Heavy meson decay constants and quark masses from lattice computations in the charm region and beyond

Glueballs and Charmonium: Lattice QCD, Experiment and Phenomenology – Workshop @ ECT*, Trento, Italy, March 2–6, 2026

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CLS based

ALPHA Collaboration

Quark flavour phenomenology and the rôle of lattice QCD

Weak interaction and (lattice) QCD

Because most extensions of the SM contain, e.g., new CP-violating phases and new quark flavour-changing interactions, New Physics effects are expected in the quark flavour sector

→ Search for BSM physics at the high-precision frontier:
Unveil deviations between SM predictions and experiment in flavour physics observables

→ Changes of quark flavour inside a hadron = Weak interaction processes

On the other hand, decays studied in accelerator experiments involve hadrons as initial states

→ Owing to confinement, QCD corrections enter decay rates Γ and can be significant

→ Low-energy, *non-perturbative* QCD effects are absorbed into / quantified by hadronic matrix elements:
Decay constants ($f_{\rm PS}$ etc.), form factors ($f_{+,0}$ etc.), bag parameters, ...
⇒ To compute them "ab-initio", *lattice QCD (LQCD)* comes into play

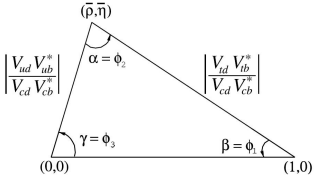
Reminder: CKM matrix in Wolfenstein parametrisation

In the SM, Cabibbo-Kobayashi-Maskawa (CKM) matrix elements encode the quark flavour dynamics (mixings, transitions) implied by weak interaction mediating quark flavour change in hadrons

$$\underbrace{\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}}_{\text{weak int.}} = V_{\text{CKM}} \underbrace{\begin{pmatrix} d \\ s \\ b \end{pmatrix}}_{\text{strong int.}} \qquad V_{\text{CKM}} = \begin{pmatrix} V_{\text{ud}} & V_{\text{us}} & V_{\text{ub}} \\ V_{\text{cd}} & V_{\text{cs}} & V_{\text{cb}} \\ V_{\text{td}} & V_{\text{ts}} & V_{\text{tb}} \end{pmatrix}$$

- 4 unknown parameters; empirically, matrix elements are largest among the diagonal
  $\rightarrow$ hierarchy gets explicit by expansion in powers of $|V_{\text{us}}| = \lambda \simeq 0.22$; e.g., to $\mathrm{O}(\lambda^4)$
- Unitarity relation $V_{\text{CKM}}^{\dagger} V_{\text{CKM}} = \mathbf{1} \Rightarrow$ Triangle representation in complex (ρ, η)–plane

$$\begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\,\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\,\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

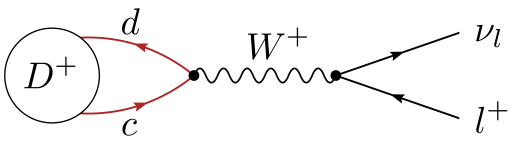


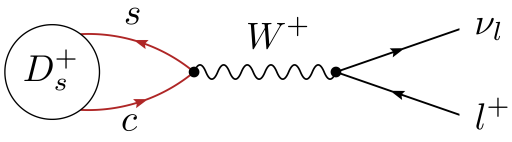
Relevance/Impact of LQCD for (heavy) flavour physics?

Heavy quark sector particularly constrains UT, where CKM/UT analyses crucially require inputs from *experiment & theory* $\leftrightarrow$ exploit SM expressions for weak decay rates of hadrons:

$$\underbrace{\left(\begin{array}{c}\text{measured}\\ \text{quantity}\end{array}\right)}_{\text{experiment}} = \left(\begin{array}{c}\text{kinematic}\\ \text{factor}\end{array}\right) \underbrace{\left(\begin{array}{c}\text{short-distance}\\ \text{factor}\end{array}\right)}_{\ni\ \text{pert. theory}} \times \underbrace{\left(\begin{array}{c}\text{QCD}\\ \text{factor}\end{array}\right)}_{\text{LQCD}:\ f_{\text{PS}}^2,\, f_+^2,\, f_0^2,\,\ldots} \times\ F(V_{\text{q}_1\text{q}_2})$$

E.g., for leptonic decays of heavy-light pseudoscalar (PS) mesons, such as $\mathrm{D}_{(\mathrm{s})}, \mathrm{B}_{(\mathrm{s})}$:





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In the D case, corresp. SM expression for the partial width, resp., experimentally measured

$$\text{decay rate}: \quad \Gamma\left(\mathrm{D_{(s)}} \to \ell\,\nu_\ell\right) = \frac{G_{\rm F}^2}{8\pi}\, m_{\mathrm{D_{(s)}}}\, m_\ell^2 \left(1 - \frac{m_\ell^2}{m_{\mathrm{D_{(s)}}}^2}\right) f_{\mathrm{D_{(s)}}}^2\, |V_{\rm cq}|^2 \qquad ,\ \mathrm{q} = \mathrm{d}, \mathrm{s}$$

$\rightarrow$ Decay constants $f_{\rm PS}$: hadronic matrix elements absorb low-energy, *non-pert.* QCD effects

$$\langle\, 0 \,|\, A_\mu \,|\, \mathrm{D_q}(p) \,\rangle = \mathrm{i}\, f_{\mathrm{D_q}}\, p_\mu$$

$$A_\mu(x) = \overline{q}(x)\gamma_\mu\gamma_5 c(x) \quad : \quad \text{axial vector current } (\mathrm{q} = \mathrm{d}, \mathrm{s})$$

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$\rightarrow$ Decay constants f_{PS}: hadronic matrix elements absorb low-energy, *non-pert.* QCD effects

$\Rightarrow$ ... obtained in LQCD from the normalisation of Euclidean two-point correlation functions

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$\rightarrow$ Theory input contributing to (quark) flavour physics, *a task for lattice QCD*

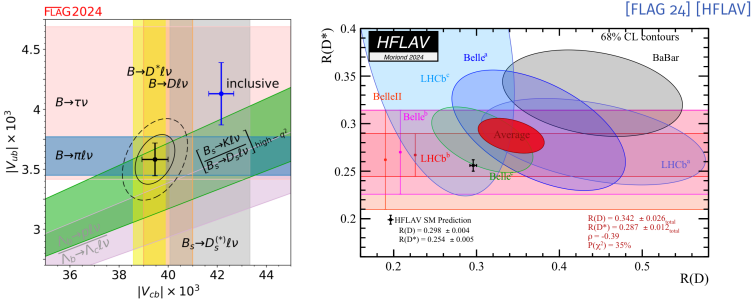
Common approach: CKM analysis of heavy-light (hl) PS-meson decays allows ...

1.) ... determining CKM matrix element $|V_{\text{hl}}|$ to (over)constrain the UT
($\leftrightarrow$ helps to test the SM resp. identities following from CKM-unitarity)

2.) ... comparing $f_{\text{PS}}|V_{\text{hl}}|$ from experiment with LQCD results of f_{PS}, if $|V_{\text{hl}}|$ is known
(e.g. via CKM unitarity, semi-leptonic decays $\leftrightarrow$ cross-checks with indep. processes)

Why D- and B-meson physics on the lattice?

- Need precise determinations of hadronic matrix elements as well as heavy quark masses (as for the latter: essential inputs to CKM phenomenology & global SM fits and, e.g., Higgs branching ratios to b- and c-quarks directly depend on them)
- Large experimental efforts: LHCb, Belle II, BESIII, ...
- B-anomalies:



- Left: Tension between inclusive and exclusive determinations of $|V_{\rm ub}|$ and $|V_{\rm cb}|$.
- Right: Between $R(D)$ and $R(D^*)$ where $R(H_c) = \frac{\mathcal{B}(H_b\to H_c\tau\nu)}{\mathcal{B}(H_b\to H_c\ell\nu)}, \quad \ell = e,\mu$

LQCD — "Ab-initio" tool to calculate SM predictions

$$\mathcal{L}_{\mathrm{QCD}}\left[g_0, m_f\right] \;=\; -\frac{1}{2g_0^2}\,\mathrm{Tr}\left\{F_{\mu\nu}F_{\mu\nu}\right\} \;+\; \sum_{f=\mathrm{u,d,s,\ldots}} \overline{\psi}_f \left\{\gamma_\mu\left(\partial_\mu + g_0 A_\mu\right) + m_f\right\}\psi_f$$

$$\underbrace{\begin{bmatrix} f_\pi, f_\mathrm{K} \\ m_\pi, m_\mathrm{K} \\ m_\mathrm{D}, m_\mathrm{B} \end{bmatrix}}_{\text{experiment}} \;\overset{\mathcal{L}_{\mathrm{QCD}}\left[g_0, m_f\right]}{\Longrightarrow}\; \underbrace{\begin{bmatrix} \Lambda_{\mathrm{QCD}} \\ M_\mathrm{u,d}, M_\mathrm{s} \\ M_\mathrm{c}, M_\mathrm{b} \end{bmatrix}}_{\text{QCD parameters (RGIs)}} \;+\; \underbrace{\begin{bmatrix} f_\mathrm{D} \\ f_\mathrm{B} \\ B_\mathrm{K}, B_\mathrm{B}, \cdots \end{bmatrix}}_{\text{predictions}}$$



$\overset{\mathcal{L}_{\mathrm{QCD}}}{\Rightarrow}$ means discretisation (regularisation) that respects: gauge invariance, locality, unitarity

- **Formulate the theory:** on a finite grid (Λ_{UV}), in finite volume (Λ_{IR}), in Euclidean space-time, as a Boltzmann distribution [→ Note: Cutoffs must be lifted at the end ...]
- Compute **expectation values** $\langle\Omega\rangle$ by sampling the QCD path integral with Markov Chain MC methods
- Contains $N_f + 1$ **bare parameters** (gauge coupling and N_f quark masses)
- Once **renormalised from hadronic input** (typ. hadron masses), all other observables become **predictions**

Outline

Quark flavour phenomenology and the rôle of lattice QCD

Challenges: Heavy flavours in LQCD

Leptonic decays of charmed mesons with $\mathrm{O}(a)$ improved Wilson quarks

Interpolating to the B-scale

The step-scaling strategy

Heavy quark masses from step-scaling

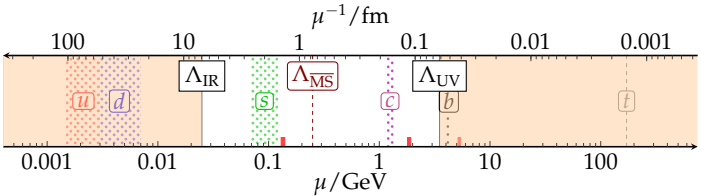
Challenges: Heavy flavours in LQCD

Lattice QCD faces multi-scale problem(s)

Predictivity in a quantum field theory relies upon a large scale ratio

interaction range $\ll$ physical length scales

momentum cutoff $\gg$ physical mass scales : $\Lambda_{\rm cut} \sim a^{-1} \gg E_i, m_j$



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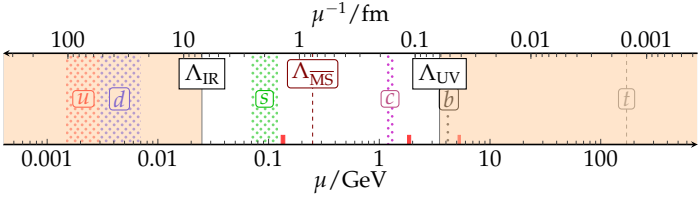


At finite lattice spacing (and finite volume), two cutoffs Λ are present:

$$\underbrace{\Lambda_{\rm IR}}\ \sim L^{-1} \ll m_\pi\,,\ \ldots\,,\ m_{\rm D}\,,\ m_{\rm B} \ll a^{-1} \sim \underbrace{\Lambda_{\rm UV}}$$

$$\left\{ {\rm O}({\rm e}^{-Lm_\pi}) \Rightarrow L \gtrsim \frac{4}{m_\pi} \sim 6\,{\rm fm} \right\} \curvearrowright\ L/a \gtrsim 120 \ \curvearrowleft \left\{ am_{\rm D} \lesssim \frac{1}{2} \Rightarrow a \approx 0.05\,{\rm fm} \right\}$$

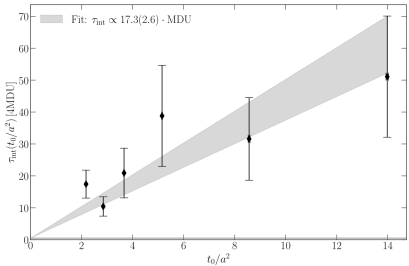
The scale of our physics question should be well separated from both, but hierarchy of disparate physical scales difficult to cover simultaneously in a numerical simulation

Multi-scale problems in LQCD: A bit closer look

$$L^{-1} \ll m_\pi \approx 135\,\mathrm{MeV} \ll am_\mathrm{h} \ll a^{-1}$$

- Cutoff effects vanish polynomially, $\propto c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + \cdots$,
  with *logarithmic corrections* [Husung et al., arXiv:1912.08498]
  - ➥ For heavy quarks h: fulfill $am_\mathrm{h} \ll 1$ for reliable continuum extrapolations!
- The cost to generate ensembles scales at least with $(L/a)^5$ ...
- ... and there is critical slowing down towards the continuum limit

  $\rightarrow$ Here: $(2+1)$–flavour LQCD ensembles by CLS (open BCs, no topology freezing!)



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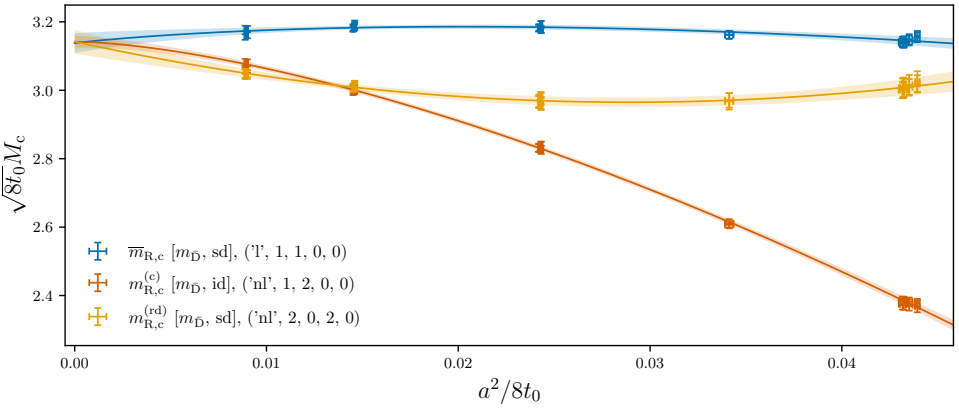
- What energy scales can be reached at physical pion mass?
  - Suppress finite-volume errors due to light π's: $Lm_\pi^\mathrm{phys} \geq 4 \Rightarrow L \geq 6\,\mathrm{fm}$ (where $T \gg L$)
  - State of the art: $L/a = 96$ at $a = 0.06\,\mathrm{fm} \Rightarrow a^{-1} \approx 3.3\,\mathrm{GeV}^{-1}$
  - Largest on the market: $L/a = 144$ at $a = 0.04\,\mathrm{fm} \Rightarrow a^{-1} \approx 4.9\,\mathrm{GeV}^{-1}$

➥ When simulating a relativistic b-quark on finest large-volume ensembles: $am_\mathrm{b} \sim \mathrm{O}(1)$!

➥ Relativ. charm: Feasible & "routinely" done, but discr. effects must be well controlled!

Charm quark mass: Continuum extrapolation



- Cutoff effects are already very relevant for the charm quark (here: computing its mass)
- Three continuum extrapolations based on state of the art large-volume simulations ($\rightarrow$ CLS ensembles) with $0.039\,\mathrm{fm} \leq a \leq 0.089\,\mathrm{fm}$ [ALPHA Collaboration, J.H. et al., 2101.02694]

Leptonic decays of charmed mesons with $\mathrm{O}(a)$ improved Wilson quarks

Sara Collins, J. H., Fabian Joswig, Simon Kuberski, Wolfgang Söldner

(Regensburg, Münster, Edinburgh, Mainz/CERN)







- D and $\mathrm{D_s}$ decay constants in $N_\mathrm{f} = 2 + 1$ QCD with Wilson fermions, arXiv:2405.04506 (JHEP 07 (2024) 090)
- D and $\mathrm{D_s}$ decay constants from $N_\mathrm{f} = 2 + 1$ lattice QCD, PoS EuroPLEx2023 (2024) 033

Decays of charmed (heavy-light) mesons

What can lattice QCD contribute here?

The *non*-perturbative physics, encoded in the hadronic matrix elements defining the ...

- Leptonic decay constants of the D– and $\mathrm{D_s}$–mesons

$$\mathrm{i}\, f_{\mathrm{D}_q}\, p_\mu \;=\; \langle\, 0 \,|\, A_\mu^{qc} \,|\, \mathrm{D}_q(p) \,\rangle \qquad\qquad q \,=\, \mathrm{d}\,,\, \mathrm{s}$$

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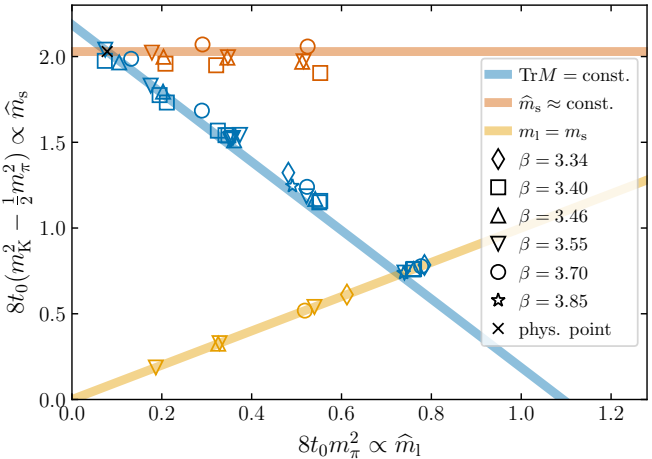
- In flavour phenomenology also of interest:
  Leptonic vector & tensor decay constants of the vecor D^*– and $\mathrm{D_s^*}$–mesons

$$\begin{aligned} m_{\mathrm{D}_q^*}\, f_{\mathrm{D}_q^*}\, \epsilon_\mu^\lambda \;&=\; \langle\, 0 \,|\, V_\mu^{qc} \,|\, \mathrm{D}_q^*(p,\lambda) \,\rangle \\ \mathrm{i}\, f_{\mathrm{D}_q^*}^{\mathrm{T}}\, (\epsilon_\mu^\lambda p_\nu - \epsilon_\nu^\lambda p_\mu) \;&=\; \langle\, 0 \,|\, T_{\mu\nu}^{qc} \,|\, \mathrm{D}_q^*(p,\lambda) \,\rangle \end{aligned}$$

  $\rightarrow$ e.g., prediction of decay rates and comparison to QCD sum rule calculations

$2+1$ flavour ensembles by CLS ("Coordinated Lattice Simulations")

Lattice action: $N_\mathrm{f} = 2+1$ non-perturbatively $\mathrm{O}(a)$ improved Wilson (sea) quarks, tree-level Symanzik-improved gauge action, open (& periodic) boundary conditions



- 49 ensembles with large statistics ($\mathrm{O}(10^{3-4})\,\mathrm{MDU}$) on three chiral trajectories:
  - $\mathrm{Tr}\,M \approx \mathrm{Tr}\,M^\mathrm{phys}$
  - $\widehat{m}_\mathrm{s} \approx m_\mathrm{s}^\mathrm{phys}$
  - $m_\mathrm{l} = m_\mathrm{s}$
- Six lattice spacings $0.039\,\mathrm{fm} \lesssim a \lesssim 0.1\,\mathrm{fm}$
- 2 physical ensembles: $m_\pi = m_\pi^\mathrm{phys}$

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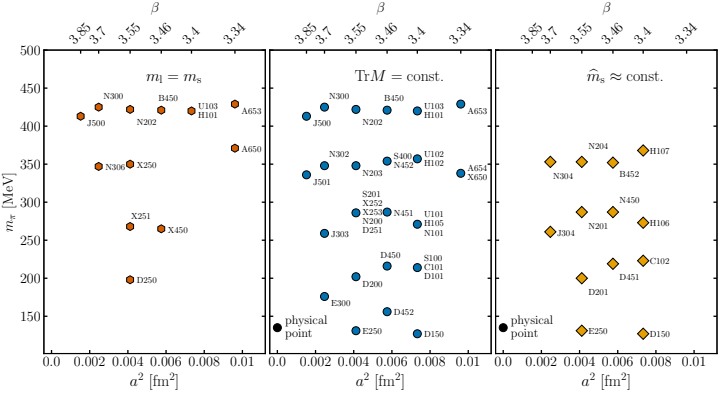

⇒ enables very controlled simultaneous chiral & continuum extrapolations towards the physical point

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  - $m_\mathrm{l} = m_\mathrm{s}$
- Partially quenched setup: 2 guessed close-to-phys. valence m_c to wrap target

Extraction of bare decay constants

In LQCD, bare (= unrenormalised) matrix elements are accessible through the spectral decomposition of 2-point correlation functions (in time $t \propto x_0 - y_0$) in respective channels:

$$C^{qc,\mathrm{I}}_{\mathrm{A}_0\tilde{\mathrm{P}}}(x_0, y_0) \;\propto\; A^{qc}_{\mathrm{A}_0\tilde{\mathrm{P}}}\, \mathrm{e}^{-m_{\mathrm{D}_q}(x_0-y_0)}$$

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$$f_{\mathrm{D}_{(q)}} \;=\; \frac{\sqrt{2}\, A^{qc}_{\mathrm{A}_0\tilde{\mathrm{P}}}}{\sqrt{A^{qc}_{\tilde{\mathrm{P}}\tilde{\mathrm{P}}} m_{\mathrm{D}_q}}} \qquad (q = \mathrm{d}, \mathrm{s})$$

$\Rightarrow$ effective (= timeslice-dependent) decay constants isolated as timeslice-averages of

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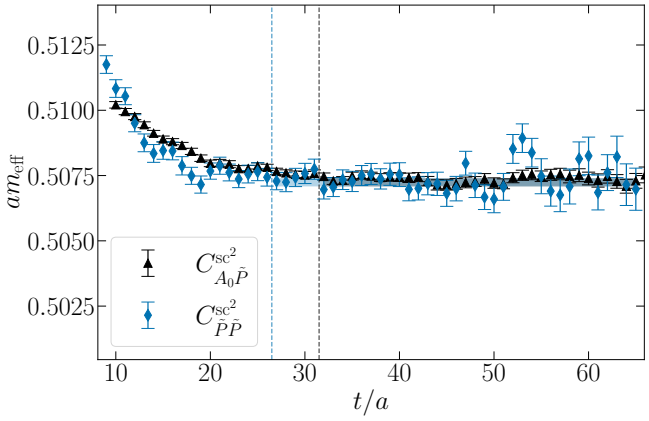
Some lattice-technicalities involved in the present calculation

- Point-to-all propagators with Gaussian smearing on APE-smoothed links (3–37 source pos. per config.)
- Region of groundstate dominance explored with two-state fits on all ensembles
  $\rightarrow$ time slices, where contribution of excited state is negligible, set final fit ranges
- Matrix elements extracted from simultaneous one-state fits

Examples of results: Effective mass & decay constant of $\mathrm{D}_{(q)}$–meson



- For strange-charm current
- From simultaneous fits to smeared-smeared and smeared-point correlation functions
- Left figure: Overlay fit result with effective masses

$N_{\mathrm{f}} = 2 + 1$ CLS ensemble E300:
$a \approx 0.05\,\mathrm{fm}$, $m_\pi \approx 174\,\mathrm{MeV}$, open boundary conditions

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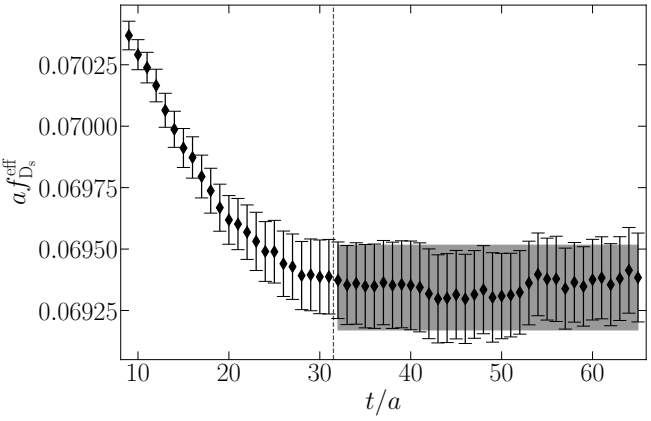

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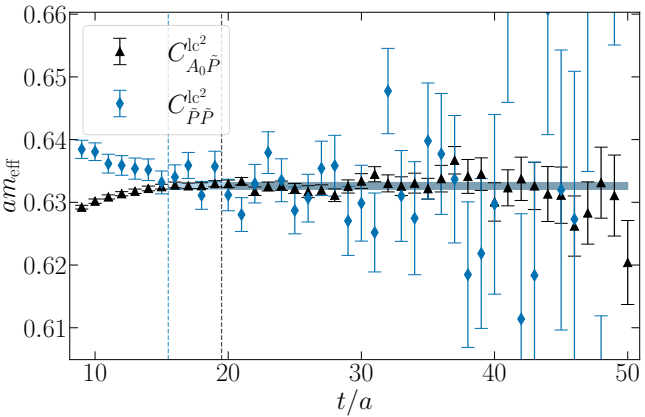
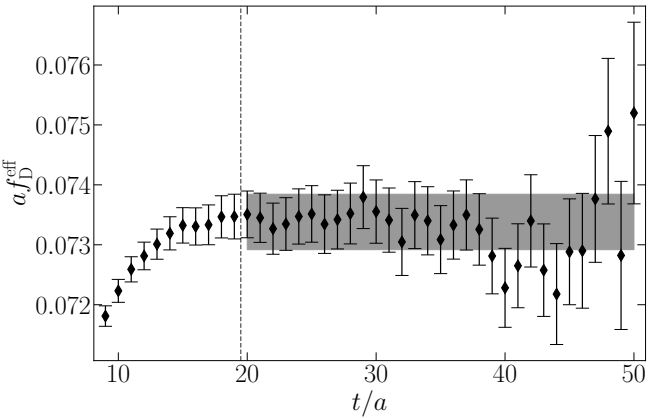


- For light-charm current
- From simultaneous fits to smeared-smeared and point-smeared correlation functions
- Left figure: Overlay fit result with effective masses

$N_{\mathrm{f}} = 2 + 1$ CLS ensemble E250:
$a \approx 0.064\,\mathrm{fm}$, $m_\pi \approx 130\,\mathrm{MeV}$, periodic boundary conditions

Examples of results: Effective mass & decay constant of $\mathrm{D}_{(q)}$–meson



- For light-charm current
- From simultaneous fits to smeared-smeared and point-smeared correlation functions
- Left figure: Effective decay constant

$N_{\mathrm{f}} = 2 + 1$ CLS ensemble E250:
$a \approx 0.064\,\mathrm{fm}$, $m_\pi \approx 130\,\mathrm{MeV}$, periodic boundary conditions

Towards physical decay constants: Renormalisation & Improvement

Lattice discretisation à la Wilson amounts to characteristic renormalisation & improvement pattern as implied by Symanzik effective theory, to remove the leading $\mathrm{O}(a)$ discretisation errors (that are a consequence of chiral symmetry violation in Wilson's LQCD):

→ e.g., cf. [Bhattacharya et al., hep-lat/0511014]

$$
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f^{\mathrm{R}}_{D_q} &= Z_{\mathrm{A}}\big(1 + b_{\mathrm{A}} a m_{\mathrm{q,cq}} + \bar{b}_{\mathrm{A}} a\,\mathrm{Tr}[M_{\mathrm{q}}]\big)\big(f^{(0)}_{D_q} + c_{\mathrm{A}}\, f^{(1)}_{D_q}\big) \;+\; \mathrm{O}(a^2) \\
f^{\mathrm{R}}_{D^*_q} &= Z_{\mathrm{V}}\big(1 + b_{\mathrm{V}} a m_{\mathrm{q,cq}} + \bar{b}_{\mathrm{V}} a\,\mathrm{Tr}[M_{\mathrm{q}}]\big)\big(f^{(0)}_{D^*_q} + c_{\mathrm{V}}\, f^{(1)}_{D^*_q}\big) \;+\; \mathrm{O}(a^2) \\
f^{\mathrm{T,R}}_{D^*_q} &= Z_{\mathrm{T}}\big(1 + b_{\mathrm{T}} a m_{\mathrm{q,cq}} + \bar{b}_{\mathrm{T}} a\,\mathrm{Tr}[M_{\mathrm{q}}]\big)\big(f^{\mathrm{T},(0)}_{D^*_q} + c_{\mathrm{T}}\, f^{\mathrm{T},(1)}_{D^*_q}\big) \;+\; \mathrm{O}(a^2)
\end{aligned}
$$

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\end{aligned}
$$

Available non-pert. determinations of renorm. factors & improvement coefficients [ALPHA Collaboration, RQCD, ...]

- Z_{A}, b_{A}, c_{A} — [1502.04999], [1604.05827], [1607.07090], [1808.09236]
- Z_{V}, b_{V}, c_{V} — [1805.07401], [1811.08209], [2010.09539]
- Z_{T}, b_{T}, c_{T} — [2012.06284], [2305.04717]; [2309.04314]
- κ_{crit} — [2211.03744]

Combined chiral & continuum extrapolation

To obtain physical results, the lattice estimates must be extrapolated to the physical point by taking the (simultaneous) limits $m_\pi \to m_\pi^{\rm phys}$, $m_{\rm K} \to m_{\rm K}^{\rm phys}$ and $a \to 0$

$\to$ position in quark mass plane parameterised by (t_0 renders everything dimensionless):

$$\phi_\Delta \equiv 8t_0\big(m_{\rm K}^2 - m_\pi^2\big) \sim m_{\rm s} - m_{\rm l}\ ,\ \phi_4 \equiv 8t_0(m_{\rm K}^2 + \tfrac{m_\pi^2}{2}) \sim 2m_{\rm l} + m_{\rm s}\ ,\ \phi_{\rm H} \equiv \sqrt{8t_0}m_{\rm H} \sim m_{\rm c}$$

where we fix the phys. charm quark mass via $m_{\rm H} = m_{\bar{D}} = \frac{2}{3}m_D + \frac{1}{3}m_{D_{\rm s}}$ (or $m_{\rm H} = \eta_{\rm c}^{\rm con}$)

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- Constraint: $f_D^{\rm R}$ and $f_{D_{\rm s}}^{\rm R}$ are implicitly degenerate in the $SU(3)$ symmetric limit
- General fit ansatz inspired by $SU(3)$ chiral perturbation theory

$$\begin{aligned}
\sqrt{8t_0}f_{D_{\rm s}}^{\rm R}(m_\pi, m_{\rm K}, 0) &= f_0 + c_1\,\phi_4 + \tfrac{2}{3}c_2\,\phi_\Delta + c_3\,(4\mu_{\rm K} + \tfrac{4}{3}\mu_\eta) + c_4\,\phi_{\rm H} \\
&\quad + c_5\,\phi_{\rm H}^2 + c_6\,\phi_\Delta\phi_{\rm H} + c_8\phi_4\phi_{\rm H} + \dots \\
\sqrt{8t_0}f_D^{\rm R}(m_\pi, m_{\rm K}, 0) &= f_0 + c_1\,\phi_4 - \tfrac{1}{3}c_2\,\phi_\Delta + c_3\,(3\mu_\pi + 2\mu_{\rm K} + \tfrac{1}{3}\mu_\eta) + c_4\,\phi_{\rm H} \\
&\quad + c_5\,\phi_{\rm H}^2 + c_7\phi_\Delta\phi_{\rm H} + c_8\phi_4\phi_{\rm H} + \dots
\end{aligned}$$

with the terms containing chiral logarithms defined as: $\mu_X = 8t_0 m_X^2 \ln(8t_0 m_X^2)$

Combined chiral & continuum extrapolation

The (leading) lattice spacing dependence at physical masses is modelled as $\propto a^2$, i.e.

$$\sqrt{8t_0}\, f^{\mathrm{R}}_{D_{(\mathrm{s})}}(m_\pi^{\mathrm{phys}}, m_{\mathrm{K}}^{\mathrm{phys}}, a) \;=\; \frac{a^2}{8t_0^*}\left(c_9 \,+\, c_{10}\,\phi_{\mathrm{H}} \,+\, \ldots\right)$$

also allowing for higher-order cutoff effects by terms: $\propto a^3, a^4$

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Some analysis details ...

- Correlated fits performed; block diagonal covariance matrices
  $\rightarrow$ 6×6 at the $SU(3)$ symmetric point, 9×9 away from it

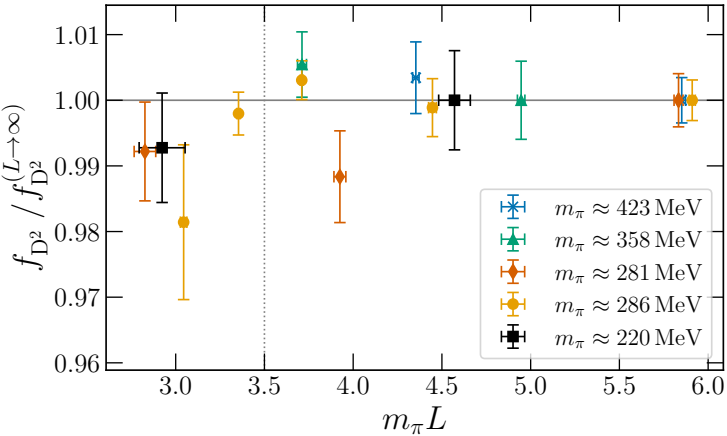
$$\chi^2 \;=\; \sum_{e=1}^{n_e} \delta f_e\, C_e^{-1}\, \delta f_e^T \qquad\qquad \delta f_e \;=\; \left(\, f(p_i;\tilde{x}_e) - y_e\,,\; \tilde{x}_e - x_e\,\right)$$

$$y_e \;=\; \left(f^{\mathrm{c1}}_{D_{\mathrm{s}}}, f^{\mathrm{c2}}_{D_{\mathrm{s}}}, f^{\mathrm{c1}}_{D}, f^{\mathrm{c2}}_{D}\right)_e \qquad\qquad x_e \;=\; \left(m_\pi, m_{\mathrm{K}}, m^{\mathrm{c1}}_{\bar{D}}, m^{\mathrm{c2}}_{\bar{D}}, 8t_0\right)_e$$

- Renormalisation and improvement parameters, as well as t_0^*/a^2, accounted for via Gaussian priors
- Convert to physical units using value for scale-setting quantity: $\sqrt{t_0^{\mathrm{phys}}} = 0.1449^{(7)}_{(9)}\,\mathrm{fm}$

[RQCD, 2211.03744]

Systematics from finite-volume effects?



- Explicit study on dedicated ensembles with varied volumes
  $\rightarrow$ no significant finite-volume effects observed for $m_\pi L \gtrsim 3.5$
- Ensembles with $m_\pi L < 3.5$ and/or $L < 2.3\,\mathrm{fm}$ excluded from final analysis

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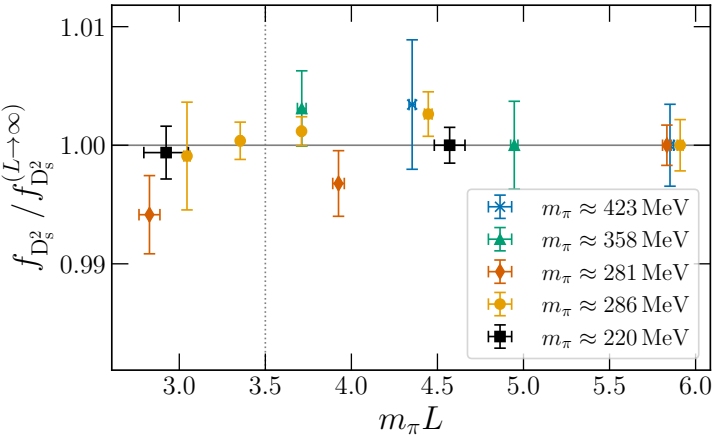

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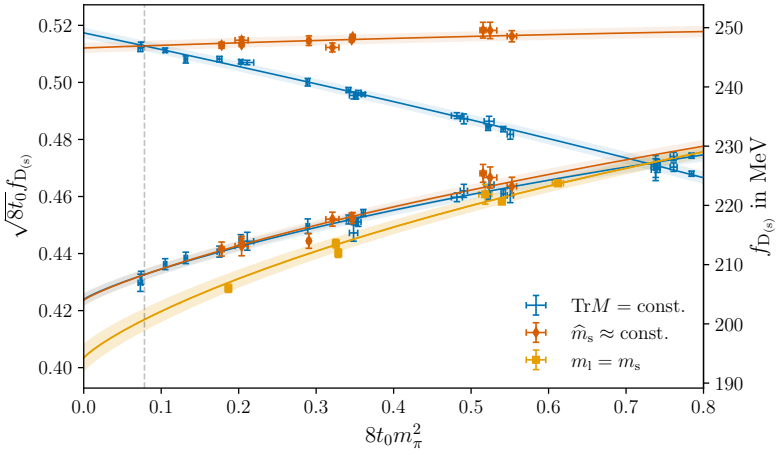
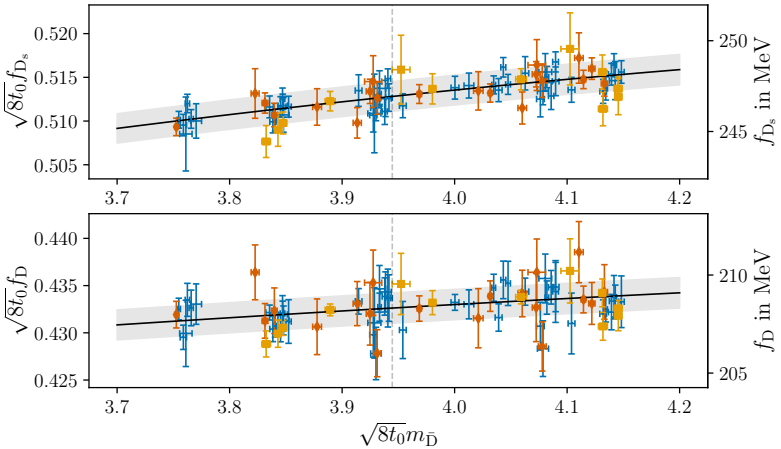
Light quark mass dependence



- Three chiral trajectories enter simultaneous fits
- Two ensembles at *physical* quark masses
- Left figure: Illustration of best fit, $\frac{\chi^2}{\mathrm{dof}} = 0.92$ (data points are projected acc. to continuum-limit fit)

Note: best fit includes $c_{11/12}\,\phi_\Delta\phi_\mathrm{H}^2 + c_{13}\,a^3 + c_{14/15}\,a^3\phi_\Delta$ (i.e., terms $\propto m_{\bar{D}}^2$ & cubic in a)

Charm quark mass dependence

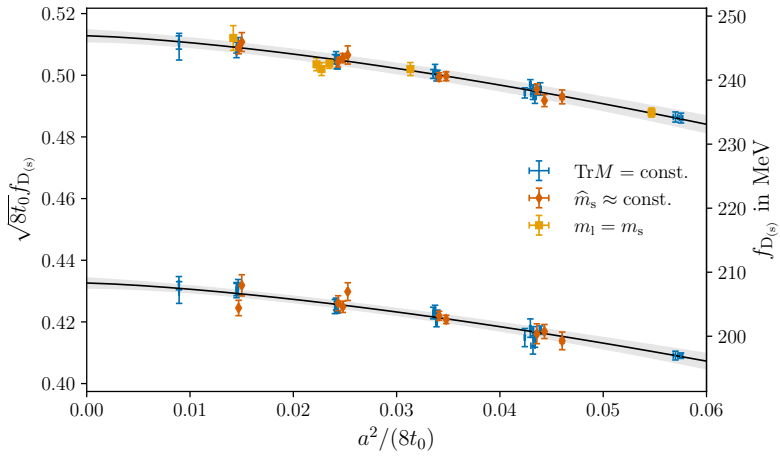


Fixing the charm quark mass (i.e., its hopping parameter in the lattice action): demand $m_{\bar{D}}$–parametrisation to enclose $\frac{2}{3}m_D + \frac{1}{3}m_{D_s} \stackrel{!}{=} m_{\bar{D}}^{\mathrm{phys}} = 1899.4(3)\,\mathrm{MeV}$

[and η_c^{con} employed as cross-check]

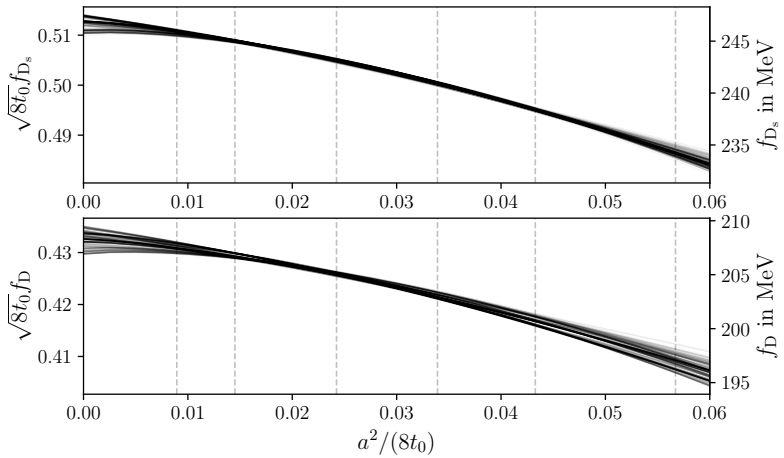
- 2 bare heavy (valence) quarks per ensemble with masses bracketing physical charm
- Global fit allows testing for curvature in the heavy (charmed) meson mass, $\phi_{\mathrm{H}} \equiv \sqrt{8t_0}m_{\mathrm{H}}$.
- Note: a small higher-order term $\propto m_{\bar{D}}^2$ is resolvable, but no cubic one

Lattice spacing dependence



- Data range covers factor of about 6 in a^2 (data points are projected acc. to fit to phys. point)
- Best fit includes higher order lattice artifacts ($\propto a^3$, see above)

Lattice spacing dependence



- Data range covers factor of about 6 in a^2 (data points are projected acc. to fit to phys. point)
- Best fit includes higher order lattice artifacts ($\propto a^3$, see above)
- Within model average: opacity $\sim$ weight
- Finding: pure a^2–fits have a small weight

Model averaging procedure

To reliably estimate the systematic error via the combined chiral & continuum extrapolations (within different fit categories w.r.t. to ansaetze, cuts in the data, etc.), a model averaging relying on the Akaike information criterion (AIC) is adopted [Jay & Neil, 2008.01069; Akaike, 1998]

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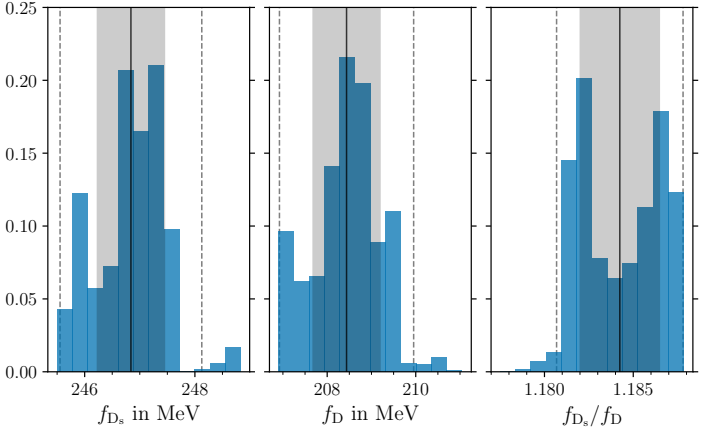
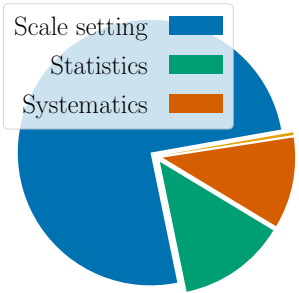


- Left: histogram of fit results, reweighted by AIC weight
  - $\Rightarrow$ system. uncertainty from model average
- $0.92 \leq \chi^2/\mathrm{dof} \leq 1.09$ in $\mathrm{O}(500)$ fits with ~ 150 naive dof
- Ratio computed from extrapolated f_D & f_{D_s}

Results & Error budget

Contributions to $(\Delta f_{\mathrm{D_s}})^2$



$f_{\mathrm{D_s}} = 246.8(1.3)\,\mathrm{MeV}$

Contributions to $(\Delta f_{\mathrm{D}})^2$



$f_{\mathrm{D}} = 208.4(1.5)\,\mathrm{MeV}$

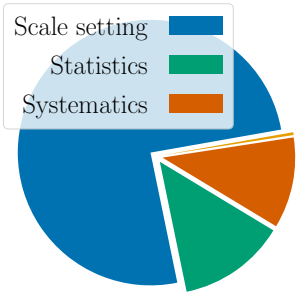
Contributions to $(\Delta f_{\mathrm{D_s}}/f_{\mathrm{D}})^2$



$f_{\mathrm{D_s}}/f_{\mathrm{D}} = 1.1842(36)$

Results & Error budget

Contributions to $(\Delta f_{\mathrm{D_s}})^2$



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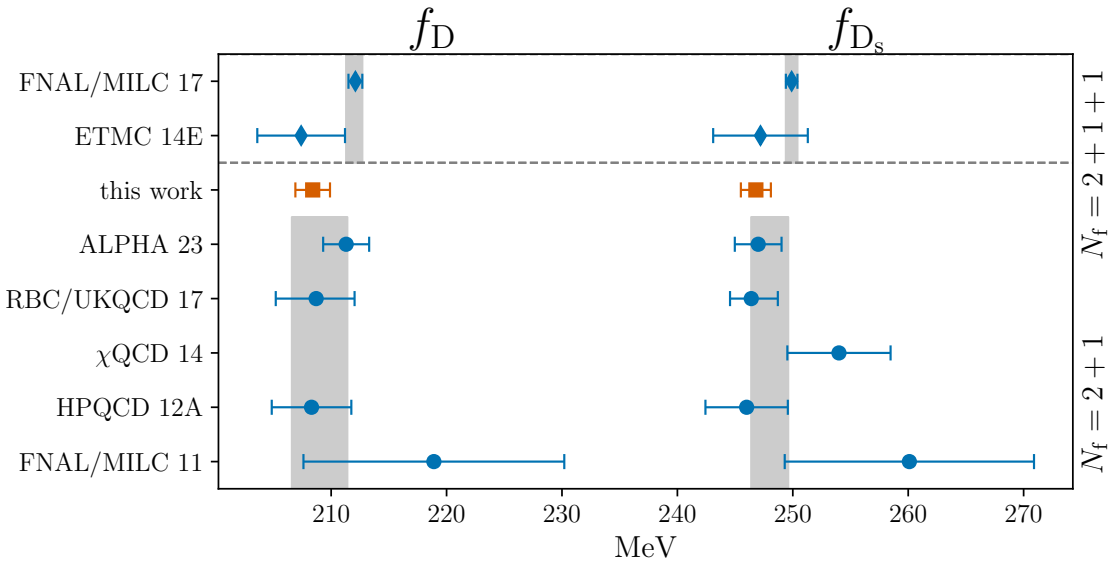
Contributions to $(\Delta f_{\mathrm{D_s}}/f_{\mathrm{D}})^2$



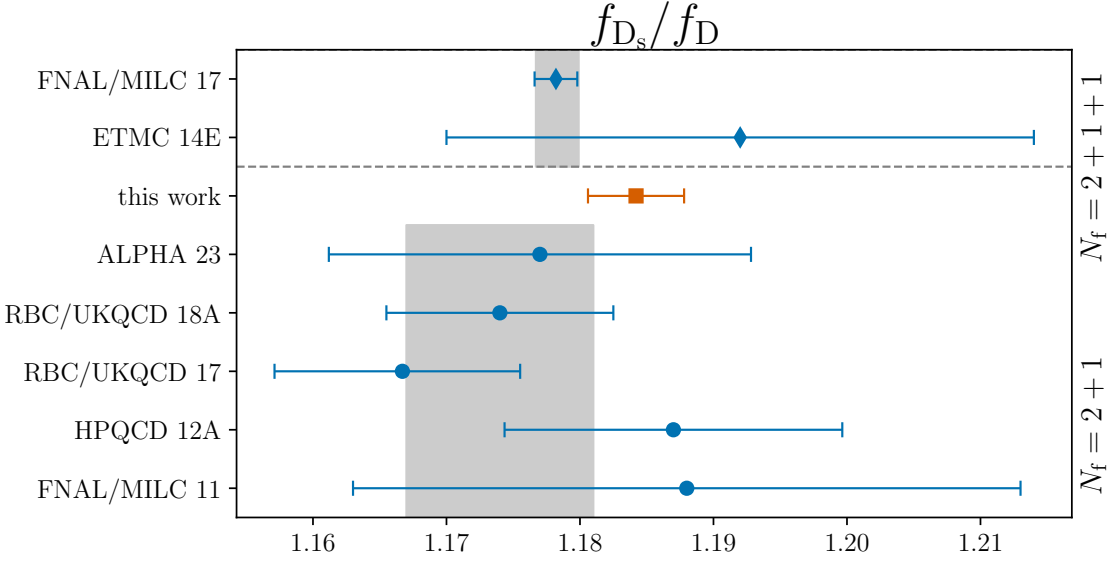
$f_{\mathrm{D_s}}/f_{\mathrm{D}} = 1.1842(36)$

- Uncertainty on decay constants limited by scale setting (with $\approx 0.6\%$ precision)
- Balanced statistical and systematic uncertainties

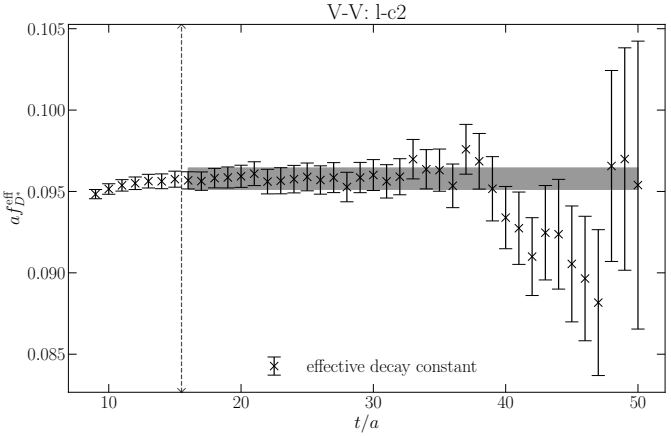
Comparison with LQCD results from other groups: Decay constants



Comparison with LQCD results from other groups: Ratio of decay constants



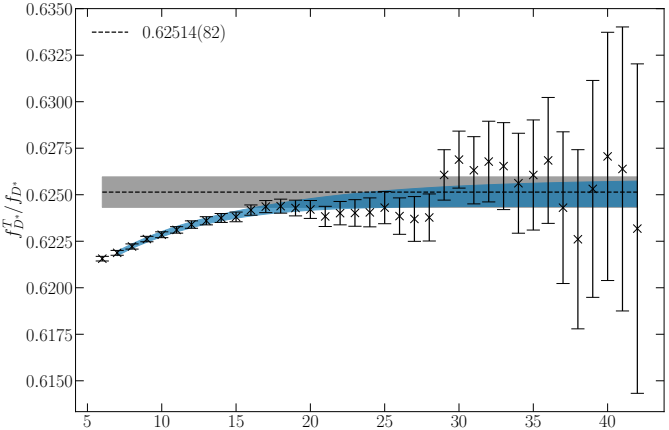
Prospects for computing vector decay constants



- Left figure: bare effective decay constant
- For light-charm vector current
- Sub-percent precision at physical light quark mass reachable
- Analysis is ongoing ...

CLS ensemble E250: $a \approx 0.064\,\mathrm{fm}$, $m_\pi \approx 130\,\mathrm{MeV}$, periodic boundary conditions

Prospects for computing tensor decay constants

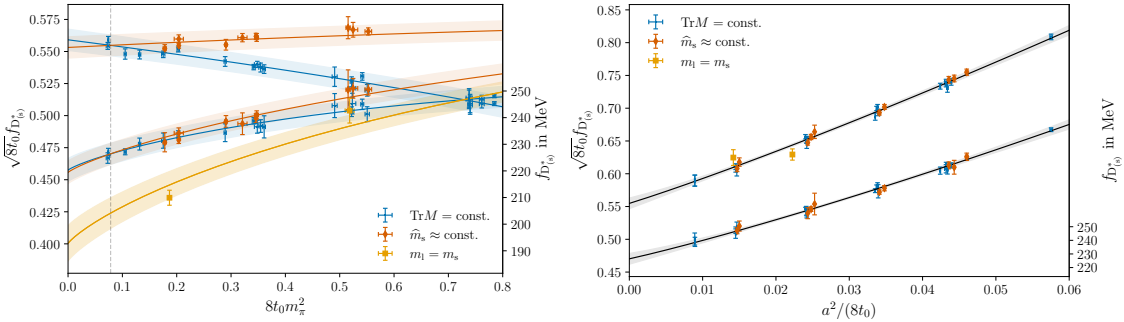


- Left figure: *preliminary* extraction of the **ratio of bare tensor and vector decay constants**
- For light-charm currents
- Statistical uncertainty of the ratio is negligible
- Analysis is ongoing ...

CLS ensemble E250: $a \approx 0.064\,\mathrm{fm}$, $m_\pi \approx 130\,\mathrm{MeV}$, periodic boundary conditions

First attempt of a joint chiral-continuum extrapolation ...

... of f_{D^*} and $f_{\mathrm{D}_\mathrm{s}^*}$ (once properly $\mathrm{O}(a)$ improved and renormalised)



- Left: light quark mass (aka m_π^2) dependence; right: lattice spacing (aka a^2) dependence
- *Preliminary* analysis including 32 ensembles, 15 (fit) parameters: $\chi^2/\mathrm{dof} \approx 1.3$
- Careful study of fit models and systematics in progress

Summary

[of this calculation]

- f_{D} and $f_{\mathrm{D_s}}$ are determined in partially quenched (2+1)–flavour QCD (lattice-discretised à la Wilson and NP'ly $\mathrm{O}(a)$ improved) with sub-percent precision, thanks to the large number of ensembles that constrain the joint chiral & continuum extrapolation tightly
- Bare decay constants at $\sim 0.5\,\%$ uncertainty, while total error is limited by scale setting
- Higher precision would call for including/computing (electromagnetic and strong) isospin-breaking corrections arising from QED and unequal up- and down-quark masses
- We are also working on the extraction of $f_{\mathrm{D}^*_{(\mathrm{s})}}$ and $f^{\mathrm{T}}_{\mathrm{D}^*_{(\mathrm{s})}}$
  - Bare decay constants also at $\sim 0.5\%$ statistical error, chiral & continuum looks promising
  - Final uncertainty again limited by scale setting, and by the Z–factor in the tensor case
    $\rightarrow$ NP determination of $c_{\mathrm{T}}, Z_{\mathrm{T}}(\mu)$ available by ALPHA Collaboration [arXiv:2309.04314]
  - Computing $f_{\mathrm{D}^*_{(\mathrm{s})}}/f_{\mathrm{D}_{(\mathrm{s})}}$ $\rightarrow$ investigate heavy quark limit [within B-physics programme]

Interpolating to the B-scale

Alessandro Conigli, Patrick Fritzsch, Antoine Gérardin, J. H., Gregorio Herdoíza, Simon Kuberski, Carlos Pena, Hubert Simma, Rainer Sommer

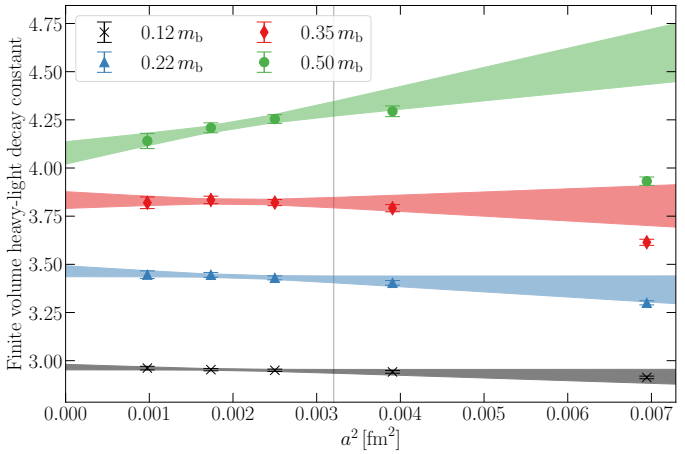
(Mainz, Dublin, Marseille, Münster, Madrid, CERN, DESY Zeuthen)

ALPHA Collaboration

CLS based

- A strategy for B-physics observables in the continuum limit, arXiv:2312.09811 (LATTICE 2023)
- m_{b} and $f_{\mathrm{B}^{(*)}}$ in $2+1$ flavour QCD from a combination of continuum limit static and relativistic results, arXiv:2312.10017 (LATTICE 2023)
- B-physics observables in the continuum from a combination of static and relativistic results, PoS EuroPLEx2023 (2024) 005
- Heavy quark masses from step-scaling (S. Kuberski @ LATTICE 2025) ...
- ... and papers in preparation

Quark-mass dependent cutoff effects



- Consider (finer than) conventional lattice spacings

$$0.031\,\mathrm{fm} \leq a \leq 0.083\,\mathrm{fm}$$

in *finite* volume

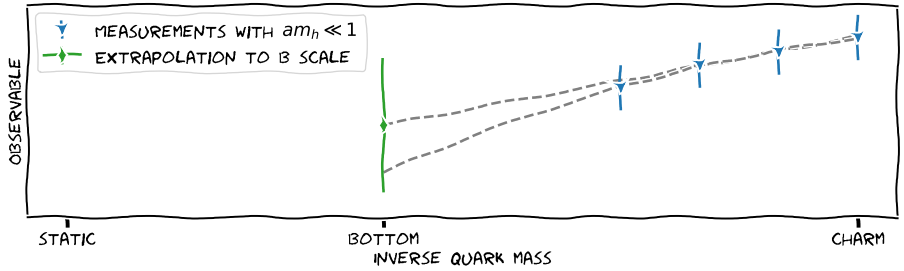
- Continuum extrapolation of the pseudoscalar heavy-light decay constant at fixed (renormalised) quark masses

- For illustration: Use three finest resolutions $\leq 0.05\,\mathrm{fm}$ to extrapolate with

$$f_{\mathrm{PS}}^{\mathrm{hl}}(a) \;=\; p_0 \,+\, p_1 \cdot a^2$$

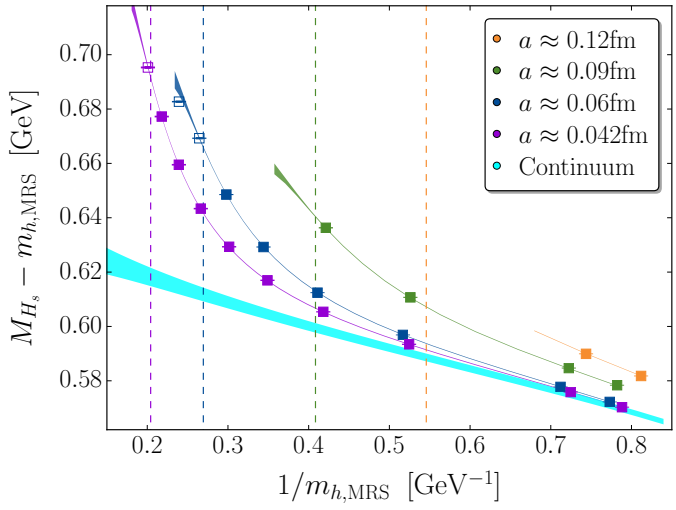
Heavy quark physics

- As argued before: Simulating relativistic bottom quarks at several resolutions is not possible in large volumes with CPU/GPU resources expected in near future!
- Long-standing, widely adopted approach: Extrapolation to the B scale ($\sim m_\mathrm{b}$)



- This is difficult/problematic, because of possibly mixing extra-/interpolations in a, $1/m_\mathrm{h}$ (and q^2, especially for semi-leptonic form factors)

Quark mass extrapolations



- An example from the literature
- From the most precise determination of m_b [Bazavov et al., 1802.04248]
- Quark mass extrapolation guided by Heavy Quark Effective Theory (HQET)
- Deviations from the turquoise band are cutoff effects of $\mathrm{O}(a^2 m^2)$

Heavy Quark Effective Theory

How to separate b-quark's mass scale from low-energy physics?

Employ effective field theories s.th. $a\,|\vec{p}| \ll 1\,,\ |\vec{p}| \ll m_{\mathrm{b}} \Rightarrow$ ✓ for num. simulations

Heavy Quark Effective Theory

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Employ effective field theories s.th. $a\,|\vec{p}| \ll 1\,,\ |\vec{p}| \ll m_{\mathrm{b}} \Rightarrow$ ✓ for num. simulations

Here: Heavy Quark Effective Theory (HQET)

- Integrate out heavy degrees of freedom of QCD Lagrangian for one heavy quark
- Expand Lagrangian and operators in (an asymptotic) power series of $1/m_{\mathrm{h}}$
- Formally: (power-counting) renormalisable effective theory $\Leftrightarrow$ Continuum limit exists

$$\mathcal{L}_{\mathrm{heavy}} = \bar{h}_v D_0 h_v - \omega_{\mathrm{kin}}\mathcal{O}_{\mathrm{kin}} - \omega_{\mathrm{spin}}\mathcal{O}_{\mathrm{spin}} + \cdots\,, \quad \mathcal{O}_{\mathrm{kin}} = \bar{h}_v \mathbf{D}^2 h_v\,, \quad \mathcal{O}_{\mathrm{spin}} = \bar{h}_v \sigma\cdot\mathbf{B} h_v$$

- Perturbative matching to QCD at order g_0^{2l} leads to power divergences (w.r.t. a) in the coefficients [Nucl.Phys.B 368 (1992) 281-292, Maiani et al.]

$$\Delta c_k \;\sim\; g_0^{2(l+1)} a^{-p} \;\sim\; a^{-p}\left[\ln(a\Lambda)\right]^{-(l+1)} \overset{a\to 0}{\to} \infty$$

due to mixing (under renormalisation) of operators differing in dimensions by p

(Non-perturbative) HQET

- $\exists$ well-defined strategy to renormalise HQET non-perturbatively via finite-volume matching with QCD and step-scaling techniques [J.H. and R. Sommer, hep-lat/0310035]
  $\rightarrow$ works but quite challenging though, since $1/m_{\mathrm{h}}$ effects are needed for precision

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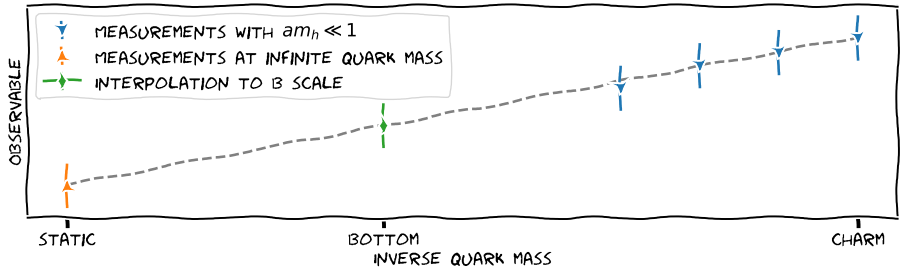

- It would be great to perform an interpolation between results in static HQET and results in relativistic QCD below m_{b} (i.e., beyond the charm region), where $am_h \ll 1$

(Non-perturbative) HQET

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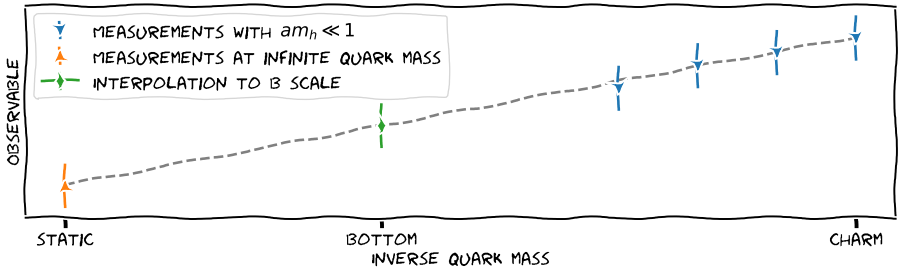

- It would be great to perform an interpolation between results in static HQET and results in relativistic QCD below m_{b} (i.e., beyond the charm region), where $am_h \ll 1$

$\rightarrow$ However: Even the static approximation requires non-trivial renormalisation and matching that would have to be **computed non-perturbatively**

Intermission: The static theory

HQET in static approximation exhibits a non-trivial renormalisation & matching pattern:

- Given the static action $\mathcal{L}_{\rm stat} = \bar\psi_{\rm h} D_0 \psi_{\rm h}$, the static quark's self-energy inside a B-meson behaves as:
$$E_{\rm stat} \;\sim\; \tfrac{1}{a}\, g_0^2 \qquad \Rightarrow \qquad E_{\rm stat} \text{ diverges as } a \to 0$$
- Renormalisation in the static effective theory thus amounts to a (power divergent) additive mass counterterm $\delta m \sim \frac{1}{a}\, g_0^2$, while matching to QCD implies finite mass shift:
$$E \;=\; E_{\rm stat} \,+\, \delta m \,+\, m_{\rm b}^{\rm finite}$$
- Heavy-light currents in static HQET are then related to the QCD ones by
$$\begin{aligned} V_k^{\rm stat} &= C_{V_k}(m_{\rm b})\, Z^{\rm stat}(g_0)\, \bar\psi_{\rm h} \gamma_k \psi_{\rm l} \\ V_0^{\rm stat} &= C_{V_0}(m_{\rm b})\, Z^{\rm stat}(g_0)\, \bar\psi_{\rm h} \gamma_0 \psi_{\rm l} \end{aligned}$$
with matching coefficients (aka "conversion functions") $C_{V_k,V_0}(m_{\rm b})$ that are $\log$-*divergent* as $m_{\rm b} \to \infty$

[E.g., see R. Sommer, arXiv:1008.0710]

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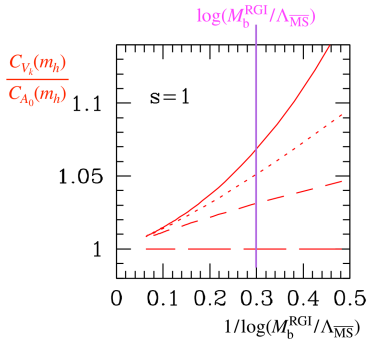

High orders of PT are known

$\rightarrow$ so why not use them in the $C_{\rm X}$ here ?

- at $m_{\rm b}$ no "asymptotic convergence"
- even more dramatic below $m_{\rm b}$ ($m_{\rm c}$!)
- Lesson:
  be careful using PT in flavour physics ...

➥ Performing an interpolation between results in static HQET and relativistic QCD below $m_{\rm b}$, where $am_{\rm h} \ll 1$, requires renormalisation & matching that would have to be calculated NP'ly !

➥ Our strategy: Cancel renormalisation & matching in **step-scaling functions**, where these pieces drop out, s.th. the **static HQET limit constraining interpolations** to the b-scale gets sound !

The step-scaling strategy

Example: Quark mass determinations — First a recap ...

- Conventionally, we fix the heavy quark mass from a heavy hadron mass $m_{\rm H}$ through

$$am_{\rm H}(m_{\rm h}) \;=\; am_{\rm H}^{\rm PDG}$$

  - $m_{\rm H}^{\rm PDG}$ is the *physical input* (continuum, infinite volume)
  - $am_{\rm H}$ is the (ideally infinite-volume) lattice estimator of this hadron mass, extracted from the exponential decay of some suitable 2-pt. correlation function

  and then compute the renormalised quark mass $m_{\rm R}(\mu)$

  $\rightarrow$ more convenient: scheme & scale indep. renormalisation group invariant (RGI) quark mass M

  [$\frac{M}{m_{\rm R}(\mu_{\rm had})}$ known for $N_{\rm f}=3$: ALPHA Collaboration, Campos et al., 1802.05243]

- E.g., from the PCAC quark mass via (recall $\partial_\mu A_\mu^a(x) = 2m_{\rm PCAC}P^a(x)$ relation):

$$m_{\rm h}^{\rm RGI} \;=\; \frac{M}{m_{\rm R}(\mu_{\rm had})}\,\frac{Z_{\rm A}}{Z_{\rm P}(\mu_{\rm had})}\,m_{\rm h}^{\rm PCAC}\left[1+(b_{\rm A}-b_{\rm P})\,am_{{\rm q},h}\right] \;+\; {\rm O}\left(a^2, a{\rm Tr}[M_{\rm q}]\right)$$

- Leading cutoff effects are ${\rm O}(a^2m_{\rm h}^2)$, where $a_{\rm max}m_{\rm c} \approx 0.4$

  $\Rightarrow$ not feasible for the bottom quark with $am_{\rm b} > 1$

Quark mass determinations: Alternative route

- Different route: Compute $m_{\mathrm{h}}^{\mathrm{RGI}}$ in **small volumes** on **fine lattices**

  $\rightarrow$ Choose $L_1 = 0.5\,\mathrm{fm}$ such that $0.15 \lesssim am_{\mathrm{b}} \lesssim 0.4$

Quark mass determinations: Alternative route

- Different route: Compute $m_{\rm h}^{\rm RGI}$ in **small volumes** on **fine lattices**

  $\rightarrow$ Choose $L_1 = 0.5\,\mathrm{fm}$ such that $0.15 \lesssim am_{\rm b} \lesssim 0.4$

- Tune the quark mass such that (where we are free to set $L_{\rm ref} \equiv L_2$)

$$\begin{aligned} L_{\rm ref} m_{\rm H}(L_1) &= L_{\rm ref} m_{\rm H}^{\rm PDG} - L_{\rm ref}\left[m_{\rm H} - m_{\rm H}(L_2)\right] - L_{\rm ref}\left[m_{\rm H}(L_2) - m_{\rm H}(L_1)\right] \\ &\equiv L_{\rm ref} m_{\rm H}^{\rm PDG} - \rho_m(L_2) - \sigma_m(L_1) \end{aligned}$$

  using the **step-scaling functions** $\rho_m(L_2)$, $\sigma_m(L_1)$, where $L_2 = 2L_1$

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  using the **step-scaling functions** $\rho_m(L_2)$, $\sigma_m(L_1)$, where $L_2 = 2L_1$

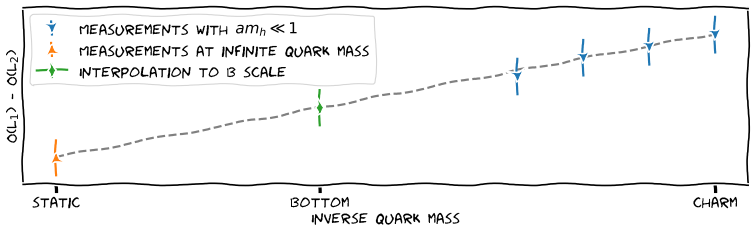
- In practice, we compute the quark mass via

$$L_{\rm ref} m_{\rm h}^{\rm RGI} = \left[L_{\rm ref} m_{\rm H}^{\rm PDG} - \rho_m(L_2) - \sigma_m(L_1)\right] \frac{m_{\rm h}^{\rm RGI}}{m_{\rm H}(L_1)}$$

  - Can simulate heavy (bottom) quarks with controlled cutoff effects in L_1
  - Suppressed cutoff effects in step-scaling functions $\rho_m(L_2)$ and $\sigma_m(L_1)$
    (welcome add. feature, as we want to take continuum limits everywhere!)

Combining relativistic heavy quarks with HQET

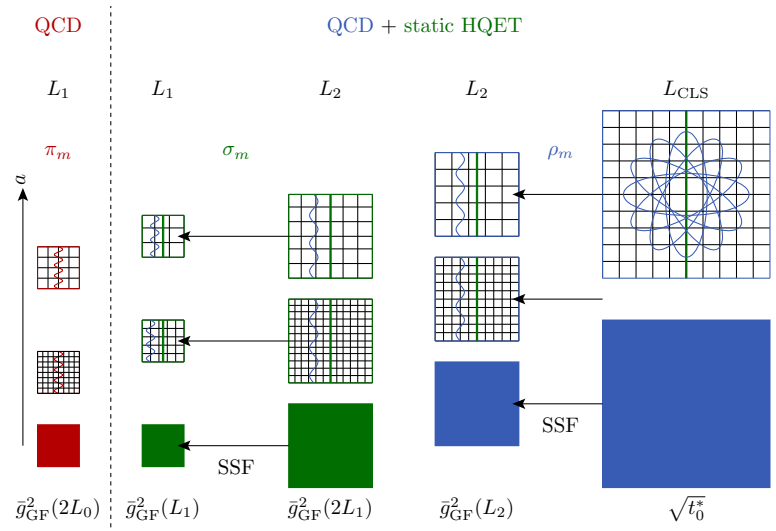
- We cannot evaluate the step-scaling functions $\rho_m(L_2)$ and $\sigma_m(L_1)$ directly at $m_\mathrm{h} = m_\mathrm{b}$, because larger volumes require coarser lattices
- But they are accessible in the **static limit of non-perturbative HQET** ➥ ρ_m^stat, σ_m^stat
  $\Rightarrow$ Additive renormalisation and matching with QCD cancels in the difference !
- Plan: Interpolate between static and relativistic heavy quarks (around & beyond charm)



[Guazzini et al., 0710.2229][ALPHA Collaboration, 2312.09811, 2312.10017] inspired by [de Divitiis et al., hep-lat/0305018]

Heavy quark masses from step-scaling

Lattice setup: Step-scaling



- Simulation of 32 ensembles in the massless Schrödinger functional (SF) $\Leftrightarrow$ LCPs [ALPHA Collaboration, 1811.02591]:
  - $L_2 = 2L_1 = 1.0\,\mathrm{fm}$
  - $L_1 = 2L_0 = 0.5\,\mathrm{fm}$
  - Renormalisation in L_0 where $\bar{g}^2_{\mathrm{GF}}(L_0) = 3.949$
- Correlation functions with heavy and static valence quarks
- Contact to the $2+1$ flavour CLS ensembles

Lattice setup: Large volume

- Determine the light-quark mass dependence on CLS ensembles,

$$\rho_m(L_2) \;=\; L_{\mathrm{ref}}\left[\, m_{\mathrm{H}} - m_{\mathrm{H}}(L_2)\,\right] \;=\; L_{\mathrm{ref}}\left[m_{\mathrm{H}} - m_{\mathrm{H}}^{\mathrm{sym}}\right] \;+\; L_{\mathrm{ref}}\left[m_{\mathrm{H}}^{\mathrm{sym}} - m_{\mathrm{H}}(L_2)\right]$$
$$\equiv\; \tau_m(m_\pi, m_K) \;+\; \rho_m^{\mathrm{sym}}(L_2)$$

  where $m_{\mathrm{H}}^{\mathrm{sym}}$ is the heavy hadron mass at the SU(3) flavor-symmetric point

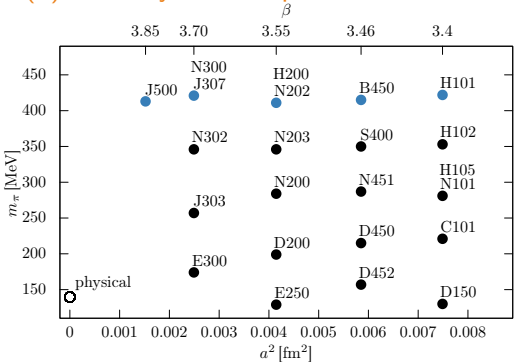
- Chiral trajectory with $2m_{\mathrm{l}} + m_{\mathrm{s}} = \mathrm{const.}$

  $\rightarrow$ tune the heavy quark mass via flavor averaged mesons:

$$m_H \;\in\; \left\{ \frac{2}{3}m_D + \frac{1}{3}m_{D_{\mathrm{s}}}\,,\; \frac{2}{3}m_B + \frac{1}{3}m_{B_{\mathrm{s}}} \right\}$$

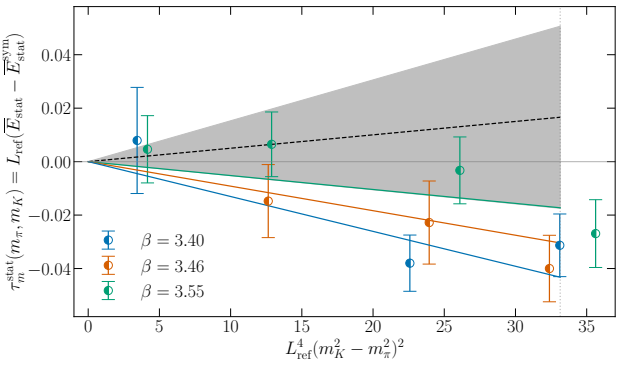
- Reach physical point via

$$\lim_{m_\pi \to m_\pi^{\mathrm{phys}}} \tau_m(m_\pi, m_K)$$



Light quark mass dependence in large volume

Determination of $\tau_m(m_\pi, m_K) = L_{\rm ref}\left[m_{\rm H}(m_\pi, m_K) - m_{\rm H}^{\rm sym}\right]$



- **Flavor averaged** meson masses:
  $$m_{\rm H} - m_{\rm H}^{\rm sym} \sim (m_{\rm s} - m_{\rm l})^2$$
- Chiral-continuum extrapolation in the static limit [ALPHA Collaboration, 2111.04693] including $m_\pi \lesssim m_\pi^{\rm phys}$ ...

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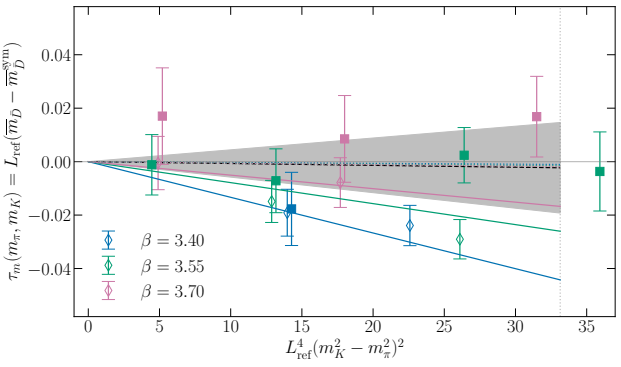

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- ... and at the charm scale using two valence actions
  - Wilson on CLS (*open*) [ALPHA Collaboration, 2101.02694]
  - Twisted-mass on CLS (*filled*) [ALPHA Collaboration, 2309.14154]

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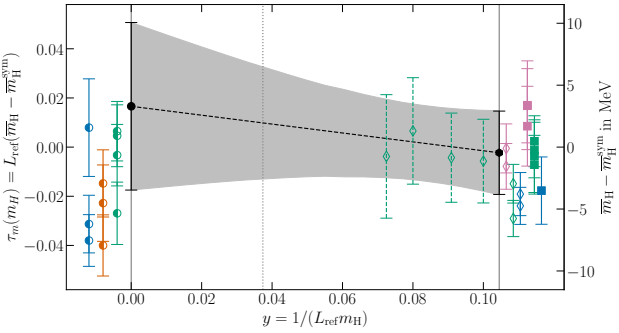

→ Heavy quark mass dependence at the physical point: $\tau_m < 10\,\mathrm{MeV}$

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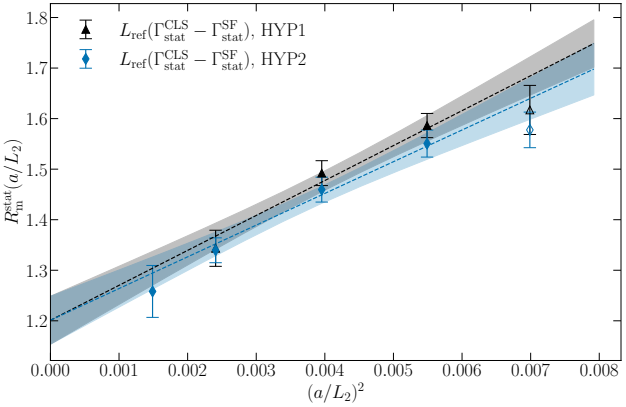
Step-scaling from CLS to 1 fm

$0.039\,\mathrm{fm} \leq a \leq 0.085\,\mathrm{fm}$

Determination of $\rho_m^{\mathrm{sym}}(L_2) = \lim_{a\to 0} R_{\mathrm{m}}(L_2, a/L_2) = L_{\mathrm{ref}}\left[m_{\mathrm{H}}^{\mathrm{sym}} - m_{\mathrm{H}}(L_2)\right]$



- Combined continuum extrapolation in the static limit using two static actions

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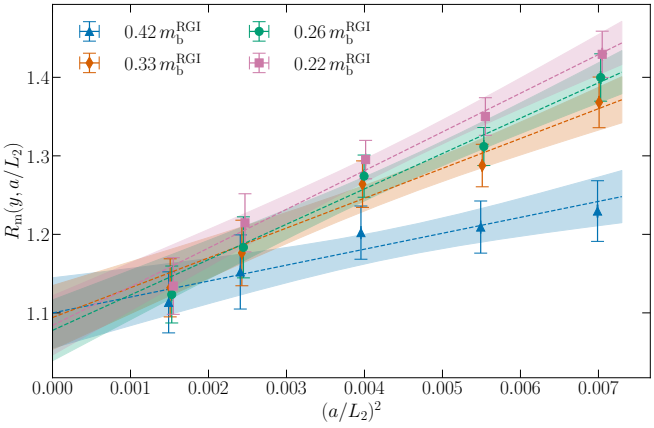

- Combined continuum extrapolation in the static limit using two static actions
- Four relativistic heavy quark masses $m_{\mathrm{h}} \geq m_{\mathrm{c}}$

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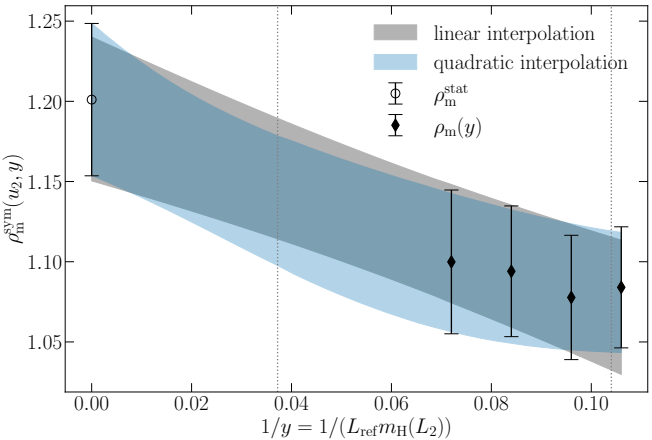
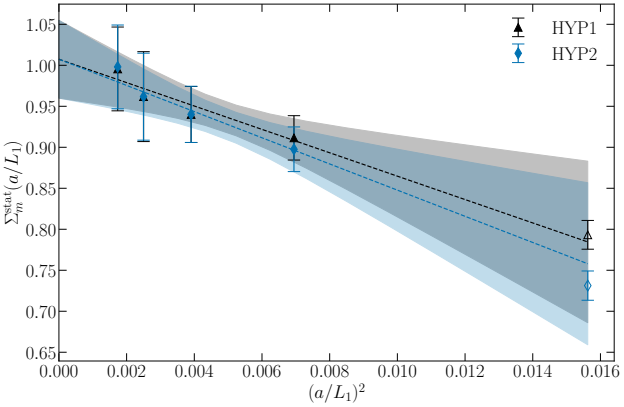


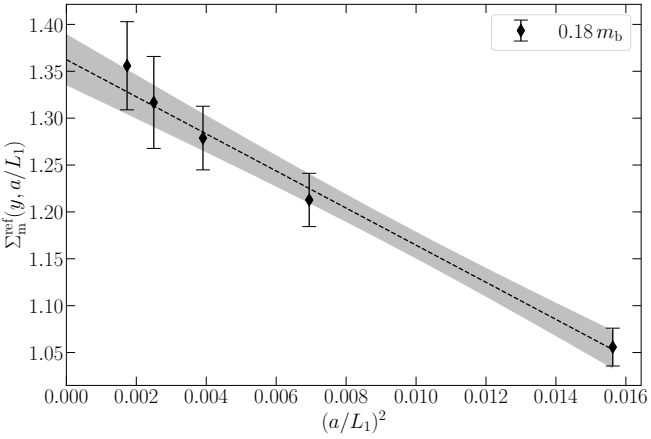
- Combined continuum extrapolation in the static limit using two static actions
- Four relativistic heavy quark masses $m_{\mathrm{h}} \geq m_{\mathrm{c}}$
- Interpolation to the bottom and charm (mass) scales

Determination of $\sigma_m(L_1) = \lim_{a\to 0} \Sigma_{\mathrm{m}}(L_1, a/L_1) = L_{\mathrm{ref}}\,[\,m_{\mathrm{H}}(L_2) - m_{\mathrm{H}}(L_1)\,]$



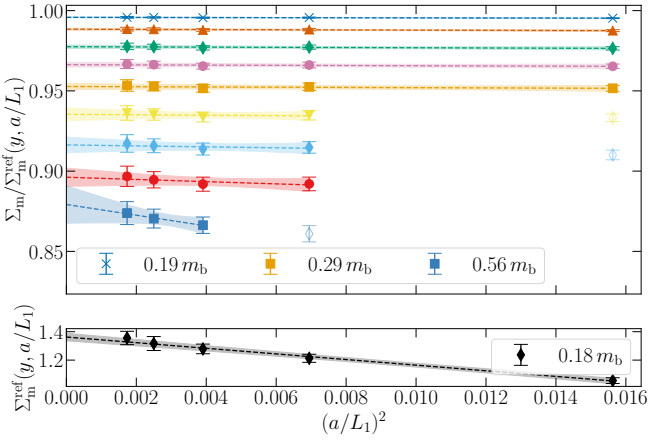
- Combined continuum extrapolation with two actions for the static quark

Determination of $\sigma_m(L_1) = \lim_{a\to 0} \Sigma_\mathrm{m}(L_1, a/L_1) = L_\mathrm{ref}\,[\,m_\mathrm{H}(L_2) - m_\mathrm{H}(L_1)\,]$



- Combined continuum extrapolation with two actions for the static quark
- For relativistic heavy quark masses:
  - Extrapolate $\Sigma_\mathrm{m}^\mathrm{ref}$ directly below the charm scale

Determination of $\sigma_m(L_1) = \lim_{a\to 0} \Sigma_\mathrm{m}(L_1, a/L_1) = L_\mathrm{ref}\,[\,m_\mathrm{H}(L_2) - m_\mathrm{H}(L_1)\,]$



- Combined continuum extrapolation with two actions for the static quark
- For relativistic heavy quark masses:
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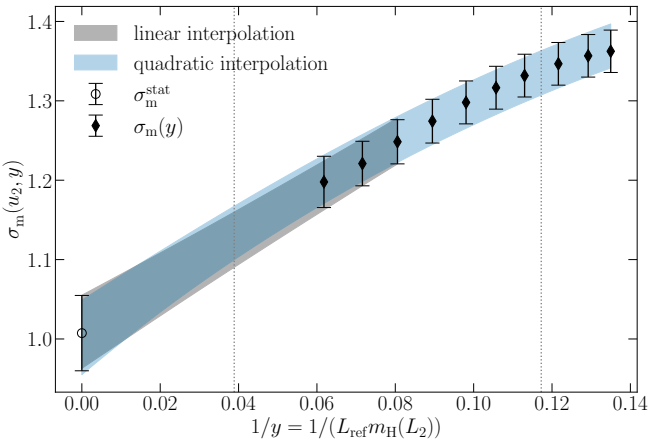
  - For larger masses: Extrapolate ratio of heavy step-scaling function to $\Sigma_\mathrm{m}^\mathrm{ref}$

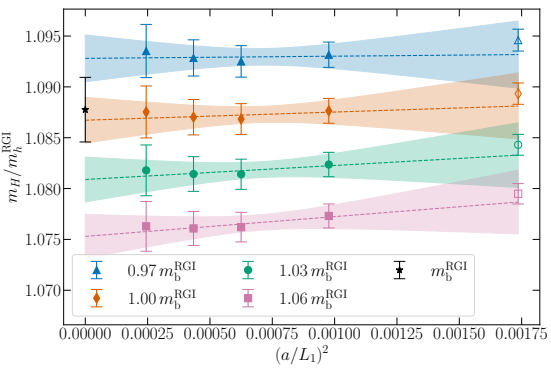
Determination of $\sigma_m(L_1) = \lim_{a\to 0} \Sigma_{\mathrm{m}}(L_1, a/L_1) = L_{\mathrm{ref}}\,[\,m_{\mathrm{H}}(L_2) - m_{\mathrm{H}}(L_1)\,]$

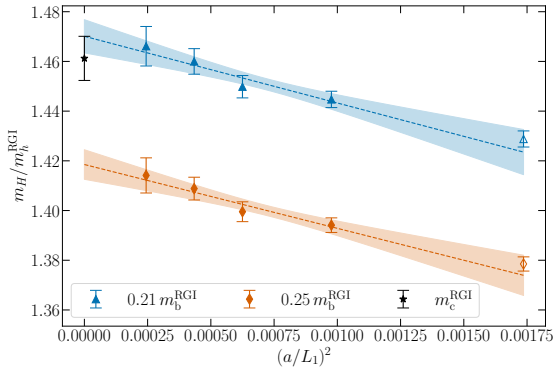


- Combined continuum extrapolation with two actions for the static quark
- For relativistic heavy quark masses:
  - Extrapolate $\Sigma_{\mathrm{m}}^{\mathrm{ref}}$ directly below the charm scale
  - For larger masses: Extrapolate ratio of heavy step-scaling function to $\Sigma_{\mathrm{m}}^{\mathrm{ref}}$
- Interpolate to bottom & charm scales and check for systematics

Quark mass determination in small volume

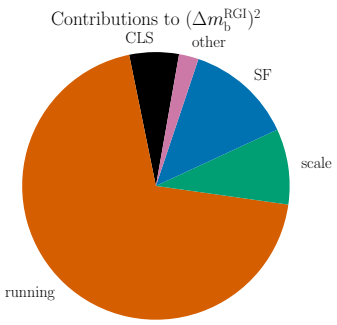
$0.0078\,\mathrm{fm} \leq a \leq 0.021\,\mathrm{fm}$

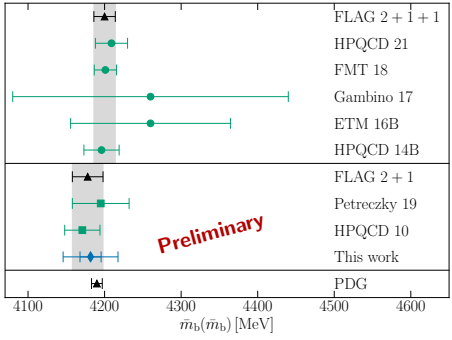




- Renormalise the quark mass at $\mu = 1/L_0 \approx 790\,\mathrm{MeV}$
- Continuum extrapolation of $\frac{m_{\mathrm{H}}(L_1)}{m_{\mathrm{h}}^{\mathrm{RGI}}}$ at the bottom (*left*) and charm (*right*) scales
- Exploration of systematics:
  - Cuts in (a/L) and $am_{\mathrm{h}}^{\mathrm{RGI}}$
  - Explore also fits with $\alpha_s(1/a)^{\Gamma} a^2$ and a^3 effects

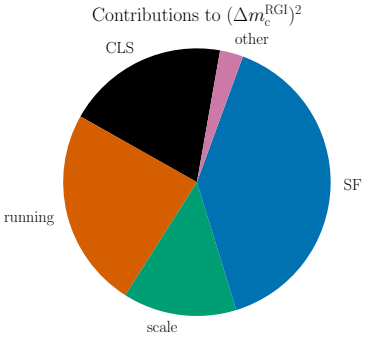
Results: Bottom quark mass

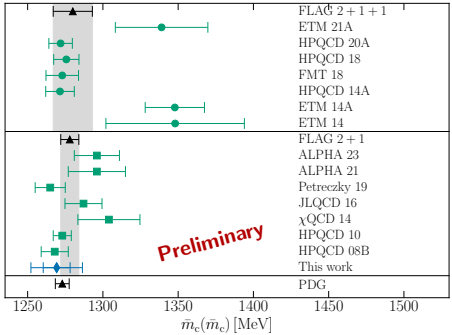




- Uncertainty of $m_{\rm b}^{\rm RGI}$ limited by $\frac{M}{m_R(1/L_0)}$ [ALPHA Collaboration, 1802.05243]: will be improved
- Systematics (continuum extrapolations, quark mass interpolations) subleading: Renormalisation is non-perturbative, uncertainties are **statistics dominated**
- Entirely different method compared to existing work / other approaches

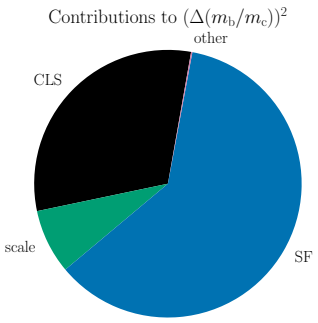
Results: Charm quark mass

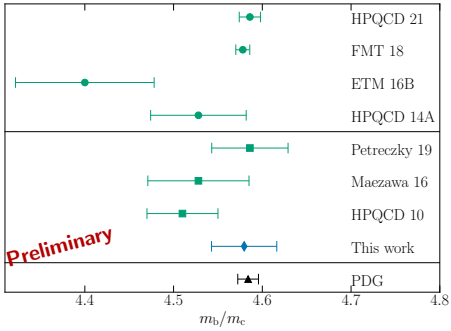




- Step-scaling not as efficient for the charm quark mass:
  Small-volume SF part contributes significantly to the overall uncertainty
- Cross-check of the step-scaling chain (but also reliable for its own)
- Result is only weakly correlated with existing CLS-based calculations (by ALPHA Collaboration, RQCD)

Results: Quark mass ratio





- Contribution from the RG running cancels in ratio $m_b^{\mathrm{RGI}}/m_c^{\mathrm{RGI}} = \overline{m}_b(\mu)/\overline{m}_c(\mu)$
- Charm and bottom quark masses are correlated, but residual uncorrelated uncertainties lead to a non-negligible total uncertainty in their ratio

Outlook

[on the 2nd part]

Step-scaling enables a precise and systematically controlled determination of the bottom (and charm) quark mass

- Last bits of data for $\rho_m^{\mathrm{sym}}(L_2)$ are being computed — results will soon be final
- Next step: Form-factors of semi-leptonic B-meson decays and $f_{B_{(s)}}$:
  - Highly relevant for phenomenology
  - Strategy is clear, step-scaling functions reliably computed in QCD and (static) HQET [ALPHA Collaboration, 2312.09811, 2312.10017]
  - Computation of light quark mass and momentum dependence ongoing
  - Ground-state form factors are challenging, because when extracting relevant matrix elements in large volume, one has to worry about contamination from excited states (especially: multi-hadron states like $B^{(*)}\pi$, dominating at large distances)
    $\rightarrow$ accessible/subtractable via HMχPT, in progress by [A. Gérardin, S. Kuberski et al.]

Backup: Towards semi-leptonics and status on f_{B^*} resp. $f_{\mathrm{B}^*}/f_{\mathrm{B}}$

Another (showcase) application: Semi-leptonic form factors

- Phenomenologically relevant: the q^2 dependence of semi-leptonic form factors.
- Form factor decomposition in the B-meson rest frame

$$(\sqrt{2}p_k^\pi)^{-1}\langle\pi(p^\pi)|V_k(0)|B(\vec{p}=0)\rangle = h_\perp(E_\pi) = h_\perp^{\text{stat}}(E_\pi) + O(1/m_\text{b})$$

- Cancel matching and renormalization for $h_\perp^{\text{stat}}$,

$$\frac{h_\perp(E_\pi)}{h_\perp(E_\pi^{\text{ref}})} = \frac{h_\perp^{\text{stat}}(E_\pi)}{h_\perp^{\text{stat}}(E_\pi^{\text{ref}})} + O(1/m_\text{b})\,.$$

- Connection with $f_{B^\star}$: Normalize to the vector decay constant

$$h_\perp(E_\pi^{\text{ref}}) = \hat{f}_\text{V}\frac{h_\perp(E_\pi^{\text{ref}})}{\hat{f}_\text{V}} = \hat{f}_\text{V}\left[\frac{h_\perp^{\text{stat}}(E_\pi^{\text{ref}})}{\hat{f}_\text{V}^{\text{stat}}} + \text{O}(1/m_\text{b})\right]$$

$\rightarrow$ Problem solved for $h_\perp$: Ratio can be computed in the static theory.
How to compute $\hat{f}_\text{V}$?

Again, the key is ... step-scaling

Step-scaling: Use the finite-volume as a tool.

- Cancel matching and renormalization via ratios of observables $O(L_2)/O(L_1)$ or differences of logs computed in two volumes :

$$\sigma_{\mathrm{V}} = \left[\log[L_{\mathrm{ref}}^{3/2}\hat{f}_{\mathrm{V}}(L_2)] - \log[L_{\mathrm{ref}}^{3/2}\hat{f}_{\mathrm{V}}(L_1)]\right]$$

- Same ansatz to cancel the additive divergence in the static energy

$$\sigma_m = L_{\mathrm{ref}}\left[m_{\mathrm{PS}}(L_2) - m_{\mathrm{PS}}(L_1)\right]$$

- Connect large-volume (CLS) ensembles with small volumes in two steps:

$$L_\infty \to L_2 = 1\,\mathrm{fm} \qquad \text{and} \qquad L_2 = 1\,\mathrm{fm} \to L_1 = 0.5\,\mathrm{fm}$$

- Small volume $L_1 = 0.5\,\mathrm{fm}$: Simulate **relativistic b quarks** with $am_{\mathrm{b}} \ll 1$

The vector meson decay constant from step-scaling

- Vector meson decay constant $\hat{f}_V = f_V\sqrt{m_V}$,

$$\hat{f}_V = \sqrt{2}\langle 0|V_k(0)|V(\vec{p}=0,k)\rangle_{\mathrm{NR}} = \hat{f}_V^{\mathrm{stat}} + O(1/m_b)\,,$$

- For the step-scaling, we define

$$\Phi_{\vec{V}}(L) \equiv \ln\left(\frac{L_{\mathrm{ref}}^{3/2}\hat{f}_V(L)}{2}\right)$$

- Compute the large-volume (physical) quantity via

$$\Phi_{\vec{V}} = \Phi_{\vec{V}}(L_1) + [\Phi_{\vec{V}}(L_2) - \Phi_{\vec{V}}(L_1)] + [\Phi_{\vec{V}} - \Phi_{\vec{V}}(L_2)]$$

- Each observable is continuum extrapolated.

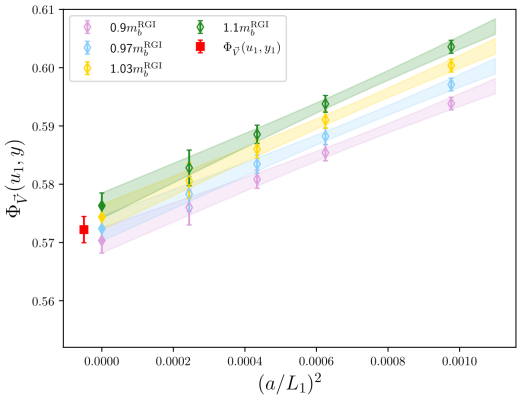
Recap of the strategy

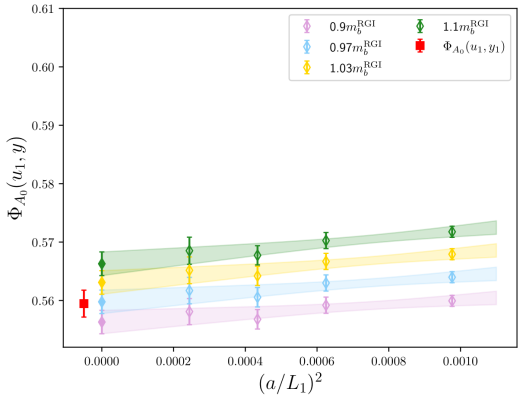
Interpolate observables to the B-scale:

- Interpolate relativistic measurements around m_{b}:
  - In small volumes: Observables such as $\hat{f}_{\mathrm{V}}$, m_B/m_{b}.
- Interpolate between the static limit and $m_h \ll m_{\mathrm{b}}$:
  - Step-scaling functions such as σ_{V}, σ_m.
  - In large volume: Ratios of observables like $h_\perp(E_\pi)/h_\perp(E_\pi^{\mathrm{ref}})$ or $h_\perp(E_\pi^{\mathrm{ref}})/\hat{f}_{\mathrm{V}}$.
- Interpolations in $1/m_h$ are performed in the continuum limit:
  - Continuum extrapolations at the B-scale only for $a^{-1} \in [9.5,\ 25]\,\mathrm{GeV}^{-1}$
  - Cutoff effects partially cancel in differences.

$$\Phi_{\vec{V}} = \Phi_{\vec{V}}(L_1) + [\Phi_{\vec{V}}(L_2) - \Phi_{\vec{V}}(L_1)] + [\Phi_{\vec{V}} - \Phi_{\vec{V}}(L_2)]$$

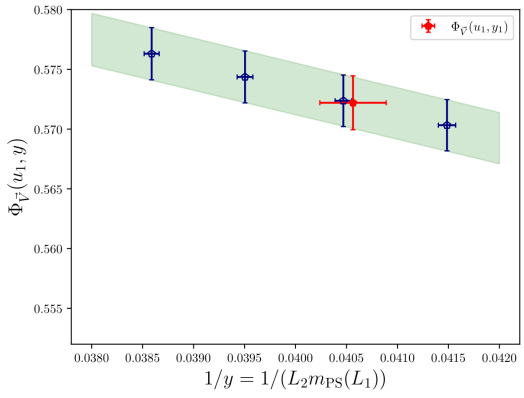
Continuum extrapolation at the bottom scale

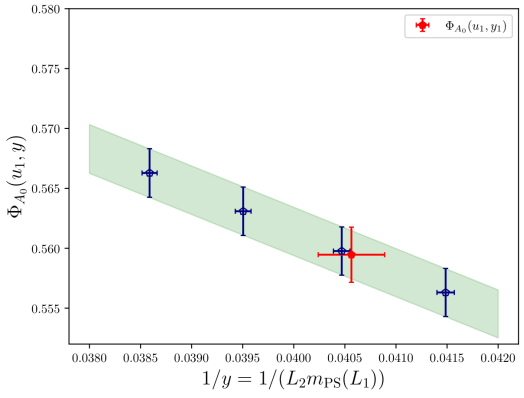




- B-physics on fine lattices in small volumes.
- Continuum extrapolations for vector (left) and axial (right) decay constants.
- Four heavy valence quark masses encompass the bottom quark mass.

Interpolation to the bottom scale

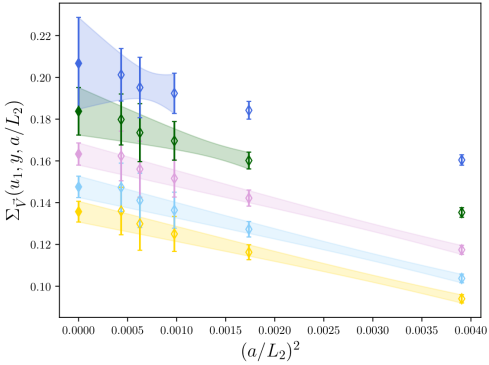


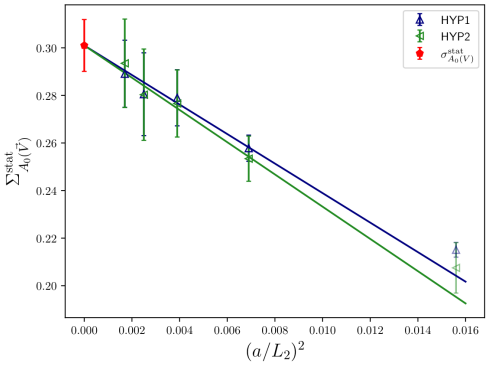


- B-physics on fine lattices in small volumes.
- Interpolate in inverse heavy-light meson mass $1/y = 1/(L_{\mathrm{ref}} m_{\mathrm{PS}}(L_1)) \propto 1/m_h$.
- Straight-forward interpolation to $m_h = m_{\mathrm{b}}$.

Step scaling from L_1 to L_2: Continuum limit

- Continuum extrapolation of relativistic and static step-scaling functions for $\Phi_{\vec{V}}$.
- $L = 0.5\,\mathrm{fm}$ to $L = 1\,\mathrm{fm}$. Only include $am_h^{\mathrm{RGI}} < 0.8$.



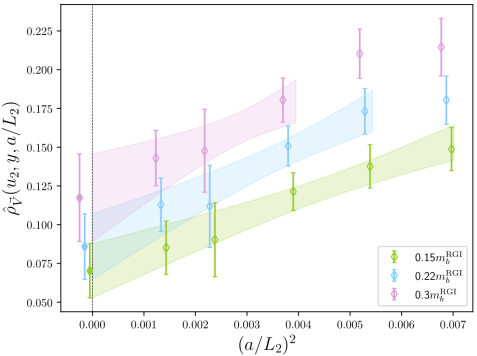


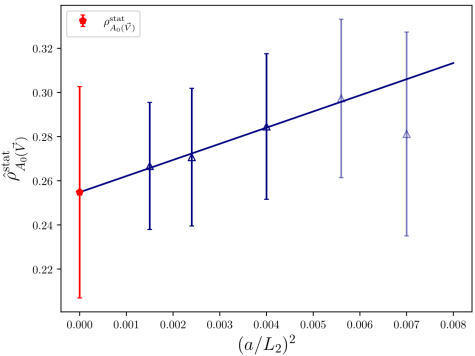
relativistic: valence masses $m_h^{\mathrm{RGI}} < 0.5\, m_b^{\mathrm{RGI}}$

static: two heavy quark actions

$L_2 \to L_\infty \equiv L_{\rm CLS}$: Continuum limit

- Continuum extrapolation of relativistic and static step-scaling functions for $\Phi_{\vec{V}}$.
- $L = 1\,\mathrm{fm}$ to $L_{\rm CLS}$. Only include $am_h^{\rm RGI} < 0.8$.



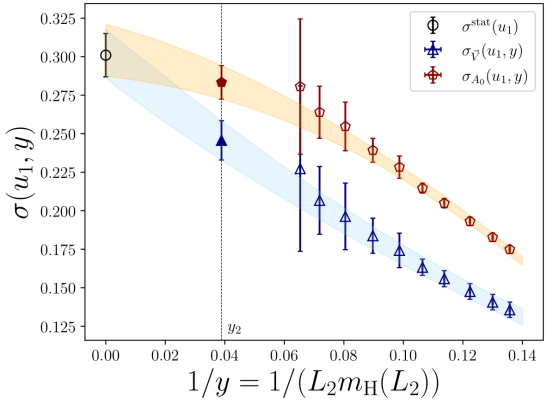


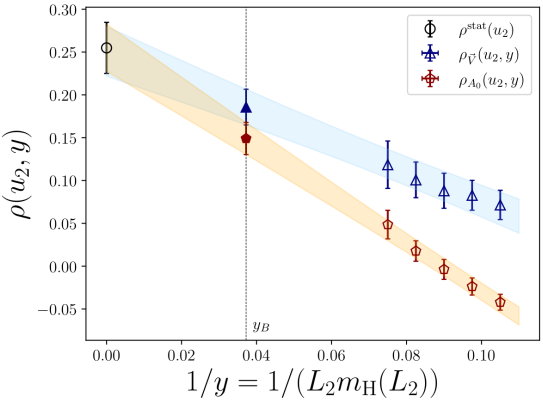
relativistic: valence masses $m_h^{\rm RGI} < 0.3\, m_b^{\rm RGI}$

static

Interpolations for decay constants

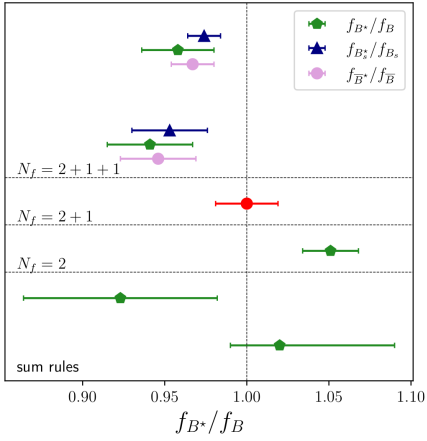
- Interpolation to $1/m_B$: highly constrained by the static result.
- Step-scaling functions of pseudoscalar Φ_{A_0} and vector $\Phi_{\vec{V}}$ decay constant have the same static limit (heavy quark symmetry).





Result for $f_{\mathrm{B}^*}/f_{\mathrm{B}}$ (preliminary)

- Combine all pieces to arrive at the final result.
- N.b.: We (currently) work at the SU(3) symmetric point.
  Expect light quark dependence in the **ratio** $f_{B^\star}/f_B$ to be small.



ETMC 1707.04529

HPQCD 1503.05762

This work (preliminary)

twisted mass 1407.1019

1410.6684

1305.5432

- Puzzling situation for the ratios $f_{B_{(\mathrm{s})}}/f_{B^\star_{(\mathrm{s})}}$.
- Systematically improvable result with competitive uncertainties.

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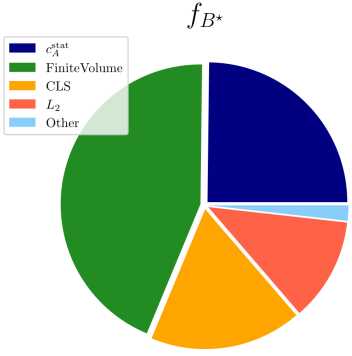

- Puzzling situation for the ratios $f_{B_{(\mathrm{s})}}/f_{B^\star_{(\mathrm{s})}}$.
- Systematically improvable result with competitive uncertainties.
- Decay constants currently at about 2.5% precision, dominated by finite-volume statistical uncertainties.