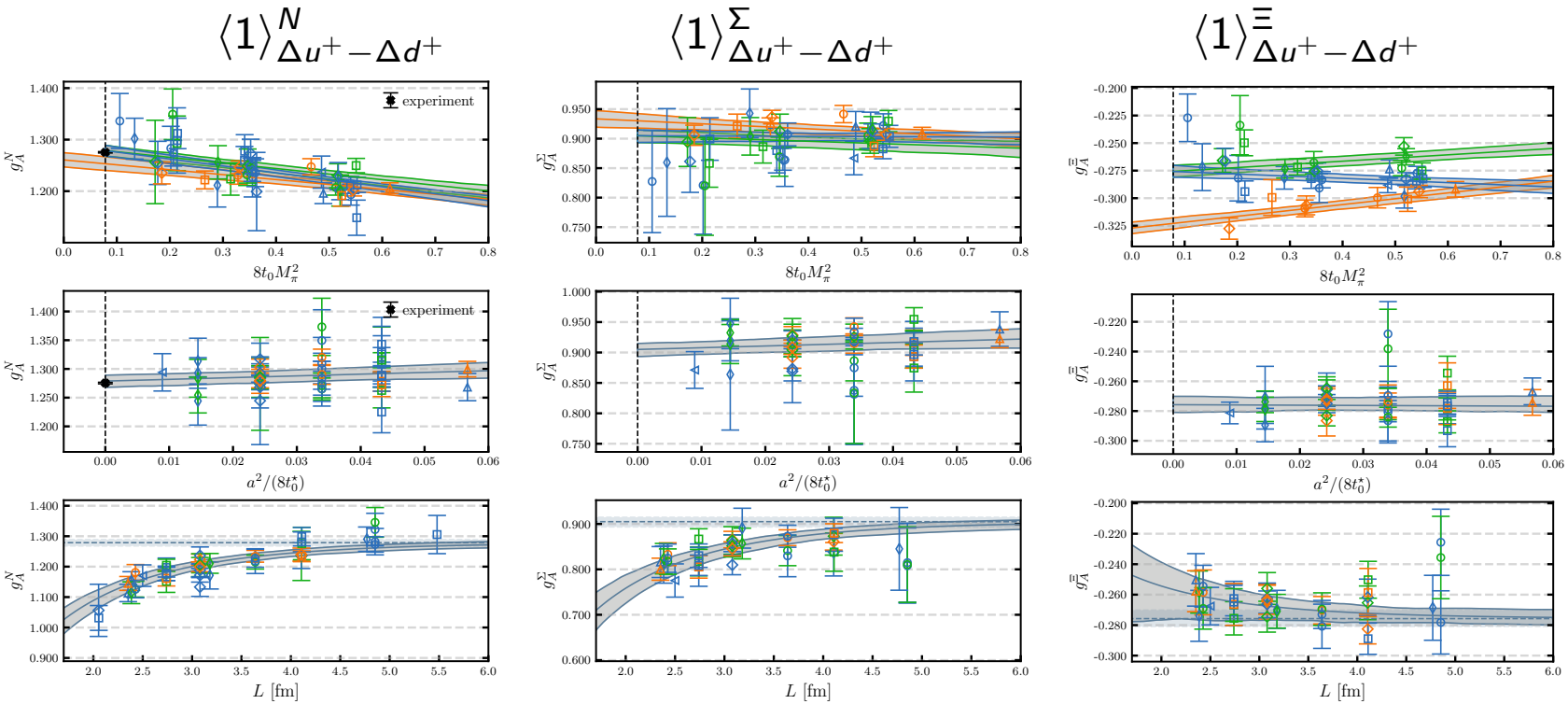
Higher orders: Taylor expansion in $m_s - m_\ell$ about a (symmetric) point $m_s = m_\ell$ see [QCDSF-UKQCD-CSSM,1909.02521].

Higher orders: BChPT see, e.g. covariant BChPT with EOMS [Ledwig et al.,1405.5456], HBChPT [Jenkins and Manohar,Phys. Lett. B 255 (1991) 558].

Simultaneous a, m_q, L extrapolation of the first moments

$$\langle 1 \rangle^B(M_\pi, M_K, L, a) = c_B + \bar{c}_B \overline{M}^2 + \delta c_B \delta M^2 + c_V M_\pi^2 \frac{e^{-LM_\pi}}{\sqrt{LM_\pi}} + O(a^2, a^2\overline{M}^2, a^2\delta M^2)$$

Correlations between $B = N$, Σ and Ξ taken into account.



Vary the fit form and data: cuts with $a < 0.1$ fm, $M_\pi < 400, 300$ MeV, $M_\pi L > 4$.

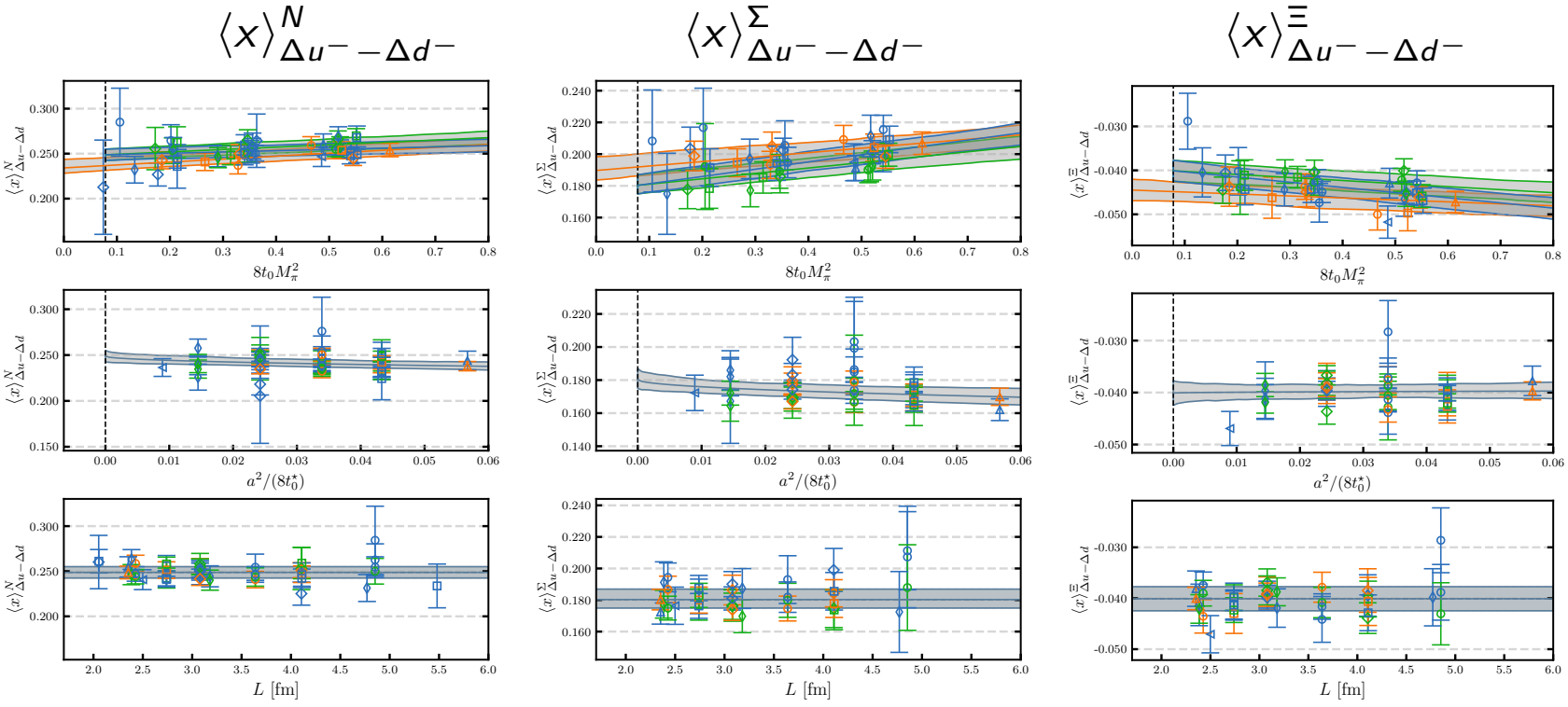
Final results: $M_\pi < 400$ MeV which means largest $M_\pi \sim 350$ MeV.

Use a modified Akaike Information Criterion (AIC) [Akaike,(1998)]: weighted average

$w_j \propto \exp\left[-\frac{1}{2}\left(\chi^2_j - N_{\mathrm{DF},j} + p_j\right)\right]$

Simultaneous a, m_q, L extrapolation of the second moments

$$\langle x \rangle^B(M_\pi, M_K, L, a) = c_B + \bar{c}_B \overline{M}^2 + \delta c_B \delta M^2 + c_V M_\pi^2 \frac{e^{-LM_\pi}}{\sqrt{LM_\pi}} + O(a, a\overline{M}^2, a\delta M^2)$$



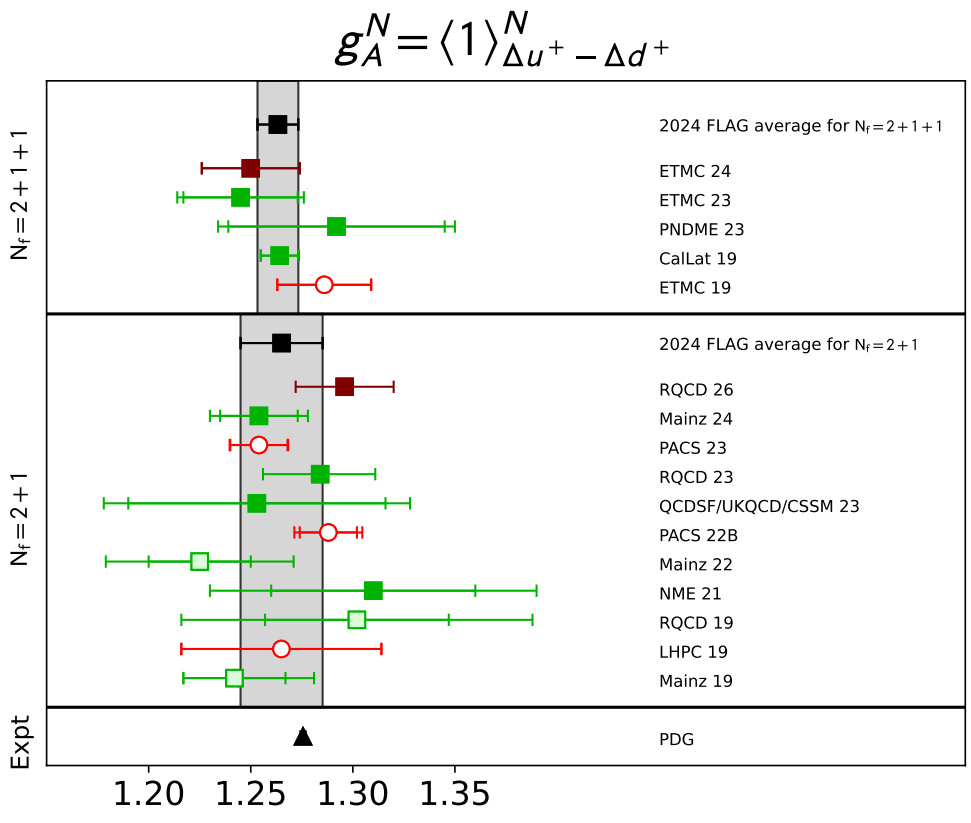
Second moments exhibit mild dependence on a and V.

Benchmarks

Charge conservation (in the isospin limit): $F_V = g_V^N = \frac{1}{2} g_V^\Sigma = g_V^\Xi = 1$.

Applies for any V and m_q in the continuum limit: $F_V = 1.0001(10)$.

Nucleon: g_A is sensitive to excited state contamination, finite volume, quark mass dependence, . . .



RQCD 26: $g_A^N = 1.296(24)$.

Most precise: CalLat 19, $g_A^N = 1.2642(93)$.

■: M_π dependence, renormalisation finite a, V under control.

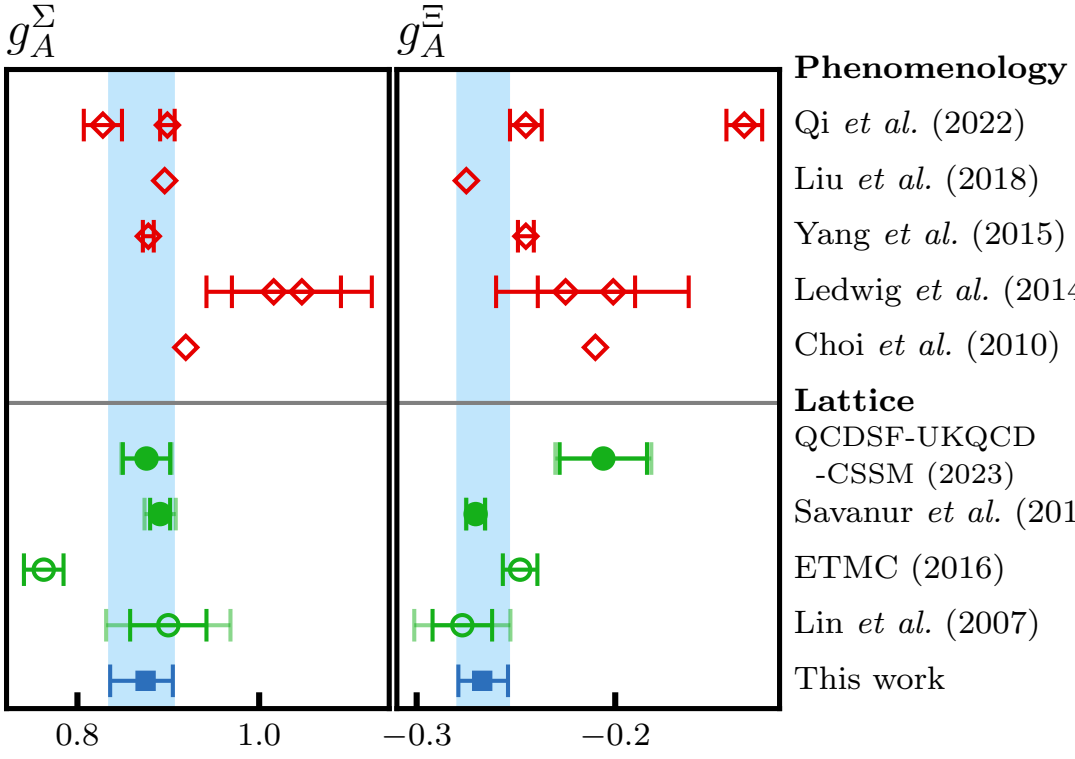
Passes Flavour Lattice Averaging Group (FLAG) criteria.

Only recent works shown and not exhaustive.

■ RQCD and new works.

Axial charge of the Σ and Ξ

[RQCD,2305.04717]



RQCD 26:

$g_A^\Sigma = 0.894(26)$,

$g_A^\Xi = -0.2705(98)$.

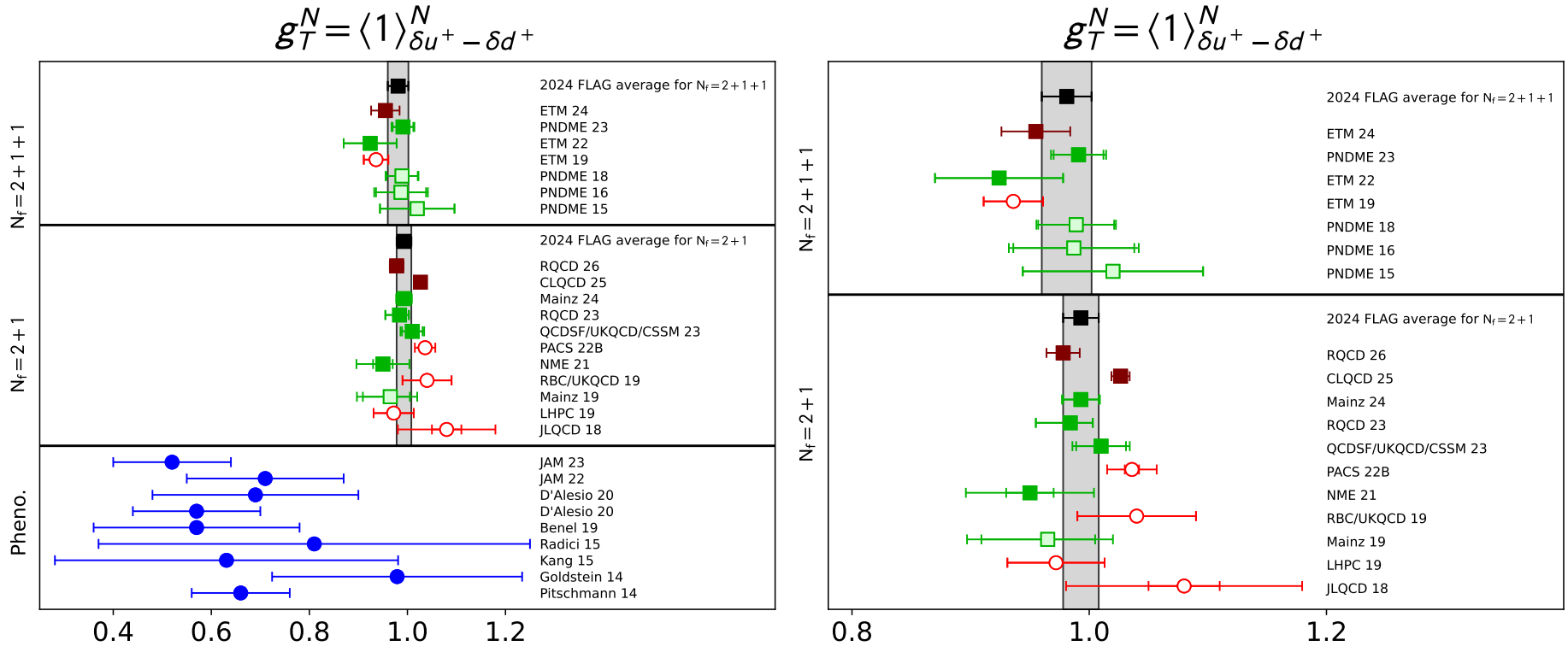
Filled symbols: a, m_q, V extrapolation performed.

Phenomenology: experimental results for $n \to p$, $\Lambda \to p$, $\Sigma \to n$ etc are fitted using e.g. quark models, chiral soliton models, covariant ChPT etc (related via SU(3) symmetry).

Invoking the isospin symmetry, results for $g_A^\Sigma = g_A^{\Sigma^+} = -\sqrt{2} g_A^{\Sigma^+,\Sigma^0}$ and $g_A^\Xi = g_A^{\Xi^0} = -g_A^{\Xi^0,\Xi^-}$ are inferred.

g_T^N in the $\overline{\text{MS}}$ scheme at $\mu = 2$ GeV

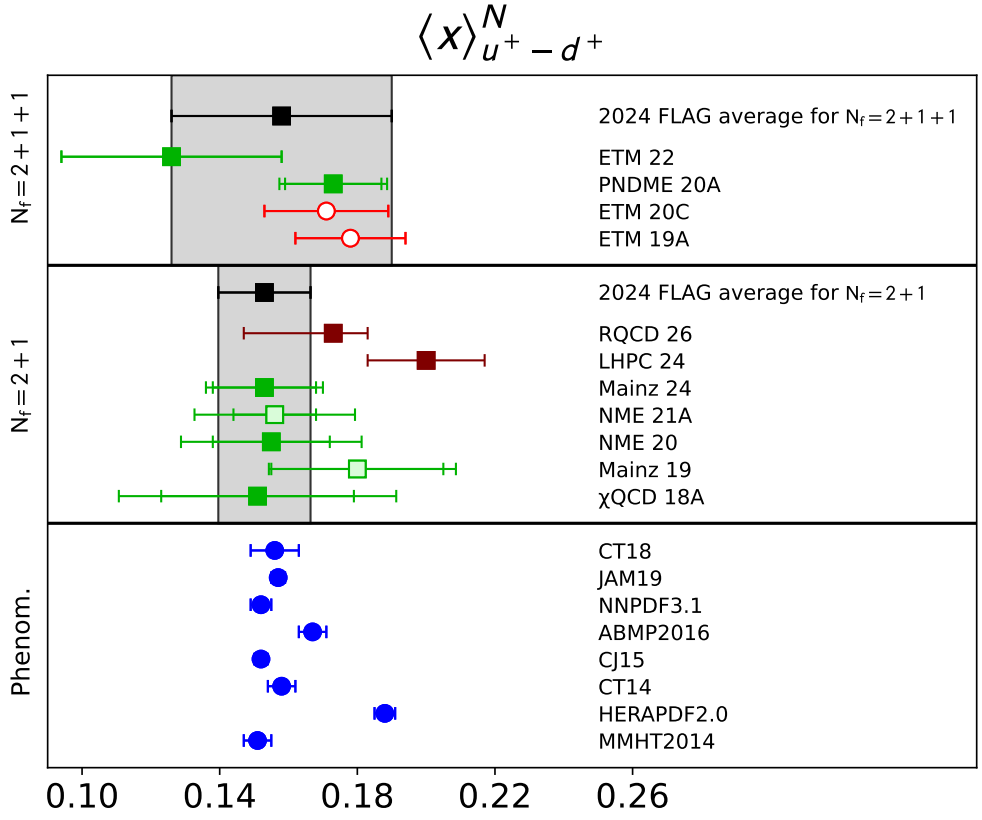
RQCD 26: $g_T^N = 0.978(14)$



Only phenomenologial results without lattice input shown above.

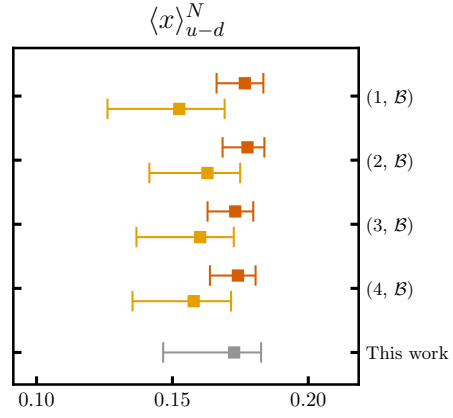
Global analysis of transversity with lattice constraints:
[JAM 18,1710.09858] (averaged g_T), [JAM 22,2205.00999] (ETMC 19, g_T) and [JAM 23,2306.12998] ($\langle 1 \rangle_{\delta u^+}$ and $\langle 1 \rangle_{\delta d^+}$ from PNDME 18 and ETMC 21).

$\langle x \rangle^N_{u^+-d^+}$ in the $\overline{\text{MS}}$ scheme at $\mu = 2$ GeV.



RQCD 26:
$\langle x \rangle^N_{u^+-d^+} = 0.173^{(10)}_{(26)}$.

Sensitive to excited states.



FLAG: $\langle x \rangle^N_{u^+-d^+} = 0.158(32)$ $(N_f = 2+1+1)$, $\langle x \rangle^N_{u^+-d^+} = 0.153(13)$ $(N_f = 2+1)$.

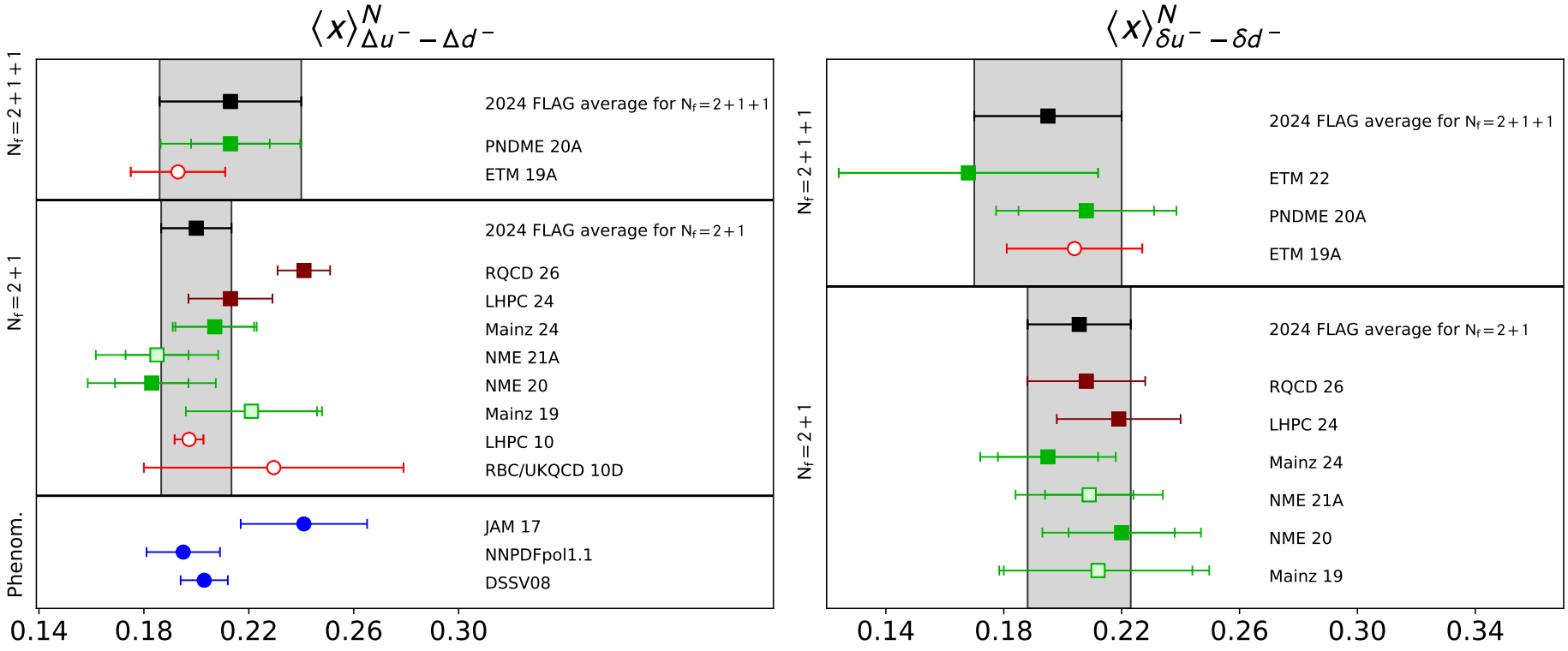
Average of $N_f = 2+1$ and $2+1+1$ results (including new LHPC and RQCD results)

$\langle x \rangle^N_{u^+-d^+} = 0.168(8)$.

Helicity and transversity second moments of the nucleon

$\overline{\text{MS}}$ scheme at $\mu = 2$ GeV

RQCD 26: $\langle x \rangle^N_{\Delta u^- - \Delta d^-} = 0.241(10)$ and $\langle x \rangle^N_{\delta u^- - \delta d^-} = 0.208(20)$



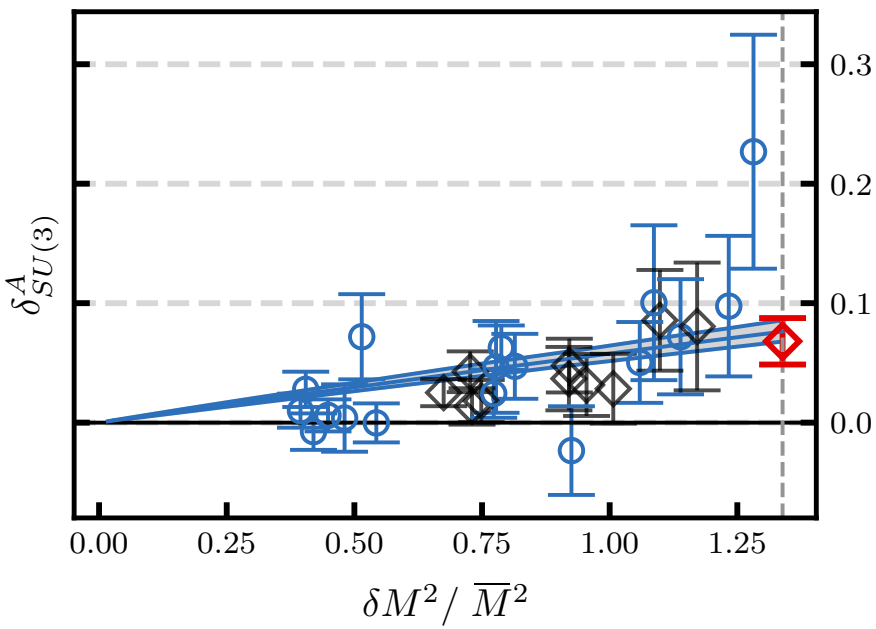
Average of $N_f = 2+1$ and $2+1+1$ results (including new LHPC and RQCD results)

$\langle x \rangle^N_{\Delta u^- - \Delta d^-} = 0.222(9)$ and $\langle x \rangle^N_{\delta u^- - \delta d^-} = 0.206(10)$.

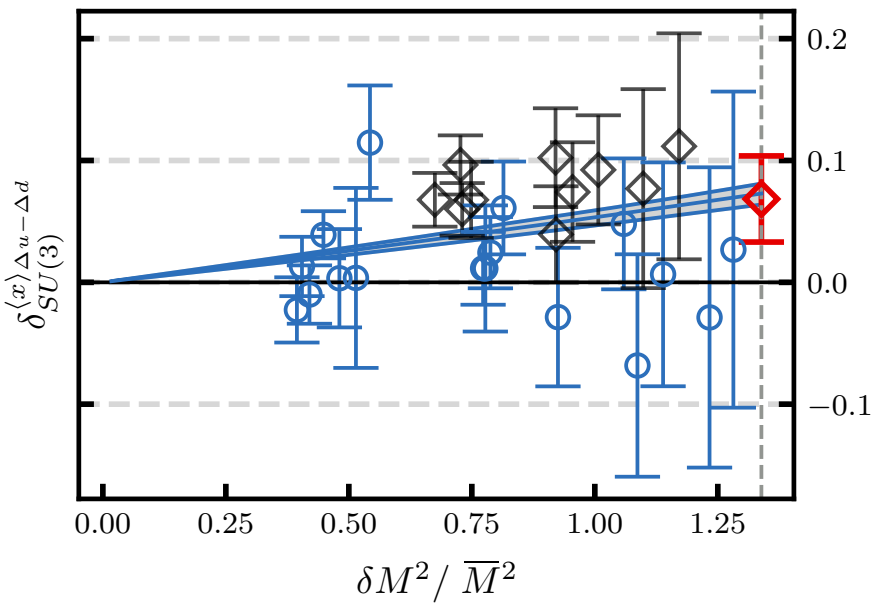
Flavour symmetry breaking

$$\delta_{SU(3)} = \frac{\langle x^n \rangle^{\Xi} + \langle x^n \rangle^{N} - \langle x^n \rangle^{\Sigma}}{\langle x^n \rangle^{\Xi} + \langle x^n \rangle^{N} + \langle x^n \rangle^{\Sigma}} \qquad \text{LO} \qquad \frac{(F_n + D_n) + (F_n - D_n) - 2F_n}{4F_n}$$

$\delta_{SU(3)}^{\langle 1 \rangle_{\Delta u^+ - \Delta d^+}}$



$\delta_{SU(3)}^{\langle x \rangle_{\Delta u^- - \Delta d^-}}$



$\delta M^2/\overline{M}^2 \propto (m_s - m_\ell)/\overline{m}$. $\delta_{SU(3)}$ **not so significant for other moments.**

◯: data on the tr(M) = const. trajectory. ♢: m_s = const., ♢ result from individual extrapolations.

See also previous work, e.g., [Lin, Orginos,0712.1214], [ETMC,1606.01650], [Savanur, Lin,1901.00018], [QCDSF,2304.02866], ...

Higher moments and twist: helicity

LO in OPE, third Mellin moments of the helicity structure functions g_1 and g_2

$$2\int_0^1 dx\, x^2 g_1(x,Q^2) = \frac{1}{2}\sum_q Q_q^2\, E_{1,2}^{(q)}\left(\frac{\mu^2}{Q^2},\alpha_s(\mu)\right)\, a_2^{(q)}(\mu)\,,$$

$$2\int_0^1 dx\, x^2 g_2(x,Q^2) = \frac{1}{3}\sum_q Q_q^2\left[E_{2,2}^{(q)}\left(\frac{\mu^2}{Q^2},\alpha_s(\mu)\right)\, d_2^{(q)}(\mu) - E_{1,2}^{(q)}\left(\frac{\mu^2}{Q^2},\alpha_s(\mu)\right)\, a_2^{(q)}(\mu)\right],$$

Wandzura-Wilczek relation: if $d_2^q \ll a_2^q$, g_2 can be obtained from g_1.

$a_2^{(q)}$, twist-2 contribution to g_1. $d_2^{(q)}$ twist-3 contribution to g_2 also at leading power in $1/Q$.

Wilson coefficients $E_{1,n}^{(q)}$ are known to 2-loop, $E_{2,n}^{(q)}$ only known at tree-level.

Both involve operators with two derivatives. Lattice: d_2 mixing with operators of lower dimension. Cancellation of $1/a$ contributions via renormalisation.

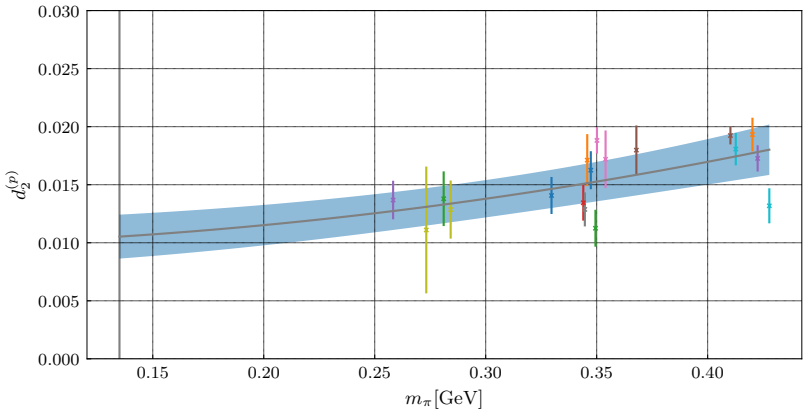
[RQCD,2111.08306] S. Bürger et al.: some approximations (no disconnected diagrams, ...), however, unlikely to be the main uncertainty.

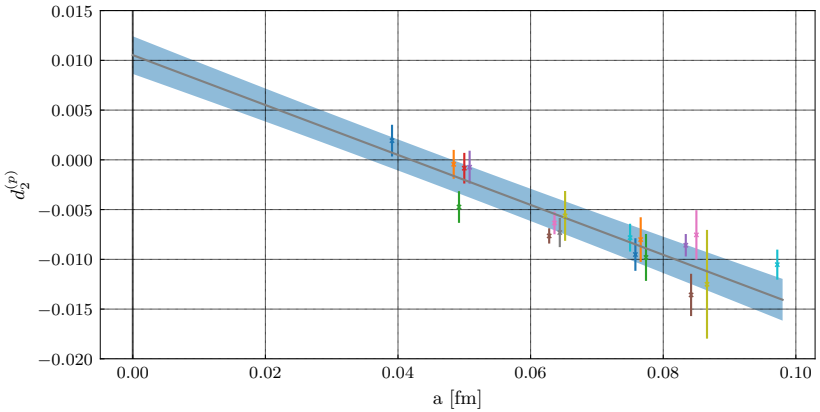
Employed a sub-set of the CLS ensembles.

Higher moments and twist: helicity

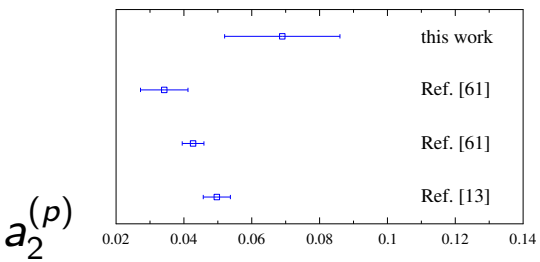
Continuum, quark mass extrapolation.

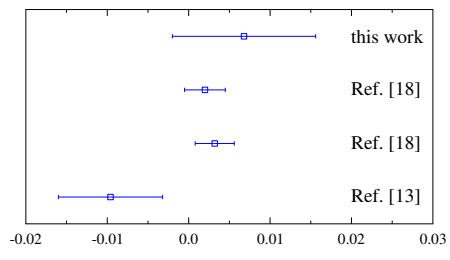
$d_2^{(p)} = \left(\frac{2}{3}\right)^2 d_2^{(u)} + \left(-\frac{1}{3}\right)^2 d_2^{(d)}, \quad d_2^{(n)} = \left(-\frac{1}{3}\right)^2 d_2^{(u)} + \left(\frac{2}{3}\right)^2 d_2^{(d)}$. Similarly for $a_2^{(p,n)}$.

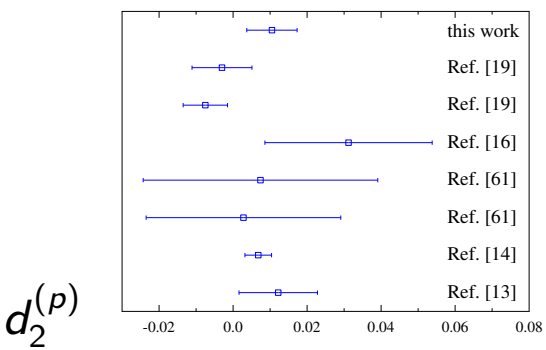


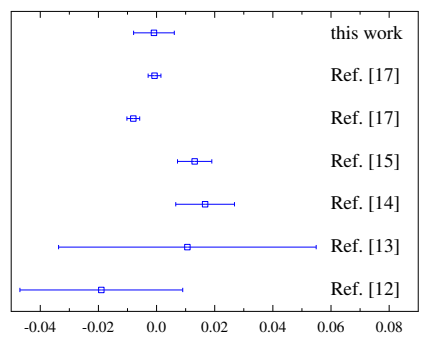


Comparison with phenomenology (some approximations also made) ($\mu = 2$ GeV)









[Ref 12, E154,hep-ex/9705017],
[Ref 13, E143,hep-ex/9705017],
[Ref 14, E155,hep-ex/0204028],
[Ref 15, JLab Hall A,nucl-ex/0405006],
[Ref 16, HERMES,1112.5584],
[Ref 17, JLab Hall A,1404.4003],
[Ref 18, JLab Hall A,1603.03612],
[Ref 19, SANE,1805.08835],
[Ref 61, Osipenko,hep-ph/0503018].

Previous direct lattice studies: [QCDSF,hep-lat/0506017], [LHPC,1001.3620], [LHPC,SESAM,hep-lat/0201021].

Indirect: isovector combination [Gao et al.,2604.00143] $d_2^{u-d} = 0.0024(46)$, statistical errors only, single m_π^{phys} ensemble.

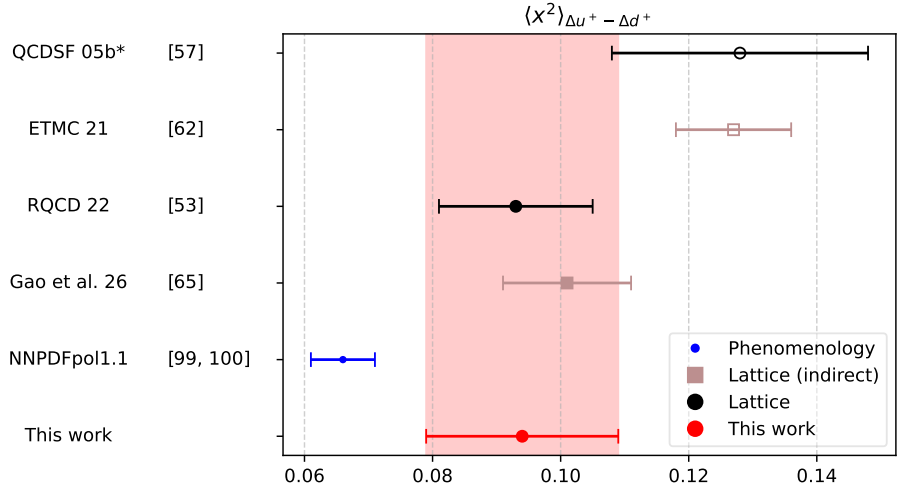
cf. RQCD: $d_2^{u-d} = 0.034(4)(11)$.

Comparison of lattice results for $\langle x^2 \rangle^N_{\Delta u^+ - \Delta d^+}$

RQCD 21: $\langle x^2 \rangle^N_{\Delta u^+ - \Delta d^+} = 0.094(12)$ ($\overline{\text{MS}}$ at $\mu = 2$ GeV).

All systematics included: multiplicative renormalisation, no disconnected contributions.

[LHPC,2605.02808]



Unpolarised and transversity third moments also computed.

[ETMC,2106.16065] $N_f = 2 + 1 + 1$ at $m_\pi = 260$ MeV, [QCDSF,hep-lat/0506017] $N_f = 2$.

See also ETMC, Kummer et al., Tues. 11:15.

Earlier work: $N_f = 2$ [LHPC, SESAM,hep-lat/0201021] and $N_f = 2 + 1$ [LHPC,1001.3620].

Summary

★ Computed the lowest two isovector moments of PDFs of the nucleon with

Unpolarised	$\langle 1 \rangle_{\Delta u^+ - \Delta d^+} \sim 0.1\%$	$\langle x \rangle_{u^+ - d^+} \sim 11\%$	
Helicity	$\langle 1 \rangle_{\Delta u^+ - \Delta d^+} \sim 2\%$	$\langle x \rangle_{\Delta u^- - \Delta d^-} \sim 4\%$	$\langle x^2 \rangle_{\Delta u^+ - \Delta d^+} \sim 13\%$
Transverse	$\langle 1 \rangle_{\delta u^+ - \delta d^+} \sim 1.4\%$	$\langle x \rangle_{\delta u^- - \delta d^-} \sim 10\%$	

Helicity twist-3 contribution d_2 found to be small.

In general, good consistency found with other lattice results.

Community averages for second moments $\sim 5\%$.

★ High statistics CLS ensembles enable all systematics (finite a, V, unphysical m_q) to be investigated and controlled.

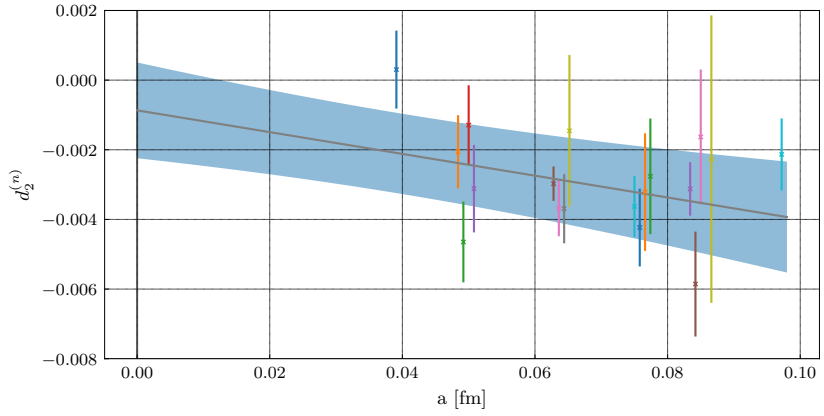
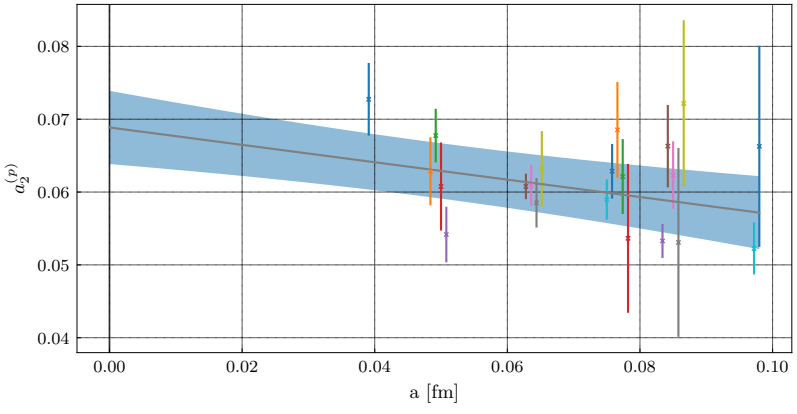
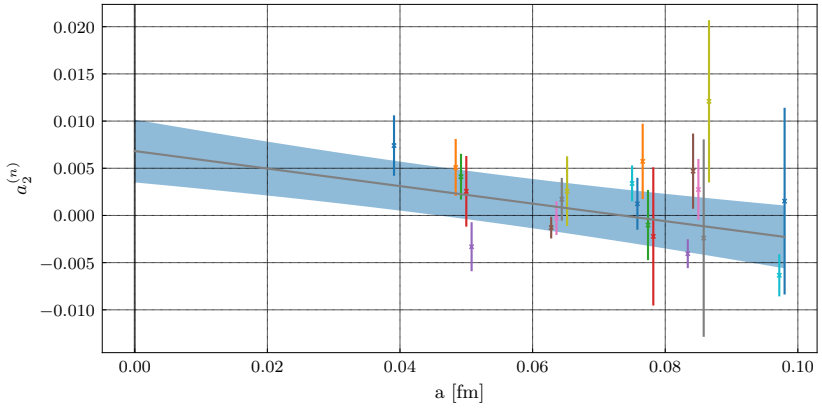
Ideal for studying flavour symmetry breaking: only significant for

$\delta_{SU(3)}^{\langle 1 \rangle_{\Delta u^+ - \Delta d^+}} = 0.068(19)$ and $\delta_{SU(3)}^{\langle x \rangle_{\Delta u^- - \Delta d^-}} = 0.069(35)$.

★ In the future,

- Extend the analysis to the quark isosinglet flavour combinations and moments of gluon PDFs.
- Calculate higher moments, improving on 2021 study.
- Analyse the off-forward limit data and compute generalised form factors.

$SU(3)$ flavour symmetry breaking for the isovector moments

$$\delta^{A}_{\mathrm{SU(3)}} = 0.068(19),$$
$$\delta^{T}_{\mathrm{SU(3)}} = -0.003(12),$$
$$\delta^{x}_{\mathrm{SU(3)}} = 0.028^{(34)}_{(61)},$$
$$\delta^{\Delta x}_{\mathrm{SU(3)}} = 0.069(35),$$
$$\delta^{\delta x}_{\mathrm{SU(3)}} = 0.020(77),$$

Continuum extrapolation of d_2^n and $a_2^{p,n}$