From Static Potentials to the Charmonium(-like) Spectrum

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Glueballs and Charmonium:
Lattice QCD, Experiment and Phenomenology
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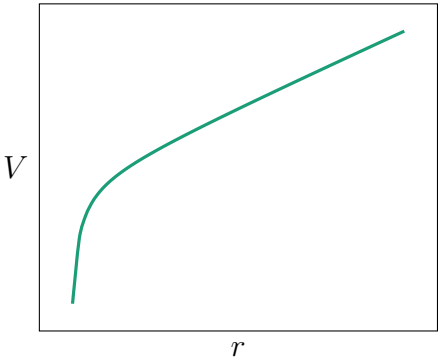
The Cornell Model of Charmonium



Limits of the potential

- $r \to 0$: attractive Coulomb
- $r \to \infty$: $N = 0$ string

Explicit expression

$$V(r) = -\frac{\kappa}{r} + \sigma r + V_0$$

[E. Eichten, K. Gottfried, T. Kinoshita, K.D. Lane and T.-M. Yan, PRD 17, 3090 (1978)]

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Modern Justification from Lattice QCD

B-O approximation

- The charm-quark mass is larger than the nonperturbative energy scale of QCD.
- Charmonium(-like) mesons can be described as bound states and resonances in static potentials.
- The static potentials can be calculated from Wilson loops at large euclidean times using lattice QCD.

Static potential Σ_g^+

- It is the lowest static potential in pure SU(3) lattice gauge theory.
- It turns out to be a perfect fit to a Cornell-like potential.

Charmonium Schrödinger Equation

Just add the kinetic energy to the potential!

$$\left(-\frac{1}{m}\nabla^2 + V_{\Sigma_g^+}(r)\right)\psi_{\Sigma_g^+}(\vec{r}) = E\psi_{\Sigma_g^+}(\vec{r})$$

Then project onto orbital angular momentum

$$\left[-\frac{1}{m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{l(l+1)}{m\,r^2} + V_{\Sigma_g^+}(r)\right] R_{\Sigma_g^+}(r) = E R_{\Sigma_g^+}(r)$$

Each solution corresponds to a multiplet of J^{PC} states

$$J = |l-s|, \ldots, l+s \qquad P = (-1)^{l+1} \qquad C = (-1)^{l+s}$$

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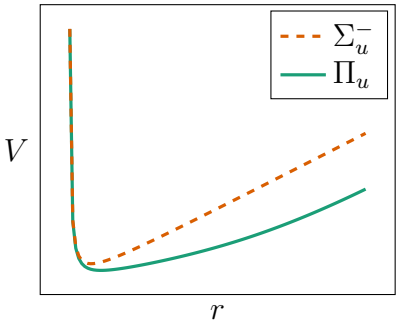
Charmonium-Hybrid Potentials



Hybrid potential Π_u

- $r \to 0$: repulsive Coulomb
- $r \to \infty$: $N = 1$ string

Hybrid potential Σ_u^-

- $r \to 0$: repulsive Coulomb
- $r \to \infty$: $N = 3$ string

The Π_u and Σ_u^- potentials are degenerate at $r \to 0$!

Unlike the charmonium potential Σ_g^+, which is a singlet at $r \to 0$...

[N. Brambilla, A. Pineda, J. Soto and A. Vairo, hep-ph/9907240]
[K.J. Juge, J. Kuti and C. Morningstar, hep-lat/0207004]

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Charmonium-Hybrid Schrödinger Equations

Equation for $P = (-1)^{l+1}$

$$\left[-\frac{1}{m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)+\frac{l(l+1)}{m\,r^2}+V_{\Pi_u}(r)\right]R_{\Pi_u}(r)=E R_{\Pi_u}(r)$$

Equation for $P = (-1)^{l}$

$$\left[-\frac{1}{m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)+\frac{1}{m\,r^2}\begin{pmatrix} l(l+1)+2 & -2\sqrt{l(l+1)} \\ -2\sqrt{l(l+1)} & l(l+1)\end{pmatrix}+\begin{pmatrix} V_{\Sigma_u^-}(r) & 0 \\ 0 & V_{\Pi_u}(r)\end{pmatrix}\right]\begin{pmatrix} R_{\Sigma_u^-}(r) \\ R_{\Pi_u}(r)\end{pmatrix}=E\begin{pmatrix} R_{\Sigma_u^-}(r) \\ R_{\Pi_u}(r)\end{pmatrix}$$

[M. Berwein, N. Brambilla, J. Tarrús Castellà and A. Vairo, 1510.04299]

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Why such Complicated Schrödinger Equations?

Can we just add the kinetic energy to each potential?

$$-\frac{1}{m}\nabla^2\psi(\vec{r}) + V_{\Pi_u}(r)\psi(\vec{r}) = E\psi(\vec{r})$$

$$-\frac{1}{m}\nabla^2\psi(\vec{r}) + V_{\Sigma_u^-}(r)\psi(\vec{r}) = E\psi(\vec{r})$$

Why such Complicated Schrödinger Equations?

Can we just add the kinetic energy to each potential? No!

$$-\frac{1}{m}\nabla^2\psi(\vec{r}) + V_{\Pi_u}(r)\psi(\vec{r}) = E\psi(\vec{r})$$

$$-\frac{1}{m}\nabla^2\psi(\vec{r}) + V_{\Sigma_u^-}(r)\psi(\vec{r}) = E\psi(\vec{r})$$

The B-O quantum numbers are conserved in the static limit

But they are generally not conserved when the $c\bar{c}$ motion is introduced.

There are non-adiabatic couplings between static potentials

Neglecting some or all of them is an additional assumption on top of the B-O approximation. Such an assumption needs to be justified!

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What Are Non-Adiabatic Coupling Terms?

Adiabatic approximation

The fast d.o.f. (the light quarks and gluons) adjust instantaneously to the change in position $\vec{r}$ of the slow d.o.f. (the $c\bar{c}$ pair). If the system is in a certain static potential for a given $\vec{r}$, it will remain there for all $\vec{r}$.

Reality

The fast d.o.f. take time to adjust to the motion of the $c\bar{c}$ pair. If the system is in a certain static potential for a given $\vec{r}$, it will find itself in a superposition of different static potentials after the $c\bar{c}$ have moved.

Non-Adiabatic Coupling Terms (NACTs)

They are the couplings between different static potentials resulting from the fast d.o.f. lagging behind the motion of the $c\bar{c}$ pair.

Can NACTs Be Neglected?

Naïve guess: single-channel approximation

The $c\bar{c}$ move so slowly that the adiabatic approximation applies and we can neglect all NACTs.

Educated guess: truncated coupled-channel approximation

For any given energy E, we can neglect NACTs to static potentials with minima much larger than E.

Dangerous NACTs: neglect at your own risk!

- NACTs between static potentials that are degenerate at $r \to 0$
- NACTs between static potentials near a narrow avoided crossing

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The B-O Approximation as a Top-Down Approach

"Bottom-up" approach

1. finite set of potentials
2. matching to static QCD
3. Schrödinger equation

"Top-down" approach

1. infinite set of potentials
2. matching to static QCD
3. truncate the potentials
4. Schrödinger equation

$$-\frac{1}{m}\nabla^2\psi(\vec{r}) + V_{\Sigma_u^-}(r)\psi(\vec{r}) \stackrel{?}{=} E\psi(\vec{r})$$

- OK in a bottom-up approach (nothing apparently wrong with it).
- NOT OK in a top-down approach (unwarranted truncation).

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The Abstract B-O Hamiltonian

$$H_{\text{B-O}}(\vec{r}) = -\frac{1}{m}\nabla^2 + V_0(\vec{r}) + \frac{1}{m}V_1(\vec{r}) + \mathcal{O}(m^{-2})$$

Static potential $V_0(\vec{r})$

- infinite-dimensional matrix
- eigenvectors depend on $\vec{r}$
- eigenvalues (static potentials) depend only on $r = |\vec{r}|$
- eigenvectors and eigenvalues labeled by:
  - B-O quantum numbers Λ_η^ϵ (e.g., $\Sigma_g^+, \Pi_u, \Sigma_u^-, \ldots$)
  - other quantum numbers (e.g., excitation level $n = 1, 2, \ldots$, flavor)

The Adiabatic B-O Hamiltonian

$$H_0(\vec{r}) = -\frac{1}{m}\nabla^2 + V_0(\vec{r})$$

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The Adiabatic B-O Hamiltonian

$$
\begin{aligned}
H_0(\vec{r}) &= -\frac{1}{m}\nabla^2 + V_0(\vec{r}) \\
V_0(\vec{r}) &= U(\vec{r}) V_0^{\mathsf{ad}}(r) U^\dagger(\vec{r}) \\
V_0^{\mathsf{ad}}(r) &= \mathrm{diag}\big(V_{\Sigma_g^+}(r), V_{\Pi_u}(r), V_{\Sigma_u^-}(r), \dots\big)
\end{aligned}
$$

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The Adiabatic B-O Hamiltonian

$$
\begin{aligned}
H_0(\vec{r}) &= -\frac{1}{m}\nabla^2 + V_0(\vec{r}) \\
V_0(\vec{r}) &= U(\vec{r}) V_0^{\mathsf{ad}}(r) U^\dagger(\vec{r}) \\
V_0^{\mathsf{ad}}(r) &= \mathrm{diag}\big(V_{\Sigma_g^+}(r), V_{\Pi_u}(r), V_{\Sigma_u^-}(r), \dots\big) \\
H_0^{\mathsf{ad}}(\vec{r}) &= U^\dagger(\vec{r}) H_0(\vec{r}) U(\vec{r}) \\
&= -\frac{1}{m}\big(\vec{\nabla} + \vec{\Pi}(\vec{r})\big)^2 + V_0^{\mathsf{ad}}(r)
\end{aligned}
$$

Definition of the non-adiabatic coupling terms (NACTs)

$$
\vec{\Pi}(\vec{r}) = U^\dagger(\vec{r}) \vec{\nabla} U(\vec{r})
$$

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General Features of the Angular NACTs

Definition of the angular NACTs

$$\Pi_\theta(\vec{r}) = \frac{1}{r} U^\dagger(\vec{r}) \partial_\theta U(\vec{r}) \qquad \Pi_\phi(\vec{r}) = \frac{1}{r \sin\theta} U^\dagger(\vec{r}) \partial_\phi U(\vec{r})$$

- The eigenvectors of the static potential (i.e., the columns of $U(\vec{r})$) depend on θ and ϕ through the B-O quantum number $\Lambda = \left| \vec{J} \cdot \hat{r} \right|$.
- The angular NACTs mix static potentials with different values of Λ.
- The angular NACTs are closely related to rotational symmetry.

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Physical Meaning of the Angular NACTs

Λ-doubling

It is a well-known experimental effect in atomic molecules showing the physical relevance of the angular NACTs.

- If one neglects these NACTs, then the B-O quantum number Λ of a system cannot change, becoming a "good" quantum number.
- Reflection symmetry then implies that any Λ energy level with $\Lambda > 0$ is a doublet of degenerate states with opposite parity.
- This degeneracy is however not realized in Nature, there is a splitting between Λ energy levels with opposite parity.
- That splitting is due to a parity-dependent mixing with $\Lambda \pm 1$ energy levels induced by the angular NACTs.

Λ-Doubling in Charmonium Hybrids

Schrödinger equation for $l = 1$, $P = +1$

$$\left[-\frac{1}{m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)+\frac{2}{m\,r^2}+V_{\Pi_u}(r)\right]R_{\Pi_u}(r)=ER_{\Pi_u}(r)$$

Schrödinger equation for $l = 1$, $P = -1$

$$\left[-\frac{1}{m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)+\frac{1}{m\,r^2}\begin{pmatrix}4 & -2\sqrt{2}\\ -2\sqrt{2} & 2\end{pmatrix}+\begin{pmatrix}V_{\Sigma_u^-}(r) & 0\\ 0 & V_{\Pi_u}(r)\end{pmatrix}\right]\begin{pmatrix}R_{\Sigma_u^-}(r)\\ R_{\Pi_u}(r)\end{pmatrix}=E\begin{pmatrix}R_{\Sigma_u^-}(r)\\ R_{\Pi_u}(r)\end{pmatrix}$$

Approximately the same low-energy spectrum only if both m and the gap between Π_u and Σ_u^- were way larger than their physical values.

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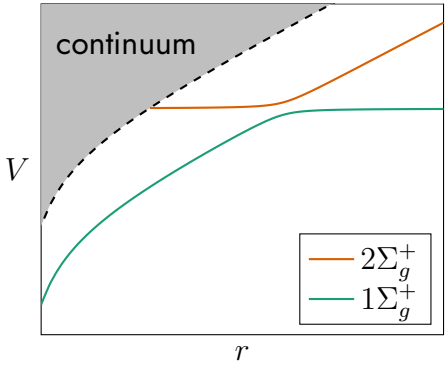
General Features of the Radial NACTs

Definition of the radial NACTs

$$\Pi_r(r) = U^\dagger(\vec{r})\partial_r U(\vec{r})$$

- The radial NACTs depend only on $r = |\vec{r}|$.
- The radial NACTs mix potentials with the same B-O quantum numbers but different excitation levels $n = 1, 2, \ldots$.
- The radial NACTs are most important for static potentials separated by a small energy gap (e.g., narrow avoided crossing).
- The radial NACTs are inherently dynamical (not symmetry-driven).

String Breaking



String-breaking NACT

- $r = 0$: exactly zero
- small r: unknown
- crossing: large
- large r: negligible

Physical meaning of the string-breaking NACTs

Charmonium-tetraquark mixing and charmonium decays into $D\bar{D}$.

[G. Bali, H. Neff, T. Düssel, T. Lippert and K. Schilling, hep-lat/0505012]

[J. Bulava, F. Knechtli, V. Koch, C. Morningstar and M. Peardon, 2403.00754]

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The Diabatic Hamiltonian

$$H_0^{\mathsf{ad}}(\vec{r}) = -\frac{1}{m}\left(\vec{\nabla} + \vec{\Pi}(\vec{r})\right)^2 + V_0^{\mathsf{ad}}(r)$$

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$$H_0^{\mathsf{ad}}(\vec{r}) = -\frac{1}{m}\left(\vec{\nabla} + \vec{\Pi}(\vec{r})\right)^2 + V_0^{\mathsf{ad}}(r)$$
$$\vec{\nabla} A(\vec{r}) = A(\vec{r})\vec{\Pi}(\vec{r})$$
$$H_0^{\mathsf{di}}(\vec{r}) = A(\vec{r}) H_0^{\mathsf{ad}}(\vec{r}) A^{\dagger}(\vec{r})$$

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\begin{aligned}
H_0^{\mathsf{ad}}(\vec{r}) &= -\frac{1}{m}\left(\vec{\nabla} + \vec{\Pi}(\vec{r})\right)^2 + V_0^{\mathsf{ad}}(r) \\
\vec{\nabla} A(\vec{r}) &= A(\vec{r})\vec{\Pi}(\vec{r}) \\
H_0^{\mathsf{di}}(\vec{r}) &= A(\vec{r}) H_0^{\mathsf{ad}}(\vec{r}) A^\dagger(\vec{r}) \\
&= -\frac{1}{m}\nabla^2 + V_0^{\mathsf{di}}(\vec{r})
\end{aligned}
$$

Diabatic potential matrix

$$
V_0^{\mathsf{di}}(\vec{r}) = A(\vec{r}) V_0^{\mathsf{ad}}(r) A^\dagger(\vec{r})
$$

Adiabatic-to-Diabatic Transformation

Implicit definition: solution of a Cauchy problem

$$\begin{cases} \vec{\nabla} A(\vec{r}) = A(\vec{r}) \vec{\Pi}(\vec{r}) \\ A(\vec{r}_0) = I \end{cases}$$

Explicit definition: path-ordered exponential

$$A(\vec{r}) = \mathcal{P} \exp \int_{\mathcal{C}} \vec{\Pi}(\vec{r}') \cdot \mathrm{d}\vec{r}'$$

$\mathcal{C}$ oriented curve with r_0 and r as start and end points

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For the Differential-Geometry Geeks

Adiabatic: decoupled potentials in a curved space

$$H_0^{\mathsf{ad}}(\vec{r}) = -\frac{1}{m}\left(\vec{\nabla} + \vec{\Pi}(\vec{r})\right)^2 + \operatorname{diag}\left(V_{\Sigma_g^+}(r), V_{\Pi_u}(r), V_{\Sigma_u^-}(r), \dots\right)$$

$$A(\vec{r}) = \mathcal{P} \exp \int_{\mathcal{C}} \vec{\Pi}(\vec{r}') \cdot \mathrm{d}\vec{r}'$$

Diabatic: coupled potentials in a flat space

$$H_0^{\mathsf{di}}(\vec{r}) = -\frac{1}{m}\nabla^2 + V_0^{\mathsf{di}}(\vec{r})$$

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Angular Momentum in Adiabatic vs. Diabatic

Adiabatic Hamiltonian

The adiabatic channels have a definite angular-momentum quantum number j only for $r \to 0$. Assigning a definite j to the adiabatic channels for arbitrary $r > 0$ is generally unwarranted.

Diabatic Hamiltonian

Matching adiabatic and diabatic in the limit $r \to 0$ (i.e., imposing $\vec{r}_0 = 0$) allows to assign a definite j to each diabatic channel at all r.

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General Expression of the Angular Dependence

Fixing $\vec{r}_0 = 0$, the dependence of $A(\vec{r})$ on the angles θ and ϕ factors out and is completely determined by rotational symmetry:

$$\big(A(\vec{r})\big)_{(j,\sigma),(n,\lambda)} = D^{j}_{\sigma,\lambda}(\phi,\theta,\psi)\big(A(r)\big)_{j,n}$$

$D^{j}_{\sigma,\lambda}(\phi,\theta,\psi)$ Wigner D-matrix element

j,σ angular-momentum quantum numbers of the channel

n,λ B-O quantum numbers (excitation and spin projection)

Angular dependence of the diabatic potential matrix

$$\big(V_0^{\mathsf{di}}(\vec{r})\big)_{(j,\sigma),(j',\sigma')} = \sum_{(n,\lambda)} D^{j}_{\sigma,\lambda}(\phi,\theta,\psi) D^{j'}_{\sigma',\lambda}(\phi,\theta,\psi)^{*} \big(A(r)\big)_{j,n}\big(A^{\dagger}(r)\big)_{n,j'} V_{(n,\lambda)}(r)$$

Reduction to a Radial Form

The diabatic Hamiltonian can be expressed in a radial form made of decoupled blocks labeled by the B-O angular momentum l.

Radial diabatic Hamiltonian

$$H_0^l(r) = -\frac{1}{m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{L_{c\bar{c}}(L_{c\bar{c}}+1)}{m\,r^2} + V_0^l(r)$$

Radial diabatic potential matrix elements

$$\left(V_0^l(r)\right)_{(j,l_{c\bar{c}}),(j',l'_{c\bar{c}})} = (-1)^{j-j'} \sum_{(n,\lambda)} \left\langle \begin{matrix} j & l \\ \lambda & -\lambda \end{matrix} \middle| \begin{matrix} l_{c\bar{c}} \\ 0 \end{matrix} \right\rangle \left\langle \begin{matrix} j' & l \\ \lambda & -\lambda \end{matrix} \middle| \begin{matrix} l'_{c\bar{c}} \\ 0 \end{matrix} \right\rangle \left(A(r)\right)_{j,n} \left(A^\dagger(r)\right)_{n,j'} V_{(n,\lambda)}(r)$$

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Diabatic Hamiltonians for Charmonium Hybrids

Radial Hamiltonian for $l = 1$, $P = +$

$$H_0^{1^+}(r) = -\frac{1}{m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{2}{m\,r^2} + V_{\Pi_u}(r)$$

Radial Hamiltonian for $l = 1$, $P = -$

$$H_0^{1^-}(r) = -\frac{1}{m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{1}{m\,r^2}\begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} + V_{\Sigma_u^-}(r) + \frac{V_{\Pi_u}(r) - V_{\Sigma_u^-}(r)}{3}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$$

[R. Oncala and J. Soto, 1702.03900]

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General Expression of the Radial Dependence

$$A(r) = \mathcal{P} \exp \int_0^r \Pi_r(r')\,\mathrm{d}r' = \exp \vartheta(r)$$

$\vartheta(r)$: antisymmetric matrix of real mixing angles

$$\big(\vartheta(r)\big)_{n,n'} = \int_0^r \big(\Pi_r(r')\big)_{n,n'}\,\mathrm{d}r'$$

Example: two-channel system

$$A(r) = \exp \begin{pmatrix} 0 & \theta(r) \\ -\theta(r) & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta(r) & \sin\theta(r) \\ -\sin\theta(r) & \cos\theta(r) \end{pmatrix}$$

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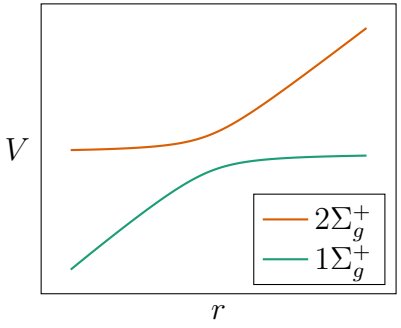
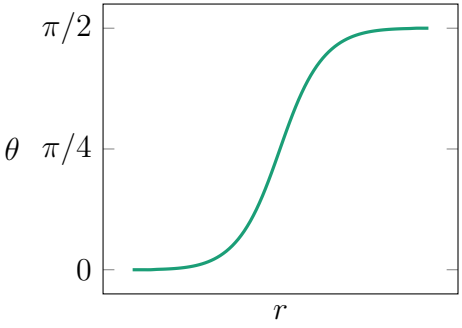
String-Breaking Mixing Angle

Static potentials



Mixing angle



[G. Bali, H. Neff, T. Düssel, T. Lippert and K. Schilling, hep-lat/0505012]

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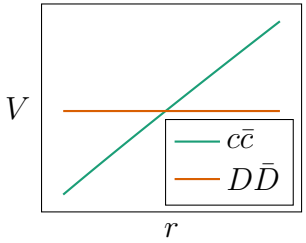
Radial String-Breaking Diabatic Potentials

General expression

$$\begin{pmatrix} \cos\theta(r) & \sin\theta(r) \\ -\sin\theta(r) & \cos\theta(r) \end{pmatrix} \begin{pmatrix} V_{1\Sigma_g^+}(r) & 0 \\ 0 & V_{2\Sigma_g^+}(r) \end{pmatrix} \begin{pmatrix} \cos\theta(r) & -\sin\theta(r) \\ \sin\theta(r) & \cos\theta(r) \end{pmatrix}$$

String-breaking model

$$\begin{pmatrix} V_{c\bar{c}}(r) & g(r) \\ g(r) & V_{D\bar{D}}(r) \end{pmatrix}$$



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Summary

- The B-O Schrödinger equation has infinitely many coupled channels, each corresponding to a different static potential.
- The static potentials are coupled by non-adiabatic coupling terms.
- Truncating to a finite number of potentials is necessary, but:
  - It is an additional assumption on top of the B-O approximation.
  - It must be justified for the energy region under study.
  - Truncating to a single static potential is generally unwarranted.
- We cannot calculate the charmoniumlike spectrum *ab initio* from the static potentials alone! We need the mixing angles as well!

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