Predicting heavy-quark exotics using a nonperturbatively tuned RHQ action

Daniel Mohler

Technische Universität Darmstadt

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Outline

1. Introduction and Motivation
2. Positive-parity B_S states from Lattice NRQCD
3. Positive-parity B_S states from the RHQ action
   - Benchmark quantities
   - Predicting exotic B^*_{s0} and B_{s1} states
4. Conclusions and Outlook

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What to call an exotic state in QCD ?

- Textbook: Quark-antiquark mesons and 3-quark baryons

- Historically, multiquark states and hybrids (made of quark and gluons) already suggested by Gell-Mann in addition.
- We are now seeing some explicitly *exotic* states – in particular with heavy quarks
- Various possible structures: regular mesons/baryons; molecules; tetraquarks/pentaquarks; hybrid hadrons; glueballs; dibaryons.

- For the purpose of this talk:
  I will also consider states with quantum numbers allowed by quark-antiquark states but unexpected properties as exotic.
  Example: B^*_{s0} and B_{s1} mesons.

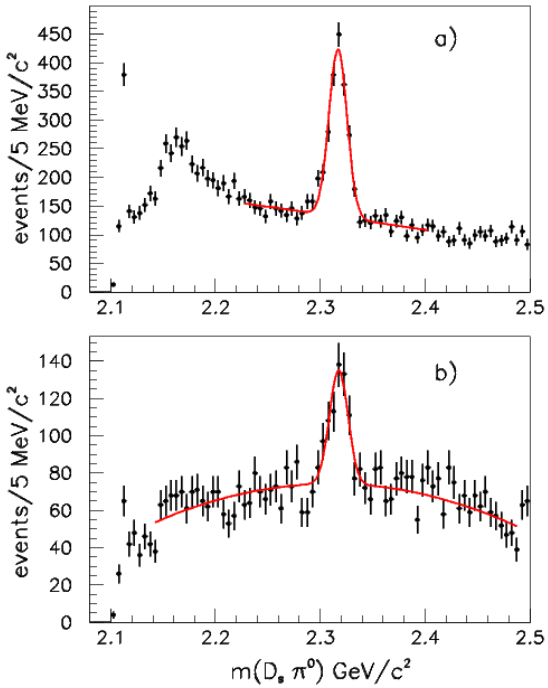
Exotic D_s and B_s candidates

Established s and p-wave hadrons:

D_s $(J^P = 0^-)$ and D_s^* (1^-)
$D_{s0}^*(2317)$ (0^+), $D_{s1}(2460)$ (1^+),
$D_{s1}(2536)$ (1^+), $D_{s2}^*(2573)$ (2^+)

B_s $(J^P = 0^-)$ and B_s^* (1^-)
?
$B_{s1}(5830)$ (1^+), $B_{s2}^*(5840)$ (2^+)

$D_{s0}^*(2317)$:
PRL 90 242001 (2003)



- Corresponding $D_0^*(2400)$ and $D_1(2430)$ are broad resonances
- Perceived peculiarity: $M_{c\bar{s}} \approx M_{c\bar{d}}$ (an old dispute; likely not the case)
- Additional exotic states are expected (in the sextet representation)

  See for example Kolomeitsev, Lutz, PLB 582, 39 (2004)

- B_s cousins of the $D_{s0}^*(2317)$ and $D_{s1}(2460)$ not (yet) seen in experiment

- Talk based on the following publications
  - Fully non-perturbative charm-quark tuning using machine learning

    R.J. Hudspith, DM, PRD 106, 034508 (2022)
  - Exotic tetraquark states with two b-quarks and $J^P = 0^+$ and 1^+ B_s states in a nonperturbatively tuned lattice NRQCD setup

    R.J. Hudspith, DM, PRD 107, 114510 (2023)
  - B-meson masses from a non-perturbatively tuned heavy-quark action

    Master Thesis by Manuel Stolz, TU Darmstadt (2025)
- The new results are still work in progress:
  Like in any Master's thesis, not all choices are perfect.
  $\rightarrow$ Stay tuned for the arXiv version of the publication.
- Goal: Decent control of systematic uncertainties without aiming for precision physics (yet?).

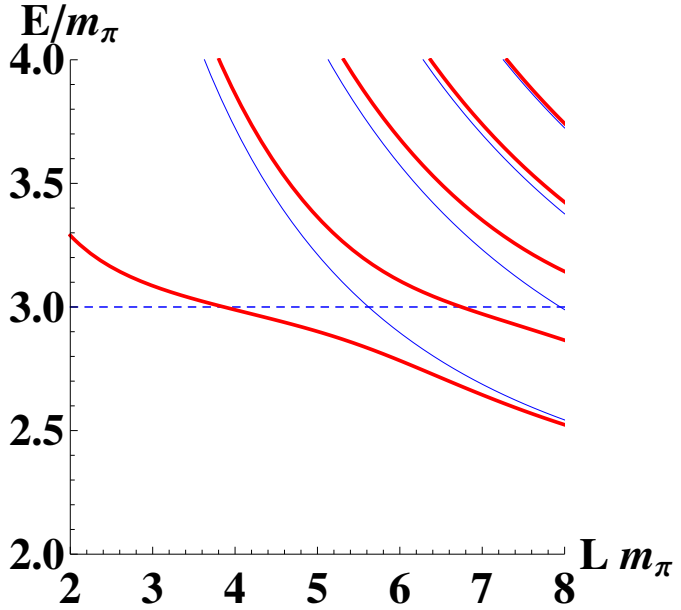
Progress from an old idea: Lüscher's finite-volume method

M. Lüscher Commun. Math. Phys. 105 (1986) 153;
Nucl. Phys. B 354 (1991) 531; Nucl. Phys. B 364 (1991) 237.

Basic observation: Finite-volume, multi-particle energies are shifted with regard to the free energy levels due to the interaction

$$E = E(p_1) + E(p_2) + \Delta_E$$

- Energy shifts encode scattering amplitude(s)
- Original method: Elastic scattering in the rest-frame in multiple spatial volumes L^3
- Coupled 2-hadron channels well understood
- $2 \leftrightarrow 1$ and $2 \leftrightarrow 2$ transitions well understood (example $\pi\pi \to \pi\gamma^*$)
- Significant progress for 3-particle scattering



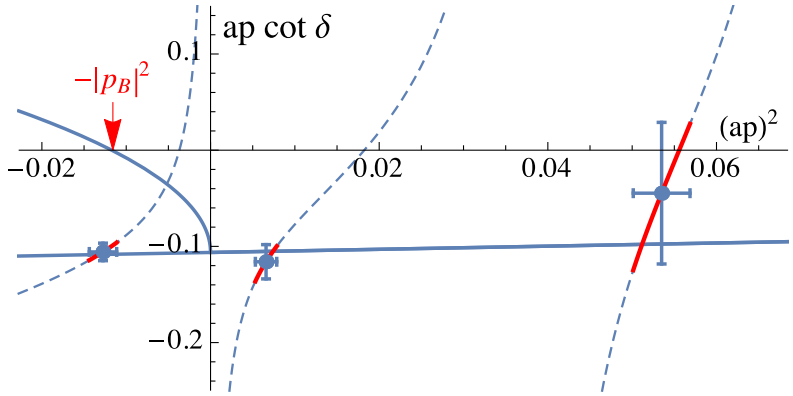
Exotic hadrons from Lattice QCD – Challenges

- Hierarchy of difficulties
  - Meson systems are simpler than baryons
    (exponentially degrading signal to noise)
  - For deeply bound states Lüscher/scattering studies not strictly necessary
  - Cost of correlation functions much larger for systems with baryons
  - Complicated scattering amplitudes need more data (volumes, frames)
    single two-hadron channel; coupled two-hadron channels; three-hadrons
- Hierarchy of projects:
  - Proof of principle
  - Explore quark mass dependence
  - Full spectroscopy calculation including continuum limit
  - Structure observables (transitions, form factors, . . .) (see Sasa's talk)
- Two examples:
  - Low-lying positive-parity B_S mesons
    Most systematics can be addressed!
  - Doubly-heavy tetraquark states

B^*_{s0} and B_{s1}: Some ancient results

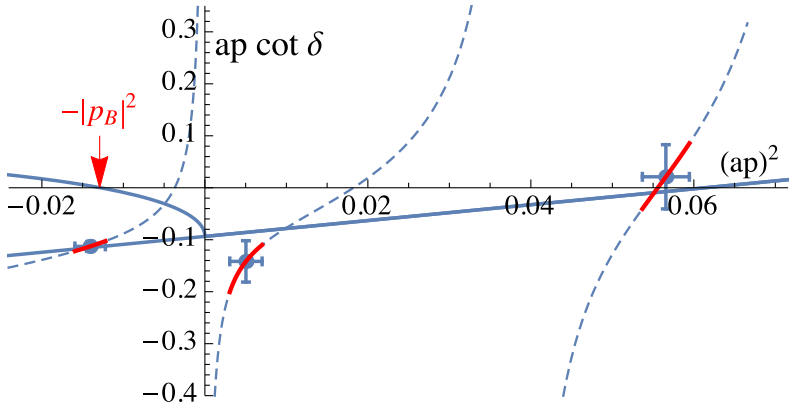
Lang, DM, Prelovsek, Woloshyn PLB 750 17 (2015)

B^*_{s0}



$$a_0^{BK} = -0.85(10)\,\text{fm}$$
$$r_0^{BK} = 0.03(15)\,\text{fm}$$
$$M_{B^*_{s0}} = 5.711(13)\,\text{GeV}$$

B_{s1}



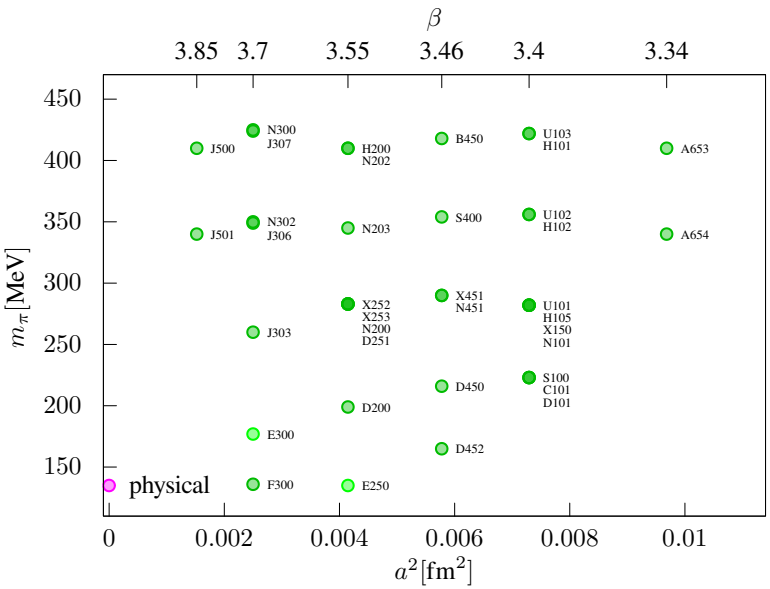
$$a_0^{B^*K} = -0.97(16)\,\text{fm}$$
$$r_0^{B^*K} = 0.28(15)\,\text{fm}$$
$$M_{B_{s1}} = 5.750(17)\,\text{GeV}$$

- Energy from the difference to the $B^{(*)}K$ threshold

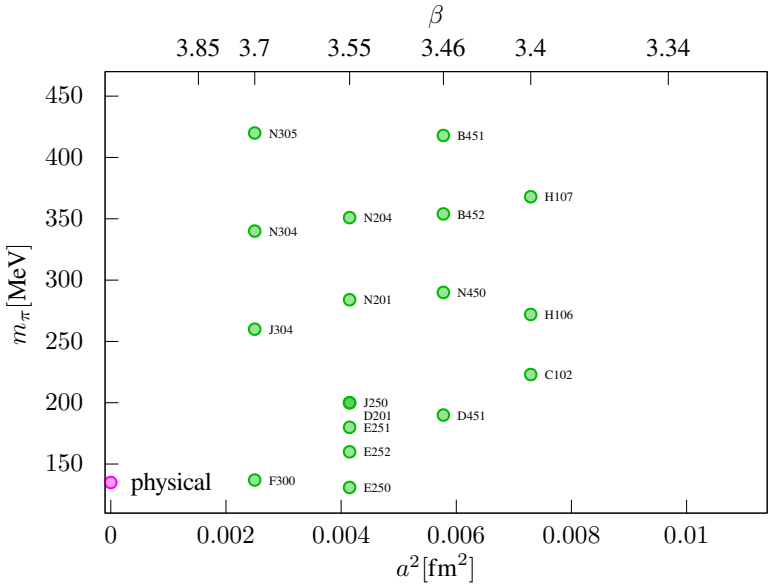
CLS gauge field ensembles (2+1 flavor, no QED, . . .)

Bruno *et al.* JHEP 1502 043 (2015); Bali *et al.* PRD 94 074501 (2016)

$Tr(M) = \text{const.}$



$m_s = \text{const.}$



plot style by Jakob Simeth, RQCD

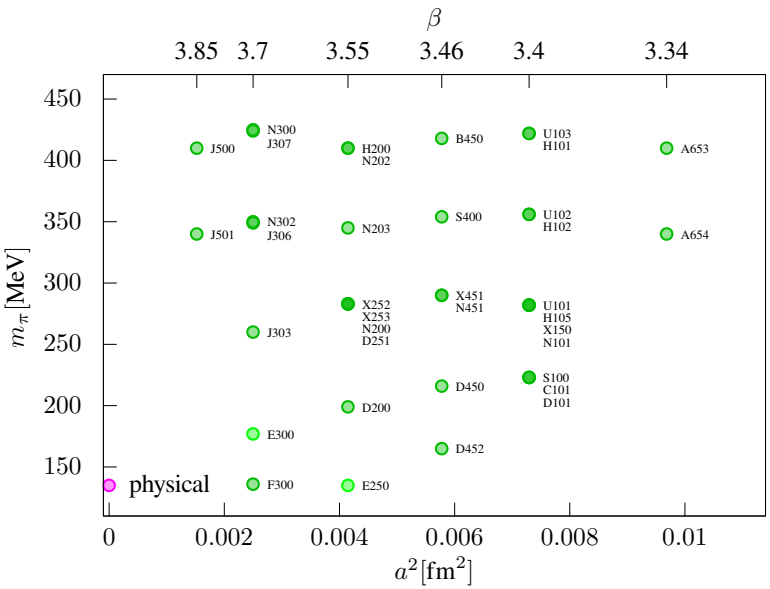
Important lattice systematics from

- Taking the *continuum limit*: $a(g, m) \to 0$
- Taking the *infinite volume limit*: $L \to \infty$
- Calculation at (or extrapolation to) physical quark masses

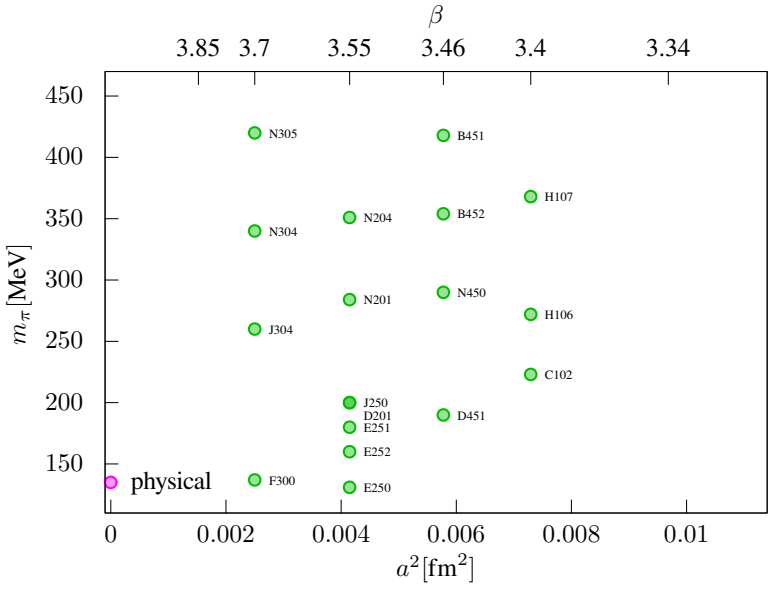
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plot style by Jakob Simeth, RQCD

Important lattice systematics from

- Taking the *continuum limit*: $a(g, m) \to 0$
- Want to exploit (power law) finite volume effects (keeping exponential effects small)
- Calculation at (or extrapolation to) physical quark masses

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NRQCD action

Typical tadpole-improved NRQCD action (here we will use n=4)

Lepage *et al.*, PRD 46, 4052-4067 (1992)

$$H_0 = -\frac{1}{2aM_0}\Delta^2,$$

$$H_I = \left(-c_1\frac{1}{8(aM_0)^2} - c_6\frac{1}{16n(aM_0)^2}\right)(\Delta^2)^2 + c_2\frac{i}{8(aM_0)^2}(\tilde{\Delta}\cdot\tilde{E} - \tilde{E}\cdot\tilde{\Delta}) + c_5\frac{\Delta^4}{24(aM_0)}$$

$$H_D = -c_3\frac{1}{8(aM_0)^2}\sigma\cdot\left(\tilde{\Delta}\times\tilde{E} - \tilde{E}\times\tilde{\Delta}\right) - c_4\frac{1}{8(aM_0)}\sigma\cdot\tilde{B}$$

$$\delta H = H_I + H_D.$$

Propagators generated through symmetric evolution equation

$$G(x,t+1) = \left(1-\frac{\delta H}{2}\right)\left(1-\frac{H_0}{2n}\right)^n \tilde{U}_t(x,t_0)^\dagger \left(1-\frac{H_0}{2n}\right)^n \left(1-\frac{\delta H}{2}\right) G(x,t).$$

- We also tune a $\mathcal{O}(v^6)$ action with tree-level coefficients for the higher order terms

Relativistic heavy quark (RHQ) action

R.J. Hudspith, DM, PRD 106, 034508 (2022)

We opted for a specific heavy-quark action with 5 parameters ($r_t = 1$)

S. Aoki, Y. Kuramashi, S. Tominaga, Prog.Theor.Phys. 109 383 (2003)

Type of action is extensively discussed in the literature, see for example

El-Khadra, Kronfeld, Mackenzie, PRD 55 3933 (1997)
Oktay, Kronfeld, PRD 78 014504 (2008)

Christ, Li, Lin, PRD 76 074505 (2007)

$$
\begin{aligned}
D_{xy} = \delta_{xy} &- \kappa_c \Big[\sum_i (r_s - \nu\gamma_i) U_i(x) \delta_{x+i,y} + (r_s + \nu\gamma_i) U_i^\dagger(x) \delta_{x,y+i} \Big] \\
&- \kappa_c \Big[(r_t - \gamma_t) U_t(x) \delta_{x+t,y} + (r_t + \gamma_t) U_t^\dagger(x) \delta_{x,y+t} \Big] \\
&- \kappa_c \Big[c_B \sum_{i,j} \sigma_{ij} F_{ij}(x) + c_E \sum_i \sigma_{it} F_{it}(x) \Big] \delta_{xy}.
\end{aligned}
\tag{1}
$$

Neural net NRQCD tuning and setup

R.J. Hudspith, DM, PRD 106, 034508 (2022)
R.J. Hudspith, DM, PRD 107, 114510 (2023)

- Calculate runs with a random distribution for the action parameters.
- Let the neural network make parameter predictions.
- Due to additive mass, we must only consider splittings $\rightarrow$ we subtract M_{η_B} from all states.
- Perform tuning at $\mathrm{SU}(3)_f$-symmetric point.
- Gauge-fixed wall sources.
- Tuning precision is about 1%.

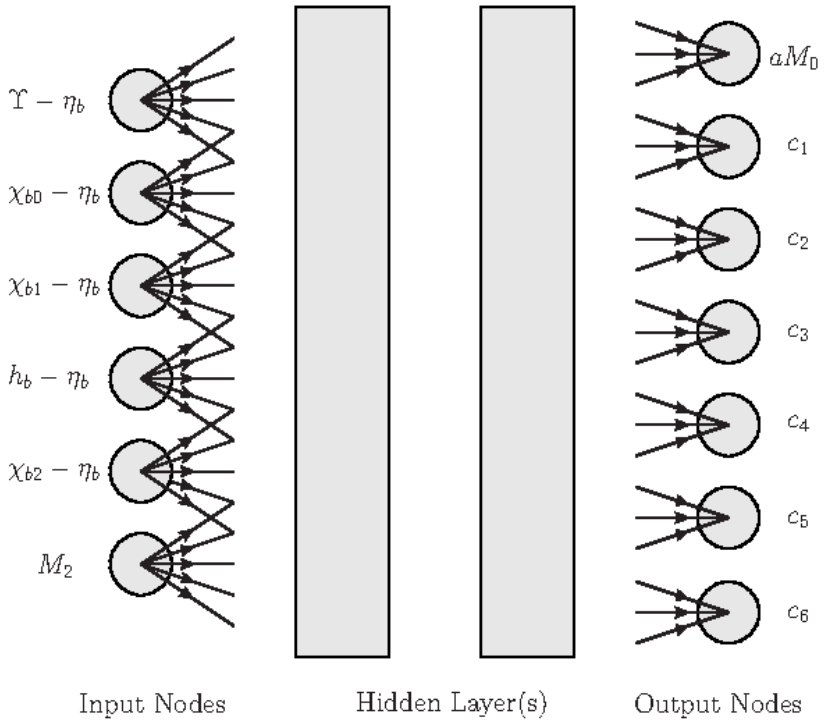


Figure: Schematic picture of our NRQCD setup

Input used for the tuning

Consider only quark-line connected parts of simple meson operators

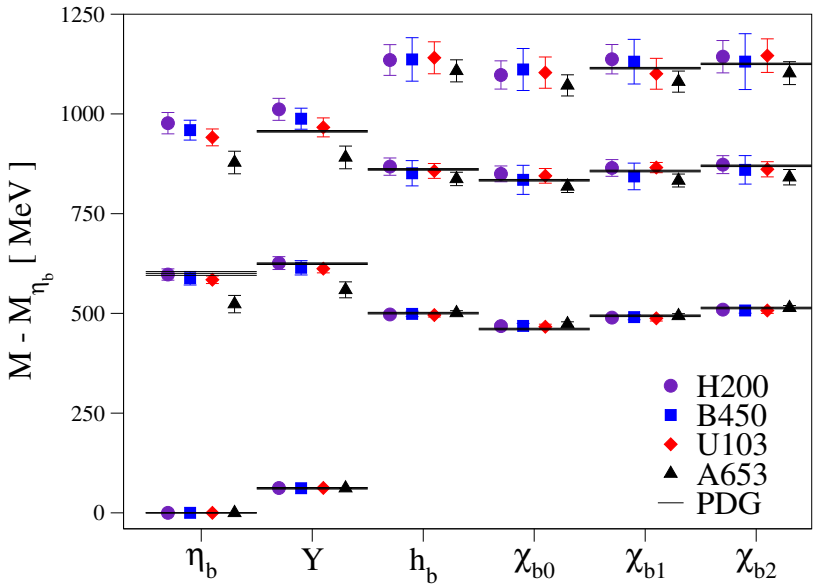
$$O(x) = (\bar{b}\Gamma(x)b)(x),$$

State	PDG mass [GeV]	$\Gamma(x)$
$\eta_b(1S)$	9.3987(20)	γ_5
$\Upsilon(1S)$	9.4603(3)	γ_i
$\chi_{b0}(1P)$	9.8594(5)	$\sigma \cdot \Delta$
$\chi_{b1}(1P)$	9.8928(4)	$\sigma_j\Delta_i - \sigma_i\Delta_j \ (i \neq j)$
$\chi_{b2}(1P)$	9.9122(4)	$\sigma_j\Delta_i + \sigma_i\Delta_j \ (i \neq j)$
$h_b(1P)$	9.8993(8)	Δ_i

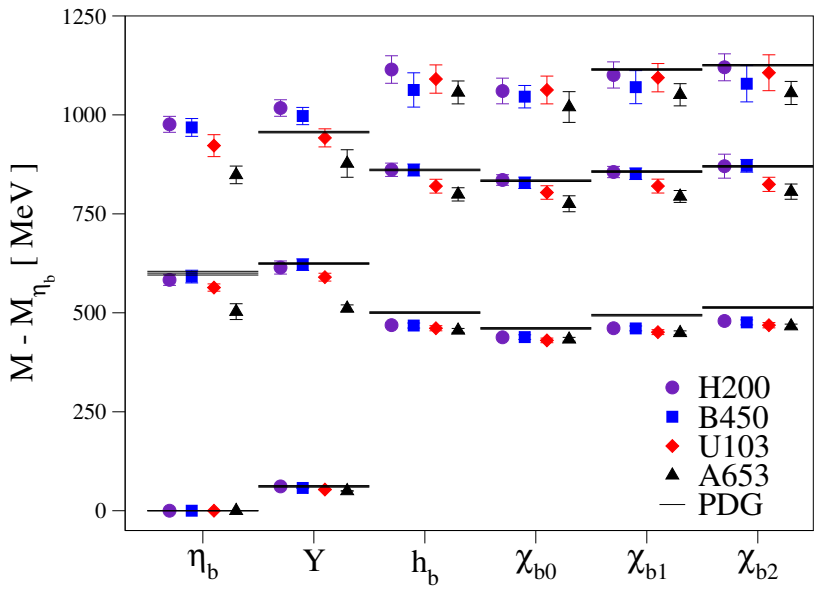
Table: Table of lattice operators used and their continuum analogs.

NRQCD Neural Net Tuning: Stable s- and p-wave bottomonia

Neural net tuning



Standard tuning



- Higher S- and P-wave states serve as a check whether our tuning leads to reasonable results.
- Main results from the lattice spacing of U103; H200 used to estimate systematics (comments on that later).

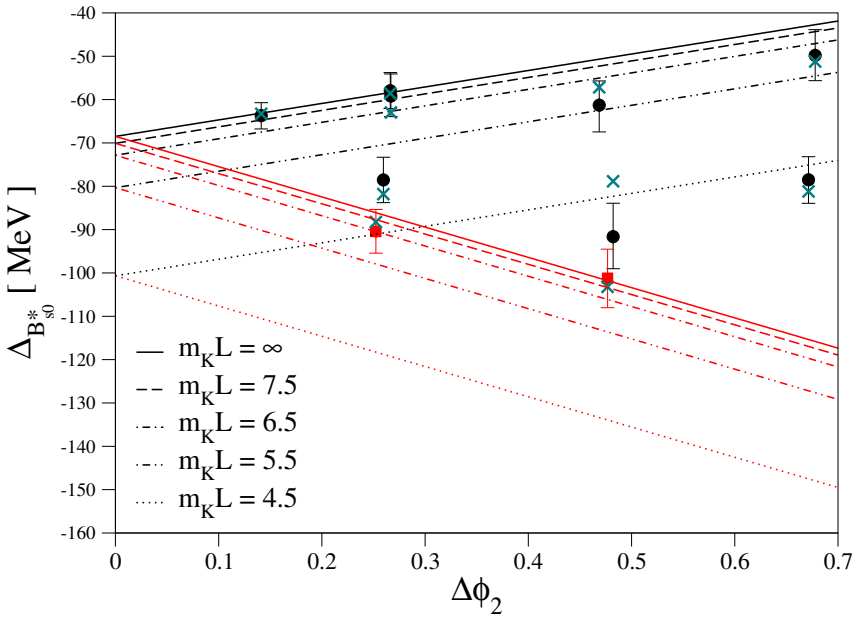
B_s: Chiral – infinite volume extrapolation

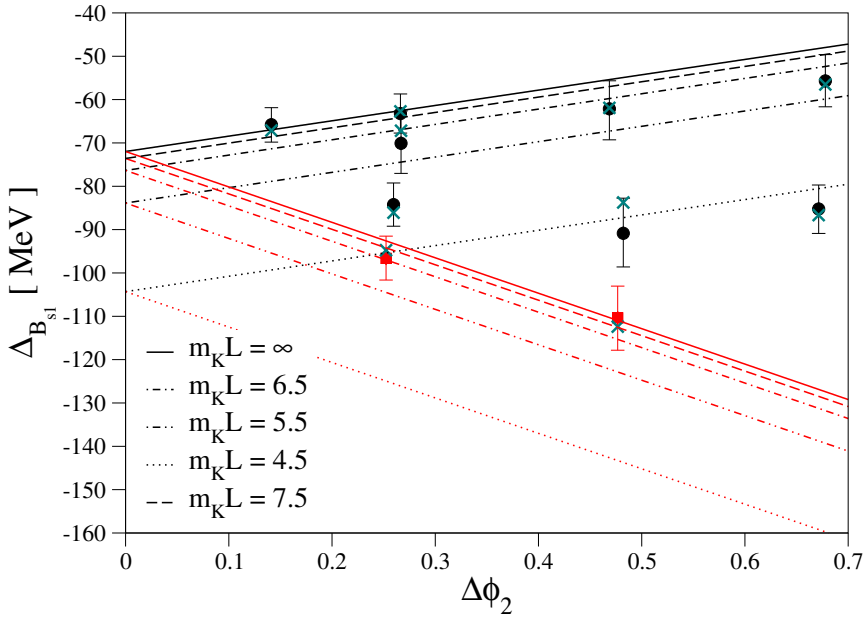
- We explore the previously predicted $J^P = 0^+$ and 1^+ bound states.
- Mainly the CLS $\mathrm{Tr}M = \text{const}$ trajectory and 2 $m_S = \text{const}$ ensembles

Combined extrapolation:

$$\Delta_{B^*_{s0}/B_{s1}}(\Delta\phi_2, m_K L, a) = \Delta_{B^*_{s0}/B_{s1}}(0, \infty, a)\left(1 + A\Delta\phi_2 + Be^{-m_K L}\right)$$

$$\Delta\phi_2 = \phi_2^{\mathrm{Lat}} - \phi_2^{\mathrm{Phys}} \quad ; \quad \phi_2 = 8t_0 m_\pi^2$$





Systematic uncertainties and final result

Resulting binding energies:

$$\Delta_{B^*_{s0}}(0,\infty,0) = -75.4(3.0)_{\text{Stat.}}(13.7)_{\text{a}}\ [\text{MeV}],$$
$$\Delta_{B_{s1}}(0,\infty,0) = -78.7(3.7)_{\text{Stat.}}(13.4)_{\text{a}}\ [\text{MeV}].$$

- Small uncertainty from statistics + combined extrapolation
- Largest systematics from usage of NRQCD/discretization effects
- Central value shifted by applying half the mass difference between H200 and U103
  In hindsight maybe a bad choice.
- All other explored uncertainties (finite volume shapes, modified quark-mass dependence, *etc.*) are small.

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Improving RHQ b-quarks

We think NRQCD for the T_{bb} and T_{bbs} is at an end.
RHQ b-tuning using the "Tsukuba" action as we did for charm.

- Again: Learn the dependence of states on parameters
- Absolute scales included
- Fixed $c^2 = 1$ to ensure relativistic nature
- 5-parameter tuning
- see large variations from 1 of r_s, ν, c_E, c_B

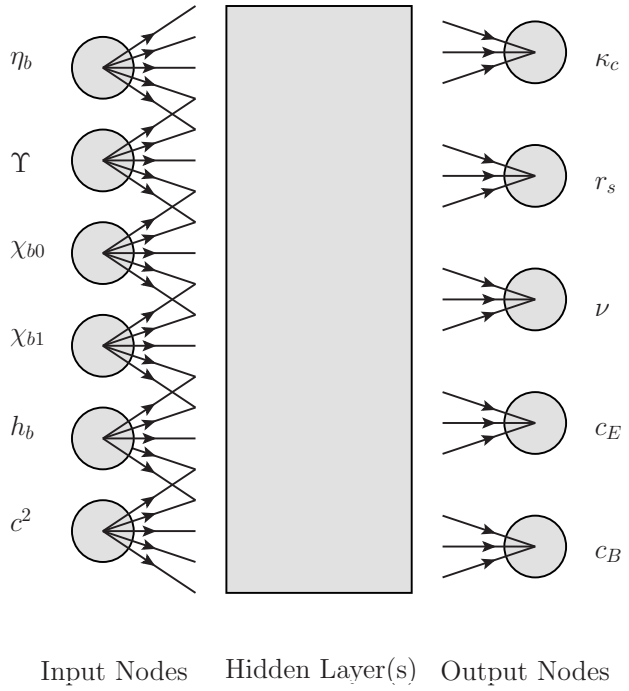


Figure: Schematic picture of our RHQ b-quark tuning

Ensembles used for the determination of the B^*_{s0} and B_{s1}

- Pion and kaon masses taken from

M. Cè *et al.* PRD 106 114502 (2022)

β	Name	$L^3 \times L_T$	T-Boundary	N_{conf}	a^{-1} [GeV]	am_π	am_K
3.34	A653	$24^3 \times 48$	Periodic	990	1.987(20)	0.21193(91)	0.21193(91)
3.34	A654	$24^3 \times 48$	Periodic	500	1.987(20)	0.16647(121)	0.22712(121)
3.34	B650	$32^3 \times 64$	Periodic	165	1.987(20)	0.12831(120)	0.23190(91)
3.40	H101	$32^3 \times 96$	Open	250	2.285(28)	0.18217(62)	0.18217(62)
3.40	H102	$32^3 \times 96$	Open	250	2.285(28)	0.15395(71)	0.19144(71)
3.40	H105	$32^3 \times 96$	Open	250	2.285(28)	0.12136(124)	0.20230(124)
3.40	N101	$48^3 \times 128$	Open	88	2.285(28)	0.12150(55)	0.20158(55)
3.40	U103	$24^3 \times 128$	Open	251	2.285(28)	0.18042(55)	0.18042(55)
3.40	X150	$40^3 \times 128$	Periodic	246	2.285(28)	0.12183(48)	0.20154(41)
3.46	B450	$32^3 \times 64$	Periodic	336	2.585(27)	0.16063(45)	0.16063(45)
3.46	X451	$40^3 \times 128$	Periodic	253	2.585(27)	0.11047(38)	0.17803(24)
3.55	D200	$64^3 \times 128$	Open	121	3.071(36)	0.06507(28)	0.15630(15)
3.55	H200	$32^3 \times 96$	Open	125	3.071(36)	0.13535(60)	0.13535(60)
3.55	N200	$48^3 \times 128$	Open	107	3.071(36)	0.09234(31)	0.15071(23)
3.55	N202	$48^3 \times 128$	Open	225	3.071(36)	0.13424(31)	0.13424(31)
3.55	N203	$48^3 \times 128$	Open	95	3.071(36)	0.11254(24)	0.14402(20)
3.70	N300	$48^3 \times 128$	Open	192	3.962(45)	0.10569(23)	0.10569(23)

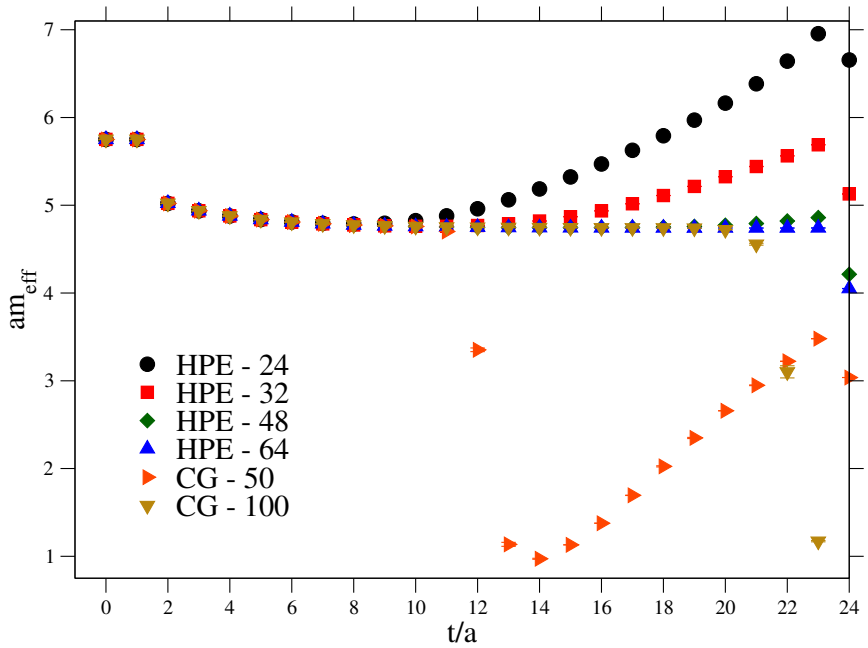
Approximating the b-quark propagators with the HPE

- Hopping parameter expansion for the propagator

$$\psi = \eta + \sum_{i=1}^{\infty} (\kappa \not{D})^i \eta$$

- With the inverse of the Clover term given by A^{-1}, one obtains

$$\psi^0 = A^{-1}\eta, \qquad \psi^i = \psi^{i-1} + (-1)^i (A_{oo}^{-1} \not{D}_{oe} \psi_e^{i-1} + A_{ee}^{-1} \not{D}_{eo} \psi_o^{i-1})$$



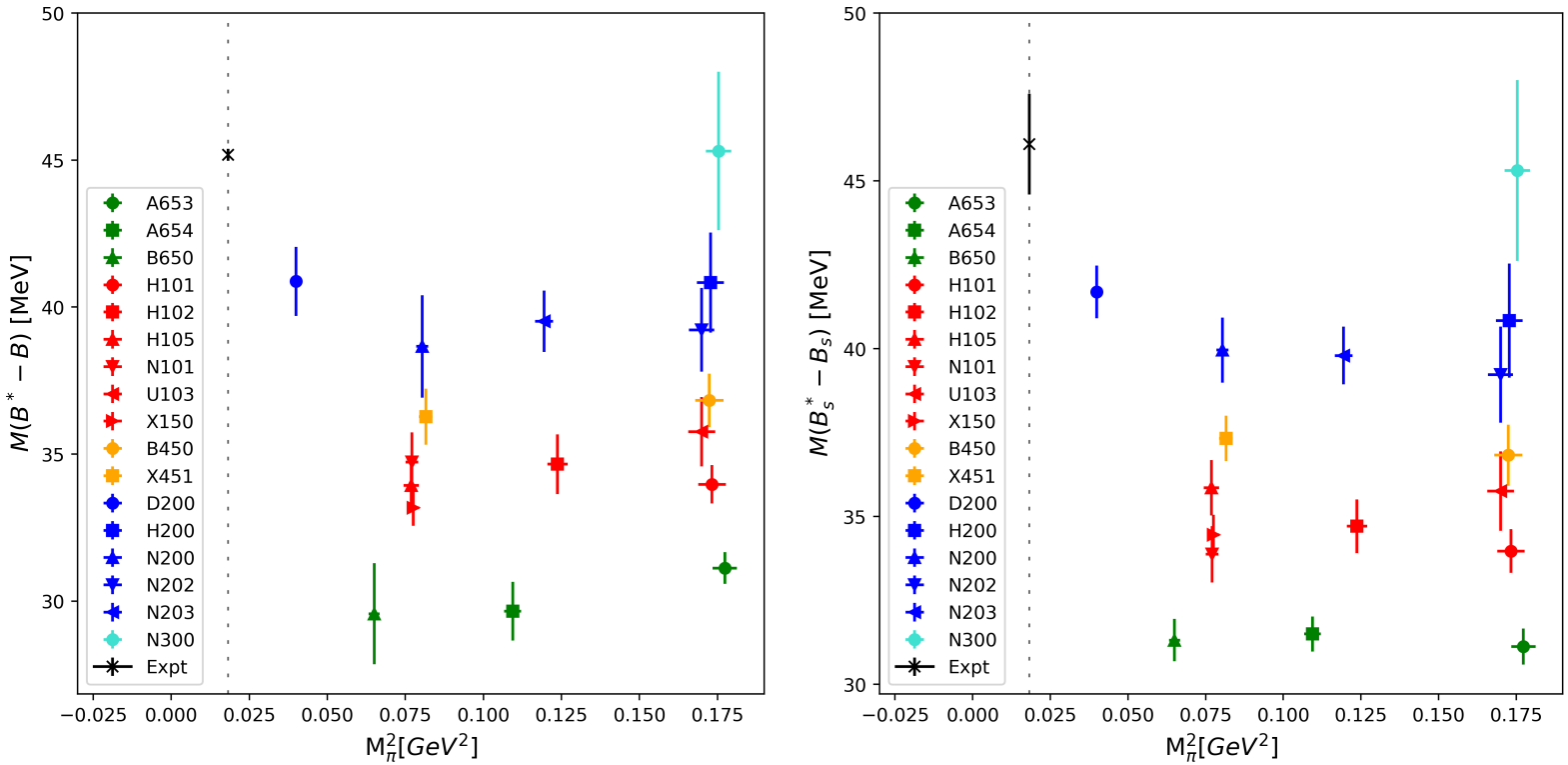
Correlator construction and # of HPE iterations

- We use Coulomb-gauge-fixed wall sources with a Gaussian source and a point-like sink
- The smearing radius used is $r \approx 0.41$ fm.

Name	β	a^{-1} [GeV]	N_{HPE}	$(r/a)^2$
A653	3.34	1.987(20)	64	32
U103	3.40	2.285(28)	94	42
B450	3.46	2.585(33)	88	54
H200	3.55	3.071(36)	96	76
N300	3.70	3.962(45)	142	126
J500	3.85	5.047(58)	200	198

Testing our RHQ b-quark action

- We pick the following benchmark quantities:
  - The B and B_s hyperfine splittings
  - The mass splittings between the B_s- and B-meson ground states
- The hyperfine splittings are very sensitive to discretization effects.



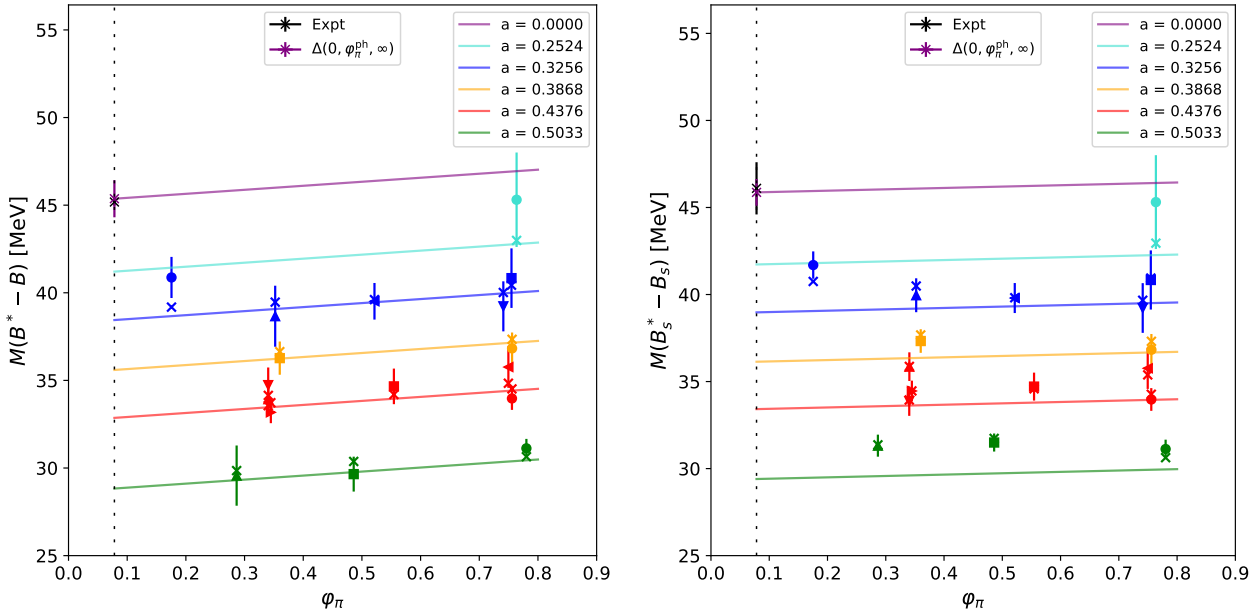
Combined chiral-continuum and infinite-volume fits

- We use the following fit Ansatz for all mass splittings

$$\Delta(a, \varphi_\pi, m_\pi L) =$$
$$\Delta(0, \varphi_\pi^{\text{ph}}, \infty)\left(1 + Aa^2 + B\left(\varphi_\pi - \varphi_\pi^{\text{ph}}\right) + C\exp(-m_\pi L) + D\exp(-m_K L) + Fa^3\right) ,$$

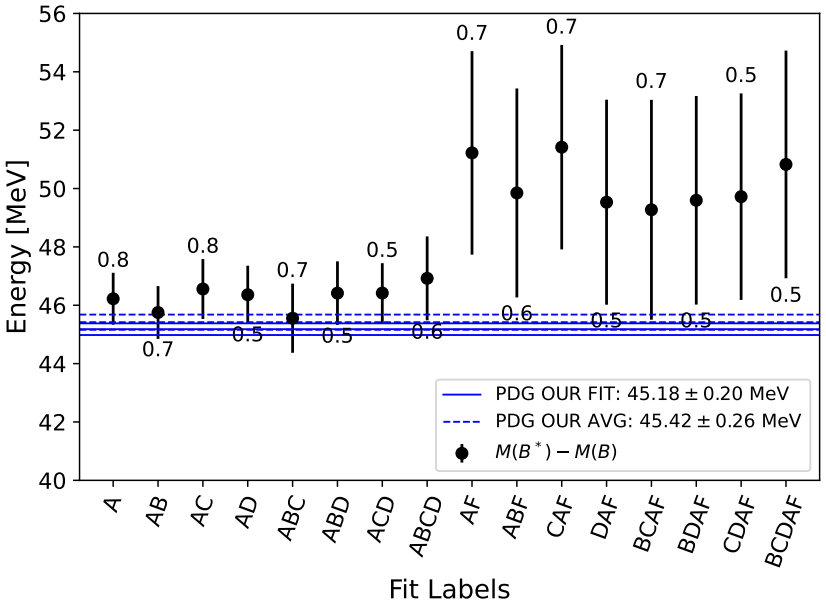
  with $\phi_\pi = 8t_0 m_\pi^2$.

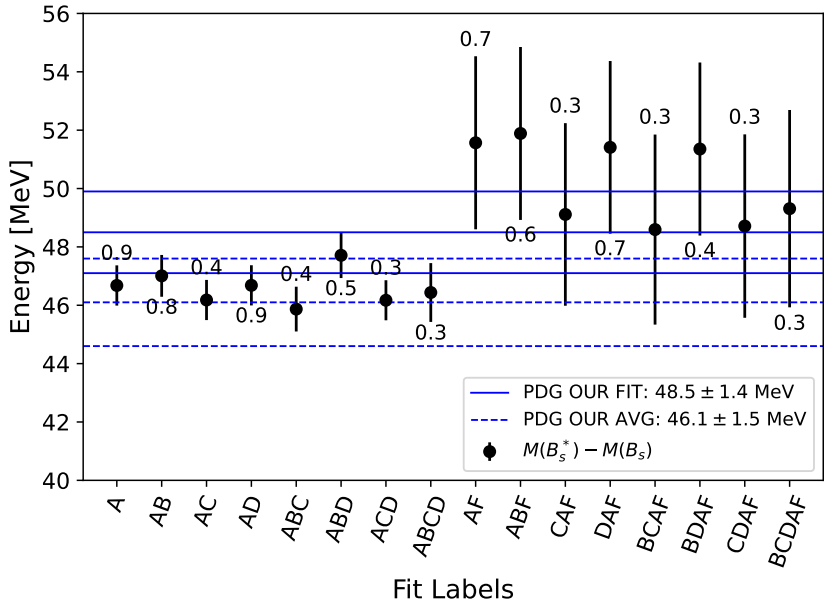
- Thesis results based on correlated differences; final results will be based on ratio fits



Variations assessed so far (thesis Manuel Stolz)

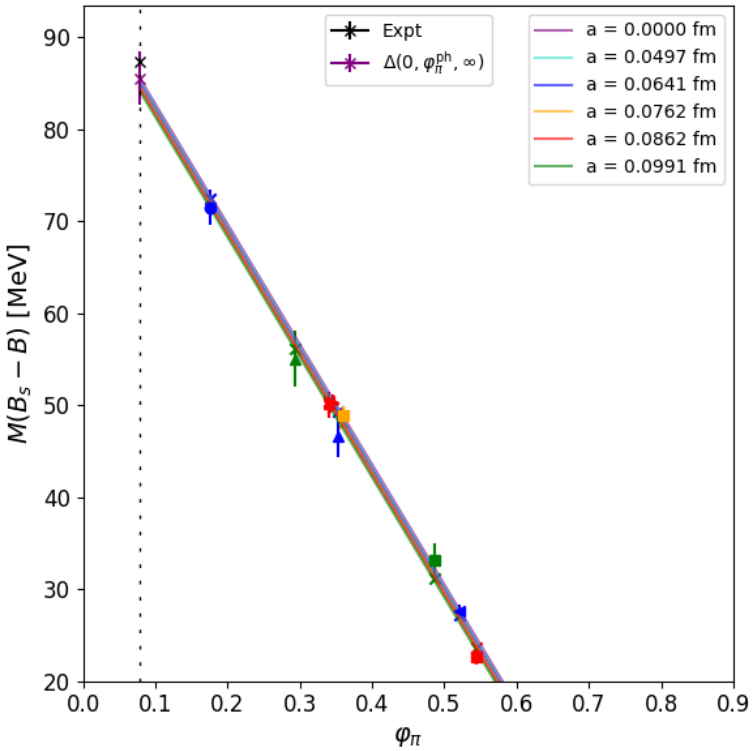
- We test infinite-volume extrapolations parameterized by an $\exp(-m_\pi L)$ or/and $\exp(-m_K L)$ term.
- We allow for the leading expected discretization errors of order a^2
  Fits with an additional term $\propto a^3$ do not really improve the fits
- We parameterize the light-quark mass-dependence with a term proportional to ϕ_2.

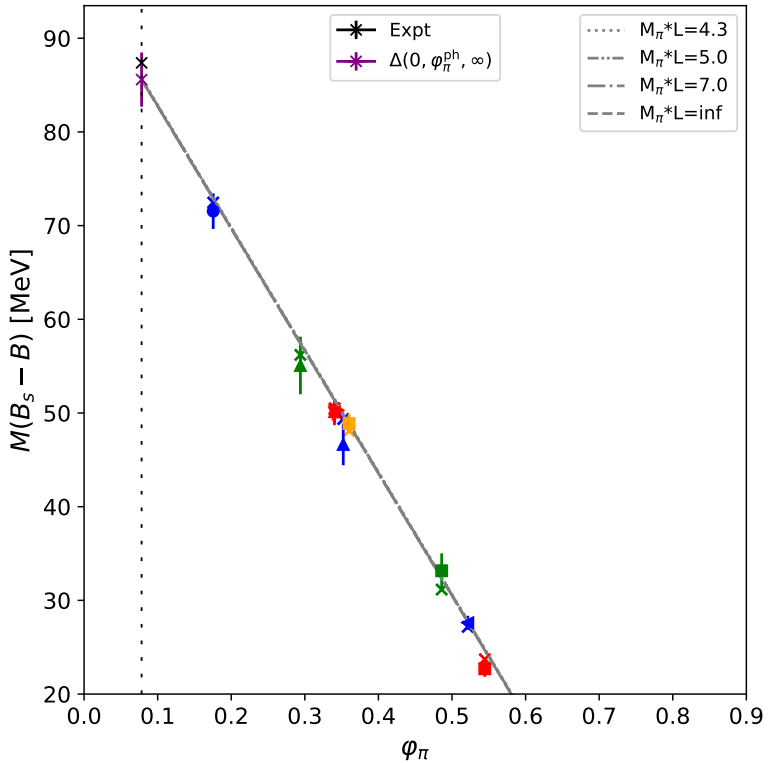




Mass splitting between the B_s and B mesons

- With our current data, we cannot correct for mistuning of the $\mathrm{Tr}(M)$ trajectory.
- Currently the splitting is not constrained to 0 at the $SU(3)$ point.
- The final results may use ratio fits.





Preliminary results (thesis of Manuel Stolz)

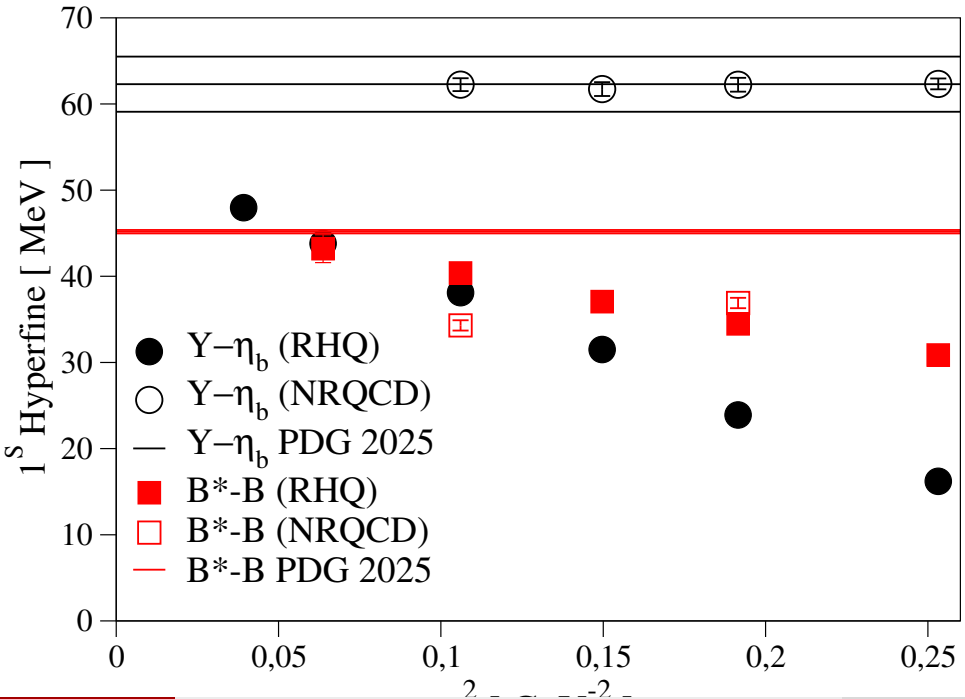
- Thesis results from a weighting by p-value of correlated mass-differences, not including a^3 terms.
- Final estimates will include further fit variations, and will determine the weighted average and its uncertainties through the AIC.
- Final fits will likely be to ratios of correlators.
- Preliminary results fully compatible with PDG averages.

Splitting	This Work	PDG "our fit"	PDG "our average"
$B_s^* - B_s$	46.6(10)	48.5(14)	46.1(15)
$B^* - B$	46.3(12)	45.18(20)	45.42(26)
$B_s - B$	85.4(31)	87.37(12)	87.42(24)

Mass splittings in MeV

Hyperfine splitting: Comparing NRQCD and RHQ data

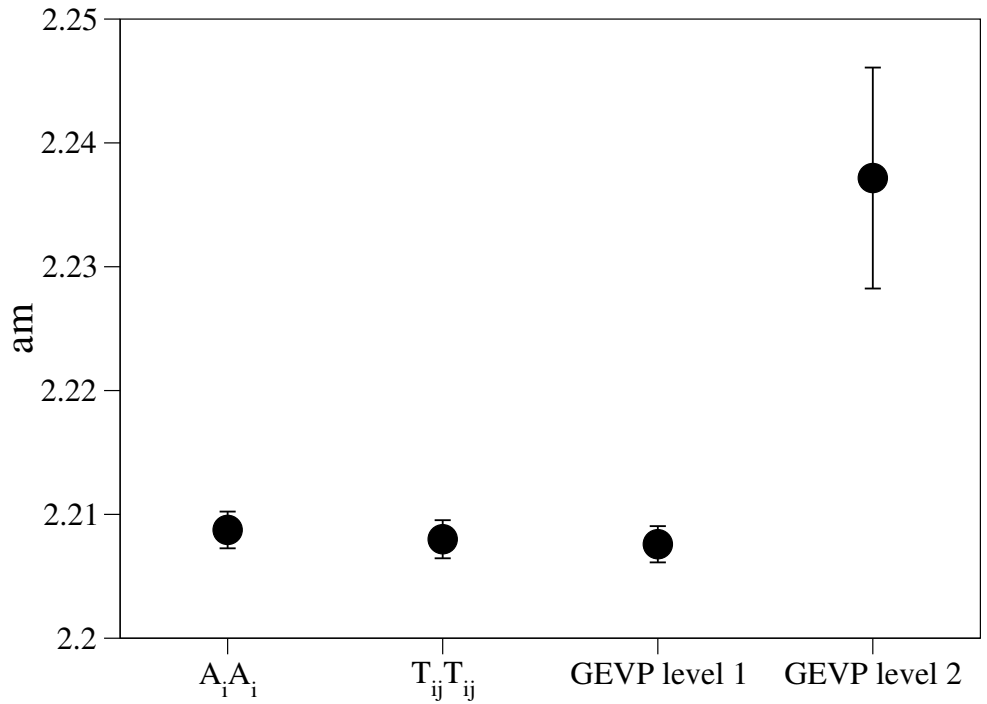
- NRQCD does not allow us to perform a continuum extrapolation.
- For a given quark-mass, we expect a window in the lattice-spacing where NRQCD (to a given order) provides a sensible approximation.
- Tuning of the RHQ action performed on the charmonium/bottomonium spectrum $\rightarrow$ The uncertainty on the experimental $\bar{b}b$ hyperfine is somewhat significant.
- Tuning based on heavy-light states significantly more complicated.



Determining the B_{s1} ground-state

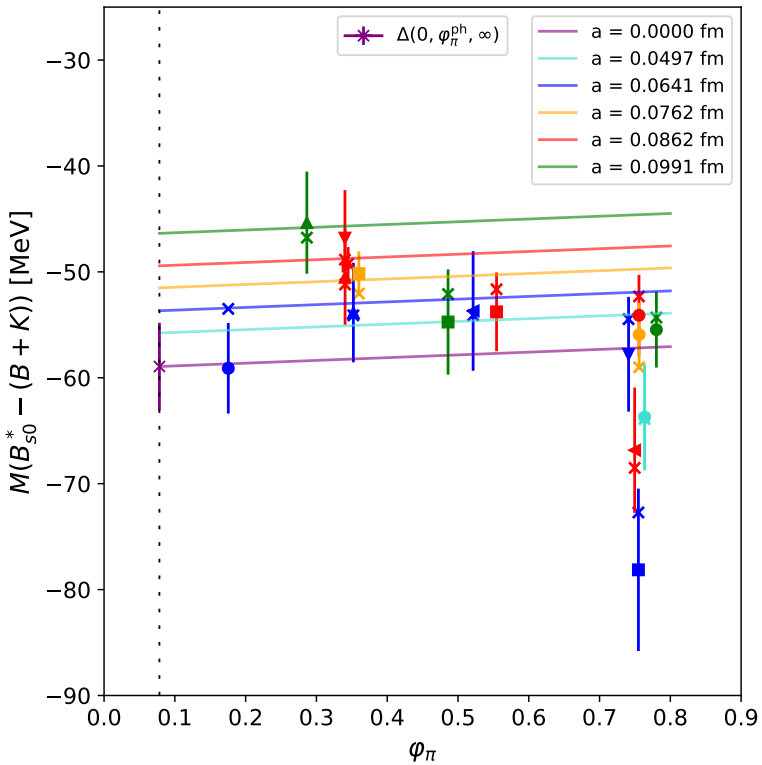
- A possible concern comes from the nearby $j = \frac{3}{2}$ B_{s1} state.
- We investigated this by use a 2×2 basis on a single ensemble.
- The splitting to thee first excited state is compatible with the naive expectation

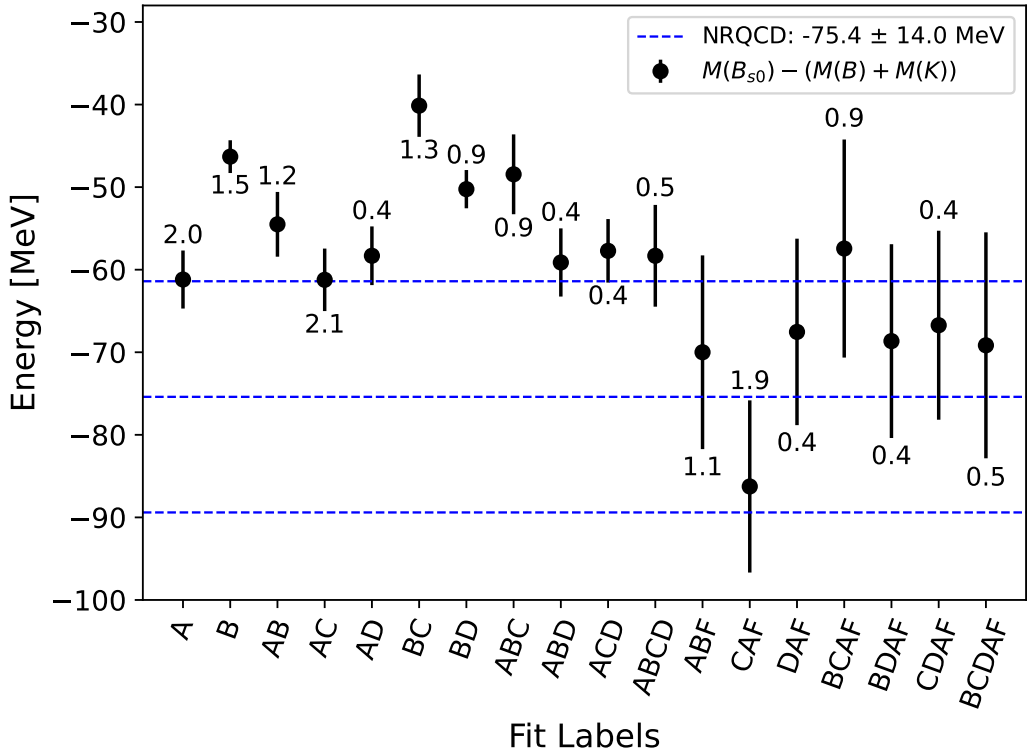
$$M_{B'_{s1}} - M_{B_{s1}} \approx 76(23)\text{MeV} \approx M_{D_{s1}(2536)} - M_{D_{s1}(2460)} \ .$$



Chiral-continuum fits and fit variations

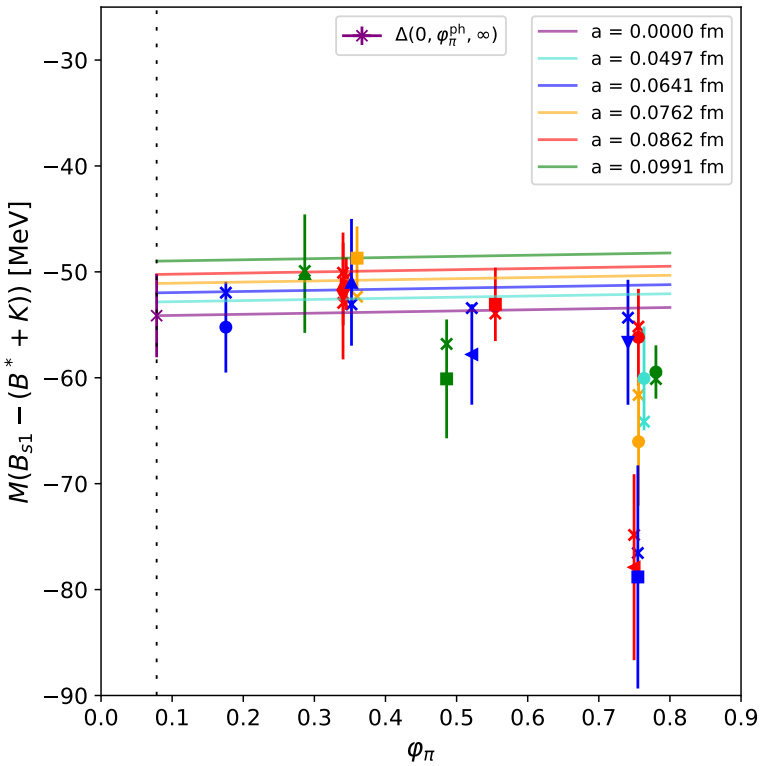
- We use the same fit forms than for our benchmark quantities
- Most fit variations result in smaller binding than with Lattice NRQCD

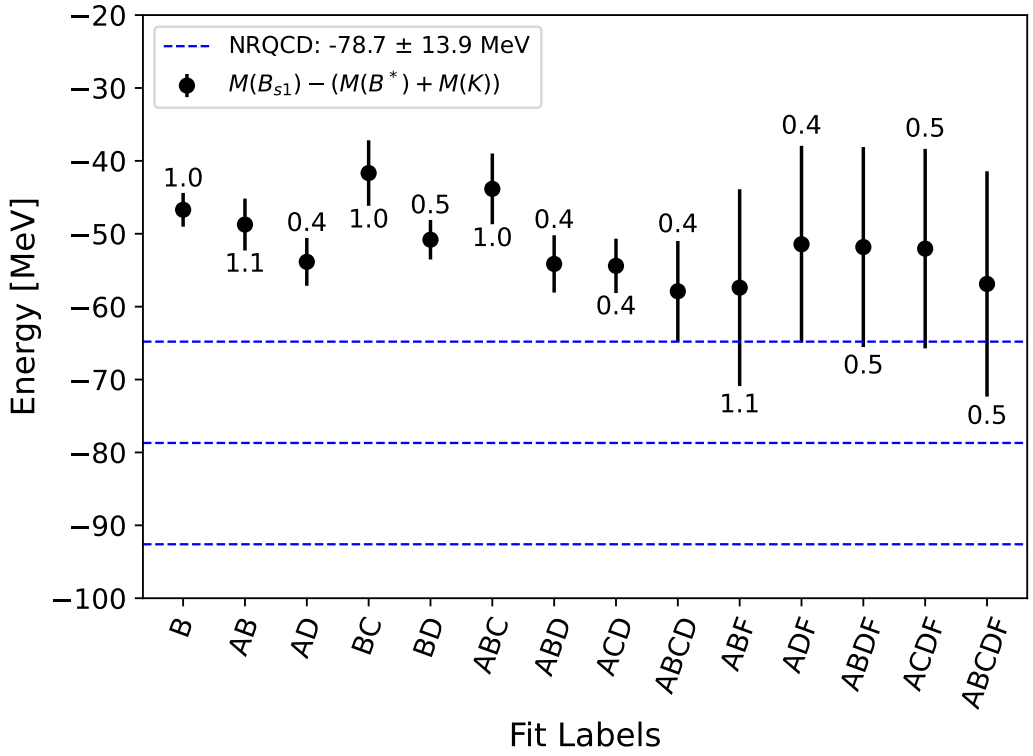




Chiral-continuum fits and fit variations

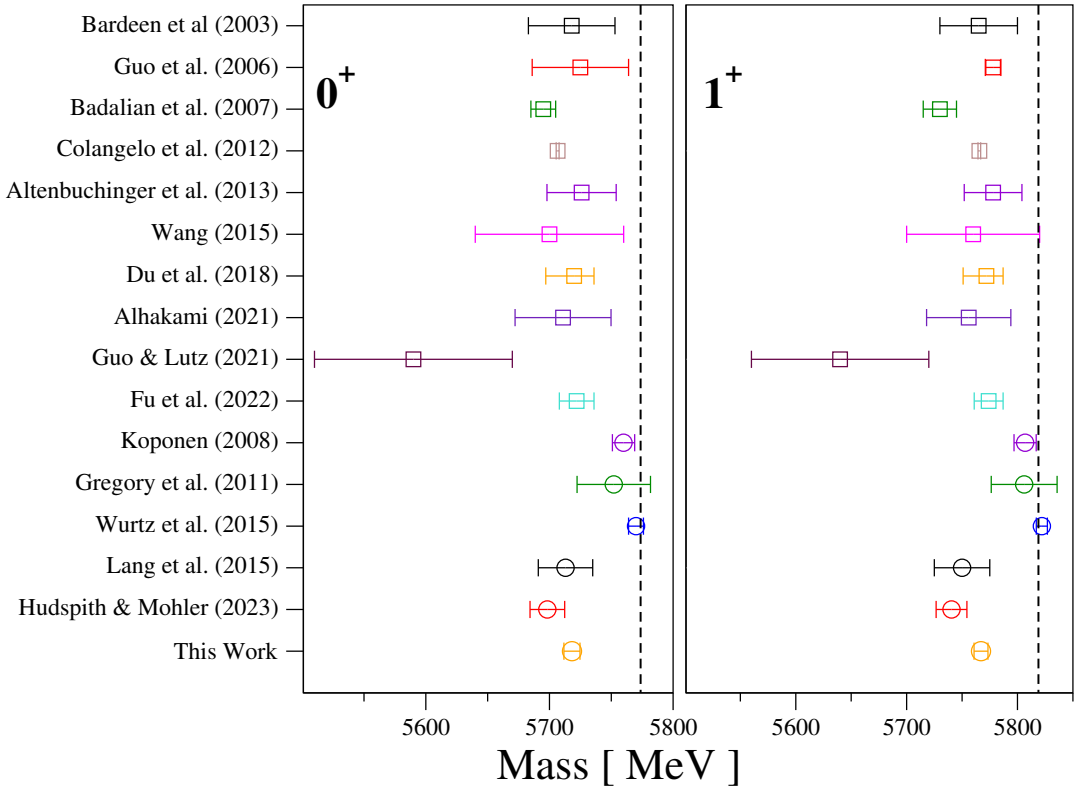
- We use the same fit forms than for our benchmark quantities
- Most fit variations result in smaller binding than with Lattice NRQCD





Preliminary results (thesis Manuel Stolz)

- Comparison to various predictions from models/ EFT + models and to previous LQCD calculations
- In general good agreement – estimates of uncertainties at varying rigor.
- Largest uncertainty for our 2023 results from use of Lattice NRQCD.



Addressing systematic uncertainties

- Exclusively using the CLS $\mathrm{Tr}(M) = \mathrm{const.}$ trajectory, we cannot correct for a mistuned $\mathrm{Tr}(M)$
  $\rightarrow$ Potentially relevant for $M_{B_s} - M_B$.
- Our final variations will also include cuts in a, $L[\mathrm{fm}]$, and pion mass.
- We will employ the *Akaike Information Criterion* for better estimates of systematic uncertainties.
  This will tend to enlarge the binding energies for the B^*_{s0} and B_{s1},
- The B_s and B hyperfine splittings will be extracted from ratio fits.
- Due to a mistake the data on ensemble A654 was not the full generated statistics.
- The final analysis will use the updated reweighting factors by Simon Kuberski (including signs).

Outline

1. Introduction and Motivation
2. Positive-parity B_S states from Lattice NRQCD
3. Positive-parity B_S states from the RHQ action
   - Benchmark quantities
   - Predicting exotic B^*_{s0} and B_{s1} states
4. Conclusions and Outlook

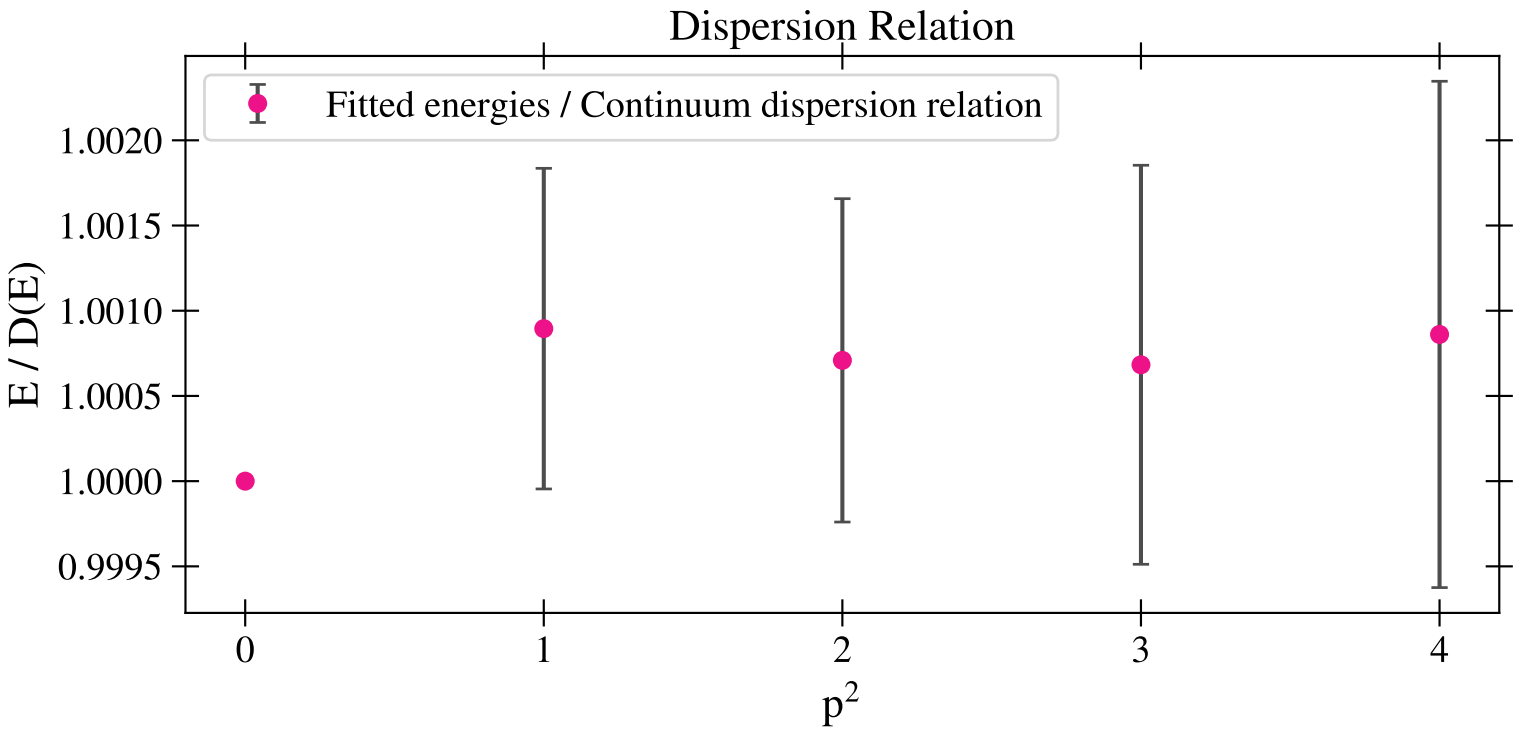
Conclusions and Outlook

- Positive-parity heavy-light mesons
  - two different calculation with full uncertainty estimate for B_0^* and B_{s1} $\rightarrow$ refined predictions for LHCb, Belle II
  - Our NRQCD determination can be significantly improved with RHQ action
  - Scattering amplitudes for the $D_{s0}^*(2317)$ and D_{s1} states using RHQ action in progress.
- Explicitly exotic heavy-quark tetraquarks
  - We also determined doubly-heavy T_{bb} and T_{bbs} tetraquarks using Lattice NRQCD
  - The community is seeing convergence for the easiest-to-compute quantities such as the T_{bb}
- Future directions
  - We will be revisiting the spectrum of B_c mesons.
  - It is worth exploring weaker-bound candidates such as T_{bc}
  - Further insight can be gained from exploring the quark-mass dependence between charm and bottom.
  - Ultimate goal: Study form factors and weak-transition amplitudes.

Backup slides

D-meson dispersion relation from the charm RHQ action

- The charm-quark action has been tuned with the low-lying charmonium spectrum.
- We need to test, that the dispersion-relation for the heavylight states looks fine.



Parameters for our b-quark action

Name	κ	C_E	C_B	r_s	ν
A653	0.03284(2)	2.315(5)	2.593(85)	3.320(8)	3.173(14)
U103	0.04363(4)	2.088(6)	2.258(10)	2.515(5)	2.532(11)
B450	0.05473(2)	1.947(3)	2.135(6)	1.981(2)	2.167(6)
H200	0.06813(14)	1.845(3)	1.932(5)	1.631(4)	1.784(3)
N300	0.08453(4)	1.707(1)	1.792(3)	1.385(2)	1.429(1)
J500	0.09756(3)	1.613(2)	1.656(4)	1.245(2)	1.274(4)

Comparison of b and c parameters - c_E and c_B

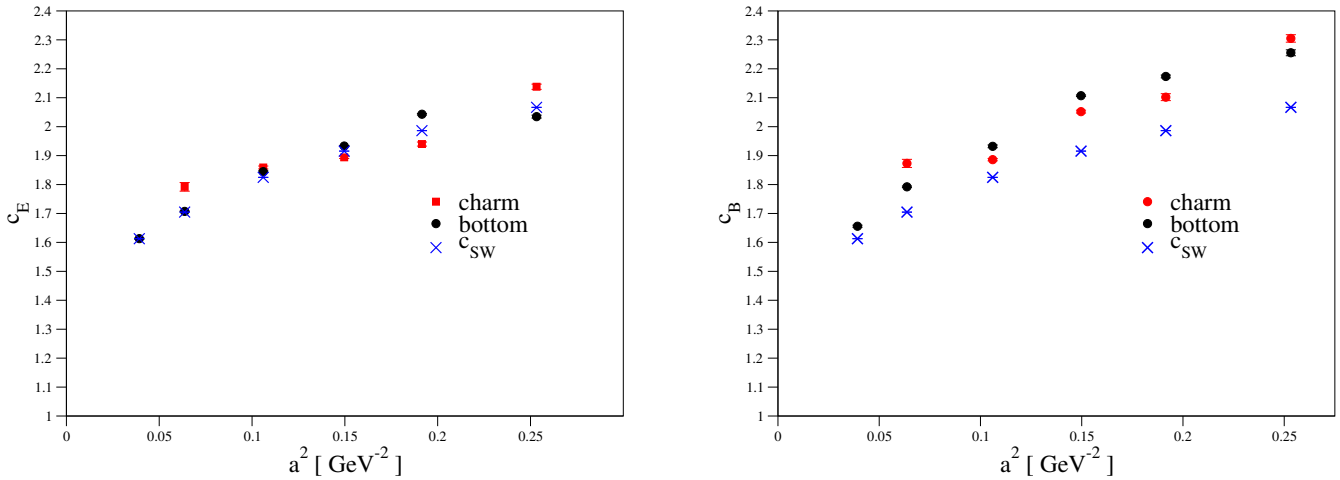


Figure: RHQ clover terms c_E and c_B for **bottom** and **charm**

As a rule of thumb $c_E \approx c_{SW}$, $c_B > c_E$. No big difference between bottom and charm!

Comparison of b and c parameters - κ, r_s, ν

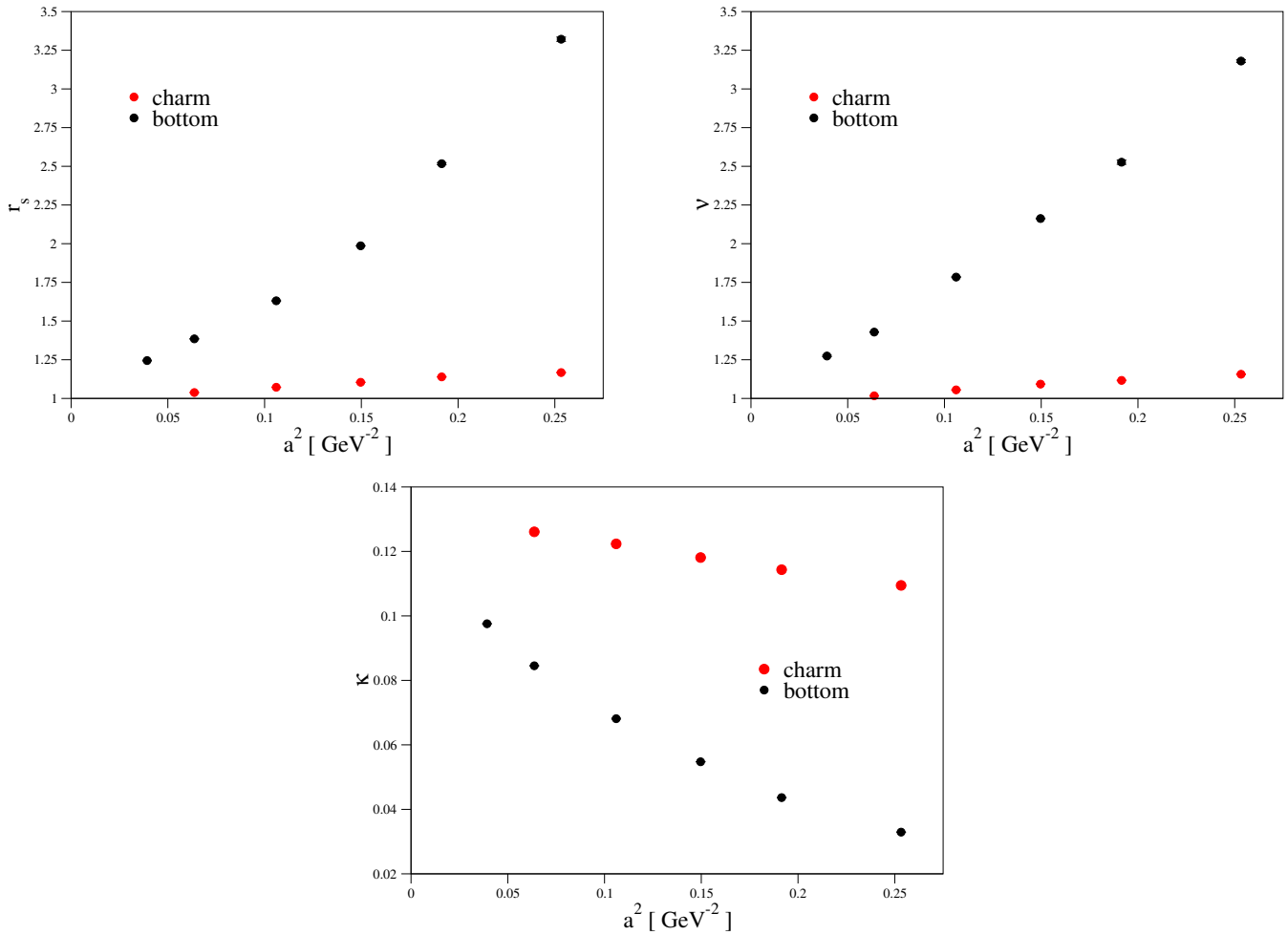
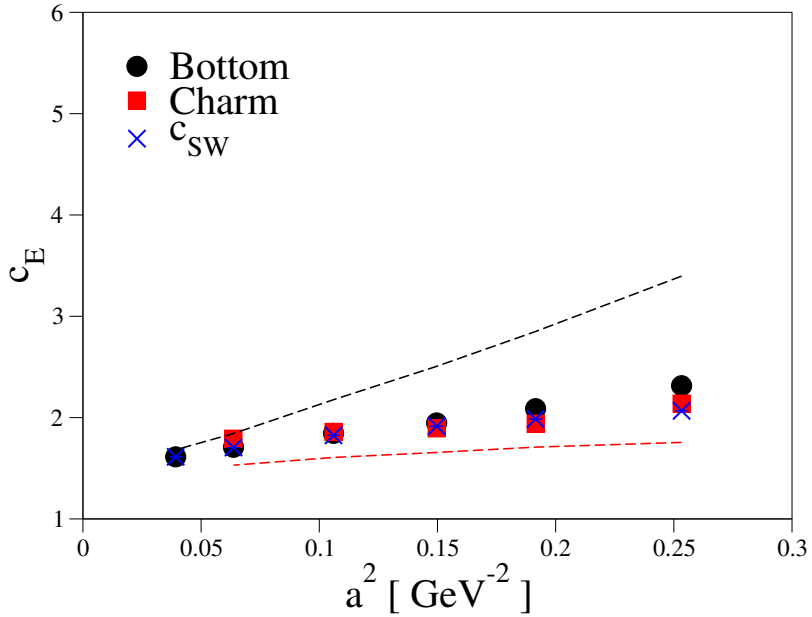
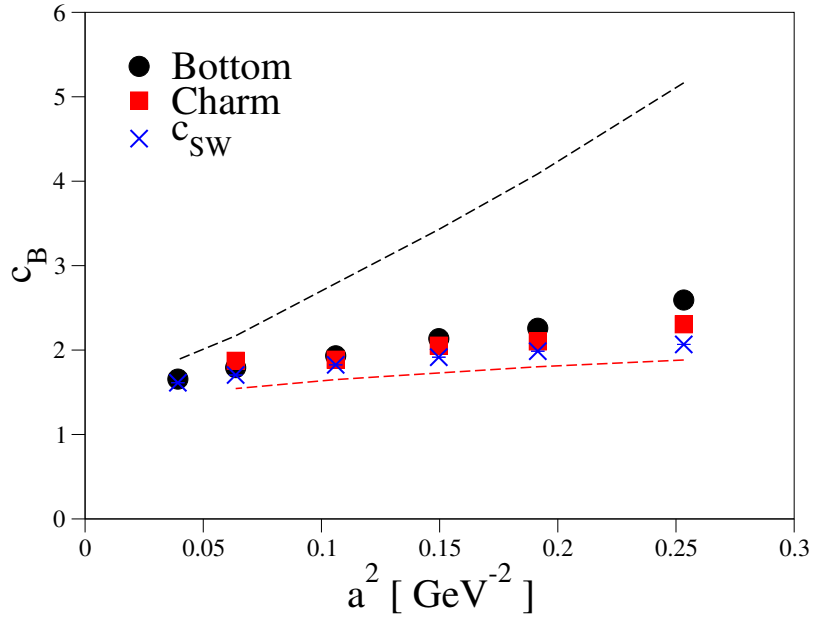


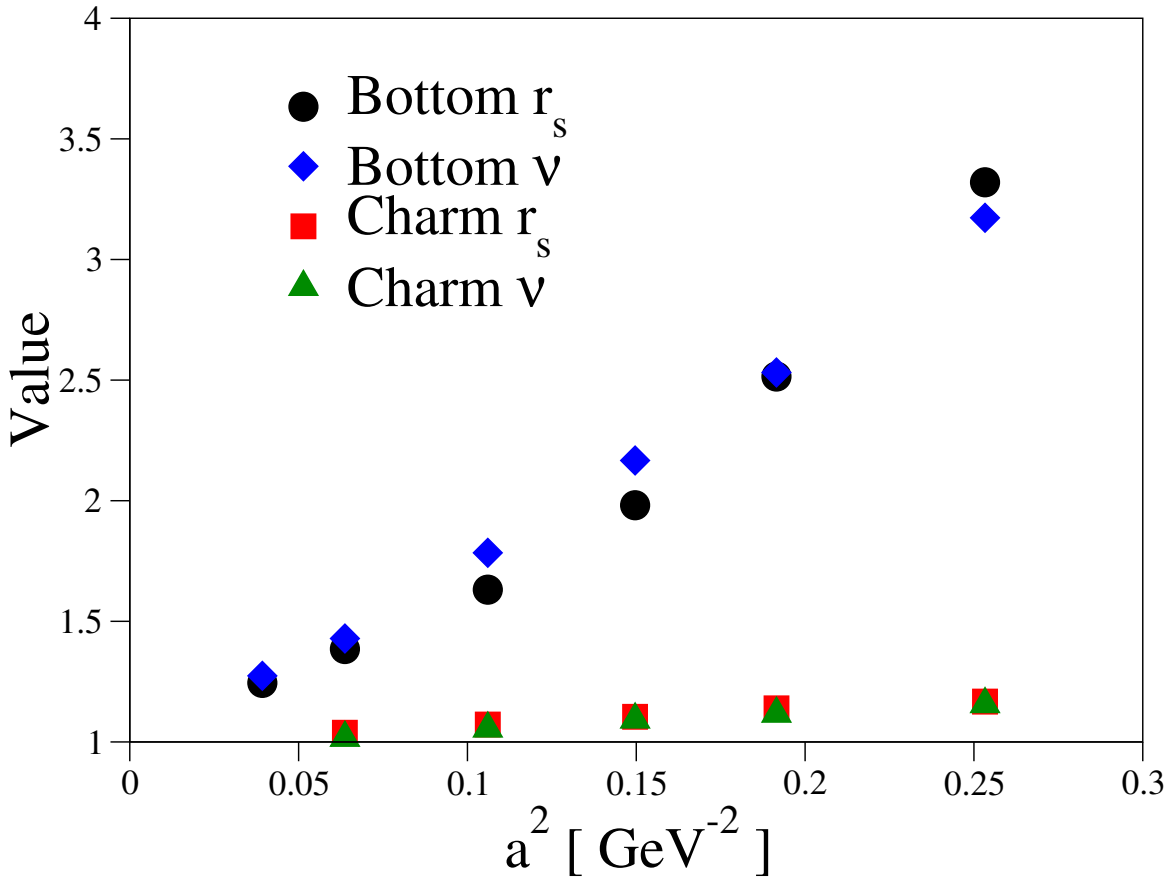
Figure: RHQ action terms r_s, ν, κ for **bottom** and **charm**

Resulting parameters of the RHQ action





Resulting parameters of the RHQ action



$D_{s0}^*(2317)$: D-meson – Kaon s-wave scattering

M. Lüscher Commun. Math. Phys. 105 (1986) 153;
Nucl. Phys. B 354 (1991) 531; Nucl. Phys. B 364 (1991) 237.

Charm-light hadrons

$D_{s0}^*(2317)^\pm$ $I(J^P) = 0(0^+)$
J, P need confirmation.

J^P is natural, low mass consistent with 0^+.

Mass $m = 2317.7 \pm 0.6$ MeV
$m_{D_{s0}^*(2317)^\pm} - m_{D_s^\pm} = 349.4 \pm 0.6$ MeV
Full width $\Gamma < 3.8$ MeV, CL = 95%

$$p \cot \delta_0(p) = \frac{2}{\sqrt{\pi} L} Z_{00}\left(1; \left(\frac{L}{2\pi} p\right)^2\right)$$
$$\approx \frac{1}{a_0} + \frac{1}{2} r_0 p^2$$

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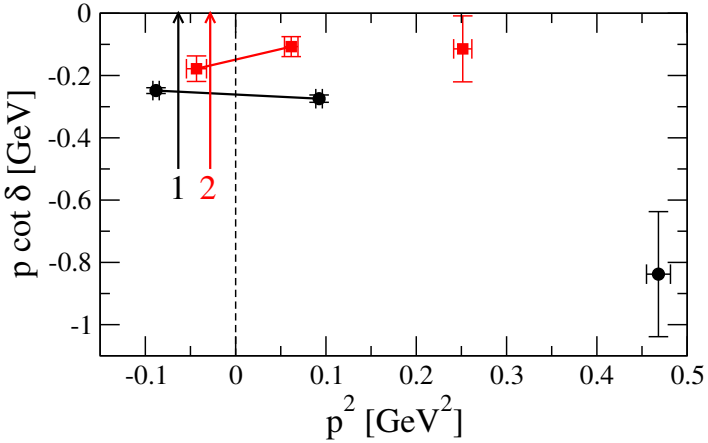
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DM *et al.* PRL 111 222001 (2013)
Lang, DM *et al.* PRD 90 034510 (2014)

Results for ensembles (1) and (2)

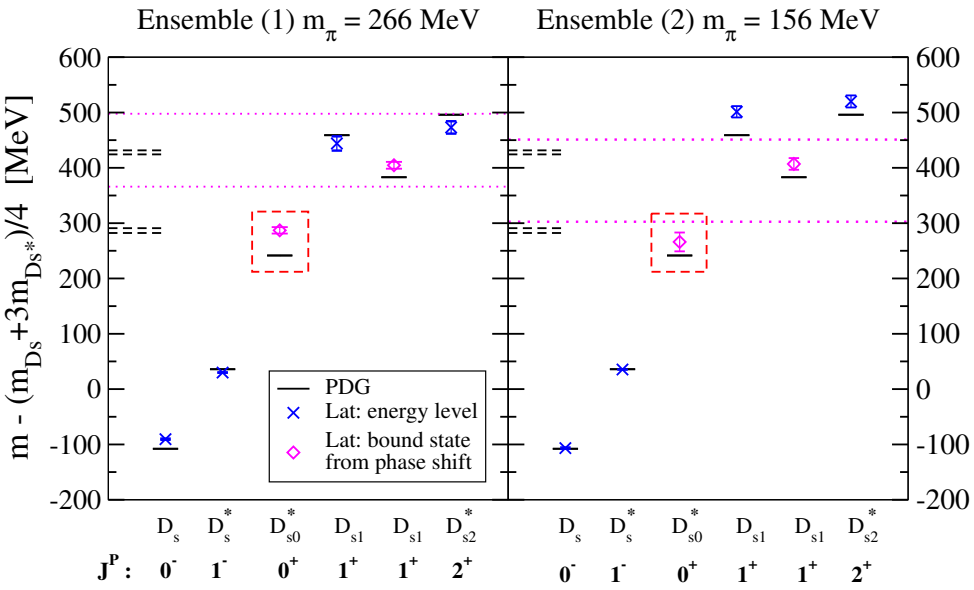


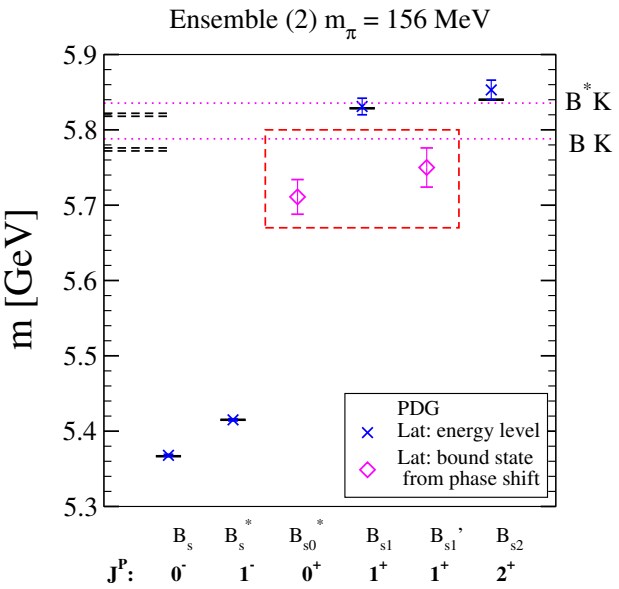
$$a_0 = -0.756 \pm 0.025\text{fm} \qquad (2)$$
$$r_0 = -0.056 \pm 0.031\text{fm}$$
$$a_0 = -1.33 \pm 0.20\text{fm} \qquad (3)$$
$$r_0 = 0.27 \pm 0.17\text{fm}$$

Positive-parity states in the D_s and B_s spectrum

DM *et al.* PRL 111 222001 (2013)
Lang, DM *et al.* PRD 90 034510 (2014)

Lang, DM, Prelovsek, Woloshyn PLB 750 17 (2015)



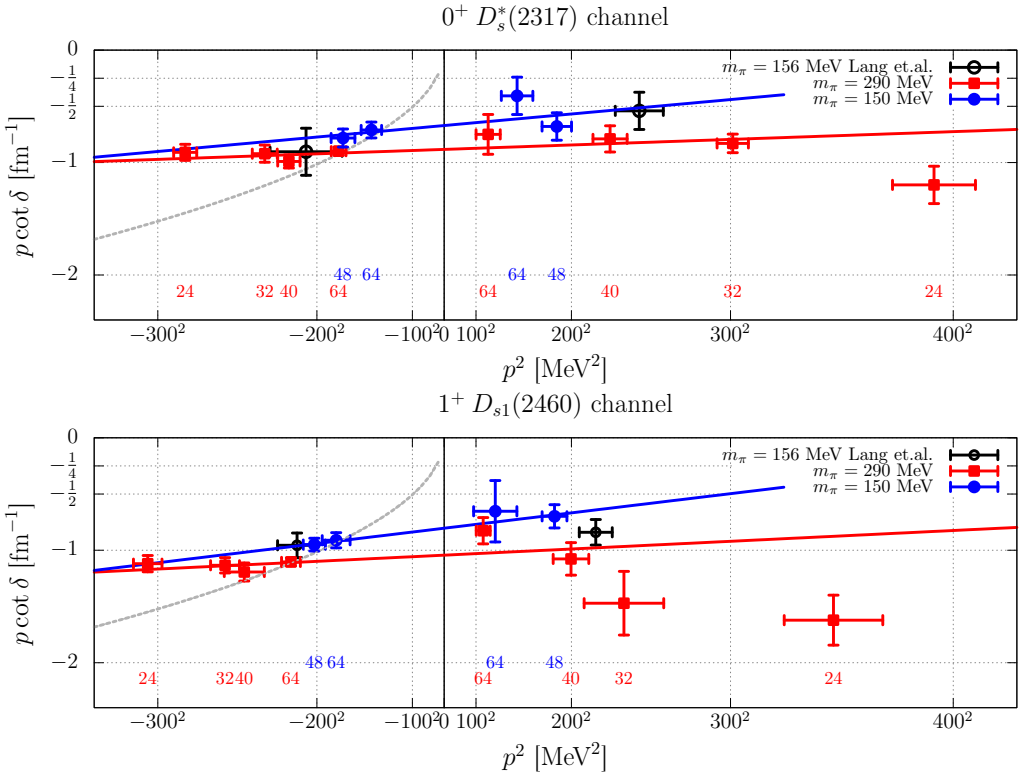


- Spectrum reliably extracted and agrees qualitatively with experiment
- Uncontrolled systematics sizable for the D_s states

- Full uncertainty estimate only for magenta B_s states
- Prediction of exotic states from Lattice QCD!

D_s results in multiple volumes from RQCD

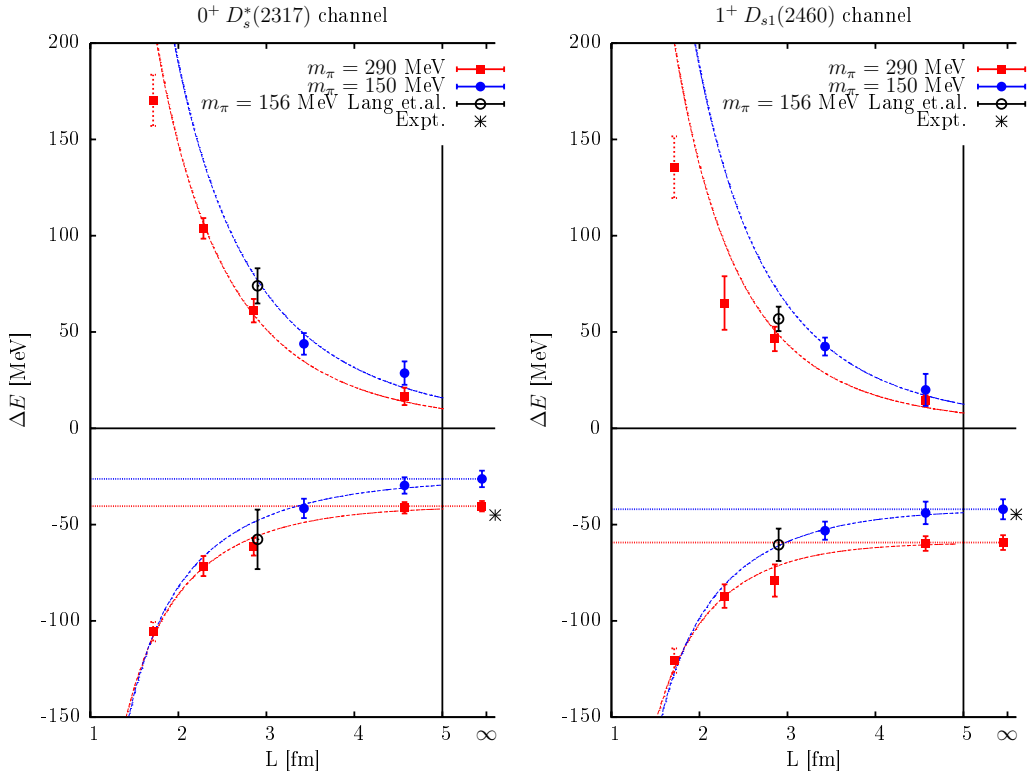
Bali, Collins, Cox, Schäfer, PRD 96 074501 (2017)



- Study with different volumes at pion masses of 150, 290 MeV
- Results confirm basic behavior seen in a single volume
- Discretization effects remain unexplored

D_s results in multiple volumes from RQCD

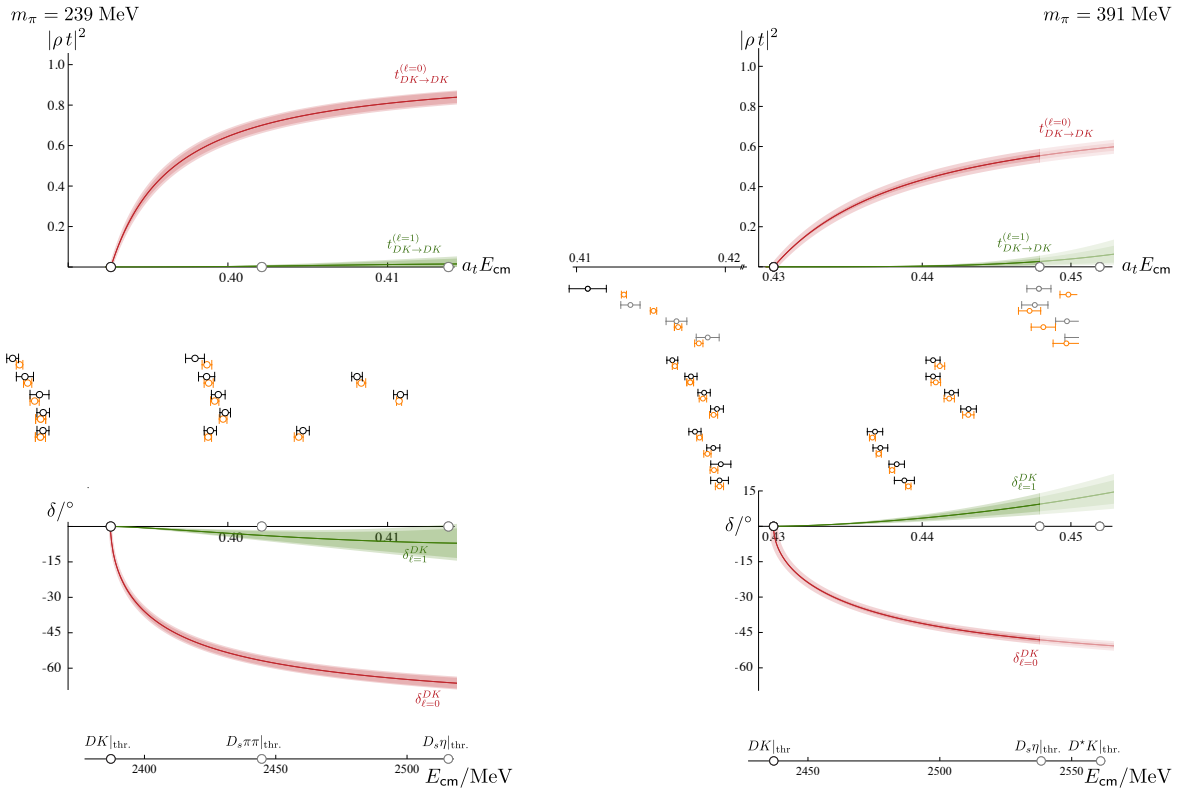
Bali, Collins, Cox, Schäfer, PRD 96 074501 (2017)



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DK and $D\bar{K}$ scattering and the $D_{s0}^*(2317)$

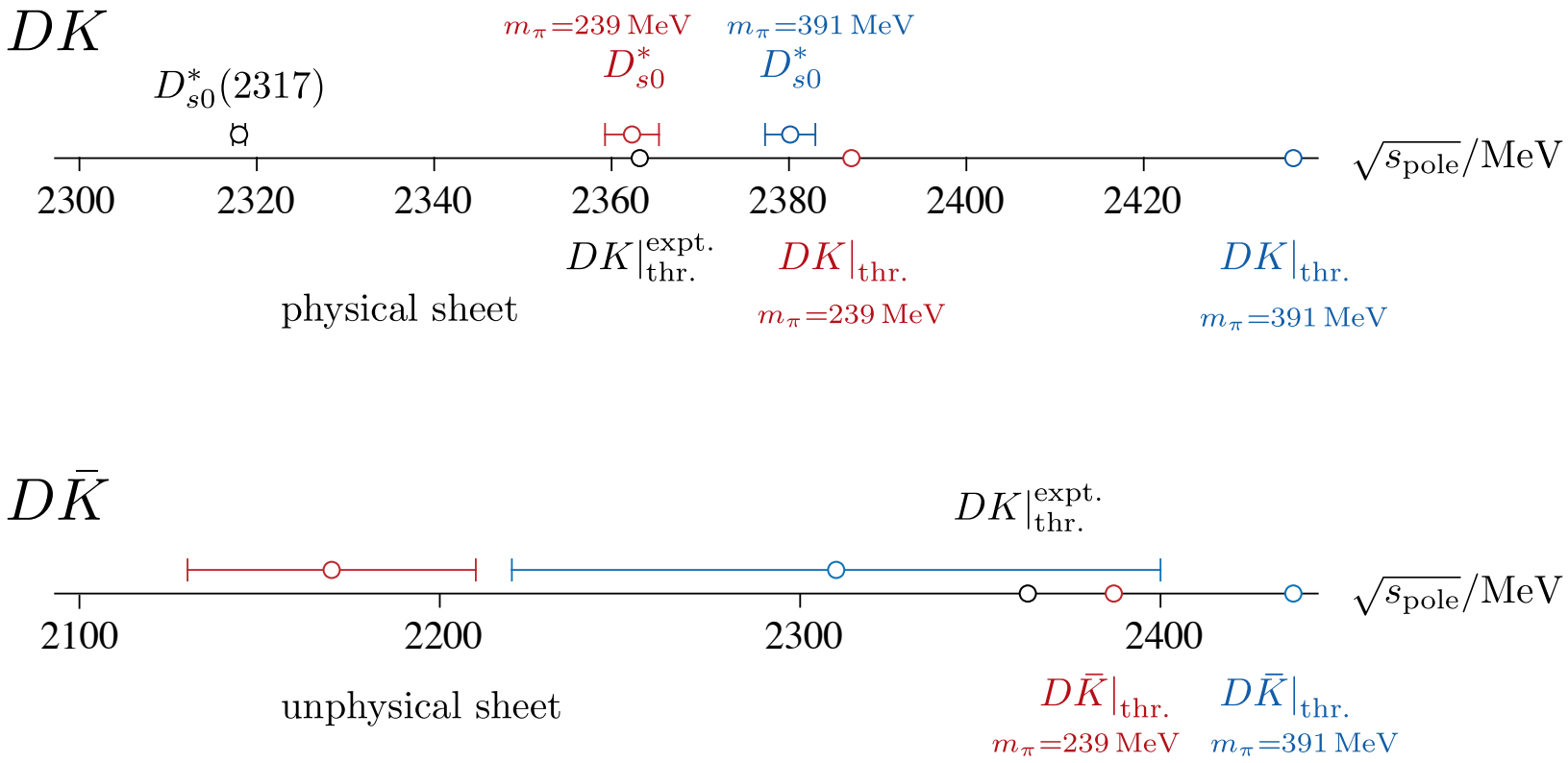
Hadron spectrum collaboration, Cheung *et al.* JHEP 02 100 (2021)



- Study uses moving frames in addition
  results in large number of energy levels at $m_\pi = 238, 391$ MeV

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Hadron spectrum collaboration, Cheung *et al.* JHEP 02 100 (2021)



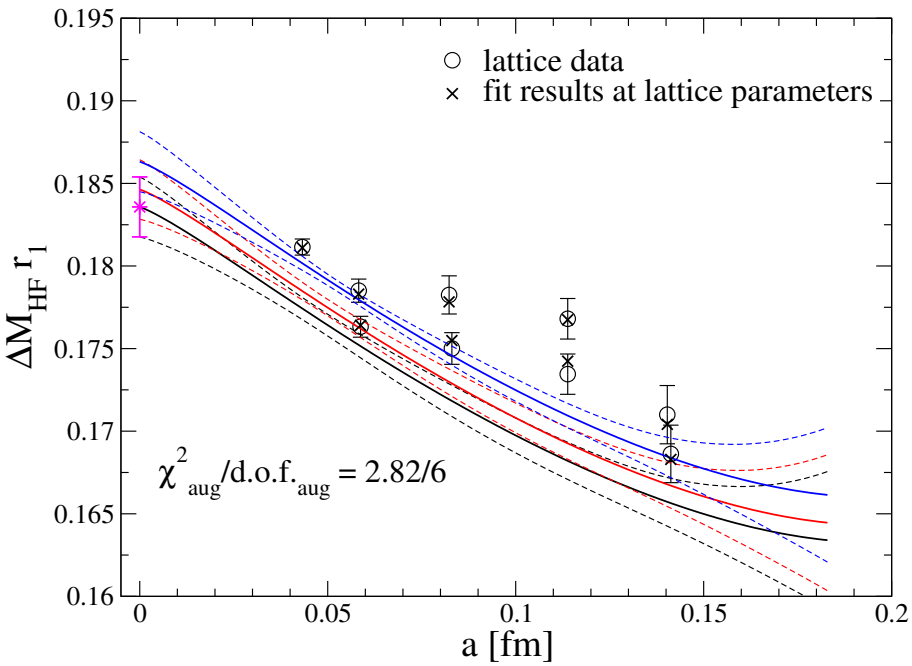
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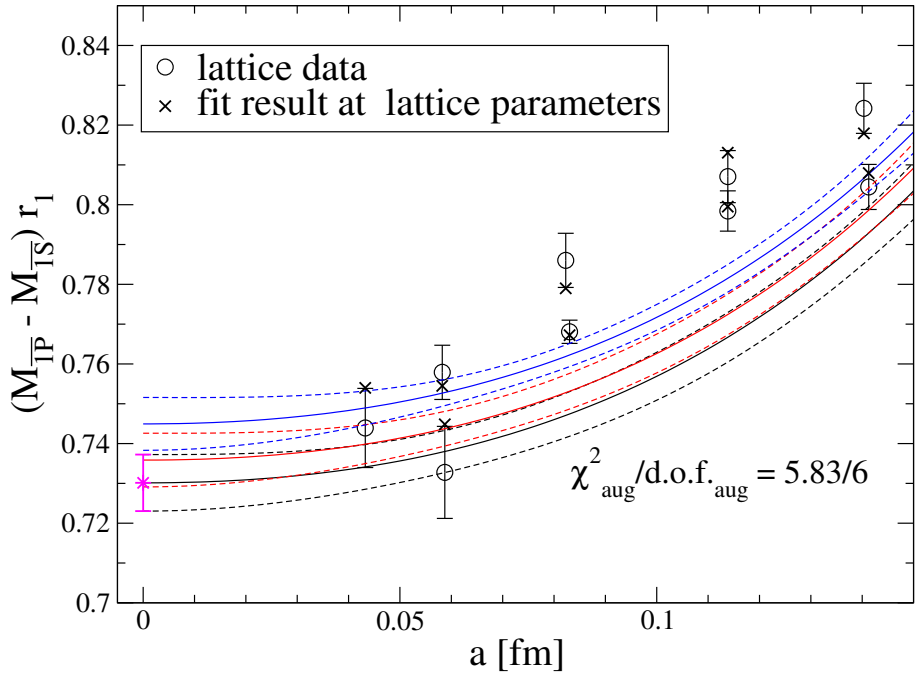
FNAL-MILC 1S hyperfine splittings and 1P-1S splittings

Fermilab Lattice and MILC Collaborations, PRD 99, 034509 (2019)

$$\Delta M_{\mathrm{HF}} = M_{J/\psi} - M_{\eta_c}$$

$$\Delta M_{\mathrm{1P\text{-}1S}} = M_{\overline{\mathrm{1P}}} - M_{\overline{\mathrm{1S}}}$$

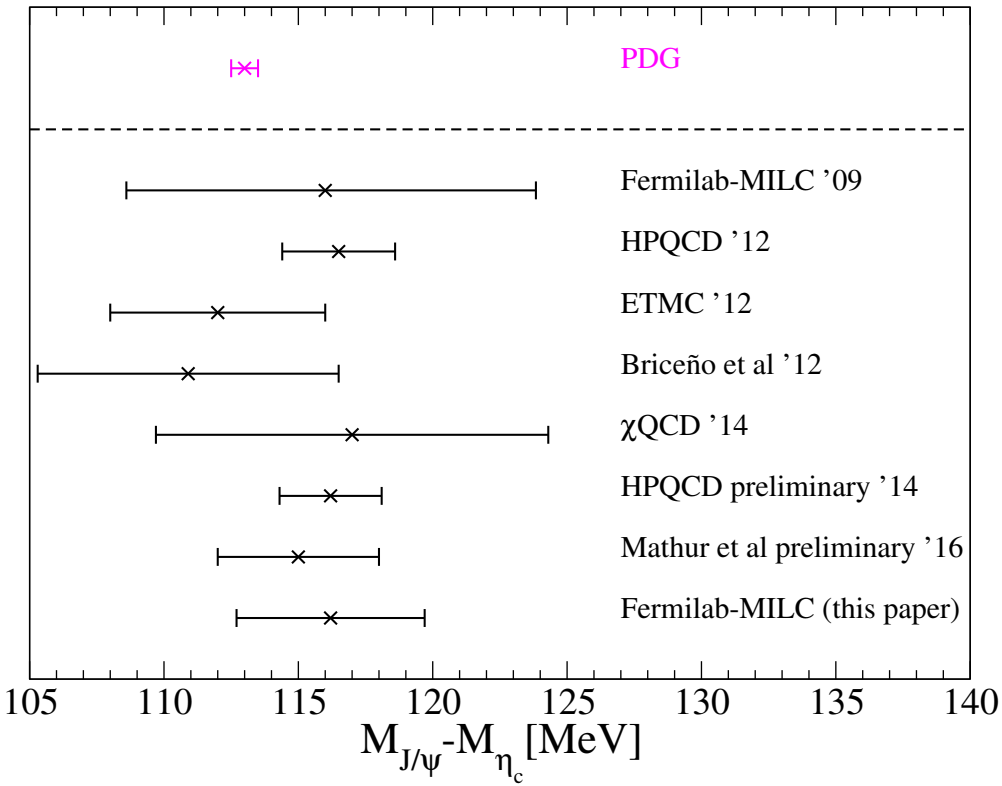




Mass difference	This analysis [MeV]	Experiment [MeV]
1S hyperfine	$116.2 \pm 1.1 \pm 3.3^{-1.5}_{-4.0}$	113.0 ± 0.5
1P1S	$462.2 \pm 4.5 \pm 3.3$	456.64 ± 0.14

A comparison of hyperfine splittings

Fermilab Lattice and MILC Collaborations, PRD 99, 034509 (2019)



- All results at physical quark masses and in the continuum limit
- Lattice numbers exclude annihilation effects
- Estimate from data expects a shift of -1.5..-4.5 MeV

Levkova and DeTar, PRD 83 074504, 2011