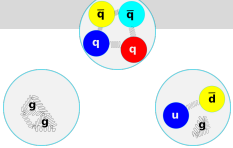
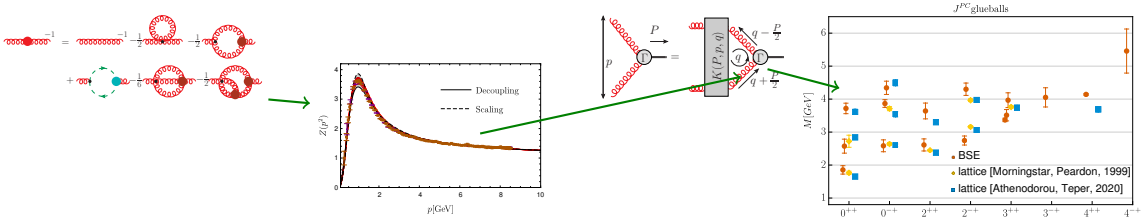
Glueballs from functional methods

Markus Q. Huber

Institute of Theoretical Physics, Justus Liebig University Giessen





Glueballs and charmonium: Lattice QCD, Experiment and Phenomenology

March 5, 2026

ECT*, Trento, Italy

→ MQH, Fischer, Sanchis Alepuz, Eur.Phys.J.C 85, 2503.03821 (2025)

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DFG Deutsche Forschungsgemeinschaft

Outline

- Functional equations: Bottom-up ($\rightarrow$ talk by Fischer) vs. top-down approaches
- Quenched glueballs: input, checks, results
- Glueballs and mesons: mixing

Glueballs

QCD: Fritzsch, Gell-Mann, Leutwyler, Phys. Lett. 47B (1973)

- Fritzsch, Gell-Mann, Proc. ICHEP (1972), hep-ph/0208010; Fritzsch, Minkowski, Nuovo Cim. A 30 (1975): “glue mesons”

  $\rightarrow$ Gauge invariant operators, e.g., $F^{\mu\nu}F_{\mu\nu}$, $F^{\mu\nu}\tilde{F}_{\mu\nu}$, ...

- Experimental challenges: see talks by Küßner, Liu, Wiedner

- Theoretical challenges...

Glueball calculations on the lattice

Nonperturbative first-principles method: lattice QCD $\rightarrow$ talk by Urrea Nino

- discretized space-time $\rightarrow$ continuum limit
- computationally expensive
- quark masses

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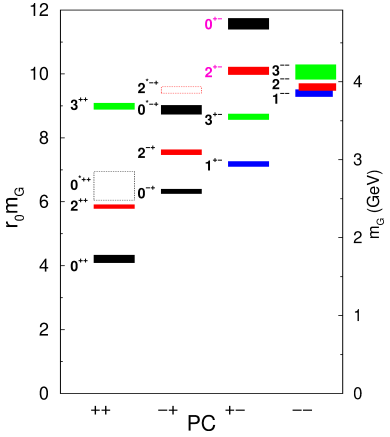
Quantitative results for pure gauge theory (no quarks):

- [Morningstar, Peardon, Phys. Rev. D60 (1999)]
- [Athenodorou, Teper, JHEP11 (2020)]
  improved statistics, more states

→ Benchmark



Glueball calculations with functional methods

- ~~discretized space-time~~ continuum
- computational complexity:
  - ~~statistical method~~
  - integral equations: complexity depends on # loops, legs, kinematic variables, ...

    simple $\leftrightarrow$ complex
- quark masses: easy to vary, chiral limit

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- Required input: correlation functions of quarks and gluons
- First-principle? That depends...

Functional methods: Overview

Systems of equations relying on effective actions:

- Functional renormalization group (FRG): effective average action Γ_k
- Dyson-Schwinger equations (DSEs): eqs. of motion of 1PI effective action
- eqs. of motion of nPI effective action, $n > 1$

} Similar eqs.

- bound state equations (BSEs) (Bethe-Salpeter, Faddeev, Yakubovsky)

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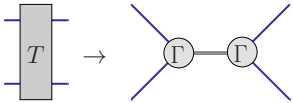
spectra

- Infinitely large systems: (in)finitely many eqs. of (in)finite size
- Nonperturbative
- Truncations
- Applied for hadron physics, QCD phase diagram, condensed matter, gravity, ...

Bound state equations

Details of derivation

n-particle bound state: Pole in a 2*n*-point function ***G***/scattering matrix ***T***.



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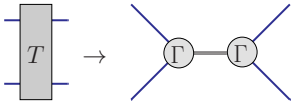
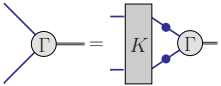


Generic forms for mesons:



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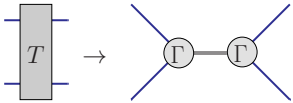
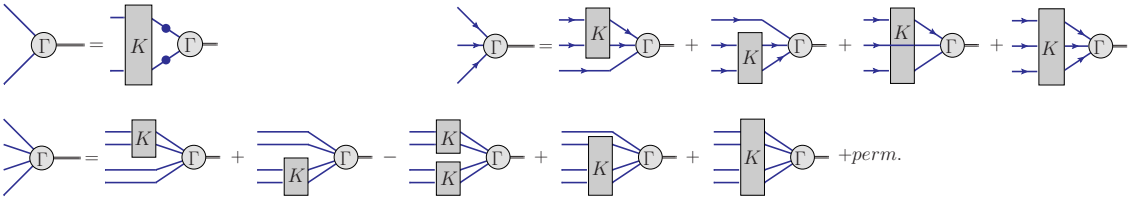


Generic forms for mesons, baryons and tetraquarks:



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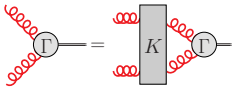

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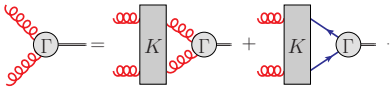

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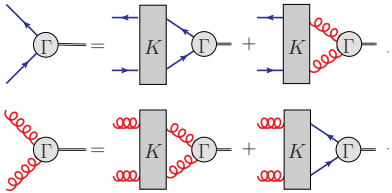

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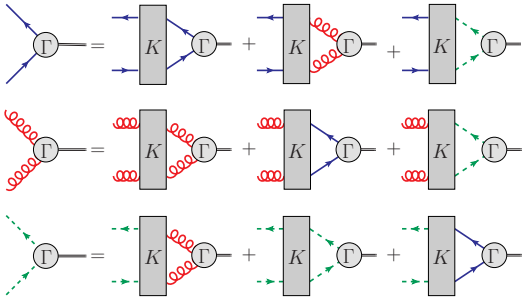


Gluons, quarks and ghosts all couple.
(Landau gauge)

Elements of a BSE

n-particle bound state: Pole in a $2n$-point function G (or the scattering matrix T).

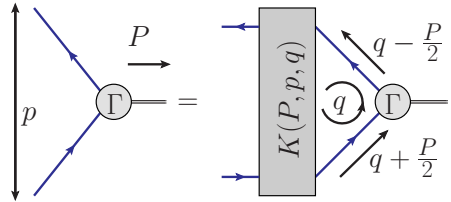
$$\Gamma = K\, G_0\, \Gamma$$

Input:

- Propagators G_0
- Kernel K

Output:

- Mass M
- Bethe-Salpeter amplitudes Γ



(quark-antiquark state)

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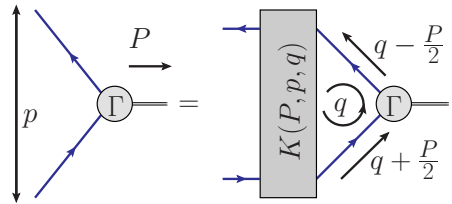

(quark-antiquark state)

Symmetry constraints: Propagators and kernels are not independent!

Relevant for QCD: Chiral symmetry in quark sector
→ pions are (almost) massless (pseudo-)Goldstone bosons

Truncations

How to tame an infinitely large system of equations?

Expansion/ordering scheme

- In UV: Loop expansion ✓
- In IR: No small parameter
  - IR analysis identifies IR leading diagrams
- Midmomentum?

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Vertex expansion:
2-, 3-, 4-point functions, ...

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- Top-down: Calculate input

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Convergence

Pragmatic approach: Enlarging the truncation until changes small enough.
→ Apparent convergence

Bottom-up approach: Rainbow-ladder

"Severe" truncations, but phenomenologically successful.

Need the gluon propagator ($Z(k^2)$) and the quark-gluon vertex ($h_i(k;p,q)$).

- $\Gamma^{a,\nu}(k;p,q) \propto \gamma^\nu h_1(k;p,q)$
- $\frac{g^2}{4\pi} Z(k^2) h_1(k;p,q) \propto \alpha(k^2)$



Iteration $\rightarrow$ only 'rainbow-like' diagrams

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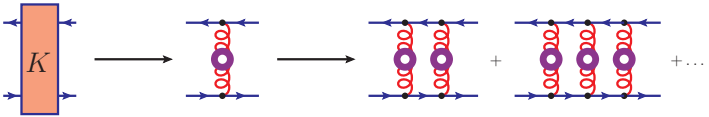
Iteration $\rightarrow$ only 'rainbow-like' diagrams

Scattering kernel: "all interactions which are two-particle irreducible with respect to two horizontal lines"

Simplest realization:



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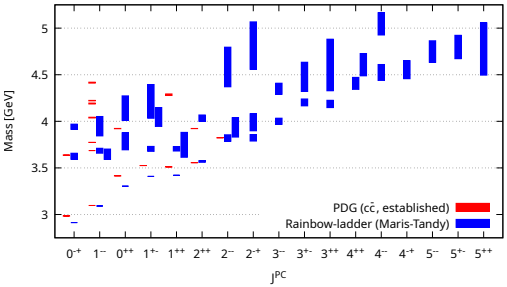
Iteration → only ’rainbow-like’ diagrams

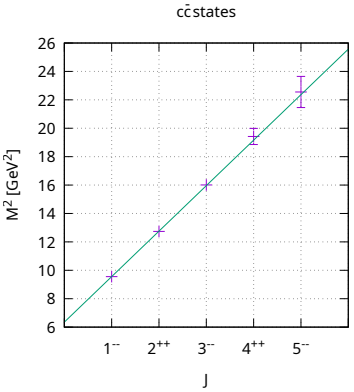
Iteration → ’ladder-like’ diagrams

’model interaction’

Rainbow-ladder: Charmonium

[Hagel, Fischer, MQH, 2510.27423]





- Good description for some states
- Regge trajectory

Model based BSE calculations for glueballs ($J = 0$)

- [Meyers, Swanson, Phys.Rev.D87 (2013)]
- [Sanchis-Alepuz, Fischer, Kellermann, von Smekal, Phys.Rev.D92, (2015)]
- [Souza et al., Eur.Phys.J.A56 (2020)]
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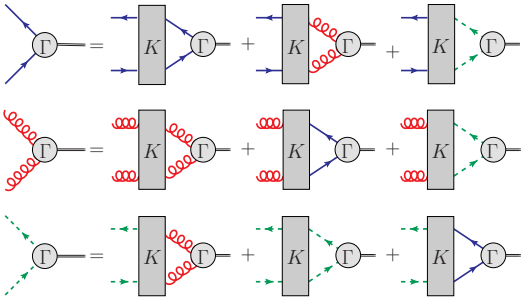
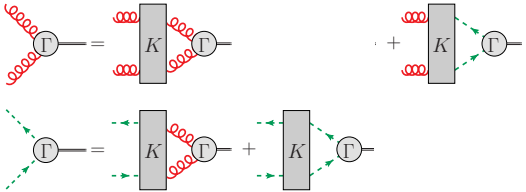
Better models? Alternative: Self-consistently calculated input.

Glueball bound state equations: Quenched



Glueball bound state equations: Quenched

Focus on pure glueballs.

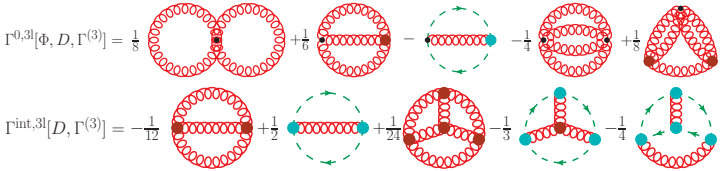


- Conceptually simpler: fewer terms, only glueballs (ghosts *do not* lead to additional states)
- Benchmark: Lattice results

3PI effective action

- 3PI: propagators and three-point functions as dynamic quantities
- Natural expansion scheme in loops.

In general: m-loop for nPI, $m \geq n$, for a consistent description.



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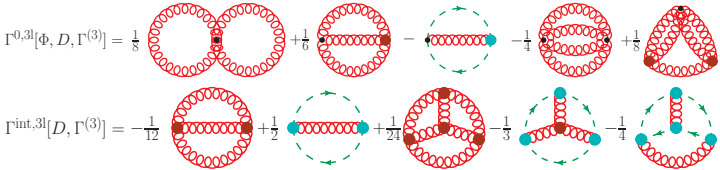

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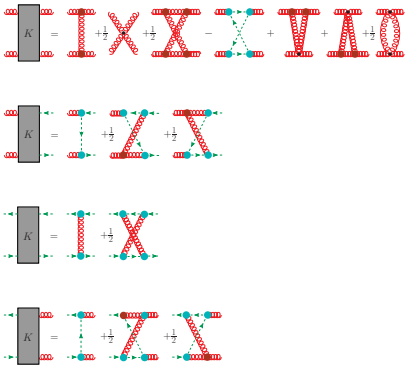
- Equations of motion from stationarity conditions → correlation functions ✓
- Kernels from stability conditions → BSEs ✓

Kernels from the 3PI 3-loop effective action

Systematic derivation from 3PI effective action [Berges, PRD70 (2004); Carrington, Guo, PRD83 (2011)].



[Fukuda, Prog. Theor. Phys 78 (1987); McKay, Munczek, Phys. Rev. D 40 (1989); Sanchis-Alepuz, Williams, J. Phys: Conf. Ser. 631 (2015); MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C80 (2020)]

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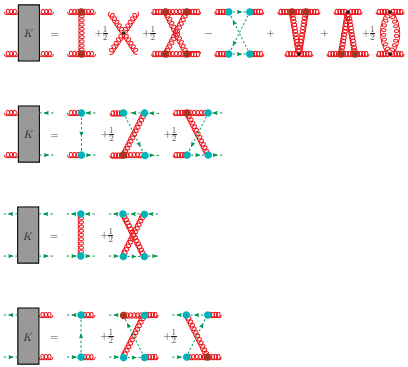



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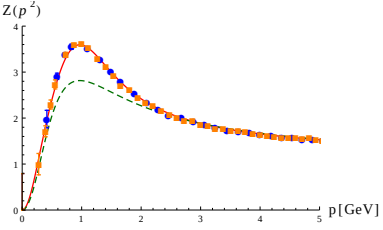
Input for bound state equations: 2013?

Propagators from 2013 with modeled vertices [Sternbeck, hep-lat/0609016; MQH, von Smekal, JHEP 04 (2013)]:

Gluon dressing function:



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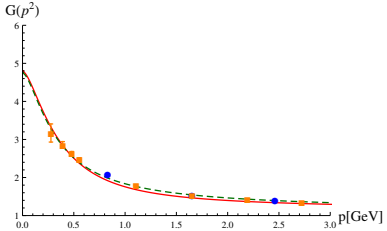

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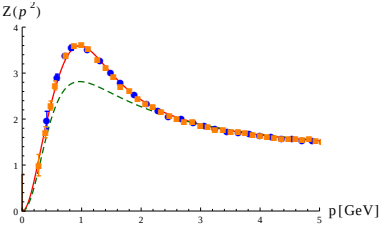
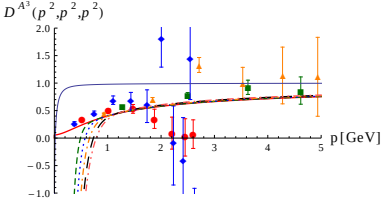


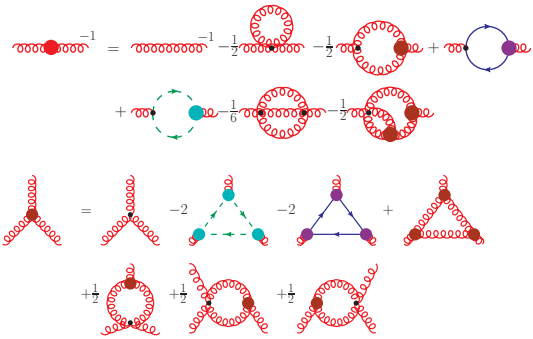
Three-gluon vertex [Cucchieri, Maas, Mendes, Phys. Rev. D 77 (2008); MQH, Phys. Rev. D 101 (2020)]:

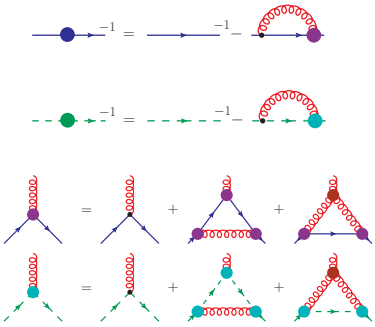


Used a three-gluon vertex model tailored to reproduce lattice results.
$\rightarrow$ Insufficient for bound states.

Equations of motion from 3PI 3-loop

Stationarity conditions: $\frac{\delta\Gamma}{\delta D} = 0$, $\frac{\delta\Gamma}{\delta\Gamma^{(3)}} = 0 \rightarrow$ Equations of motion (already truncated):





$\rightarrow$ Self-consistent treatment of propagators and three-point functions. Including two-loop diagrams.

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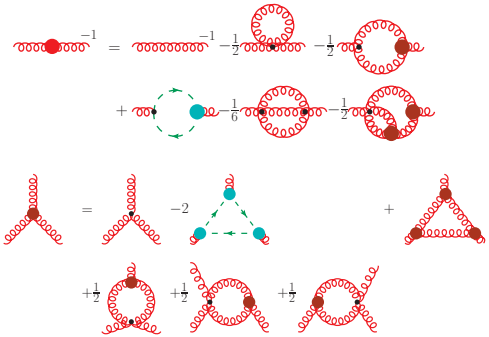





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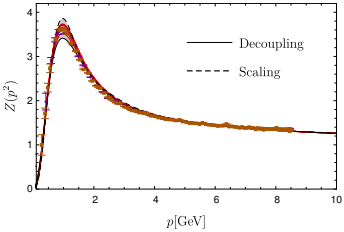

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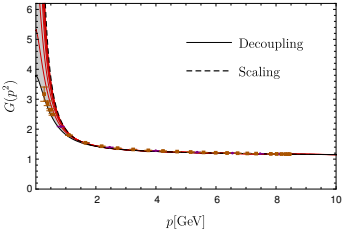

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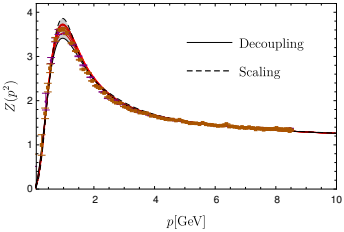
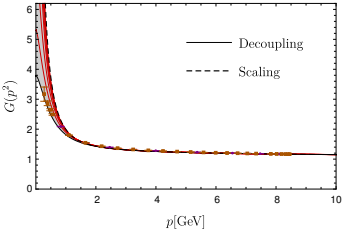
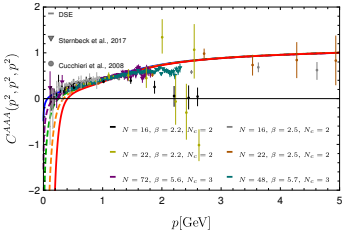


Ghost dressing function:



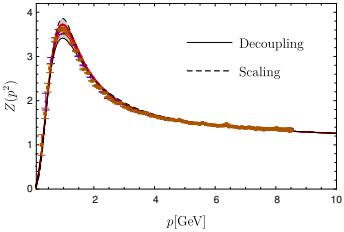
Three-gluon vertex [Cucchieri, Maas, Mendes, Phys. Rev. D 77 (2008); Sternbeck et al., Pos LATTICE2016; MQH, Phys. Rev. D 101 (2020)]:



Input for bound state equations: 2020?

[MQH, Phys. Rev. D 101 (2020)]

Gluon dressing function:



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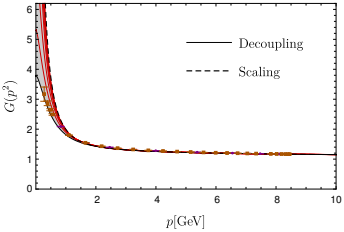

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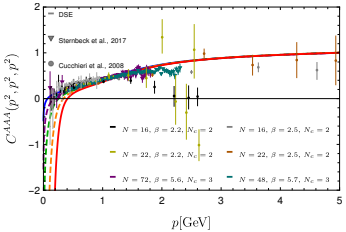


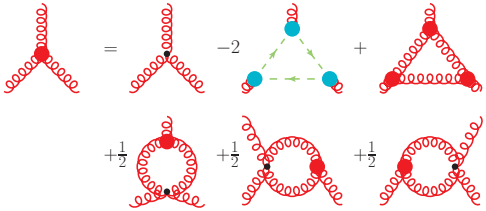
Still truncations in equations. . .

How far should we trust the results?

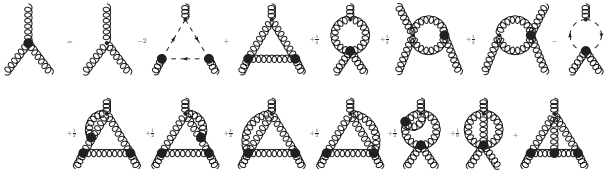
Three-gluon vertex: 1PI vs. 3PI

[MQH, Fischer, Sanchis Alepuz, Eur.Phys.J.C 85 (2025)]

3PI:

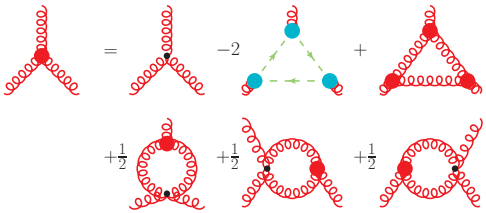


1PI:

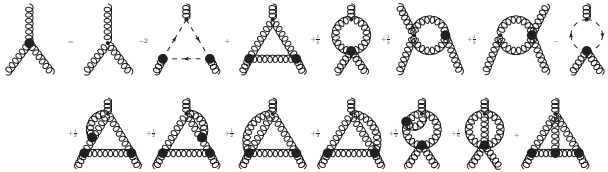


Three-gluon vertex: 1PI vs. 3PI

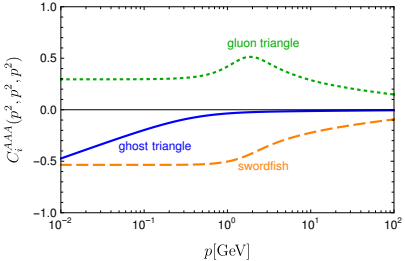
3PI:

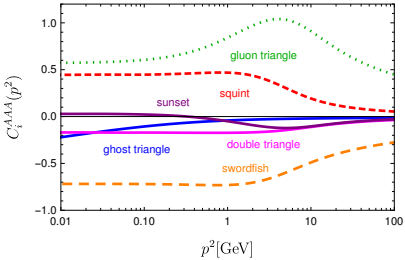


1PI:



[MQH, Fischer, Sanchis Alepuz, Eur.Phys.J.C 85 (2025)]





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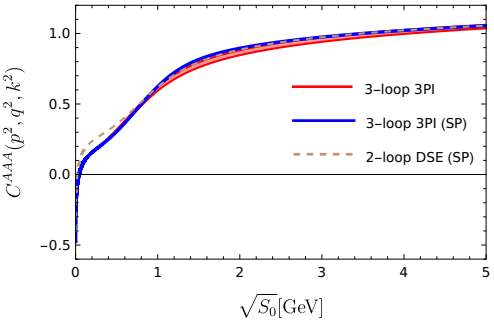
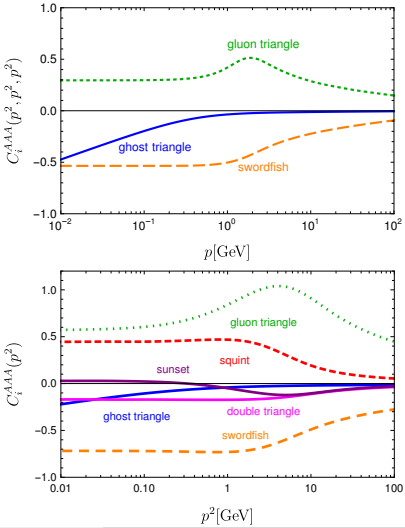
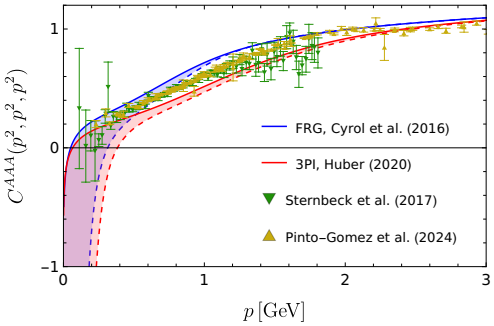




- Different cancellations for 1PI and 3PI with similar results
- Tiny angular dependence (red band) $\rightarrow$ approximations possible

Further checks: Three-point functions

- Agreement between lattice, FRG, DSE and 3PI.



[Pinto-Gómez, de Soto, Rodríguez-Quintero, Phys.Lett.B 838 (2023); Sternbeck et al., Proc.Sci. LATTICE2016 (2017); Cyrol et al., Phys.Rev.D 94 (2016); MQH, Phys.Rev.D101 (2020); MQH, Fischer, Sanchis-Alepuz, 2503.03821]

- Three-gluon vertex (4 transverse dressings): Tree-level dressing dominant, others subleading [Eichmann, Williams, Alkofer, Vujinovic, Phys.Rev.D89 (2014); Pinto-Gómez et al., 2208.01020] ✓

More checks: Four-point functions

A new benchmark quantity as new results become available: the four-gluon vertex

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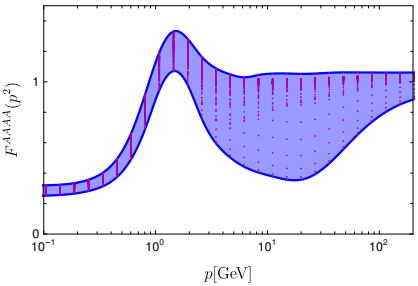
Four-gluon vertex from DSE using 3PI input: [MQH, Phys. Rev. D 101 (2020)]



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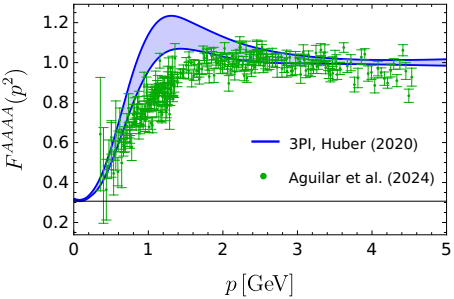
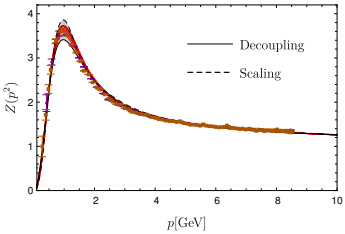


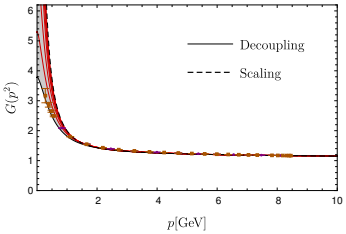
Shown: Subset of one vanishing gluon momentum.

Input

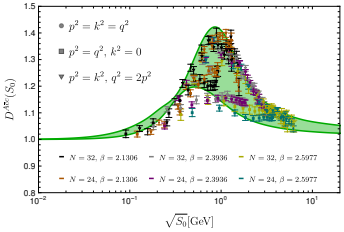
Propagators: Full

[MQH, Phys. Rev. D 101 (2020)]

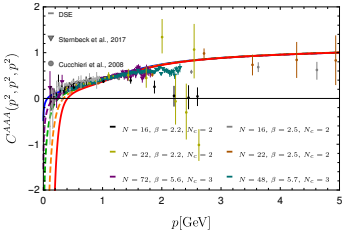




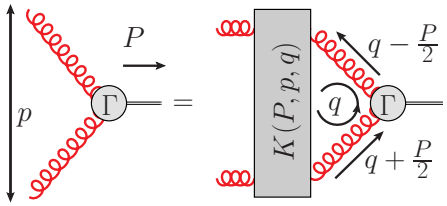
Ghost-gluon vertex: Full



Three-gluon vertex: Leading tensor, full kinematics



Solving a BSE



(pseudoscalar glueball)

$$\lambda(P)\Gamma(P) = \mathcal{K} \cdot \Gamma(P)$$

$\rightarrow$ Eigenvalue problem for $\Gamma(P)$:

1. Solve for $\lambda(P)$.
2. Find P with $\lambda(P) = 1$.
   $\Rightarrow M^2 = -P^2$

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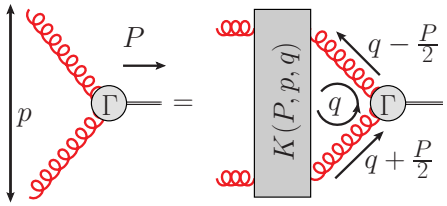

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However:

Propagators are probed at $\left(q \pm \frac{P}{2}\right)^2 = \frac{P^2}{4} + q^2 \pm \sqrt{P^2\, q^2}\cos\theta = -\frac{M^2}{4} + q^2 \pm i\, M \sqrt{q^2}\,\cos\theta$

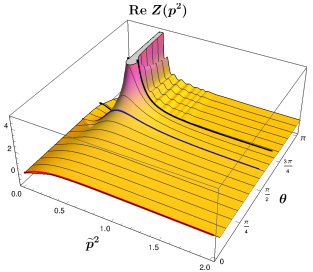
→ Complex for $P^2 < 0$!

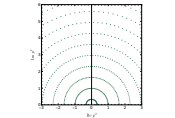
Time-like quantities ($P^2 < 0$) → Correlation functions for complex arguments.

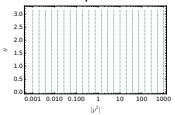
Correlation functions in the complex plane

Gluon

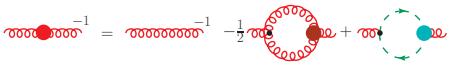
[Fischer, MQH, Phys.Rev.D 102 (2020)]







Simpler truncation:



Correlation functions in the complex plane

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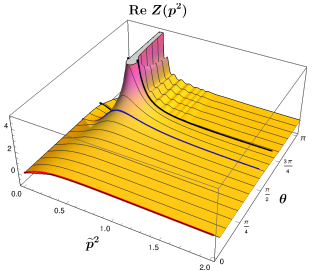

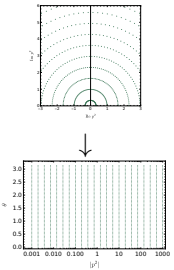

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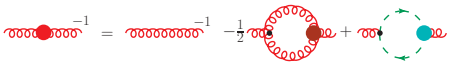
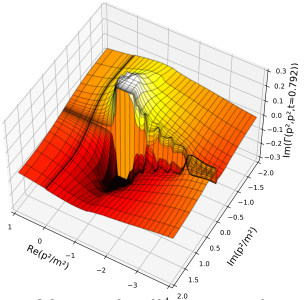


Vertex (scalar toy theory)

[MQH, Kern, Alkofer, Phys.Rev.D107 (2023)]



- Numerically more demanding due to calculations close to cuts.

Extrapolation of $\lambda(P^2)$

Extrapolation method

- Do purely Euclidean calculation.
- λ for time-like P^2 using Schlessinger's continued fraction method (superior to default Padé approximants) [Schlessinger, Phys.Rev.167 (1968)]
- Average over extrapolations using subsets of points for error estimate

$$f(x) = \cfrac{f(x_1)}{1 + \cfrac{a_1(x-x_1)}{1+\cfrac{a_2(x-x_2)}{1+\cfrac{a_3(x-x_3)}{\cdots}}}}$$

Coefficients a_i can be determined such that $f(x)$ exact at x_i.

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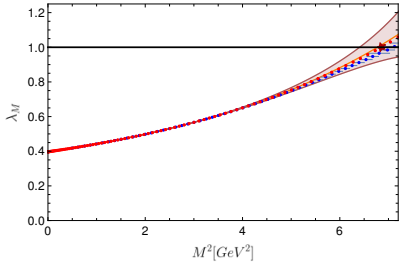
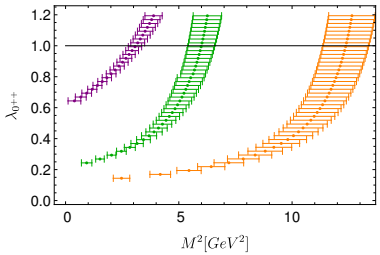
Test extrapolation for solvable system:
Heavy meson [MQH, Sanchis-Alepuz, Fischer, Eur.Phys.J.C 80 (2020)]

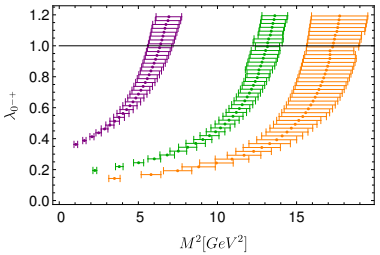


Extrapolation for glueball eigenvalue curves

0^{++}:



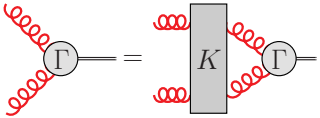
0^{-+}:

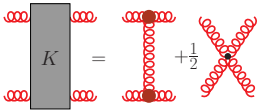


Several curves: ground state and excited states.

Approximations in kernels

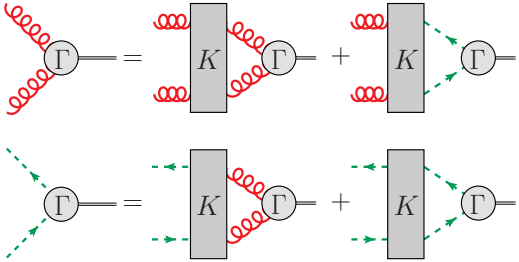
- No ghosts (exact for *P*-odd states)

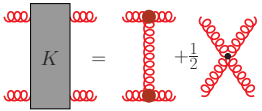




Approximations in kernels

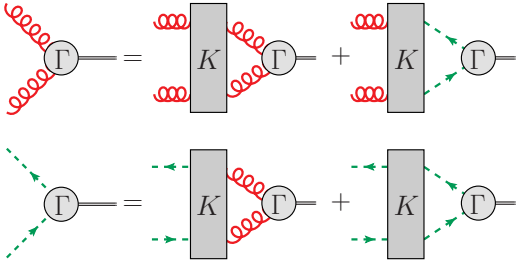
- No ghosts (exact for P-odd states)
- Only one-particle exchanges (2020)

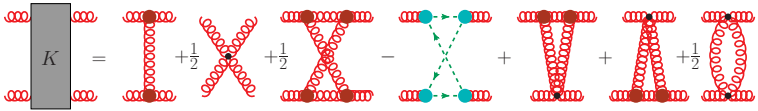




Approximations in kernels

- No ghosts (exact for P-odd states)
- Only one-particle exchanges (2020)
- Gluonic two-loop diagrams (2025)

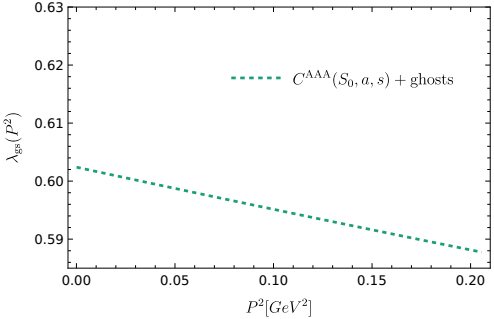




Effect of approximations (for 0^{++})

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C 85 (2025)]

Ground state:



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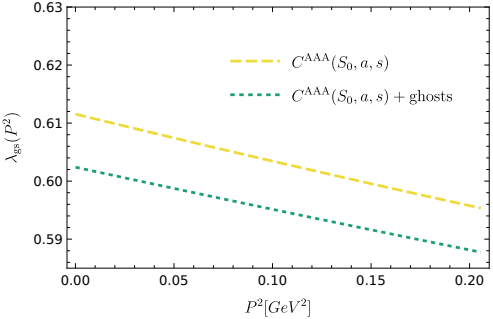

- Ghost diagrams essential: $\Delta m \approx 200 - 350\,\text{MeV}$

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[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C 85 (2025)]

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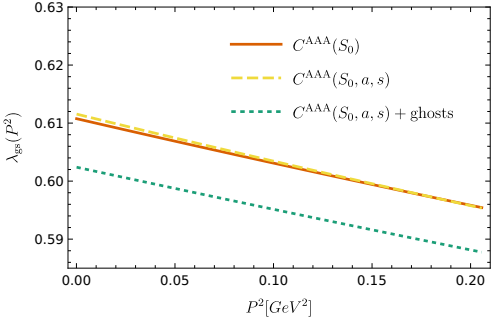

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- Small influence of three-gluon kinematics (S_0 only vs. S_0, a, s) $\rightarrow \Delta m \approx 50\,\text{MeV}$

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[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C 85 (2025)]

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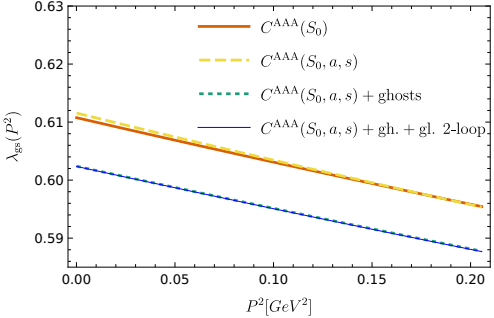

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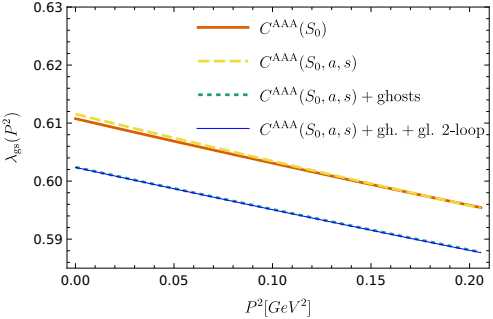
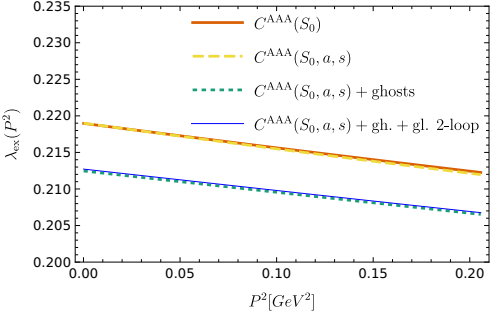


First excited state:



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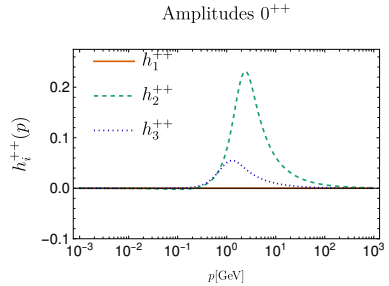
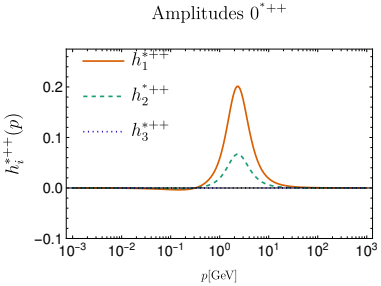
Amplitudes

Information about significance of single parts.

Ground state scalar glueball:



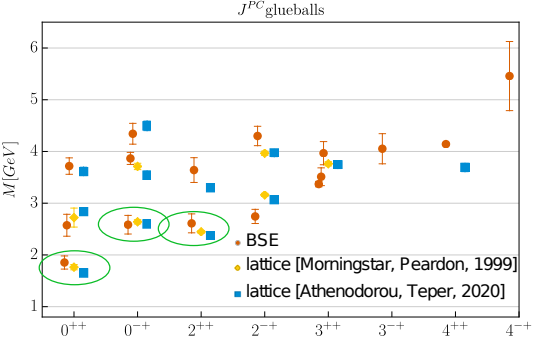
Excited scalar glueball:

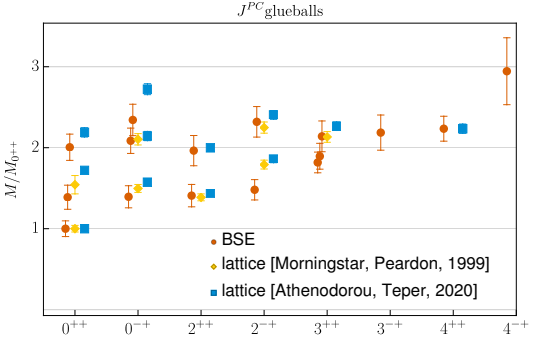


$\rightarrow$ Amplitudes have different behavior for ground state and excited state. Useful guide for future developments.

$\rightarrow$ Meson/glueball amplitudes: Information about mixing.

Glueball results



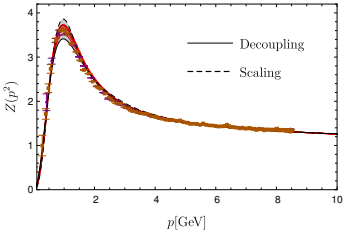


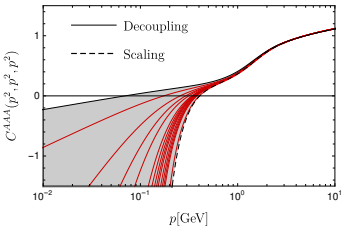
[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C81 (2021)]

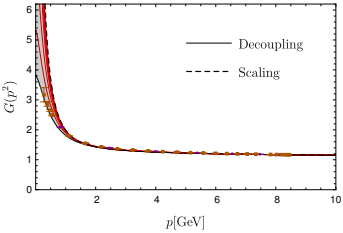
- Agreement with lattice results
- Consensus: **0^{++}** is lightest state
- Next: 0^{-+}, 2^{++}, 0^{++*}

- Cf. experimental states $f_0(1370)$, $f_0(1500)$, $f_0(1710)$ as scalar and $X(2370)$ as pseudoscalar glueball candidates

Family of solutions







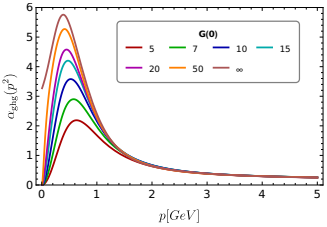
- Family of solutions [von Smekal, Alkofer, Hauck, PRL79 (1997); Aguilar, Binosi, Papavassiliou, Phys.Rev.D 78 (2008); Boucaud et al., JHEP06 (2008); Fischer, Maas, Pawlowski, Ann.Phys. 324 (2008); Alkofer, MQH, Schwenzer, Phys. Rev. D 81 (2010)]
- Nonperturbative completions of Landau gauge [Maas, Phys. Lett. B 689 (2010)]?

Spectrum for family of solutions

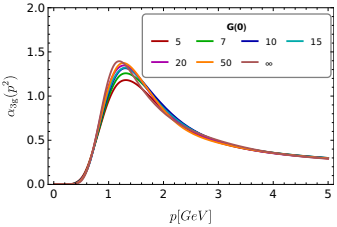
Couplings:

[MQH, Fischer, Sanchis-Alepuz, Eur.Phys.J.C 85]

$$\alpha_{3g}(p^2) = \alpha(\mu^2)\left[D^{3g}(p^2)\right]^2\left[Z(p^2)\right]^3$$



$$\alpha_{ghg}(p^2) = \alpha(\mu^2)\left[D^{ghg}(p^2)\right]^2 Z(p^2)\left[G(p^2)\right]^2$$



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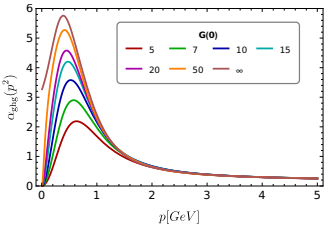

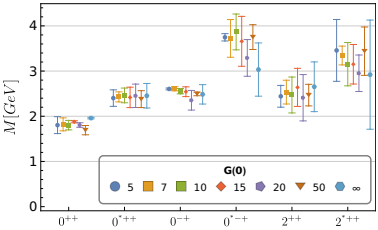

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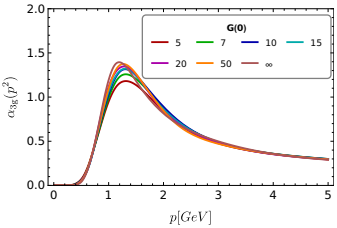
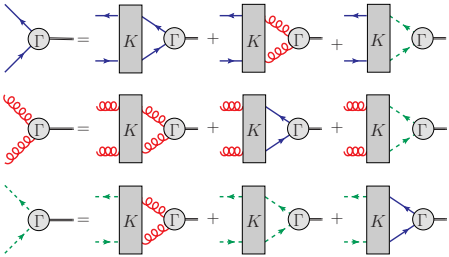
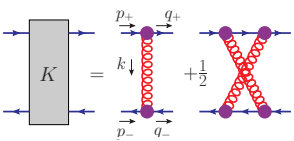


- Invariant spectrum → Nonperturbative gauge hypothesis supported.
- Invariance lost when deforming the input. → Consistency relevant.

Glueballs and mesons



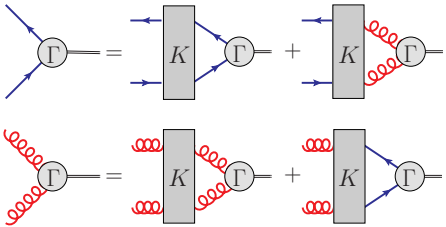


- Quark propagator and quark-gluon vertex required
- Back-coupling $\rightarrow$ complete system of (at least) 6 correlation functions

A peek at mixing

Pseudoscalars 0^{-+} with model quark-gluon vertex (variant of [Fischer, Watson, Cassing, PRD 72 (2005)], strength only in tree-level, depends on Yang-Mills part):

$$\Gamma_\mu(k;p,q) = \gamma_\mu \boldsymbol{A}\left(\frac{k^2+p^2+q^2}{3}\right) \boldsymbol{G}^2\left(\frac{k^2+p^2+q^2}{3}\right)$$



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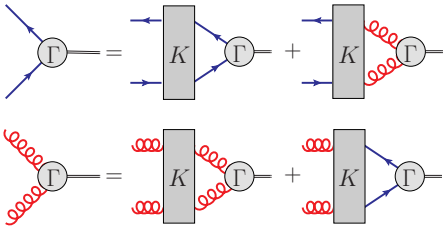

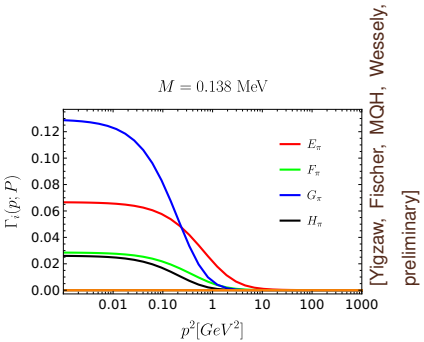

- Individual states have clearly dominant amplitudes.
- Effect of further dressings to be explored.

Full QCD



Quenched QCD: Base camp
$\rightarrow$ Further explorations

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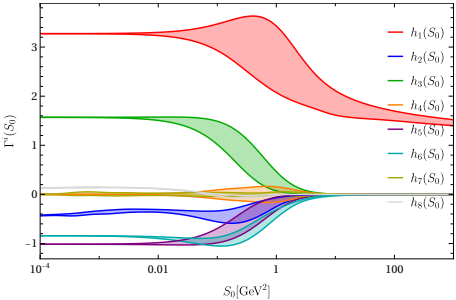
$\rightarrow$ Further explorations

- Input: Quark propagator, quark-gluon vertex



Full QCD



Quenched QCD: Base camp
$\rightarrow$ Further explorations

- Input: Quark propagator, quark-gluon vertex
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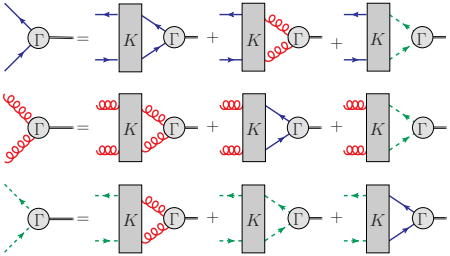

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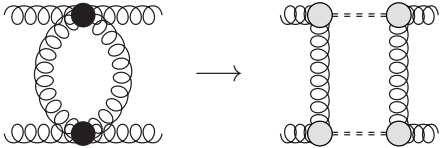

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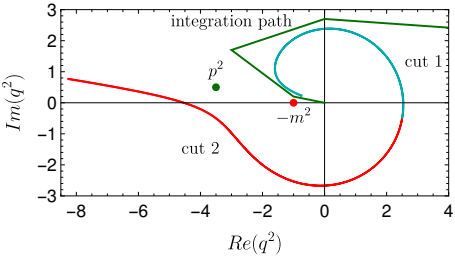
  - Improve techniques (complex plane!)



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Quenched QCD: Base camp
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- Input: Quark propagator, quark-gluon vertex
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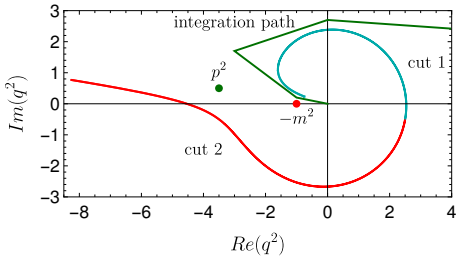
  - Interplay propagators, vertices, amplitudes



Summary and conclusions

- Functional methods are versatile tools:
  - Qualitative: Study mechanisms, e.g., structure of tetraquarks
  - Quantitative: tailored for certain sectors; large sets of correlation functions, tests necessary

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Glueballs

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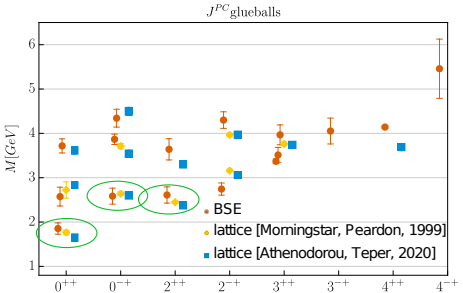
- Quantitative agreement between lattice and functional methods for quenched system



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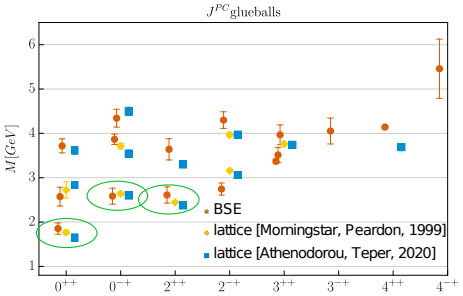

- Benchmarks and tests $\rightarrow$ Apparent convergence
  - Correlation functions stable under extensions, agreement between 3 functional methods and lattice
  - BSEs: two-loop diagrams irrelevant ($0^{\pm+}$, 2^{++})

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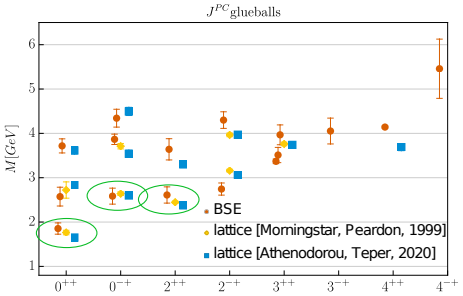

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Thank you for your attention.

Derivation of bound state equations I

For more details see [Eichmann, Sanchis-Alepuz, Williams, Alkofer, Fischer, Prog.Part.Nucl.Phys. 91 (2016)]

n-particle bound state

Pole in a $2n$-point function.

For simplicity here $n = 2$.

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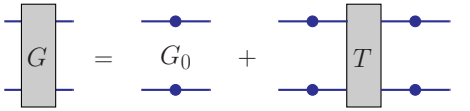
Full 4-point function:

$$G = G_0 + G_0\, T\, G_0$$



disconn. connected part

$\rightarrow$ scattering matrix T (amputated, conn. part of G)

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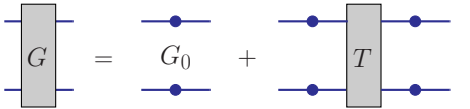

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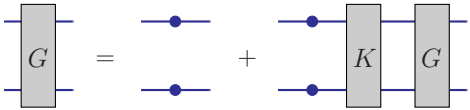
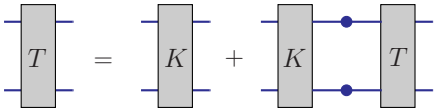
Dyson equations: nonperturbative resummations! Compare:

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots = 1 + x\, f(x) = 1 + x + x^2\, f(x)$$

$$G = G_0 + G_0\, K\, G$$



$$T = K + K\, G_0\, T$$



Scattering kernel K: 2-particle irreducible with respect to horizontal quarks lines (rest created by iteration)

Derivation of bound state equations II

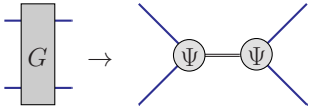
Back

On-shell: at pole position $P^2 = -M^2$

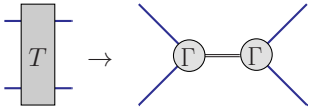
Pole position: mass M

Residues: $\Gamma\,\overline{\Gamma}$, $\Psi\,\overline{\Psi}$

$$G \to \frac{\Psi\,\overline{\Psi}}{P^2 + M^2}$$



$$T \to \frac{\Gamma\,\overline{\Gamma}}{P^2 + M^2}$$



Bethe-Salpeter amplitude Γ is the amputated wave function, $\Psi = G_0\Gamma$.

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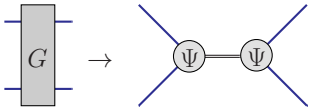

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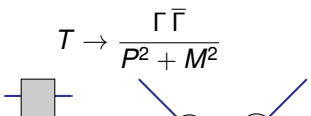

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Pole position: mass M

Residues: $\Gamma\,\overline{\Gamma}$, $\Psi\,\overline{\Psi}$

Plug into Dyson equations: homogeneous Bethe-Salpeter equations

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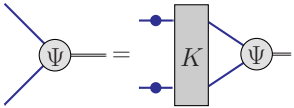

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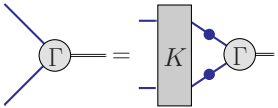
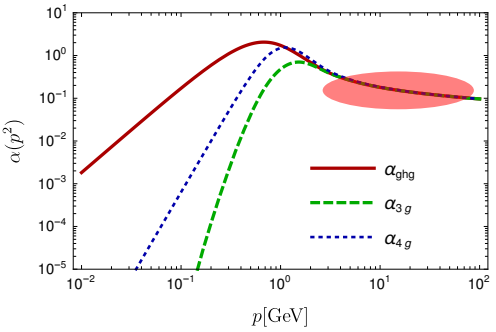


Gauge invariance

[MQH, Phys. Rev. D 101 (2020)]

Couplings can be extracted from each vertex.

- Slavnov-Taylor identities (gauge invariance): Agreement perturbatively (UV) necessary.
  [Cyrol et al., Phys.Rev.D 94 (2016)]
- Difficult to realize: Small deviations $\rightarrow$ Couplings cross and do not agree.
- Here: Vertex couplings agree down to GeV regime (IR can be different).



Hadrons as bound states

Hadron masses from correlation functions of gauge invariant operators.

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Examples:

$J^{PC} = 0^{-+}$ meson $\rightarrow O(x) = \overline{\psi}(x)\gamma_5\psi(x)$ $\rightarrow \overline{\psi}^2\psi^2$

$J^{PC} = 0^{++}$ glueball $\rightarrow O(x) = F_{\mu\nu}(x)F^{\mu\nu}(x)$ $\rightarrow A^4, A^5, A^6, A^7, A^8$

$$\boxed{D(x-y) = \langle O(x)O(y)\rangle}$$

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$$\boxed{D(x-y) = \langle O(x)O(y)\rangle}$$

Lattice: Mass from exponential Euclidean time decay

$$\lim_{t\to\infty} \langle O(x)O(0)\rangle \sim e^{-t\,M}$$

Hadrons as bound states

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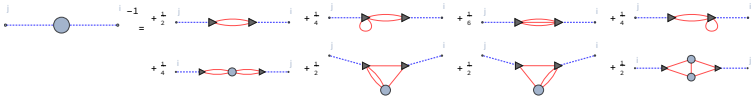
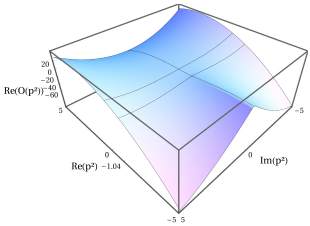
Functional approach:

Glueball:



+ 3-loop diagrams [MQH, Cyrol, Pawlowski, Comput.Phys.Commun. 248 (2020)]

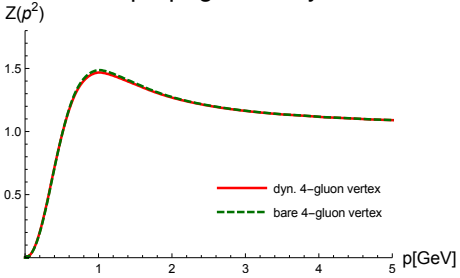


Leading order: [Windisch, MQH, Alkofer, Phys.Rev.D87 (2013)]

Not feasible. (For now?)

More checks: Four-point functions II

- Four-gluon vertex: Influence on propagators tiny for $d = 3$ [MQH, Phys.Rev.D93 (2016)] ✓



More checks: Four-point functions II

- Four-gluon vertex: Influence on propagators tiny for $d = 3$ [MQH, Phys.Rev.D93 (2016)] ✓
- Two-ghost-two-gluon vertex with 25 dressings [MQH, Eur. Phys.J.C77 (2017)]: ✓
  (FRG: [Corell, SciPost Phys. 5 (2018)])

