Inequivalence of sets of mutually unbiased bases shapes quantum uncertainty

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Mutually Unbiased Bases (MUBs)

Mutually unbiased bases capture the very essence of the *complementarity* of quantum obersvables.

Definition

Given two d-dimensional normal operators $\hat{A} = \sum_j a_j |a_j\rangle\langle a_j|$ and $\hat{B} = \sum_k b_k |b_k\rangle\langle b_k|$, we say that their eigenbases (orthonormal and complete) are *mutually unbiased* (MU) if

$$|\langle a_j | b_k \rangle| = 1/\sqrt{d} \quad \forall\, j,k \in [0,d). \tag{1}$$

Maximal sets of MUBs

If the dimension of the Hilbert space is the power of a prime ($d = p^N$) then sets of $d+1$ MUBs are shown to exist by explicit construction. These sets are *maximal*: no further MUBs can be included.
If $d \neq p^N$ much less is known: for sure at least three MUBs can be found.

Mutually Unbiased Bases (MUBs)

Constructions of maximal sets

- Heisenberg-Weyl group
- identities from number theory
- Latin squares
- properties of Galois fields

Classification of MUBs

Within a set of MUBs, the transformations that map bases into each other are represented by *complex Hadamard matrices*. The classification of all MUBs in a given dimension follows from the classification of Hadamard matrices.

Equivalence classes of MUBs

Equivalent sets

Given two sets of MUBs $\{H_k^{(1)}\}_{k=1}^n$ and $\{H_k^{(2)}\}_{k=1}^n$, with $n \leq d+1$, we say that they are *equivalent*, $\{H_k^{(1)}\}_{k=1}^n \sim \{H_k^{(2)}\}_{k=1}^n$, if and only if there is a unitary U, a set of diagonal unitaries $\{D_k\}_{k=1}^n$ and a set of permutations $\{P_k\}_{k=1}^n$ such that

$$\{U H_k^{(1)} P_k D_k\}_{k=1}^n = \{H_k^{(2)}\}_{k=1}^n. \tag{2}$$

This equivalence relation is the most general preserving mutual unbiasedness, as defined in Eq. (1).

Standard form

The equivalence under unitaries and permutations allow to recast any set of MUBs in the so-called *standard form*, where one of the basis is represented by the identity $\mathbb{1}$ (computational basis) and another one by the Fourier matrix F.

Equivalence classes of MUBs

For $2 \leq d \leq 5$, a *unique*, up to equivalence, complete classification of MUB sets is known. We have that

- all bases are equivalent to the Fourier basis
- there is a unique maximal set of MUBs
- for $d = 2, 3, 5$, pairs of MUBs are all equivalent to the pair $\{\mathbb{1}, F\}$.

P. O. Boykin, M. Sitharam, P. H. Tiep, and P. Wocjan, *Quantum Inf. Comp.* **7**, 371 (2007)

S. Brierley, S. Weigert and I. Bengtsson, arXiv:0907.4097 [math-ph] (2010)

Inequivalence of sets of MUBs

Even for small dimensions ($d \leq 5$), subsets of the maximal set of MUBs may be *inequivalent*: there is no unitary or permutation that can map a set into the other.

- In $d = 4$, there is a one-parameter family of Fourier bases $F(\phi)$, resulting in a continuous one-parameter family of inequivalent pairs $\{H_j(\phi), H_k(\phi)\}$ and in a three-parameter family of inequivalent triplets $\{H_j(\phi), H_k(\phi,\theta), H_l(\phi,\gamma)\}$.
- In $d = 5$, there are two sets of inequivalent triplets.
- For $d > 5$, the structure of inequivalence sets gets more involved.

S. Brierley, S. Weigert and I. Bengtsson, arXiv:0907.4097 [math-ph] (2010)

Consequences

Inequivalence of sets of MUBs has an effect on many different quantum platforms and protocols, which have been, in turn, exploited to classify the inequivalent sets in large dimensions.

- quantum random access codes (qRACs)

  E. A. Aguilar, J. J. Borkała, P. Mironowicz and M. Pawłowski, *Phys. Rev. Lett.* **121**, 050501 (2018)

- noise robustness

  S. Designolle, P. Skrzypczyk, F. Fröwis and N. Brunner, *Phys. Rev. Lett.* **122**, 050402 (2019)

- entanglement detection

  B. C. Hiesmayr, D. McNulty, S. Baek, S. Singha Roy, J. Bae, and D. Chruściński, *New J. Phys.* **23**, 093018 (2021)

- estimation fidelities

  H. Zhu, *PRX Quantum* **3**, 030306 (2022)

  W.-Z. Yan, Y. Li, Z. Hou, H. Zhu, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, *Phys. Rev. Lett.* **132**, 080202 (2024)

Uncertainty relations

Heisenberg-Robertson uncertainty relations

$$\mathrm{var}_\psi(\hat{A})\mathrm{var}_\psi(\hat{B}) \geq \frac{1}{4}\left|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle\right|^2, \tag{3}$$

Maassen-Uffink entropic uncertainty relations (EURs)

$$H_\alpha(\hat{A}|\psi) + H_\beta(\hat{B}|\psi) \geq -\log_2 \max_{a,b}\left[|\langle a|b\rangle|^2\right], \quad \alpha^{-1}+\beta^{-1}=2 \tag{4}$$

Shannon entropy

Case $\alpha = \beta = 1$:

$$H(\hat{X}|\psi) \equiv -\sum_{j=0}^{d-1} p_j \log_2 p_j, \tag{5}$$

with $p_j = |\langle x_j|\psi\rangle|^2$.

Uncertainty relations

Maximally incompatible measurements

The Maassen-Uffink EURs are tight for MUBs. A pair of MUBs identifies *maximally incompatible projective measurements*: perfect knowledge of one implies maximal uncertainty on the other:

$$H_\alpha(\hat{A}|\psi) + H_\beta(\hat{B}|\psi) \geq \log_2 d. \tag{6}$$

The bound is saturated only by states belonging to the MUBs: the EUR is *maximally strong*.

More than two measurements

$$\sum_{k=1}^{m} H(\hat{A}_k|\psi) \geq f(d,m) \qquad\qquad m \leq d+1$$

Few analytic tight bounds are known, for specific d and m.

Numeric tight bounds, for $3 \leq d \leq 5$

A. Riccardi, C. Macchiavello and L. Maccone, *Phys. Rev. A* **95** (2017)

Consider the sets of MU observables $\{\hat{A}_j\}_{j=1}^{m}$ with $\hat{A}_1 = \mathbb{1}, \hat{A}_2 = \hat{F}, \hat{A}_{k+2} = \hat{U}^k \hat{F}$, with $\hat{U}$ a diagonal matrix depending on $\omega = \exp(i2\pi/d)$.

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$d=3$	$\log_2 3$	3	4	-	-
$d=4$	2	3	5	7	-
$d=5$	$\log_2 5$	$4.43 \vee 2\log_2 5$	6.34	8.33	10.25

Remark

The existence of uncertainty relations for sets of measurements is a necessary and sufficient condition for the complementarity of the pertaining observables.

Numeric tight bounds, for $3 \leq d \leq 5$

L. Serino, G. Chesi, B. Brecht, L. Maccone, C. Macchiavello and C. Silberhorn, *Phys. Rev. Research* **7** (2025)

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In $d=5$, $\hat{U} = \mathrm{diag}(1, \omega, \omega^4, \omega^4, \omega)$. Take $A = \hat{A}_1$, $B = \hat{A}_2$, $C = \hat{A}_3$, $D = \hat{A}_4$, $E = \hat{A}_5$, $F = \hat{A}_6$.

$$\mathscr{S}_1 \equiv \{ABC, ABF, BEF, ADE, BCD, CEF, CDF, BDE, ACE, ADF\};$$
$$\mathscr{S}_2 \equiv \{ABD, ABE, ACF, BCE, DEF, CDE, BDF, ACD, BCF, AEF\}.$$

Inequivalence of MUB sets shapes uncertainty

L. Serino, G. Chesi, B. Brecht, L. Maccone, C. Macchiavello and C. Silberhorn, *Phys. Rev. Research* **7** (2025)

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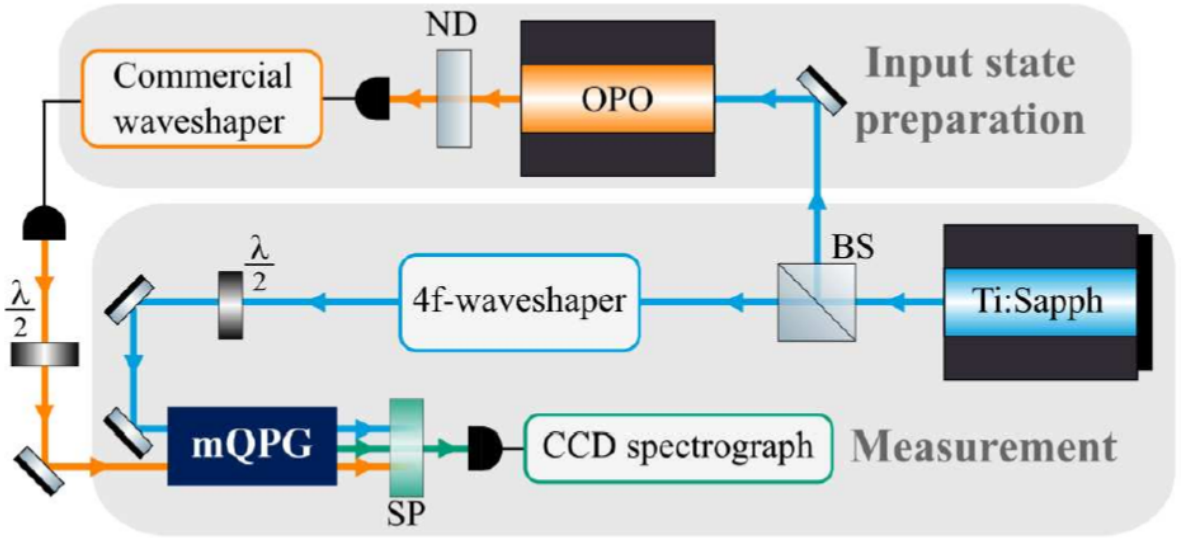
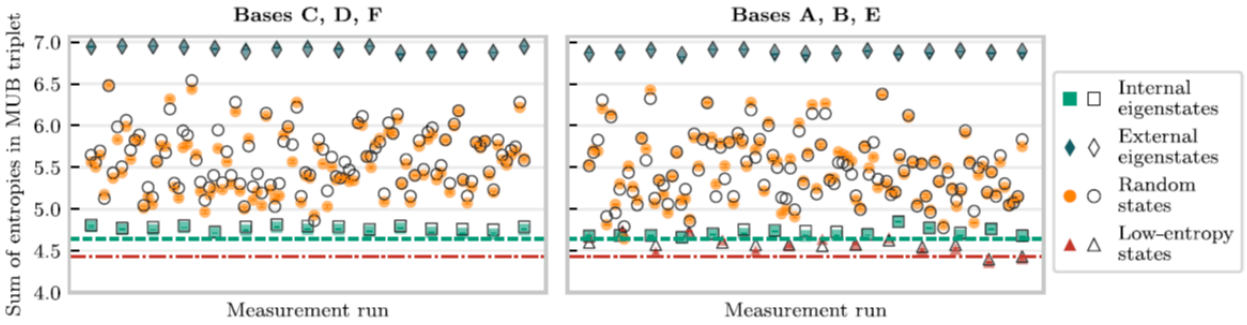
Effect on complementarity properties

Due to the existence of inequivalent sets of MUBs, the complementarity properties of quantum observables can depend on the choice of the complementary bases: *complementarity-based complementarity*. A "second-level" of complementarity, being the first one identified by mutual unbiasedness.

Experimental verification

L. Serino et al., *Phys. Rev. Research* **7** (2025)
L. Serino et al., *Phys. Rev. A* **113** (2026)





Final remarks

Conclusion

We have shown, both analytically and experimentally, that the inequivalence of sets of MUBs shapes a rich structure of complementarity properties.

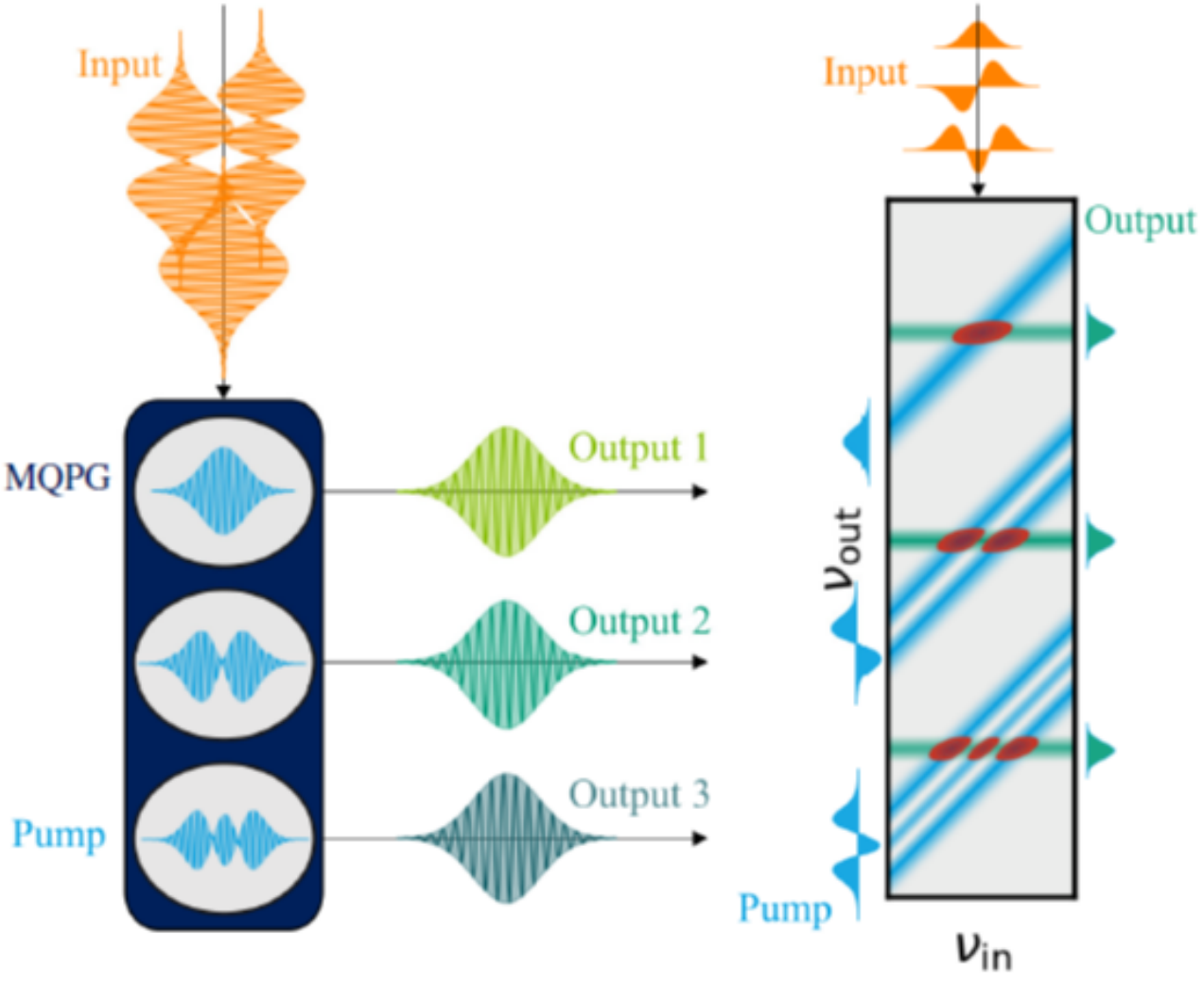
Outlooks

- The lower bounds identifying these EURs are numeric: analytic derivation?
- Inequivalence sets for pairs and triples of MUBs in $d = 4$.
- For $d \geq 3$, the EURs for maximally incompatible measurements are not *maximally strong*. Beyond MUBs: can a notion of *joint unbiasedness* be defined to capture the complementarity of sets of bases larger than pairs?
- Extension to POVMs.
- Side information: extension to conditional entropies and Rènyi entropies. Applications to cryptography.

Thank you for your attention!

Experiment details

L. Serino et al., *PRX QUANTUM* **4** (2023)
L. Serino et al., *Phys. Rev. A* **113** (2026)



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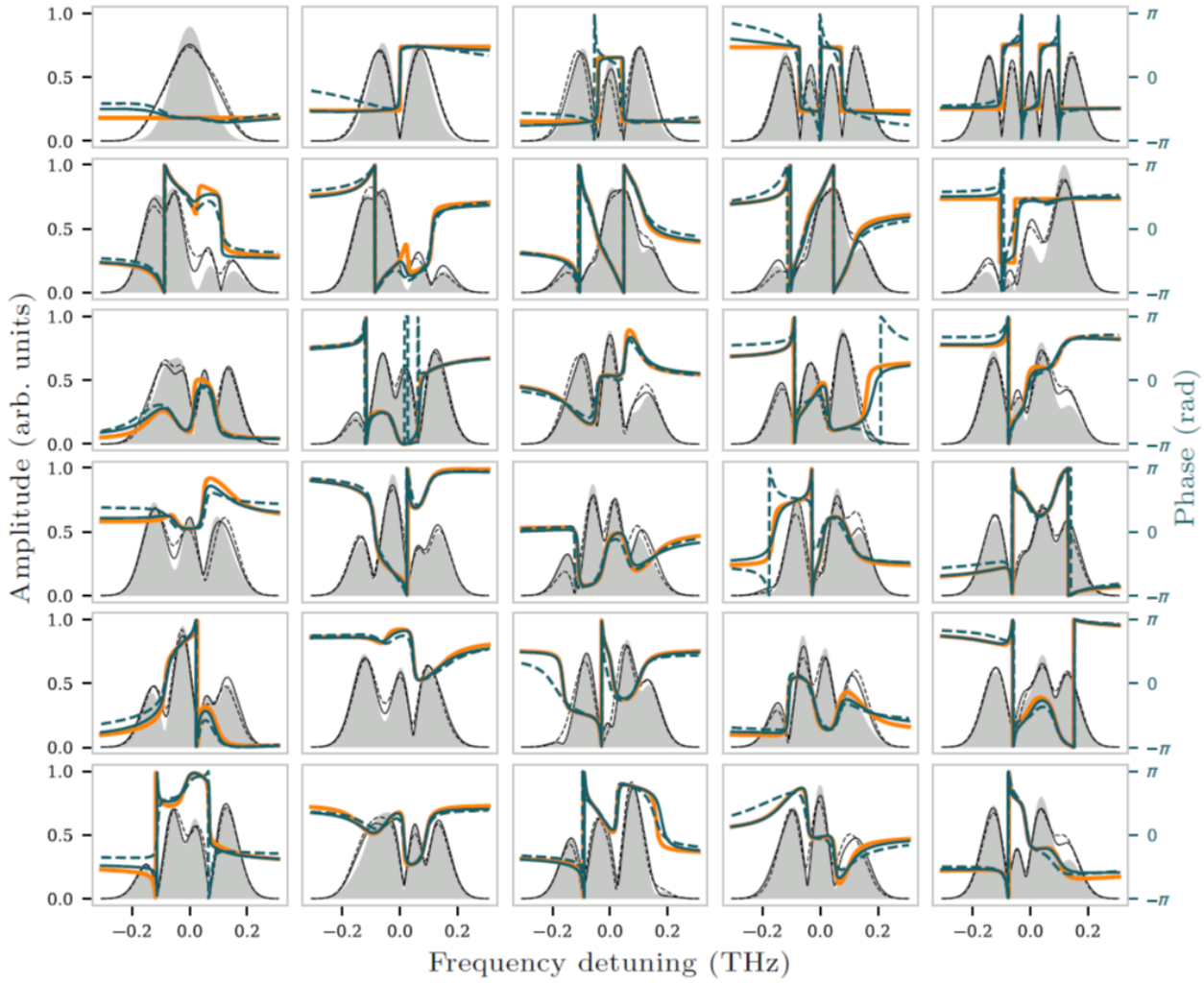
L. Serino et al., *PRX QUANTUM* **4** (2023)
L. Serino et al., *Phys. Rev. A* **113** (2026)