Conformal moments of the DVCS coefficient functions: Method and two-loop results

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based on

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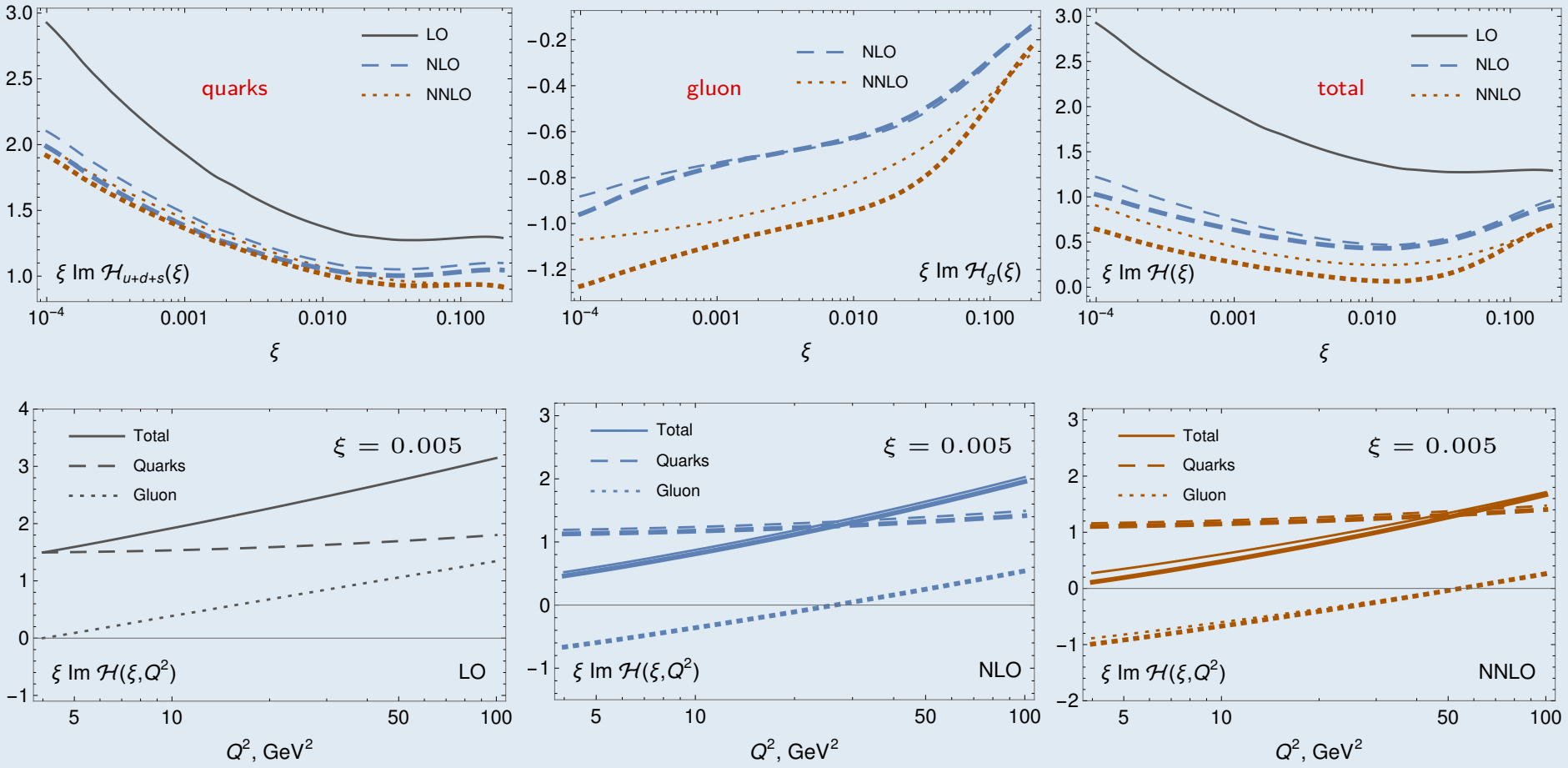
Push for high theory precision:

1. Towards NNLO accuracy
   - Two-loop coefficient functions for DVCS
   - Three-loop evolution equations for GPDS (flavor nonsinglet)
2. Resummation of threshold logarithms in DVCS and DDVCS
3. Kinematic power corrections $(\sqrt{-t}/Q)^k$, $(m/Q)^k$
4. **Bottleneck: Implementation**

Imaginary part of the Compton form factor $\mathcal{H}$, $t = -0.1$ GeV^2



GK-model, normalized at input scale $\mu^2 = 4$ GeV^2 to HERAPDF20 (thin lines) and ABMP16 (thick)

— the gluon contribution is large and negative, enhanced at NNLO

VB, Yao Ji, J. Schoenleber, PRL 129 (2022) 17, 172001

Two strategies

1. Convolution in momentum fraction space

$$\mathcal{H} = \sum_q \int_{-1}^{1} \frac{dx}{\xi} C_q\left(x/\xi, \mu^2/Q^2\right) H_q(x,\xi,t,\mu) + \int_{-1}^{1} \frac{dx}{\xi^2} C_g\left(x/\xi, \mu^2/Q^2\right) H_g(x,\xi,t,\mu),$$

2. Mellin representation in terms of conformal moments

$$\mathcal{H} \sim \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\xi} \frac{(i\xi)^{-N}}{\cos\frac{\pi N}{2}} C_q(N, Q^2/\mu^2) H_q(N,\xi,t,\mu) + \ldots$$

$$C_q(N) = \frac{1}{4}\int_{-1}^{1} dx\,(1-x^2)\,C_{N-1}^{3/2}(x)\,C_q(x)\,,$$

$$C_g(N) = \frac{3}{16}\int_{-1}^{1} dx\,(1-x^2)^2\,C_{N-2}^{5/2}(x)\,C_g(x)$$

— Analytically continued integrals of momentum space CFs with Gegenbauer polynomials

- Convenient variable

$$z = \frac{1}{2}(1-x)\,, \qquad \bar{z} = 1 - z = \frac{1}{2}(1+x)$$

In terms of this variable, the tree level CFs are given by

$$C_q^{(0)}(z) = \frac{e_q^2}{2}\left(\frac{1}{z} - \frac{1}{\bar{z}}\right) \quad \Leftarrow \text{ vector}$$

$$\widetilde{C}_q^{(0)}(z) = \frac{e_q^2}{2}\left(\frac{1}{z} + \frac{1}{\bar{z}}\right) \quad \Leftarrow \text{ axial vector}$$

and the Gegenbauer moments take the form

$$C_q(N) = 2\int_0^1 dz\, z\bar{z}\, C_{N-1}^{3/2}(1-2z)\, C_q(z)$$

$$C_g(N) = 6\int_0^1 dz\, z^2\bar{z}^2\, C_{N-2}^{5/2}(1-2z)\, C_g(z)$$

- At tree level (LO)

$$C_q(N) = \frac{1+(-1)^N}{2}e_q^2\,, \qquad \widetilde{C}_q(N) = \frac{1-(-1)^N}{2}e_q^2\,.$$

General decomposition

- quarks

$$C_q = e_q^2\, C_{q,\mathsf{NS}} + \Sigma_Q\, C_{q,\mathsf{PS}}\,, \qquad \Sigma_Q = \frac{1}{2}\sum_{q'} e_{q'}^2$$

$$C_{q,\mathsf{NS}} = C_{q,\mathsf{NS}}^{(0)} + a_s C_F C_{q,\mathsf{NS}}^{(1)} + a_s^2 C_F \left(C_F C_{q,\mathsf{NS}}^{(2)\mathsf{F}} + \frac{1}{N_c} C_{q,\mathsf{NS}}^{(2)\mathsf{NP}} + \beta_0 C_{q,\mathsf{NS}}^{(2)\beta}\right) + \dots\,, \qquad a_s = \frac{\alpha_s}{4\pi}$$

$$C_{q,\mathsf{PS}} = a_s^2 C_F C_{q,\mathsf{PS}}^{(2)} + \dots\,,$$

gluons

$$C_g = \Sigma_Q \left\{ a_s C_g^{(1)} + a_s^2 \left(C_F C_g^{(2)\mathsf{F}} + C_A C_g^{(2)\mathsf{A}}\right) + \dots \right\}$$

- The CFs become increasingly complicated in higher orders, e.g.

$$
\begin{aligned}
C_{q,\mathrm{NS}}^{(2)\mathrm{NP}} &= 6(1-2z)\Big\{\mathrm{H}_{010}-\mathrm{H}_{001}+\mathrm{H}_{110}-\mathrm{H}_{101}+\zeta_2\Big(\mathrm{H}_0+\mathrm{H}_1\Big)-3\zeta_3\Big\}\\
&+12\Big(\mathrm{H}_{10}-\mathrm{H}_{01}-\mathrm{H}_0-\mathrm{H}_1+\zeta_2\Big)+\frac{3}{\bar z}\mathrm{H}_0+\frac{3}{z}\mathrm{H}_1\\
&+\Big\{\frac{1}{z}\Big(12\zeta_3-\frac{3}{2}\zeta_2^2-\frac{5}{2}\zeta_2-\frac{73}{24}\Big)-\frac{3}{z}\mathrm{H}_{0100}-\Big(\frac{2}{z}-\frac{1}{\bar z}\Big)\mathrm{H}_{0010}+\Big(\frac{4}{z}-\frac{1}{\bar z}\Big)\mathrm{H}_{0001}\\
&-\Big(\frac{2}{z}-\frac{1}{\bar z}\Big)\mathrm{H}_{0110}+\Big(\frac{3}{z}-\frac{2}{\bar z}\Big)\mathrm{H}_{0101}-\Big(\frac{2}{z}-\frac{1}{\bar z}\Big)\mathrm{H}_{0011}-\frac{5}{\bar z}\mathrm{H}_{001}+\frac{5}{\bar z}\mathrm{H}_{010}\\
&+\Big(\frac{1}{\bar z}\Big(\zeta_2-\frac{9}{2}\Big)+\frac{1}{z}\Big(\frac{4}{3}-2\zeta_2\Big)\Big)\mathrm{H}_{00}-\Big(\frac{2}{z}\Big(\zeta_2-1\Big)-\frac{1}{\bar z}\Big(\zeta_2+\frac{7}{6}\Big)\Big)\mathrm{H}_{01}\\
&+\Big(\frac{1}{\bar z}\Big(\frac{19}{6}+5\zeta_2-3\zeta_3\Big)+\frac{1}{z}\Big(7\zeta_3-\frac{16}{9}\Big)\Big)\mathrm{H}_0-(z\leftrightarrow\bar z)\Big\}
\end{aligned}
$$

where $\mathrm{H}_{\vec m}\equiv\mathrm{H}_{\vec m}(z)$ are harmonic polylogarithms

- Wanted: analytic expressions for Gegenbauer moments of these functions

On an intuitive level

use Dirac bra-ket notation

$$M_N[f] = 2\int_0^1 dz\, f(z)\, z\bar{z}\, C_{N-1}^{3/2}(1-2z) \quad \leftrightarrow \quad \langle f|N\rangle$$

- Main idea: the "states" $C_{N-1}^{3/2} \leftrightarrow |N\rangle$ are eigenfunctions of $SL(2)$-invariant operators

$$\mathbb{H}|N\rangle = E(N)|N\rangle$$

Then

$$M_N[H^\dagger f] = \langle \mathbb{H}^\dagger f|N\rangle = \langle f|\mathbb{H}|j\rangle = E(N)\langle f|N\rangle$$

- If we know Gegenbauer moment of f and $E(N)$, we can calculate a Gegenbauer moment of $H^\dagger f$

$\Rightarrow$ start with the simplest f (tree-level CF) and generate a complete set of functions of the required algebraic complexity by applying several $SL(2)$ invariant operators $\mathbb{H}_\kappa$, $\kappa = 1, 2, \ldots$

$\Rightarrow$ Expand C_q (and C_g) in terms of this complete set of functions

On a technical level

- Problem: Invariant operators in momentum representation are complicated

  $\Rightarrow$ go over to position space $z, \bar{z} \mapsto w_1, w_2$

- $SL(2)$ invariant operators: commute with the generators of collinear conformal transformations

$$S_- = -\partial_{w_1} - \partial_{w_2}$$
$$S_0 = w_1\partial_{w_1} + w_2\partial_{w_2} + 2,$$
$$S_+ = w_1^2\partial_{w_1} + w_2^2\partial_{w_2} + 2(w_1 + w_2)$$

$$\underbrace{[S_0, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_0}_{\text{SL(2) algebra}}$$

- An $SL(2)$ invariant operator can be written as

$$[\mathbb{H}^{(h)}\varphi](w_1, w_2) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, h(\tau)\varphi(w_{12}^\alpha, w_{21}^\beta)\,, \qquad \tau = \frac{\alpha\beta}{\bar{\alpha}\bar{\beta}}$$

$$w_{12}^\alpha = w_1\bar{\alpha} + w_2\alpha$$
$$\bar{\alpha} = 1 - \alpha$$

Eigenfunctions

$$[\mathbb{H}^{(h)}\Psi_N](w_1,w_2) = E_N^{(h)}\Psi_N(w_1,w_2)$$

- Since $[\mathbb{H}^{(h)}, S^2] = 0$ they can be diagonalized simultaneoulsy
  $\Rightarrow$ Eigenfunctions of the quadratic Casimir S^2 will diagonalize $\mathbb{H}^{(h)}$ as well

A possible choice

$$\Psi_N^p(w_1,w_2) = \int_0^1 dz\, e^{-ip(w_1 z + w_2 \bar{z})}\, z\bar{z} C_{N-1}^{3/2}(1-2z)$$

$$\underbrace{\partial_{w_1}\partial_{w_2} w_{12}^2}_{S^2}\,\Psi_N^p(w_1,w_2) = N(N+1)\Psi_N^p(w_1,w_2)$$

$$\underbrace{(-\partial_{w_1} - \partial_{w_2})}_{S_-}\,\Psi_N^p(w_1,w_2) = ip\Psi_N^p(w_1,w_2)$$

The corresponding eigenvalues

$$E_N^{(h)} = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, h(\tau)\,(1-\alpha-\beta)^{N-1}$$

We are almost there!

- Since the eigenvalues do not depend on p, any linear combination of Ψ_N^p is an eigenfunction as well; we can use any function $f(z)$ instead of the exponential

$$\Psi_N^{[f]}(w_1, w_2) = \int_0^1 dz\, f(w_1 z + w_2 \bar{z})\, z\bar{z} C_{N-1}^{3/2}(1-2z)$$

Thus

$$[\mathbb{H}^{(h)} \Psi_N^{[f]}](w_1, w_2) = E_N^{(h)} \Psi_N^{[f]}(w_1, w_2) \qquad \Psi_N^{[f]}(w_1 = 1, w_2 = 0) = M_N[f]$$

It follows that the Gegenbauer moments of $f(z)$ and

$$f^h(z) = [\mathbb{H}^{(h)} f](z) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, h(\tau)\, f(z\bar{\alpha} + \bar{z}\beta)$$

are related to each other as

$$\boxed{M_N[f^h] = E_N^{(h)} M_N[f]}$$

- Thus the mapping $\mathrm{T}_h : f \mapsto f^h$ allows one to construct functions with known Gegenbauer moments using position-space kernels $h(\tau)$

- Simple(st) example: Start with

$$f(z) = \underbrace{\frac{1}{z} - \frac{1}{\bar{z}}}_{\text{LO CF, vector}} \qquad M_N[f] = \underbrace{1}_{\text{even N}}$$

- Take $h(\tau) = 1$

$$E_N^{(h=1)} = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, h(\tau)\, (1-\alpha-\beta)^{N-1} = \frac{1}{N(N+1)}$$

$$f^{h=1}(z) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta\, h(\tau)\, f(z\bar{\alpha} + \bar{z}\beta) = -\left(\frac{\ln z}{\bar{z}} + \frac{\ln \bar{z}}{z}\right)$$

- Thus

$$\boxed{M_N\left[\frac{\ln z}{\bar{z}} + \frac{\ln \bar{z}}{z}\right] = -\frac{1}{N(N+1)}}$$

- How many independent functions do we need?
  - Weight one: 2
  - Weight two: 4 $\Leftarrow$ One-loop CFs
  - Weight three: 8
  - Weight four: 16 $\Leftarrow$ Two-loop CFs
  - ...

! A product of invariant operators is an invariant operator

$$h(\tau) = h_1(\tau) \otimes h_2(\tau)\,, \qquad E_N^{(h)} = E_N^{(h_1)} E_N^{(h_2)}$$

only a few new kernels needed for higher weights

$h(\tau)$	$E_N^{(h)}$
$\widehat{\mathbb{H}}$	$2S_1$
1	$1/N/(N+1)$
$1-\tau$	$(-1)^N\,(2S_{-2}+\zeta_2)$
$\frac{1}{2}(1-\tau)\ln\left(\frac{1-\tau}{\tau}\right)$	$(-1)^N\left(2S_{-2,1}-S_{-3}+\frac{1}{2}\zeta_3\right)$
$\frac{1}{2}\frac{1-\tau}{\tau}\ln(1-\tau)$	$S_3-\zeta_3$
$\frac{1-\tau}{2}\left(\mathrm{Li}_2(\tau)+\frac{1}{2}\ln^2(\tau)\right)$	$(-1)^N\left(S_{-4}+\frac{7}{20}\zeta_2^2\right)$
$\frac{1}{4}\frac{1-\tau}{\tau}\left(\mathrm{Li}_2(\tau)+\frac{1}{2}\ln^2(\tau)\right)$	$S_{1,3}-\frac{1}{2}S_4-\zeta_3S_1+\frac{3}{10}\zeta_2^2$
$\frac{1-\tau}{4}\ln^2(1-\tau)$	$(-1)^N\left(4S_{1,1,-2}-2S_{2,-2}-2S_{1,-3}+S_{-4}-\zeta_2S_2-\zeta_3S_1+2\zeta_2S_{1,1}+\frac{1}{5}\zeta_2^2\right)$

Table: Invariant kernels and their eigenvalues. All harmonic sums have argument N

Example:

$$C_{q,\mathsf{NS}}^{(2)\mathsf{NP}} = \frac{6\,(4S_{-2}+3)}{(N-2)(N+3)} - 4(2S_{1,3}-S_4) - 4S_{-4} - 8S_{-2}^2 + 16(2S_{1,-2}-S_{-3})S_1 + 24S_1S_3 + 4\zeta_2 S_{-2}$$
$$+ 8\zeta_2 S_1^2 - 52\zeta_3 S_1 + 8S_{-2} + \frac{16}{3}S_1^2 + \frac{64}{9}S_1 + \frac{2}{5}\zeta_2^2 + 54\zeta_3 - \frac{10}{3}\zeta_2 - \frac{73}{12} + H^2\Big(32S_{-2}+10\Big)$$
$$+ H\left(8(2S_{-2,1}-S_{-3}) - 40S_{-2}S_1 - 12S_3 + 36\zeta_3 - 8\zeta_2 S_1 + 52S_{-2} - \frac{76}{3}S_1 + \frac{1}{9}\right)$$
$$+ L\left(-8(2S_{1,-2}-S_{-3}) - 8S_3 - \frac{16}{3}S_1 + 8HS_{-2} + \frac{20}{3}H + 1\right)$$

$$L = \ln Q^2/\mu^2 \qquad\qquad H = \frac{1}{N(N+1)}$$

- We have calculated Gegenbauer moments for all two-loop CFs
  - quark CFs: nonsinglet and pure singlet, vector and axial vector
  - gluon CFs
  - gluon transversity
- Important property: All Gegenbauer moments satisfy the reciprocity relation

  — their asymptotic expansion at $N \to \infty$ is symmetric under the interchange $N \leftrightarrow -1-N$.
- Main calculational tool: HyperInt Maple package by E. Panzer