Glueball hunting on the lattice

Juan Andrés Urrea-Niño

Glueballs and Charmonium: Lattice QCD, Experiment and Phenomenology, ECT*.
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Why lattice QCD?

It allows first-principles calculations for states made up of the fundamental fields in the QCD lagrangian: quarks and **gluons**.

Glueball calculations have been done almost ever since the field was invented.

- Early works of M. Teper, C. Michael, B. Berg, A. Billoire, C. Morningstar, G. Bali, M. Peardon,... in the 1980's - 1990's.

Advantages: "Nature is not set, we set it."

- ✓ Systematically improvable approach to the continuum.
- ✓ Control over physics parameters: number of quark flavors, bare quark masses, etc...

Disadvantages: "Nature is not set, we set it."

- ! Statistical uncertainties can be large for some measurements.
- ! Systematics such as finite volume, non-zero lattice spacing, etc...

How do we study glueballs in lattice QCD?

Basically, the same way we study any other hadron.

1. **Formulate QCD on a Euclidean lattice of size $N_t \times N_L^3$.**
   - Quark fields live at the lattice sites as $\psi(\vec{x}, t)$.
   - Gauge fields live in the links connecting neighboring sites as $U_\mu(\vec{x}, t)$.
   - Lattice action $S[\psi, \bar{\psi}, U]$ encodes their dynamics.

Many things happen when we go from the continuum to a hypercubic lattice: we create a **gap** between experimental results and lattice results! For example...

$\rightarrow$ **Angular momentum must be reconsidered!**

Continuum has $SO(3)$: $J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, ...$
Cubic lattice has $\mathbb{O} \subset SO(3)$ ("Cubic group"): $R = A_1, G_1, T_1, H, E, T_2, A_2, G_2$

Experimentalist: "I saw a lattice QCD paper claiming their lightest glueball state has lattice angular momentum $R^{PC} = A_1^{++}$. How do I relate this to a J?"

Fixed lattice R	Continuum J content	Fixed lattice R	Continuum J content
A_1	$0, 4, 6, 8, \ldots$	G_1	$\frac{1}{2}, \frac{7}{2}, \frac{9}{2}, \ldots$
A_2	$3, 6, 7, 9, \ldots$	G_2	$\frac{5}{2}, \frac{7}{2}, \frac{11}{2}, \ldots$
E	$2, 4, 5, 6, \ldots$	H	$\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$
T_1	$1, 3, 4, 5, \ldots$		
T_2	$2, 3, 4, 5, \ldots$		

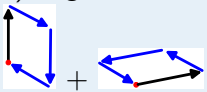
At **finite** lattice spacing a we can only fix R, but at small a:

- ✓ An A_1 state alone is probably a $J = 0$, while nearly degenerate A_1, E, T_1, T_2 states could be a $J = 4$.
- × $\chi_{c0}\,(J = 0)$, $\chi_{c1}\,(J = 1)$ and $\chi_{c2}\,(J = 2)$ are very close in nature! Without enough **precision**, a lattice QCD measurement of these could mislabel as a single $J = 4$. Argument borrowed from HadSpec.

Stronger statements require continuum extrapolation $a \to 0$ or other spin-identification techniques (See Hadron Spectrum Collaboration).

2 Build gauge-invariant operators with fixed quantum numbers.

A pure glueball *operator* $\mathcal{O}(\vec{x}, t)$ could be made of products of spatial gauge links (3×3 matrices), e.g.

$\mathcal{O}(\vec{x}, t) = \mathrm{Tr}($  + all other possible cubic transformations).

- ✓ Projection to 0 spatial momentum $\mathcal{O}(\vec{p} = 0, t) = \sum_{\vec{x}} \mathcal{O}(\vec{x}, t)$.
- ✓ Sum over all transformations gives us $R = A_1$.
- ✓ Sum (difference) of parity partners fixes $P = \pm$, real (imaginary) part fixes $C = \pm$.
- ✓ Absence of valence quark fields makes it a flavor singlet.

In the underlying Hilbert space:

$$\hat{\mathcal{O}}^\dagger |\Omega\rangle = \sum_{n=0}^{\infty} \langle n| \hat{\mathcal{O}}^\dagger |\Omega\rangle |n\rangle , \text{ with } |n\rangle \text{ the finite-volume energy eigenstates.}$$

Rule of thumb: make $\mathcal{O}(\vec{x}, t)$ "physical-looking" to have large overlap onto states of interest.

3 Calculate Euclidean temporal two-point correlation function.

Path integral given by

$$C(t) = \langle \mathcal{O}(t)\bar{\mathcal{O}}(0) \rangle = \frac{1}{\mathcal{Z}} \int d\psi d\bar{\psi} dU \; \mathcal{O}(t)\bar{\mathcal{O}}(0) \; e^{-S[\psi,\bar{\psi},U]}$$

- ✓ Fermion integration done analytically.
- ✓ Gauge field integral approximated via Monte-Carlo.

$$C(t) \approx \frac{1}{N} \sum_{i=1}^{N} A[U_i](t)$$

- $A[U](t)$: Quantity depending only on gauge field U.
- U_i: Gauge configurations sampled from e^{-S}.

Spectral Decomposition

$$C(t) = \sum_{n=0}^{\infty} \langle \Omega | \hat{\mathcal{O}} | n \rangle \langle n | \hat{\mathcal{O}}^\dagger | \Omega \rangle \, e^{-E_n t}$$
$$\overset{t\to\infty}{\approx} \underbrace{| \langle 0 | \hat{\mathcal{O}}^\dagger | \Omega \rangle |^2 e^{-E_0 t}}_{\text{Ground State Dominance}}$$

Effective Mass Extraction

$$am_{eff} = \ln\left(\frac{C(t)}{C(t+a)}\right) \overset{t\to\infty}{\approx} aE_0$$

Given a basis of N operators $\mathcal{O}_i$, we have a fixed recipe:

1. Calculate correlation matrix with entries

$$C_{ij}(t) = \left\langle \mathcal{O}_i(t) \bar{\mathcal{O}}_j(0) \right\rangle$$

2. Solve generalized eigenvalue problem (GEVP)

$$C(t) v_n(t, t_0) = \lambda_n(t, t_0) C(t_0) v_n(t, t_0)$$

3. Extract optimal correlator and operator for energy eigenstate n as

$$\lambda_n(t, t_0) \overset{t \to \infty}{\approx} a_n e^{-m_n t}$$

$$\mathcal{O}^{(n)\dagger} |\Omega\rangle = \sum_{i=1}^{N} v_n^{(i)}(t, t_0) \mathcal{O}_i^\dagger |\Omega\rangle \overset{t \to \infty}{\propto} |n\rangle$$

Seems straightforward, but there are caveats!

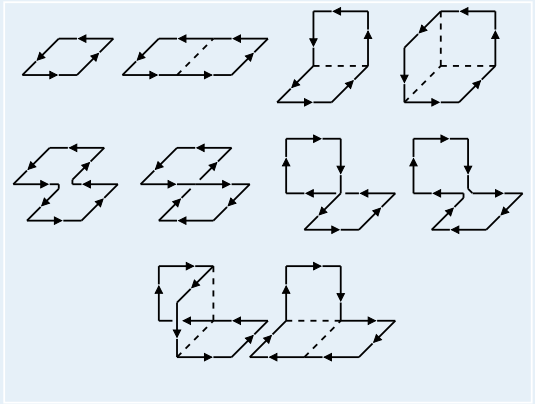
How are lattice glueballs usually built?

Different loop shapes:

- ✓ Varying size and spatial structure.
- ✓ Single and double windings are linearly independent.
- ✓ Different shapes can access different lattice angular quantum numbers.

Link Smearing:

- ✓ Reduce UV fluctuations.
- ✓ Many options in the market: APE, HYP, improved-APE, Stout,etc...



Figures: M. Peardon *et al.*, Phys. Rev. D 60, 034509, B. Lucini *et al.*, J. High Energ. Phys. 2010, 119 (2010)

A first approach: pure-gauge setup

A world without quarks: Yang-Mills Lagrangian $\propto \mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu})$

Advantages

On the physics side:

- ✓ Glueballs are the energy eigenstates.
- ✓ No sea quarks: No mixing or decays.

On the calculation side:

- ✓ No quarks $\rightarrow$ Monte-Carlo sampling is cheap $\rightarrow$ Large statistics.
- ✓ Structure of action $S[U]$ and observables $O[U]$ facilitates noise-reduction techniques: Multi-Level sampling.
- ✓ Good setup for algorithmic testing.

Systematic studies of glueballs with different quantum numbers were done in this setup where **large statistics** and **multiple lattice spacings** are not too costly.

The quenched glueball spectrum: e.g. Peardon & Morningstar (1999)

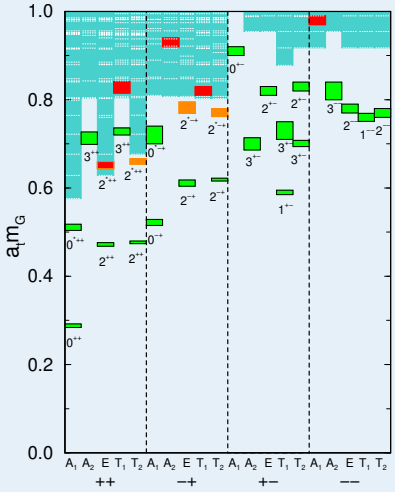


Figure: Fixed lattice spacing.

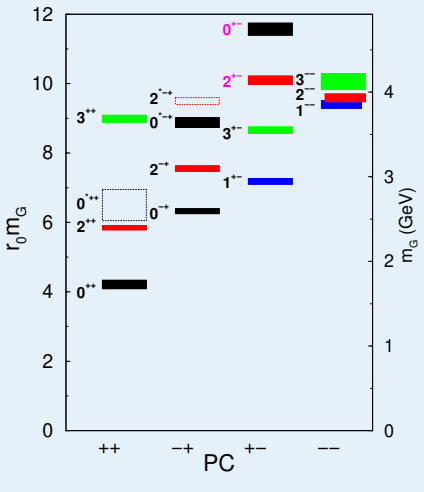
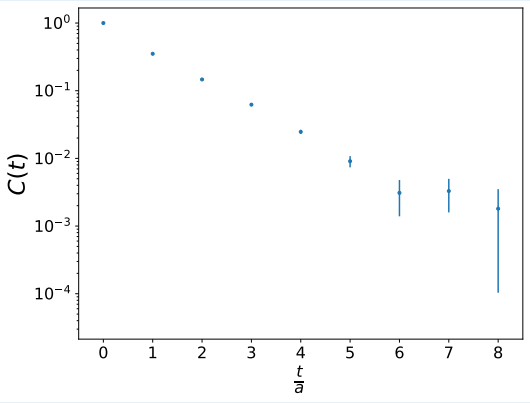


Figure: Continuum extrapolation

Behind the scenes

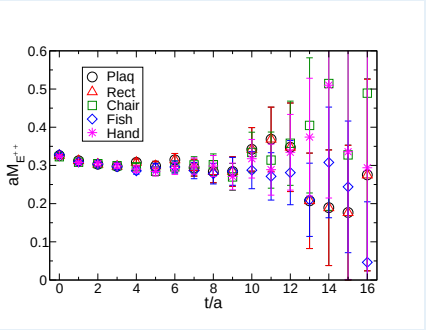
Signal-to-noise ratio decays exponentially with the glueball mass!



- × Signal survives only until $t = 6a$ using **9000** configurations for average.
- × Forces us to rely on ground-state dominance at **very early** time separations.
- × Fighting an **exponential signal loss** with brute force statistics is not sustainable.
- ! Alternative approaches should exploit the underlying physics!

The GEVP is a glass cannon

- × If operators don't have large enough $\langle n| \hat{\mathcal{O}}^\dagger |\Omega\rangle$, state $|n\rangle$ will be missed, e.g. when
  - ! Resolving single-, double-, triple-, etc... hadron states. See Fernando's talk!
  - ! Resolving contributions from high J.
- × Large statistical uncertainties and lack of sufficient linear independence between operators make the computation difficult.



- ✓ Link smearing smooths out UV-fluctuations and increases overlap with low-lying states.
- × Smearing radius grows and different shapes become close to degenerate → Basis size *effectively* reduced.

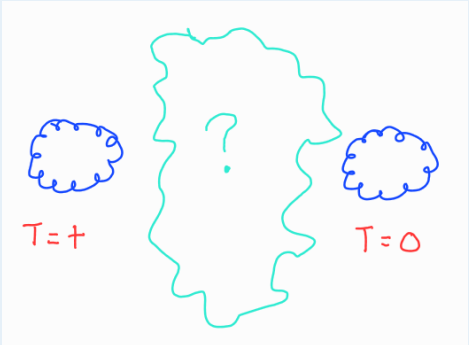
K. Sakai & S. Sasaki, Phys. Rev. D 107, 034510

A qualitative perspective of the StN problem **for 3D operators**:

$$C(t) = \frac{1}{Z} \int dU \; \mathcal{O}(t) \cdot \bar{\mathcal{O}}(0) \cdot e^{-S[U]}$$

- Operators are restricted to times t and 0 after fermion integration.
- $S[U]$ sums over all space-time points.

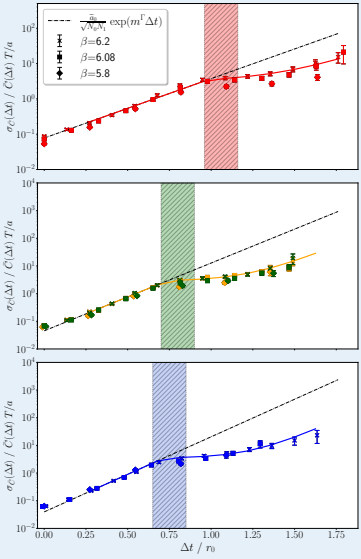
! Temporal correlations between $\mathcal{O}(t)$ and $\bar{\mathcal{O}}(t)$ happens only through the action $S[U]$ via background fluctuations.



× Background fluctuations are hard to sample, we need large statistics.

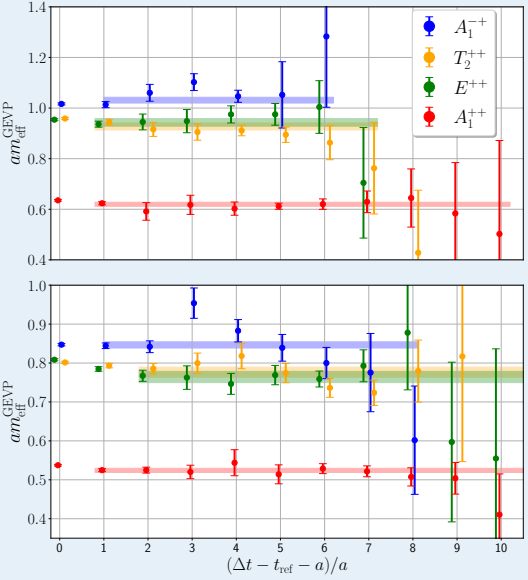
× No free lunch: Cheap to measure but very noisy.

Multi-Level Sampling: e.g. Barca, Schaefer *et al.* (2024)



- ✓ Based on (partial) factorization in time of the action and the correlation function.
- ✓ Scaling $\frac{\delta C(t)}{C(t)} \propto \frac{1}{\sqrt{N_0 N_1^2}} e^{-mt}$ is reached after large enough time separation. Up to boundary effects.
- ✓ Improvement appears independent of YM action coupling and different quantum numbers benefit! $A_1^{++}(0^{++})$, $E^{++}(2^{++})$, $T_2^{++}(2^{++})$ and $A_1^{-+}(0^{-+})$

Better signal-to-noise scaling means **longer** time separations are reachable! L. Barca *et al.*, Phys.Rev.D 110 (2024) 5, 054515



- ✓ More points of effective masses are available.
- ✓ Plateau determinations are more reliable!

Energy extraction is more reliable!

- ✓ Looks promising for applying when there are quarks!
- ✓ Could help with other glueball-related calculations, e.g. form factors. R. Abbott *et al.*, Phys.Rev.Lett. 136 (2026) 4, 041901

Full QCD: 0^{++}, 2^{++}, 0^{-+} are the most interesting

A more dense spectrum, e.g. 0^{++}

- Light meson flavor singlets, e.g f_0.
- Charmonium, e.g. χ_{c0}.
- Possible glueballs.
- Two-meson states with and without relative momentum, e.g. $\pi(-\vec{p})\pi(\vec{p})$
- Other multi-particle states...

All these states can **share lattice quantum numbers**.

Setup is very important!

- Flavor symmetry, e.g. $SU(2)$ for u,d quarks, $SU(3)$ for u,d,s, etc...
- Tunable pion mass to open/close decay channels.
- Quantized spatial momentum: $\vec{p} = \frac{2\pi}{aN_L}(p_x, p_y, p_z)$.

To reliably resolve a state, we need to resolve everything below it.

Additional operators

- Single-meson operators: $\mathcal{O}_{\bar{q}q}(\vec{p}=0,t) = \sum_{\vec{x}} \bar{q}(\vec{x},t)\gamma_5 q(\vec{x},t)$
- Two-meson operators: $\mathcal{O}_{\bar{q}q\cdot\bar{q}q}(\vec{p}=0,\vec{k},t) = \mathcal{O}_{\bar{q}q}(-\vec{k},t)\mathcal{O}_{\bar{q}q}(\vec{k},t)$, ...
- ...

$$C(t) = \begin{pmatrix} \langle \mathcal{O}_{\bar{q}q}(t)\bar{\mathcal{O}}_{\bar{q}q}(0)\rangle & \langle \mathcal{O}_{\bar{q}q}(t)\bar{\mathcal{O}}_{g}(0)\rangle & \left\langle \mathcal{O}_{\bar{q}q}(t)\bar{\mathcal{O}}_{\bar{q}q\cdot\bar{q}q}(\vec{k}_i,0)\right\rangle \\ \langle \mathcal{O}_{g}(t)\bar{\mathcal{O}}_{\bar{q}q}(0)\rangle & \langle \mathcal{O}_{g}(t)\bar{\mathcal{O}}_{g}(0)\rangle & \left\langle \mathcal{O}_{g}(t)\bar{\mathcal{O}}_{\bar{q}q\cdot\bar{q}q}(\vec{k}_i,0)\right\rangle \\ \left\langle \mathcal{O}_{\bar{q}q\cdot\bar{q}q}(\vec{k}_i,t)\bar{\mathcal{O}}_{\bar{q}q}(0)\right\rangle & \left\langle \mathcal{O}_{\bar{q}q\cdot\bar{q}q}(\vec{k}_i,t)\bar{\mathcal{O}}_{g}(0)\right\rangle & \left\langle \mathcal{O}_{\bar{q}q\cdot\bar{q}q}(\vec{k}_i,t)\bar{\mathcal{O}}_{\bar{q}q\cdot\bar{q}q}(\vec{k}_i,0)\right\rangle \end{pmatrix}$$

for $q = u, d, .., c$ and different spatial momenta $\vec{k}_i$ relevant up to a given threshold.

Easier said than done thanks to the quarks!

Quarks appear in the action $S[\bar{\psi}, \psi, U]$ via the **Dirac** operator matrix D.

- ! **Huge**: $4N_L^3N_t \times 4N_L^3N_t$, e.g. $(30 \times 10^6) \times (30 \times 10^6)$.
- ! **Very sparse**: nearest-neighbor coupling in space-time in the Wilson formulation we use.

Sampling for Monte-Carlo

- ! Distribution is $\propto det(D_{q_1})det(D_{q_2})...e^{-S[U]}$: $det(D)$ is unfeasible directly!
- ! Updates require solving "a few" linear systems $Dx = b$: Fast linear solvers available.

Observables for correlations involving quarks

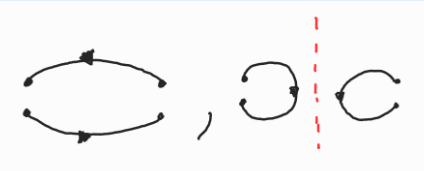
- ! Our approach to calculating these correlations requires $4 \times N_v \times N_t$ linear solves $D(U)x = b$ for each sample U, with $N_v = 100 - 200 \rightarrow \approx 80 \times 10^3$ inversions.
- × High statistics are expensive!

Cooperation with numerical analysts is very important here!

The Mixing Problem

Different terms in our mixing matrix have different signal quality, e.g.

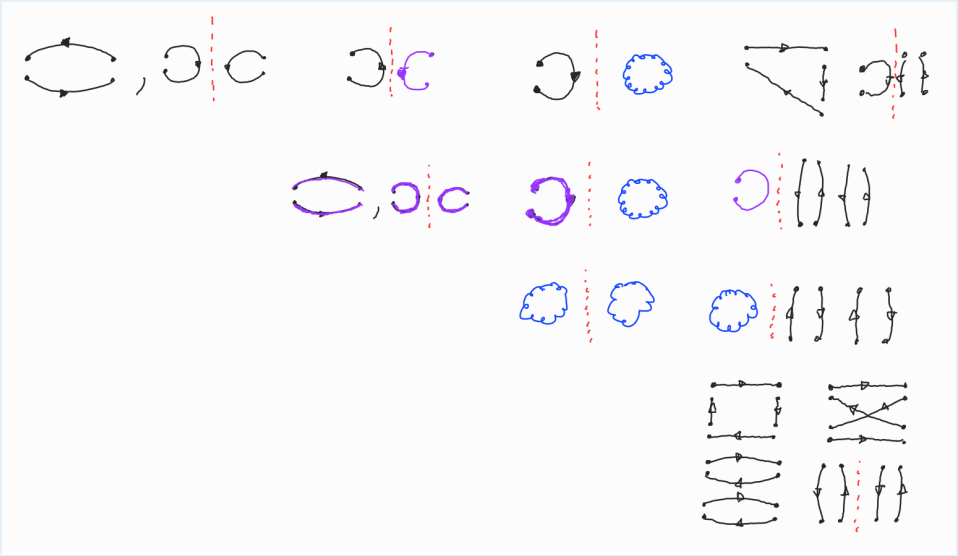
$$\langle \mathcal{O}_{\bar{q}q}(t)\bar{\mathcal{O}}_{\bar{q}q}(0)\rangle = -\mathrm{Tr}\left(\gamma_5 D^{-1}[t,0]\gamma_5 D^{-1}(0,t)\right) + \mathrm{Tr}\left(\gamma_5 D^{-1}(t,t)\right)\mathrm{Tr}\left(\gamma_5 D^{-1}(0,0)\right)$$



- ✓ Connected contribution **explicitly** relates times t and 0 at observable level.
- × Disconnected contribution **implicitly** relates times t and 0 since D^{-1} is dense and connects all space-time points.

Quality often goes: Connected >> Disconnected > Fully factorized (glueball-like)...

... and we combine them all in our correlation matrix:



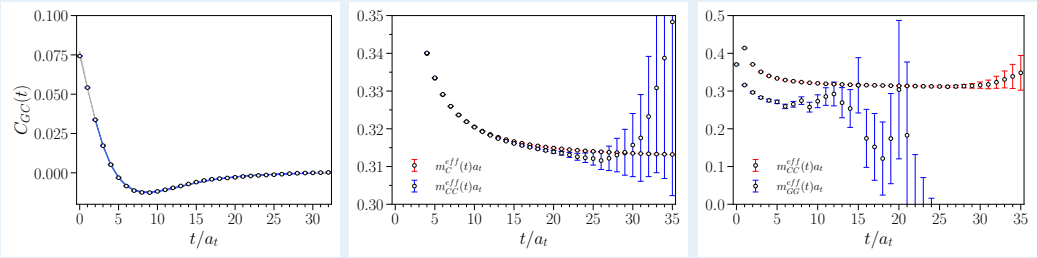
What to look out for?

? Is the spectrum saturated? → Will additional operators reveal intermediate states?

? How big are the mixing off-diagonal correlations? → Do operators decouple?

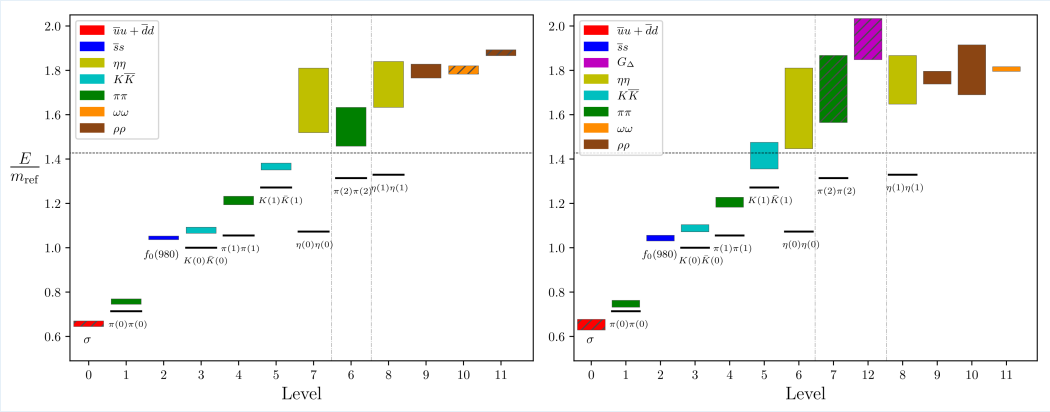
? Composition of the states? → Which types of operators best sample them?

E.g. R. Zhang *et al.*, Phys. Lett. B 827 (2022) 136960

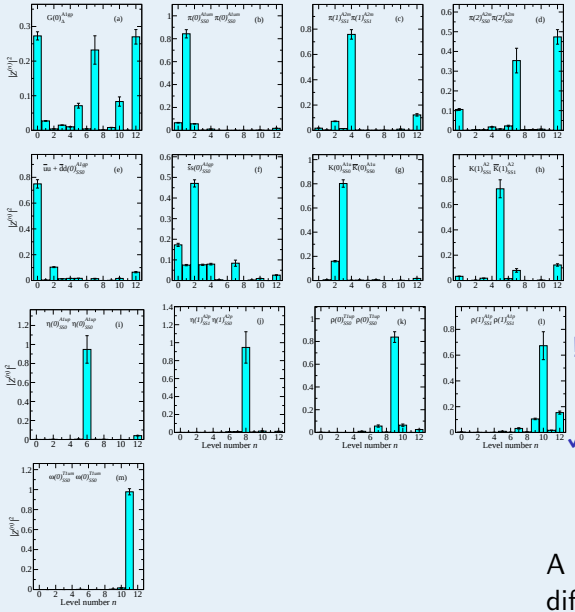


- 2 degenerate charm quarks, $M_{J/\Psi} \approx 3.068$ GeV.
- Glueball and charmonium operators mix!

R. Brett *et al.*, AIP Conf. Proc. 2249, 030032 (2020)



- 2 (u,d) + 1(s) quarks with $m_\pi \approx 390$ MeV.
- One- and two-meson operators, latter with different relative momenta $\vec{k}$.
- Adding a gluonic operator did not reveal an *intermediate* state. **Why?**



- Overlaps $\langle n| \hat{\mathcal{O}}_i^\dagger |\Omega\rangle$ reveal composition of states.

! Glueball operator G_Δ is "spread out" among different states.

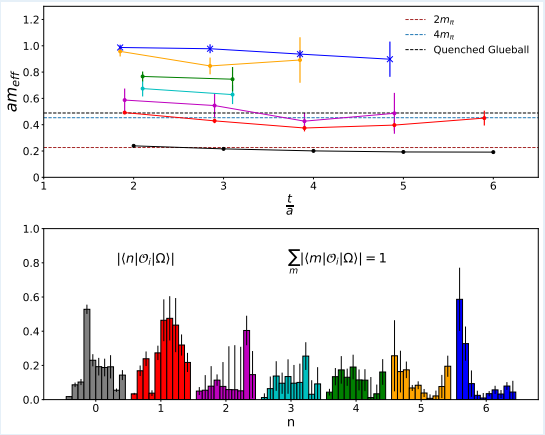
! Contrasts with "delta-like" overlaps of other operators.

!! Being purely gluonic **does not** mean it cannot mix with "mesonic" states.

✓ If our operators have "enough" overlap with a state, we can resolve it.

A dense spectrum makes calculations difficult!

J. A. Urrea-Niño *et al.*, PoS LATTICE2024 (2025) 080



- 3 (u,d,s) + 1 (c) with $m_\pi \approx 420$ MeV.
- 3 $\bar{c}c$, 5 $\bar{l}l$, 1 $\pi(\vec{0})\pi(\vec{0})$, 1 glueball-like operator, all $A_1^{++}(0^{++})$

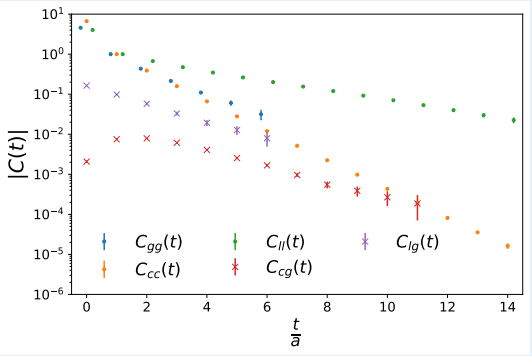
✓ Inclusion of close-to-physical charm quark.

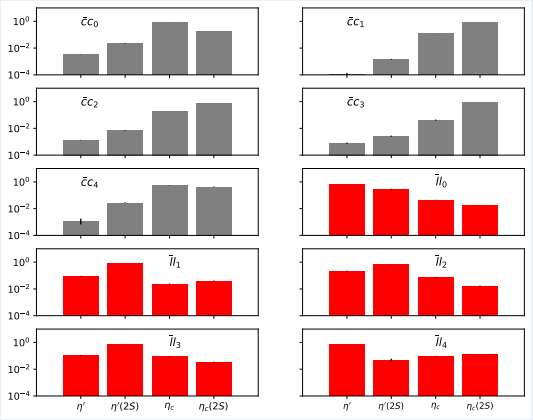
✓ Two-particle operator reveals additional state → Delta-like overlaps.

✓ Glueball operator is "spread out" among low-lying states.

× Again a dense spectrum...

J. A. Urrea-Niño *et al.*, Phys. Rev. D 112, 074502 (2025)





- Same setup as above but $A_1^{-+}(0^{-+})$.
- Light meson, charmonium and glueball operators.
- Direct measurement of effects of mixing in η_c mass: negative shift.

For a close-to-physical glueball calculation:

- × Dense flavor-singlet light meson spectrum.
- × Charmonium flavor-singlet spectrum.
- × Multi-hadron states with different relative momenta.
- × Large statistics are costly with quarks in the mix.
- × Pure-gauge glueball predictions place them above ground states: we need a GEVP.
- × Our operators must have good overlap with the relevant states.

Too many issues to tackle at the same time: we need to go piece by piece, e.g.

1. Explore multi-level in pure-gauge for gluonic observables. L. Barca *et al.*, Phys. Rev. D 110 (2024) 5, 054515.
2. Extend it to mesonic observables in quenched setup, i.e. valence quarks but no sea quarks. L. Barca *et al.*, Phys. Rev. D 113 (2026), 034505
3. Apply it in full QCD, i.e. with dynamical quarks. Work in progress...

Our current approach

Put ourselves in a convenient setup: $N_f = 2$ degenerate quarks with $m_\pi \approx 2.2$ GeV

- ✓ 0^{++} pure-gauge glueball prediction would be stable.
- ✓ We allow for mixing with mesons in $SU(2)_F$ iso-singlet ($I = 0$) channel.

Test our machinery: Use glueball-like $\mathcal{O}_i$ which can

- ✓ Model a "physical" glueball, e.g. 2-gluon, 3-gluon, etc...
- ✓ Probe continuum-like properties, e.g. $J = 0, 4, ...$

Our goals:

- ✓ Determine the spectrum using meson and glueball operators to saturate the (1-particle) spectrum.
- ✓ Identify the composition of the states in terms of mesonic and gluonic operators.
- ✓ Identify the dominant J at our fixed lattice spacing.
- ✓ Learn what works: "Future methods for studying confined gluons in QCD"

"Qualitative Features of the Glueball Spectrum", R. L. Jaffe *et al.*, Ann. of Phys. 168, 344-376 (1988)

- Mass of hadron $\propto$ mass-dimension of simplest operator which creates it.
- 2-gluon glueballs: $\mathrm{Tr}\,(F_{\mu\nu}F_{\alpha\beta})$, $J^{PC} = 0^{++}, 0^{-+}, 2^{++}, 2^{-+}$
- 3-gluon glueballs: $\mathrm{Tr}\,(F_{\mu\nu}F_{\alpha\beta}F_{\delta\sigma})$, $J^{PC} = 0^{++}, 1^{+\pm}, 2^{+\pm}, 3^{+-}$

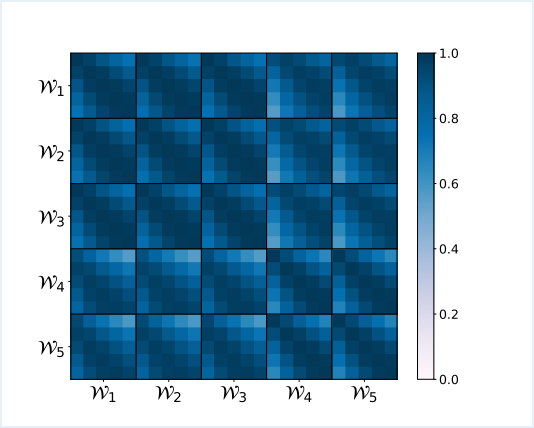
We can adapt this approach for our glueball operators:

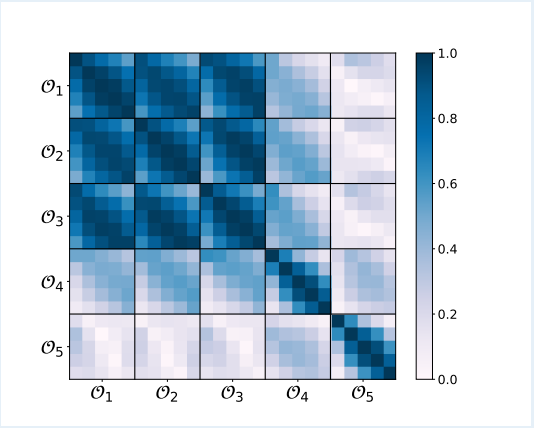
- Trace of product of $\mathcal{B}_i = \epsilon_{ijk}F_{jk}$ and its gauge-covariant derivatives D_i. Y. Chen *et al.*, hep-lat/0512033.
- Operators designed to have fixed J in $SO(3)$. J. J. Dudek *et al.*, Phys. Rev. D 82, 034508 (2010)

Advantages:

- ✓ Intuitive glueball-like construction!
- ✓ Sample non-trivial spatial structure of underlying eigenstates.

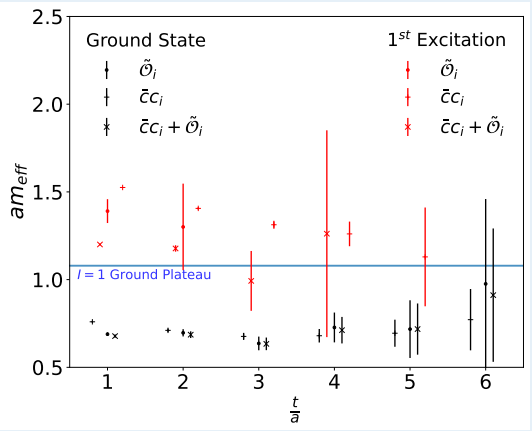
Correlation matrix at fixed t:





! Spatial Wilson loops $\mathcal{W}_i$ do not "remember" continuum J.

✓ Our choice of operators $\mathcal{O}_i$ "remember" continuum $J \rightarrow$ Easier identification!

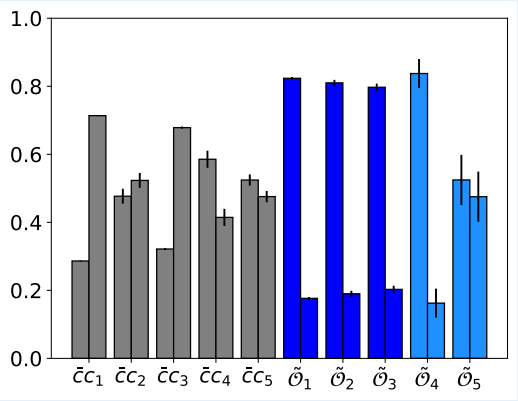


- ✓ Both meson $\bar{c}c$ and glueball-like $\mathcal{O}_i$ operators "see" common states.
- ✓ We combine them in a GEVP:
- ✓ 1^{st} excitation needs $\bar{c}c$ operators.

Comparison with iso-vector ($I = 1$):

- ! Disconnected contributions make the difference.
- ! $I = 0$ ground state is much lighter than $I = 1$ ground state: Our "glueball".
- ! $I = 0$ excitation is close to $I = 1$ ground state.

Overlaps: $\langle n| \bar{c}c^\dagger |\Omega\rangle$ and $\langle n| \hat{\mathcal{O}}_i^\dagger |\Omega\rangle$ for $n = 0, 1$ $I = 0$ energy eigenstates:



- ✓ Both types of operators have overlaps with both energy eigenstates!
- ✓ Charmonium operators with different structure have different "preference".
- ✓ "$J = 0$" glueball operators strongly prefer ground state!

$\rightarrow |0\rangle$ is a mix with a **strong** gluonic component, $|1\rangle$ is a mix with a **small** gluonic component.

Conclusions

Glueball hunting on the lattice is...

- ✓ ... a systematic approach to understand these and other hadrons and their dynamics from first principles.
- ✓ ... relevant given the different glueball candidates put forward by experimental results $\rightarrow$ Focus on 0^{++}, 0^{-+}, 2^{++}.
- ✓ **flexible:** we can work in simplified setups to focus on specific dynamics
  - Pure-gauge: simplest and cheapest but not very physical.
  - Control over quark masses and flavors: allow/forbid multi-particle decays.
- × **difficult:** physics and lattice problems
  - Finite-volume spectrum, finite lattice spacing,...
  - Glueballs mix with other hadrons and are unstable.
  - Signal-to-noise problem.
  - How do good glueball operators look like?

Ongoing efforts to address all of these problems!

Importance of good operators

- ✓ Obvious benefit: resolve states missing due to incomplete basis.
- ✓ More subtle but important benefits: better resolve already seen states.

The case of $1^{-+}(T_1^{-+})$ exotic. Is it a hybrid?:

- HadSpec has 18 operators of the form $\bar{q}\Gamma D_i ... D_j q$.
  - ! One is $\epsilon_{ijk}\bar{q}\gamma_j\mathbb{B}_k q$, with $\mathbb{B}_i = \epsilon_{ijk}D_jD_k \overset{A_\mu \to 0}{=} 0$, a "hybrid" operator.
  - ! One is $\bar{q}\gamma_0 D_i q$, which is not a "hybrid" operator.
  - ! The "hybrid" operator much better resolves the ground state. See also Sinéad's slides.

We don't **need** $\bar{q}\mathbb{B}_i q$ to create a T_1^{-+} state, but this choice couples better to the state assigned to be 1^{-+}. L. Liu *et al.*, JHEP 07 (2012) 126