Do near-threshold molecular states mix with neighbouring $\bar{Q}Q$ states?

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Introduction

- Quenched quark model predicts towers of radially excited states with different quantum numbers J^{PC}
- Appearance of quark model states near strong thresholds is accidental
- In molecular model, exotic hadrons are naturally attracted to strong thresholds
- Number of strong thresholds exceeds number of exotic candidates

  $\Longrightarrow$ Selection rule for thresholds?
- Quark model and molecular states co-exist in spectrum of hadrons

  $\Longrightarrow$ What do they know about each other?

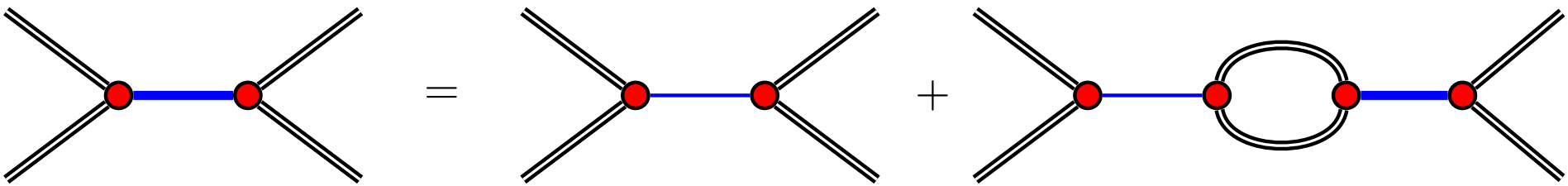
Compact state coupled to hadronic channel

Consider compact quark model state ψ_0 coupled to S-wave hadronic channel H_1H_2

$$|\Psi\rangle = \begin{pmatrix} \lambda|\psi_0\rangle \\ \chi(\boldsymbol{k})|H_1H_2\rangle \end{pmatrix} \qquad H = \begin{pmatrix} E_0 & f(k) \\ f(k) & \frac{k^2}{2\mu} \end{pmatrix} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

The transition form factor varies at momenta $k \simeq \beta$ with β inverse range of force

The full $H_1H_2 \to H_1H_2$ scattering amplitude through state Ψ obeys the equation



with the (nonrelativistic) mass operator

$$\Sigma(E) = \int \frac{d^3p}{(2\pi)^3} \frac{f^2(p)}{\frac{p^2}{2\mu} - E - i\epsilon}$$

Compact state coupled to hadronic channel

Suppose, there is a bound state and expand the scattering amplitude near this pole

$$T(E) = \frac{1}{E-E_0} + \frac{1}{E-E_0}\Big[-\Sigma(E)\Big]\frac{1}{E-E_0}\ldots = \frac{1}{E-E_0+\Sigma(E)} \underset{E\to -E_B}{=} \frac{Z}{E+E_B}$$

where the renormalisation factor Z is related to derivative of the loop at the pole

$$\frac{1}{Z}-1 = \Sigma'(-E_B) = \int \frac{d^3p}{(2\pi)^3}\frac{f^2(p)}{\left(\frac{p^2}{2\mu}+E_B\right)^2} \underset{\gamma\ll\beta}{=} \frac{\mu^2 g_0^2}{2\pi\gamma}+\ldots \implies g_0^2 = \frac{2\pi\gamma}{\mu^2}\left(\frac{1}{Z}-1\right)$$

where $g_0 = f(0)$ and $\gamma = \sqrt{2\mu E_B}$. Then for the effective coupling we find

$$\frac{g_{\text{eff}}^2}{4\pi} = \frac{1}{4\pi}\underbrace{\left(\sqrt{2m_1}\sqrt{2m_2}\sqrt{2M}\right)^2}_{\text{relativistic factor}}\underbrace{Zg_0^2}_{g_{\text{NR}}^2} = 4M^2\left(\frac{\gamma}{\mu}\right)(1-Z)$$

With natural identification $Z=\lambda^2$, normalisation of w.f. follows: ${}_B\langle\Psi|\Psi\rangle_B = 1$

Composite or elementary?

If effective range expansion is applicable

$$T^{-1}(E) \simeq a^{-1} + \frac{1}{2} r k^2 - ik$$

then Z can be related to scattering length a and effective range r:

$$a = -\frac{2(1-Z)}{(2-Z)} \frac{1}{\sqrt{2\mu E_B}} + O\left(\frac{1}{\beta}\right) \qquad r = -\frac{Z}{(1-Z)} \frac{1}{\sqrt{2\mu E_B}} + O\left(\frac{1}{\beta}\right)$$

β ($\gg p$) — (inverse) range of force

(Weinberg'1960s)

$$\boxed{Z = 1 - \frac{1}{\sqrt{1 + 2r/a}}}$$

Elementary (confined) state

- Two near-threshold poles

Composite (molecular) state

- One near-threshold pole

$\Longrightarrow$ pole counting rules (Morgan'1992)

Interplay of different dynamics

$$|\Psi\rangle = \begin{pmatrix} \lambda|\psi_0\rangle \\ \chi(\boldsymbol{k})|H_1H_2\rangle \end{pmatrix} \qquad H = \begin{pmatrix} E_0 & f(k) \\ f(k) & \frac{k^2}{2\mu} + V_{H_1H_2} \end{pmatrix}$$

Scattering amplitude $H_1H_2 \to H_1H_2$ via $V_{H_1H_2}$ (parameterised)

$$T_V(E) \approx \frac{1}{-\gamma_V - ik} \qquad k = \sqrt{2\mu E}$$

Full scattering amplitude $H_1H_2 \to H_1H_2$ — solution of LSE (obtained)

$$T(E) = \frac{1}{4\pi^2\mu} \frac{E - \overbrace{\left(E_f - \frac{1}{2}g_f\gamma_V\right)}^{E_C}}{(E - E_f)(\gamma_V + ik) + \frac{i}{2}g_f\gamma_V k} \quad \Longrightarrow \quad \boxed{\text{3 poles in } k\text{-plane}}$$

Position of bare ($g_f \propto f(0) \to 0$) poles:

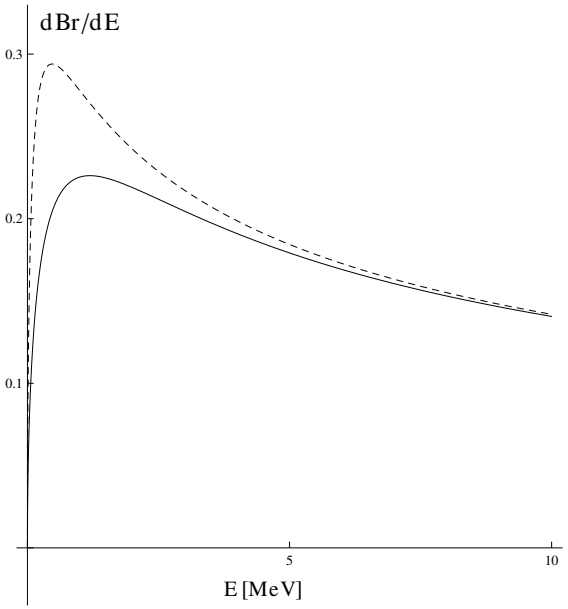
$$k_1 = i\gamma_V \qquad k_{2,3} = \pm\sqrt{2\mu E_f}$$

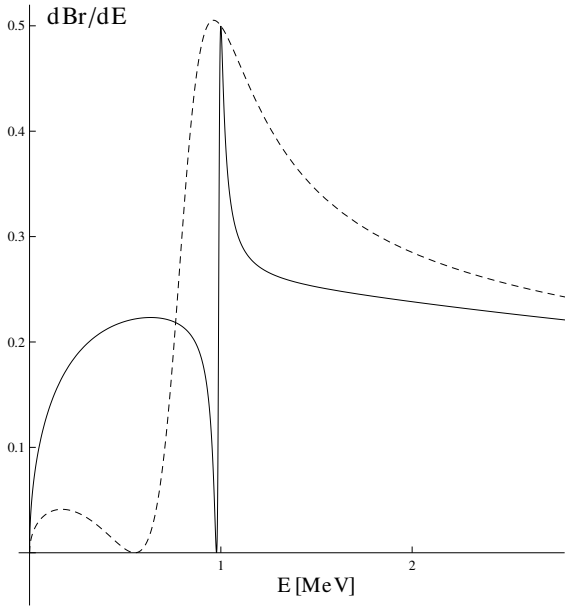
(Baru et al.'2010)

Interplay of different dynamics

$|E_C| \gg |E_f|$ $|E_C| \sim |E_f|$

ERE works in broad range ERE breaks near threshold

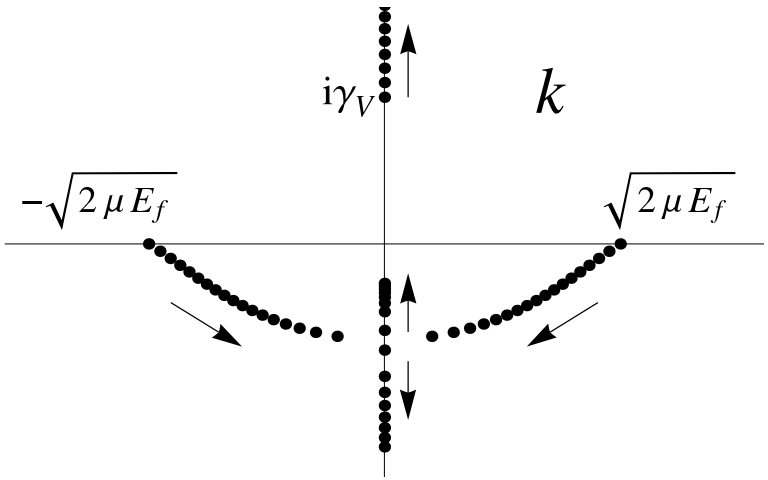


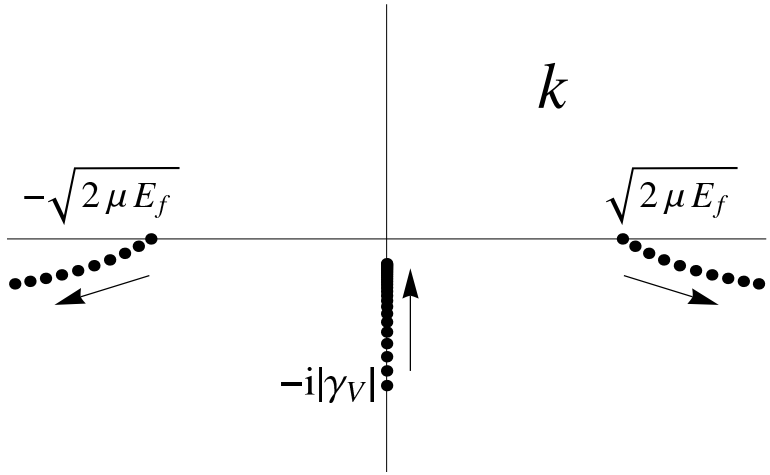


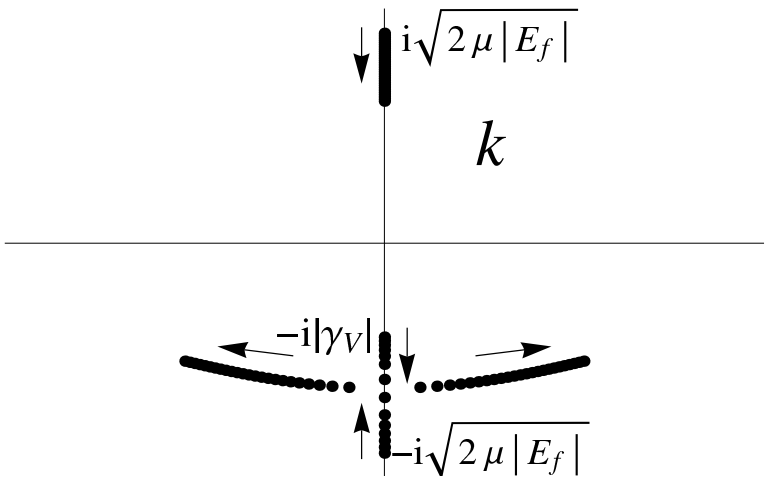
For bound state, expression for Z-factor not relying on ERE reads

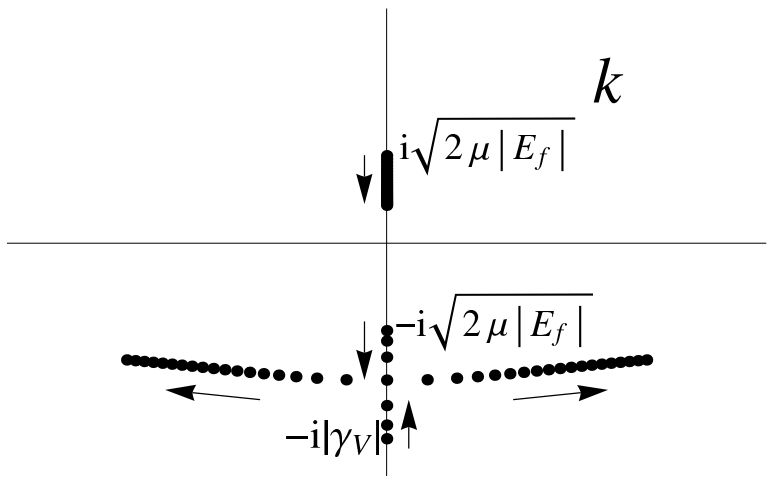
$$\boxed{\frac{Z}{1-Z} = \frac{2E_B}{E_B + E_C}\left(1 - \frac{\gamma_\nu}{\gamma}\right)} \qquad E_C = E_f - \frac{1}{2} g \gamma_\nu$$

Poles motion

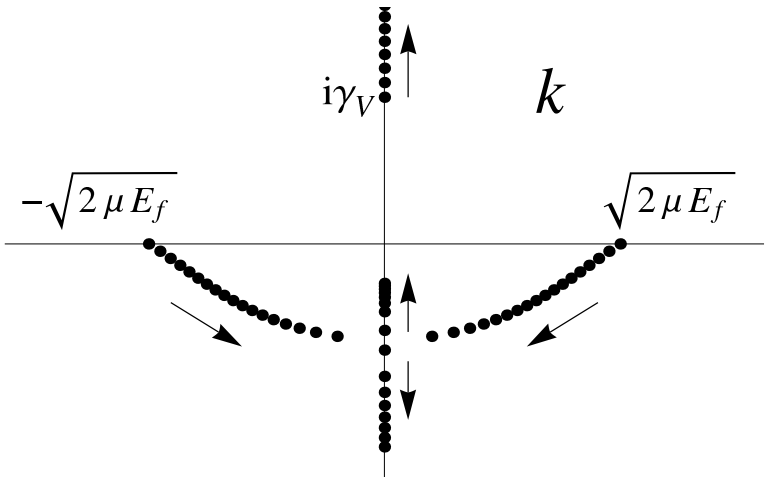


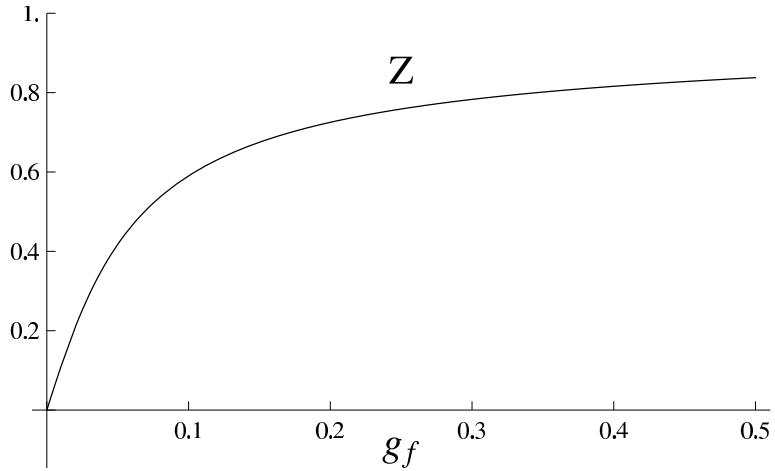


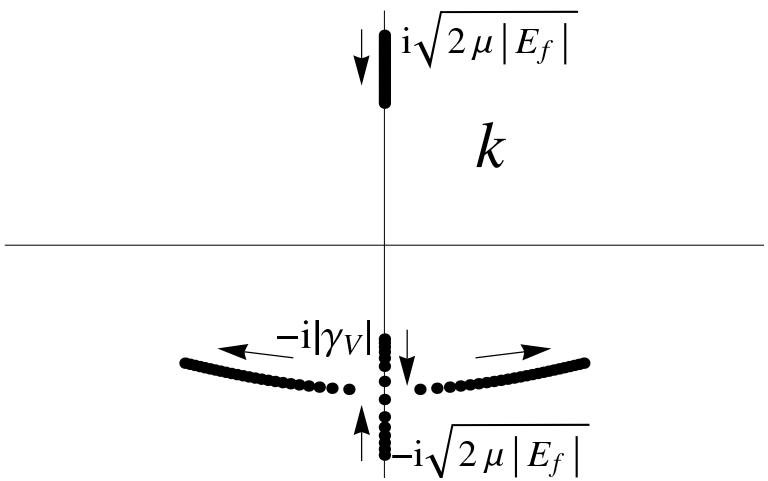


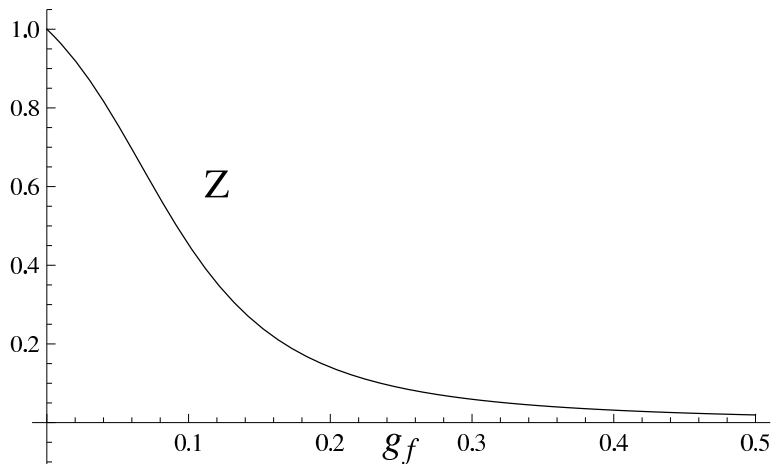


Z-factors


$i\gamma_V$
k
$-\sqrt{2\mu E_f}$
$\sqrt{2\mu E_f}$


1.
0.8
0.6
0.4
0.2
Z
0.1
0.2
0.3
0.4
0.5
g_f


$i\sqrt{2\mu|E_f|}$
k
$-i|\gamma_V|$
$-i\sqrt{2\mu|E_f|}$


1.0
0.8
0.6
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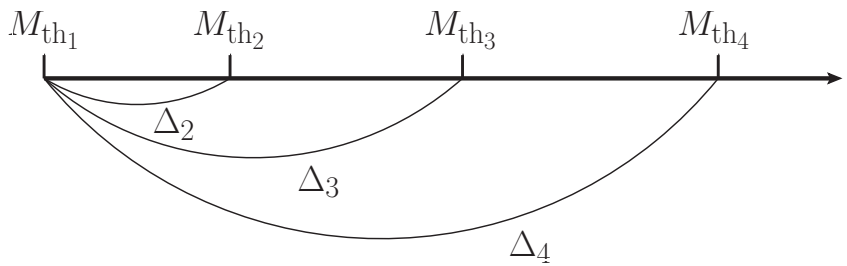
0.2
Z
0.1
0.2
g_f
0.3
0.4
0.5

Generalisation to multiple hadronic channels



$$|\Psi\rangle = \begin{pmatrix} \lambda|\psi_0\rangle \\ \chi_1(\boldsymbol{p})|H_{11}H_{12}\rangle \\ \chi_2(\boldsymbol{p})|H_{21}H_{22}\rangle \\ \cdots \end{pmatrix} \qquad H = \begin{pmatrix} E_0 & f_1 & f_2 & \cdots \\ f_1 & H_{h_1} & V_{12} & \cdots \\ f_2 & V_{21} & H_{h_2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$H_{h_i}(\boldsymbol{p},\boldsymbol{p}') = \left(\Delta_i + \frac{p^2}{2\mu_i}\right)\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}') + V_{ii}(\boldsymbol{p},\boldsymbol{p}')$$

For two channels V_{ij} $(i,j=1,2)$ is parametrised through the singlet and triplet inverse scattering lengths γ_s and γ_t:

- γ_s governs the position of the zero E_C
- γ_t governs the relevance of the term $k_1 k_2$

Generalisation to multiple hadronic channels

$$T_s = \frac{1}{2}(T_{11}+T_{22})+T_{12} = \frac{(E-E_C)\,(2\gamma_t + i(k_1+k_2))}{4\pi^2\mu\; D(E)}$$

$$T_t = \frac{1}{2}(T_{11}+T_{22})-T_{12} = \frac{2\gamma_s(E-E_f)+i(k_1+k_2)(E-E_C)}{4\pi^2\mu\; D(E)}$$

$$T_{st} = \frac{1}{2}(T_{11}-T_{22}) = \frac{i(k_2-k_1)(E-E_C)}{4\pi^2\mu\; D(E)}$$

$$D(E) = \gamma_s\Big(2\gamma_t + i(k_1+k_2)\Big)(E-E_f) - \Big(2k_1k_2 - i\gamma_t(k_1+k_2)\Big)(E-E_C)$$

$$E_C = E_f - \frac{1}{2}g\gamma_s$$

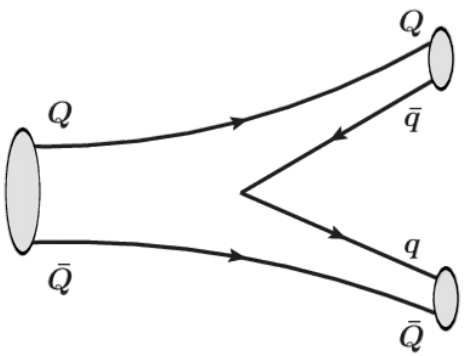
(Artoisenet et al.'2010, Hanhart et al.'2011)

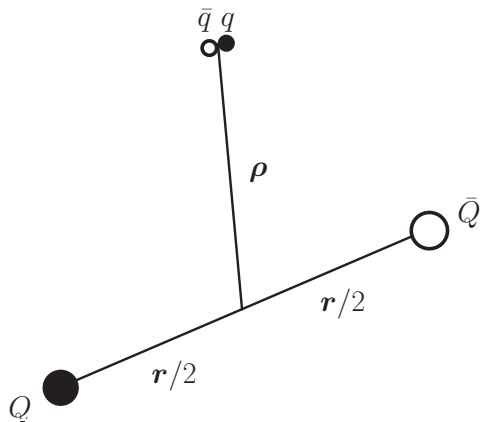
Multiple quark model states

Consider quark model states $R_n = (\bar{Q}Q)_n$ interacting through $\varphi\bar{\varphi}$ pairs ($\varphi = \bar{q}Q$)

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\varphi)^2 - m^2\varphi^2\right] + \left(\varphi \to \bar{\varphi}\right) + \sum_{n=1}^{N}\left[\frac{1}{2}(\partial_\mu R_n)^2 - \frac{1}{2}M_n^2 R_n^2 + g_n(\bar{\varphi}\varphi)R_n\right]$$

Interaction proceeds through light-quark ($\bar{q}q$) pair creation





Phenomenological coupling constants come as overlaps of wave functions

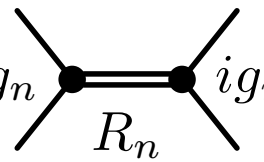
$$g_n \propto \int d^3r d^3\rho\, \psi_{R_n}(\boldsymbol{r})\psi^*_{\varphi\bar{\varphi}}(-\boldsymbol{r}/2)\psi^*_{\varphi}(\boldsymbol{r}/2+\boldsymbol{\rho})\psi^*_{\bar{\varphi}}(\boldsymbol{r}/2-\boldsymbol{\rho})$$

(Hammer,Hanhart,AN'2016, Hanhart,AN'2022)

Solution of LSE equation

The φ-$\bar{\varphi}$ scattering potential through s-channel exchange of quark model states

$$V(s) = \sum_{n=1}^{N} \; ig_n \;\; ig_n = -(\sqrt{2m})^4 \sum_{n=1}^{N} \frac{g_n^2}{s - M_n^2}$$



The resulting Lippmann–Schwinger equation is solvable analytically

$$T(s) = V(s) + V(s)G(s)T(s) \quad \Longrightarrow \quad \boxed{T^{-1}(s) = V^{-1}(s) - G(s)}$$

where $G(s)$ is loop function; its nonrelativistic form (relevant near threshold) reads

$$G_{\rm nr}(k) = \frac{1}{16\pi m}(\gamma + ik) \qquad s = 4(k^2 + m^2)$$

Poles come as solutions of equation

$$4m^2 G(s) \left[\sum_{n=1}^{N} \frac{1}{s - M_n^2}\right] = -\frac{1}{g^2} \quad \underset{g\to\infty}{\Longrightarrow} \quad \underbrace{G(s)}_{\text{Molecule}} \times \underbrace{P_{N-1}(s)}_{\text{Compact states}} = 0$$

Residues

In the vicinity of pole s_p

$$V^{-1}(s_p) - G(s_p) = 0 \qquad s_p = 4(k_p^2 + m^2)$$

and the amplitude takes the form

$$T(s) = \frac{1}{V^{-1}(s) - G(s)} \underset{s \to s_p}{\approx} \frac{1}{(d/ds)[V^{-1}(s) - G(s)]|_{s=s_p}\,(s - s_p)} = \frac{\mathrm{Res}(s_p)}{s - s_p}$$

$$\mathrm{Res}(s_p) = -\frac{8k_p}{G'(k_p) + V^{-2}(k_p)V'(k_p)} = -\frac{8k_p}{i/(16\pi m) + g^2 G^2(k_p)\underbrace{(V'(k_p)/g^2)}_{O(g^0)}}$$

Compact state poles from $P_{N-1}(s) = 0$

$$\mathrm{Res}_p \underset{g\to\infty}{\propto} \frac{1}{g^2} \to 0$$

Dressed compact states decouple

Exceptional pole from $G(s) = 0$

$$\mathrm{Res}_p \underset{g\to\infty}{\to} g_{\mathrm{eff}}^2 = \text{const}$$

$$\boxed{\frac{g_{\mathrm{eff}}^2}{4\pi} = 32m\gamma} \Longleftarrow \text{Weinberg formula}$$

A toy model

Employ a toy model for bare compact states consistent with linear Regge trajectories

$$M_n^2 = (2M_Q)^2 + \sigma n$$

with M_Q heavy quark mass and σ string tension. Consider n-independent couplings

$$g_n = g = \text{const}$$

How does n-th pole $s_n = M_n^2 + \delta s_n$ deviate from its bare position ($|\delta s_n| \ll M_n^2$)?

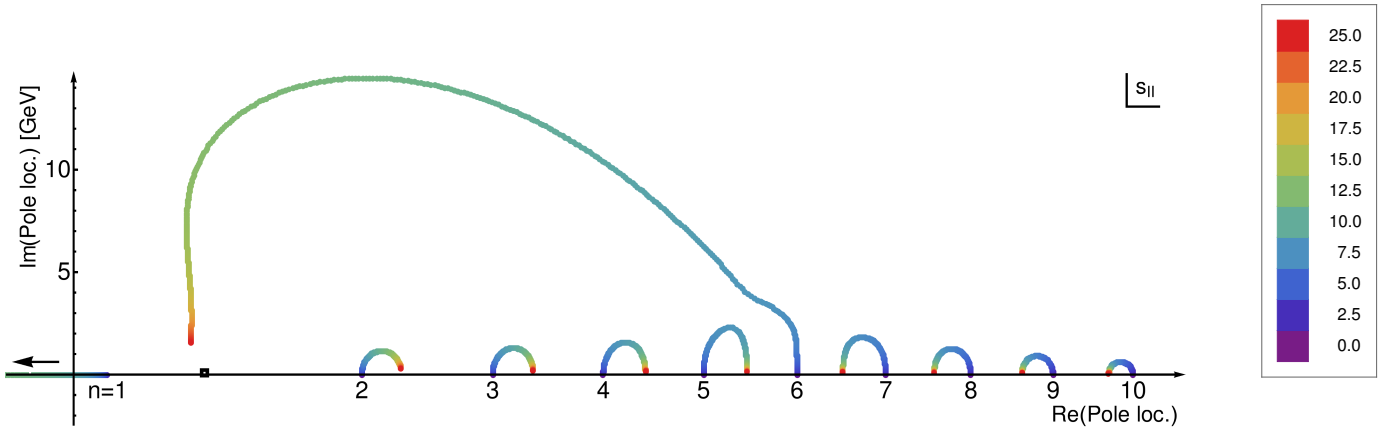
$$4m^2 G(M_n^2)\left(\frac{1}{\delta s_n} - \frac{1}{\sigma}\underbrace{\left[\psi(N+1-n) - \psi(n)\right]}_{\text{Changes sign at } n \simeq N/2}\right) = -\frac{1}{g^2}$$

where it was used that

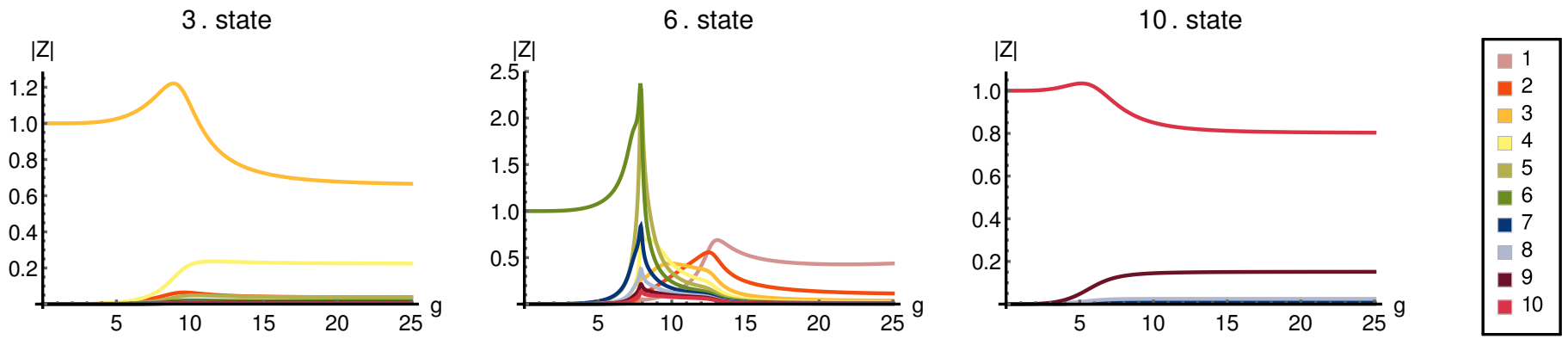
$$\sum_{i=1}^{n-1} \frac{1}{s - M_i^2} = \frac{1}{\sigma}\left[\psi(n) + \gamma_E\right] \qquad \sum_{i=n+1}^{N} \frac{1}{s - M_i^2} = -\frac{1}{\sigma}\left[\psi(N-n+1) + \gamma_E\right]$$

and $\psi(z)$ is polygamma function and $\gamma_E \approx 0.57$

Poles motion in complex s-plane



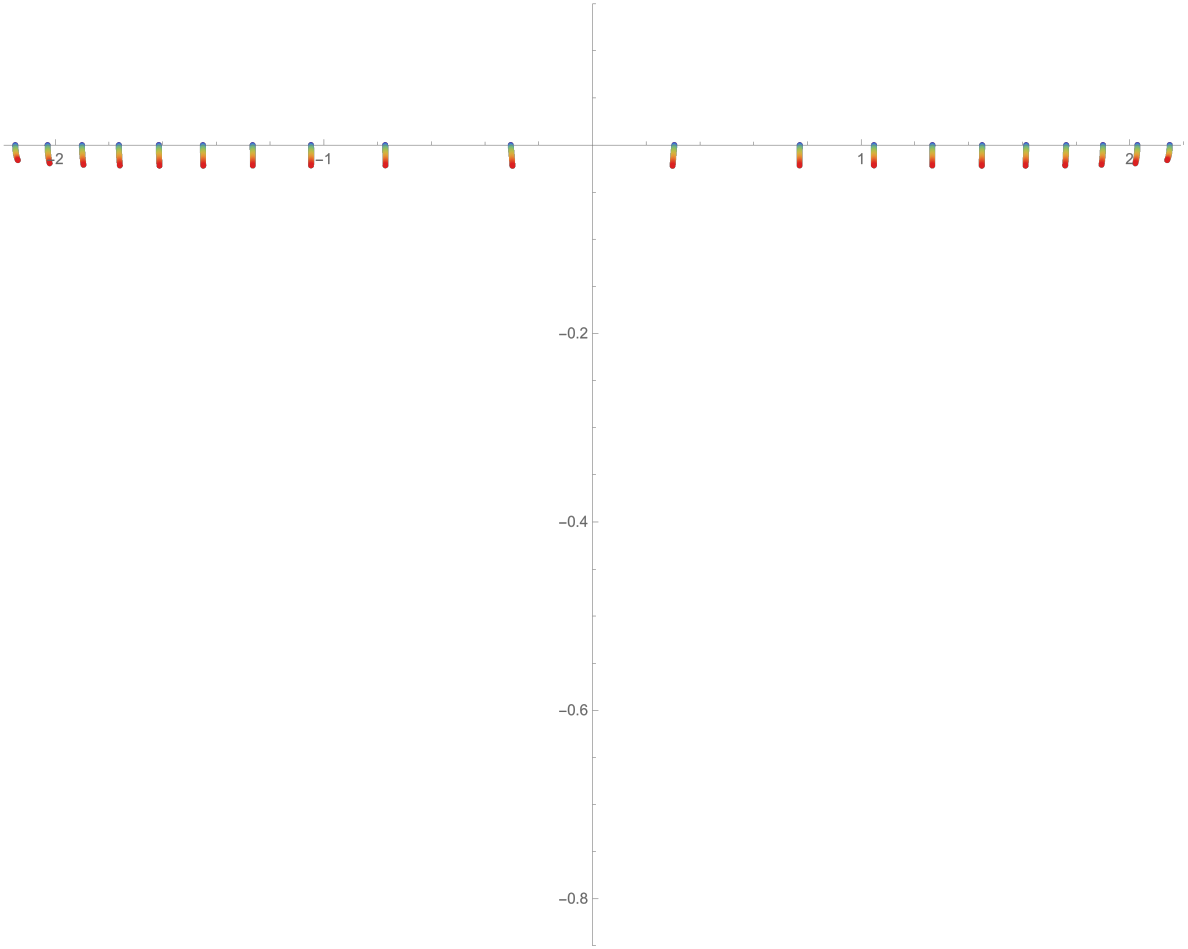
$$\text{Res}\left(T(s_p)\right) = \sum_{n,m} g_n g_{n'} Z_{nn'} \qquad \mathcal{G}_{nn'}(s \to s_p) = \frac{Z_{nn'}}{s - s_p} + \text{regular terms}$$



(Hammer,Hanhart,AN'2016)

Poles motion in complex k-plane

Consider $N = 10$ above-threshold ($M_n > 2m$) bare $\bar{Q}Q$ states and $\gamma > 0$



$g \sim 0$

Poles motion in k-plane

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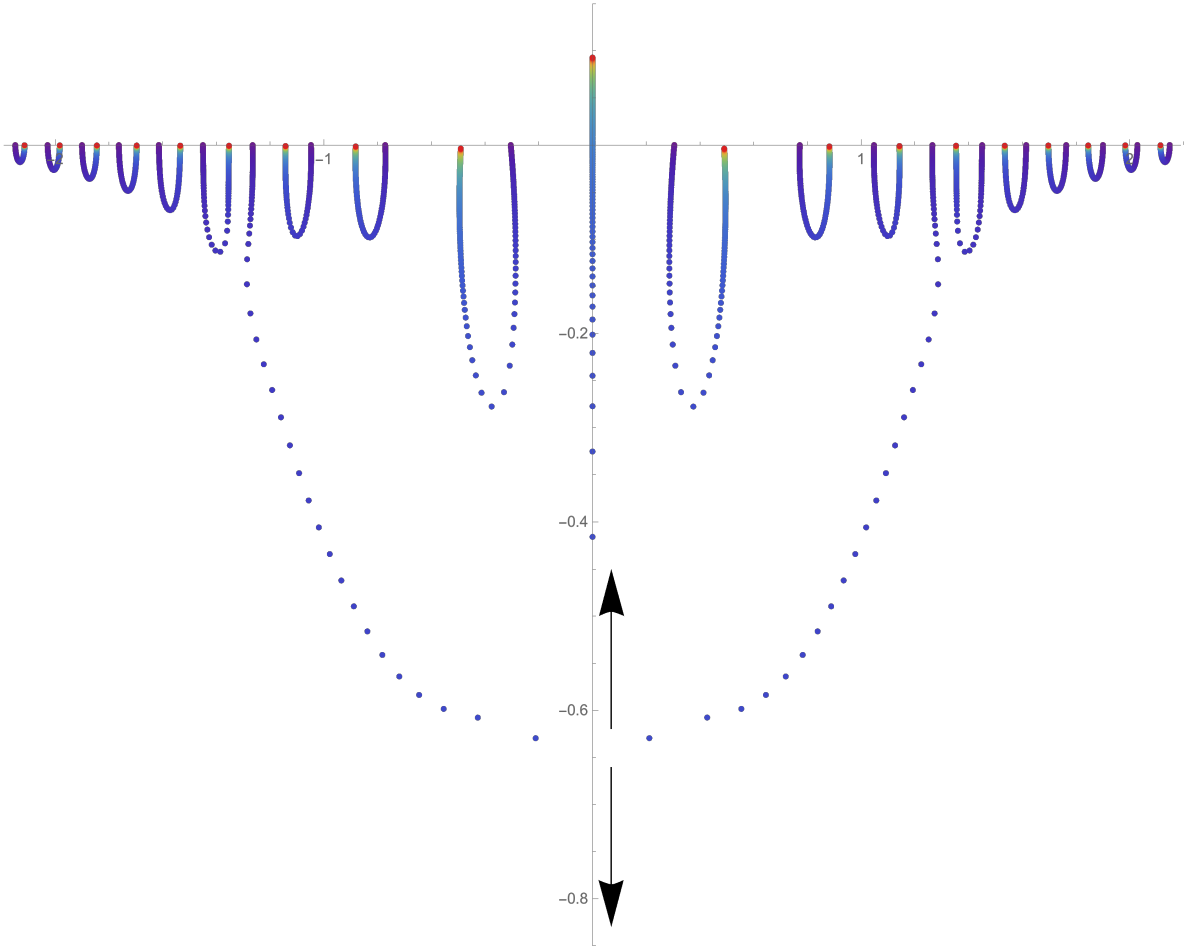
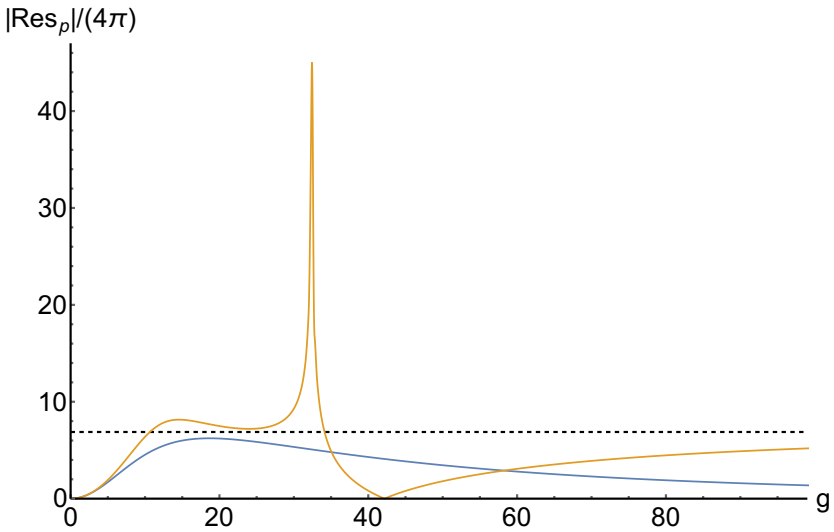


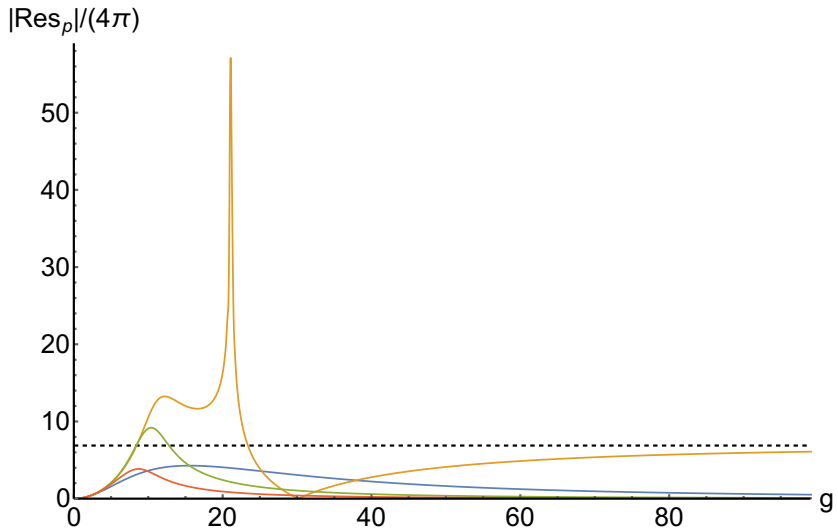
$g \to \infty$

Residues

$N = 2$



$N = 4$



- Yellow curve — exceptional (collective) state
- Other curves — compact states

Residue of collective state demonstrates nontrivial behaviour with peak and zero

$\Longrightarrow$ Subject to further studies

(Hanhart,AN'2022)

Strong coupling limit

Re-write equation for poles ($g_n = g$ for simplicity) as

$$\underbrace{1 + m^2 g^2 \left[\sum_{n=1}^{N} \frac{1}{k^2 - \Delta_n^2} \right]}_{V(k)} G(k) = 0 \qquad \Delta_n = \frac{1}{2}\sqrt{M_n^2 - 4m^2}$$

Order-of-magnitude estimate for potential:

$$\Delta_{n+1}^2 - \Delta_n^2 \simeq \Lambda_{\rm QCD}^2 \quad \Longrightarrow \quad V \simeq \frac{m^2 g^2}{\Lambda_{\rm QCD}^2}$$

Order-of-magnitude estimate for loop function:

$$G \simeq \frac{\gamma}{m}$$

Strong coupling regime:

$$V(k)G(k) \gg 1 \quad \Longrightarrow \quad \frac{m^2 g^2}{\Lambda_{\rm QCD}^2} \times \frac{\gamma}{m} \gg 1 \quad \Longrightarrow \quad \boxed{g \gg \frac{\Lambda_{\rm QCD}}{\sqrt{m\gamma}} \propto \frac{1}{\sqrt{M_Q}}}$$

Conclusions

- In strongly coupled systems effects of re-scattering (unitarisation) are essential
- Interplay of compact quark model states with hadronic states may be nontrivial
- As the coupling with continuum channel(s) increases (moderate couplings)
  - Compact states get dressed and acquire moderate finite widths
  - One state demonstrates "collective" behaviour and travels a long path
- In the strong coupling limit
  - Trajectories of compact state poles bend and re-approach real axis
  - Dressed compact states become nearly stable and decouple
  - Exceptional ("collective") state turns to a near-threshold molecule
  - Molecular and compact states do not "talk" to each other
- Strong coupling regime is easier reached for heavy quark flavours
- Loop operator for "free" field φ defines molecular pole in strong coupling regime

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Thank you for your attention!