Anti-Collinear resummation: from JIMWLK to BFKL (and hopefully back!)

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High-energy (Regge) limit of QCD

Let us introduce the problem of High-Energy (Regge) limit of QCD on a particular example of a process – *production of Mueller-Navelet dijets* at hadron collider (X =anything):

$$p(P_1) + p(P_2) \to jet(k_{J1}) + X + jet(k_{J2}),$$

where $M_{JJ}^2 = (k_{J1} + k_{J2})^2 \gg |\mathbf{k}_{J1}|^2$ or $|\mathbf{k}_{J2}|^2 \gg \Lambda_{\rm QCD}^2$ (**bold–⊥-vectors!**).

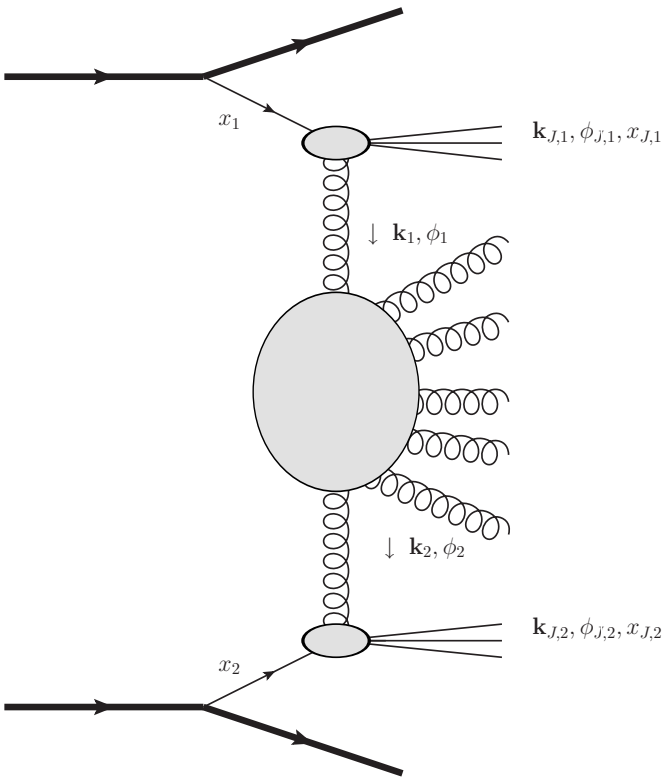
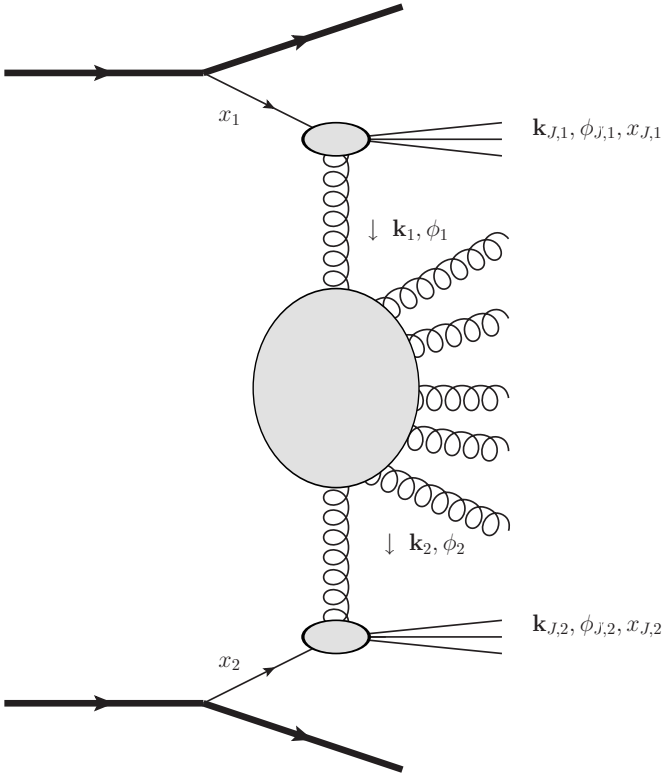


Figure from [hep-ph/1302.7012] .

- $d\sigma = \underbrace{\mathrm{PDF}_i \otimes \mathrm{PDF}_j}_{\text{“non-perturbative”}} \otimes \underbrace{d\hat{\sigma}_{ij}}_{\text{perturbative}}$,
- Fixed-order: $d\hat{\sigma}$ known up to NNLO in α_s,
- Regge limit: $Y = \ln \frac{M_{JJ}}{\sqrt{|\mathbf{k}_{J1}||\mathbf{k}_{J2}|}} \gg 1$, the $d\hat{\sigma}$ recieves corrections $\sim (\alpha_s Y)^n$, which should be resummed.
- The *Balitsky-Fadin-Kuraev-Lipatov (BFKL)* equation allows to rigirously resum them in the LLA ($\sum_n (\alpha_s Y)^n$) and NLLA ($\sum_n \alpha_s (\alpha_s Y)^n$).

BFKL equation



Gluons in t-channel are **Reggeised**.

- Up to NLLA, the partonic cross-section is:

$$d\hat{\sigma}_{ij} = \int d^2\mathbf{k}_1 d^2\mathbf{k}_2\, C_i(\mathbf{k}_1) G(Y, \mathbf{k}_1, \mathbf{k}_2) C_j(\mathbf{k}_2),$$

where $C_i(\mathbf{k})$ – *impact-factors*, G – *BFKL Green's functon*. From now on: $\int d^2\mathbf{k}_1 \to \int\limits_{\mathbf{k}}$.

- The latter satisfies the BFKL equation:

$$\frac{\partial}{\partial Y} G(Y, \mathbf{k}_1, \mathbf{k}_2) = \int\limits_{\mathbf{q}} K(\mathbf{k}_1, \mathbf{q}) G(Y, \mathbf{q}, \mathbf{k}_2),$$

The BFKL kernel is computed perturbatively:

$$K(\mathbf{k}, \mathbf{q}) = \frac{\alpha_s(\mu) N_c}{\pi} \left(K_0(\mathbf{k}, \mathbf{q}) + a_s(\mu) K_1(\mathbf{k}, \mathbf{q}) + \dots \right),$$

where $a_s = \alpha_s/(4\pi)$. The K_0 is known since [BFKL '76, '78] , the K_1 is also known in QCD [Fadin, Lipatov, '98] and $\mathcal{N} = 4$ SYM [Kotikov, Lipatov, 2000] . The K_2 is known in $\mathcal{N} = 4$ SYM [Caron-Huot, Herranen, 2016] and is being computed now in QCD [Gardi, Del Duca, Fadin, Papa, Caola, Falcioni, Byrne,Fucilla,...]

("Generalised") Characteristic functon

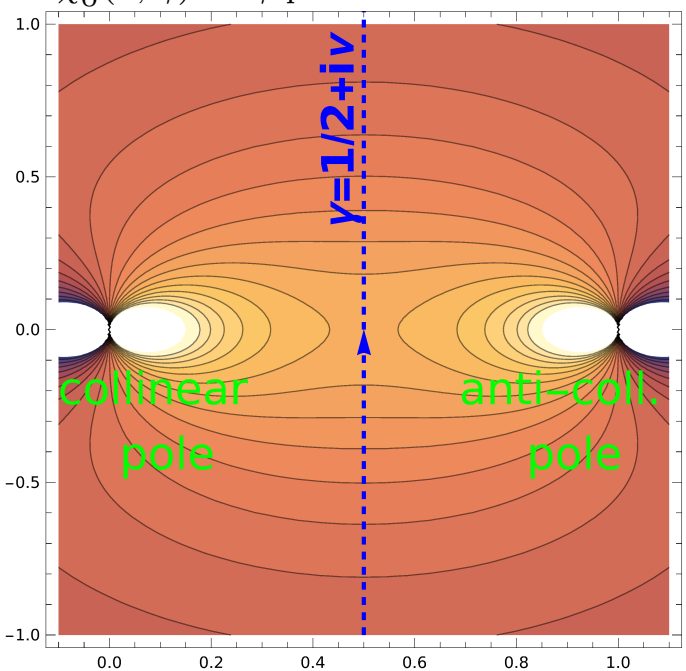
A useful mathematical tool is the *BFKL characteristic function*:

$$\int\limits_{\mathbf{q}} K(\mathbf{k},\mathbf{q})(\mathbf{q}^2)^\gamma e^{in\phi_\mathbf{q}} = \frac{\alpha_s(\mathbf{k}^2)N_c}{\pi}\chi(n,\gamma)(\mathbf{k}^2)^\gamma e^{in\phi_\mathbf{k}},$$

where $\chi(n,\gamma) = \chi_0(n,\gamma) + a_s(\mathbf{k}^2)\chi_1(n,\gamma) + \ldots$

Then $G(Y,\mathbf{k}_1,\mathbf{k}_2) \propto \sum\limits_n \int\limits_\gamma \exp\left[Y\frac{\alpha_s N_c}{\pi}\chi(n,\gamma)\right]\left(\frac{\mathbf{k}_1^2}{\mathbf{k}_2^2}\right)^\gamma e^{in\phi}.$

Re $\chi_0(0,\gamma)$ in γ-plane:



$\chi(0,1/2+i\nu)$, dashed line – LO, dash-dotted line – NLO, solid line – a version of resummation for leading collinear poles:

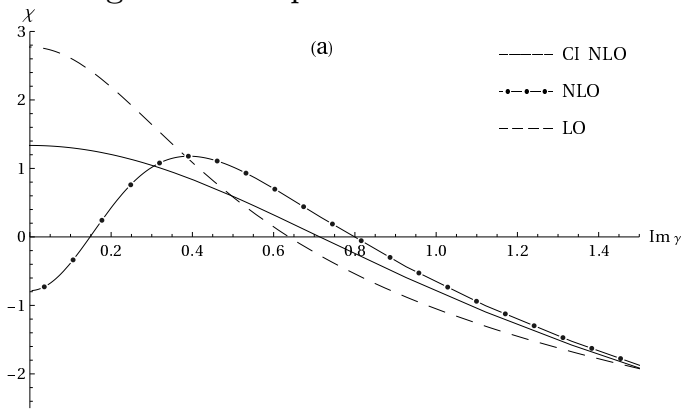


Figure from [Chernyshev,MN,Saleev, 2025]

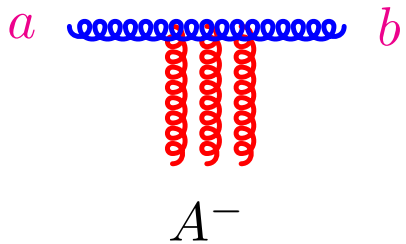
Corrections to χ are large and mostly come from poles at $\gamma = 0, 1$!

Let's "rise the sea level" *

Scattering amplitude of a gluon on a dense target (e.g. a nucleus A):

$$g_a(k) + A \to g_b(k') + A,$$

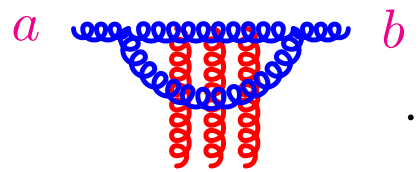
in the Regge limit ($k^+ \to \infty$) is described in the eikonal approximation, the S-matrix is:



$$= S_{ab}(\mathbf{x}) = \exp\left[ig \int_{-\infty}^{+\infty} dx^+ T^c A_c^-(x^+, 0^-, \mathbf{x})\right]_{ab}.$$

If the projectile gluon is boosted ($k^+ \to e^{\delta Y} k^+$) then the radiative correction includes:

$$\alpha_s \delta Y \times$$



.

Which leads to the JIMWLK-evolution for the projectile $\mathcal{O}$.

* A. Grothendieck, "Recoltes et Semailles"

JIMWLK equation

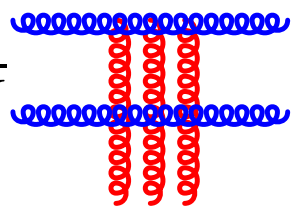
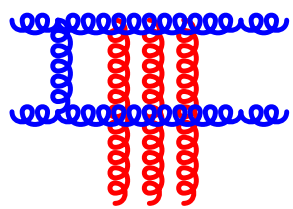
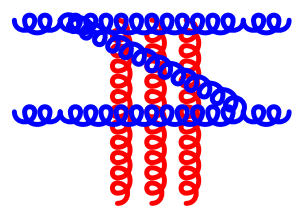
The operator of the projectile evolves with rapidity as:

$$\frac{\partial}{\partial Y}\mathcal{O} = \hat{H}_{JIMWLK}\mathcal{O},$$

where:

$$\hat{H}_{JIMWLK} = \frac{\alpha_s(\mu^2)}{2\pi^2}\int_{\mathbf{x},\mathbf{y},\mathbf{z}} K(\mathbf{x},\mathbf{y},\mathbf{z})\left[J_L^a(\mathbf{x})J_L^a(\mathbf{y}) + J_R^a(\mathbf{x})J_R^a(\mathbf{y}) - S^{ab}(\mathbf{z})\left(J_L^a(\mathbf{x})J_R^b(\mathbf{y}) + J_L^a(\mathbf{y})J_R^b(\mathbf{x})\right)\right],$$

with $K(\mathbf{x},\mathbf{y},\mathbf{z}) = (\mathbf{x}-\mathbf{z})\cdot(\mathbf{y}-\mathbf{z})/[(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2]$

$\frac{\partial}{\partial Y}$  $=$  $+$  $+\ldots$

If the target is **not** dense, the JIMWLK picture reduces to BFKL and Wilson lines S^{ab} can be expanded in the Reggeized gluon fields $\alpha^a(\mathbf{x})$

[Lipatov '95;...;Kovner, Lublinsky, 2005;...;Caron-Huot 2012;...] :

$$S^{ab}(\mathbf{x}) = \exp\left[igT^c\alpha_c(\mathbf{x})\right]^{ab},\ (ig)J^a_{(L,R)} = \left[F(\pm iT^c\alpha^c(\mathbf{x}))\right]^{ab}\frac{\delta}{\delta\alpha^b(\mathbf{x})},$$

with $F(x) = x/(1-e^{-x})$.

(anti-)Collinear resummation in JIMWLK (gluons)

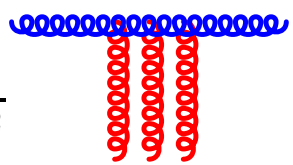
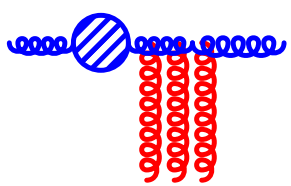
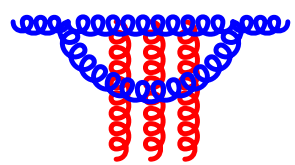
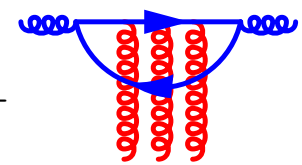
In [Kovner,Lublinsky,Skokov 2023] the method to resum large corrections to JIMWLK Hamiltonian, arising when $Q_T \gg Q_P$ was derived. One evolves the "bare" Wilson lines (S^{ab}) at the scale Q_T down to the scale Q_P ($\Rightarrow \mathbb{S}^{ab}_{Q_P}$):

$$\frac{\partial}{\partial \ln Q^2}\mathbb{S}_{ab}(\mathbf{x},Q) = -a\mathbb{S}_{ab}(\mathbf{x},Q)$$

$$-\frac{a}{\beta_0}\int\limits_0^1 d\xi \int\limits_0^{2\pi}\frac{d\phi}{2\pi}\Big[2N_c p_{gg}(\xi)\mathbb{D}_{ab}\left(\mathbf{x}+(1-\xi)Q^{-1}\mathbf{n}_\phi, \mathbf{x}-\xi Q^{-1}\mathbf{n}_\phi, Q\right)$$

$$+2T_F n_F p_{qg}(\xi)\mathbb{D}^{(F)}_{ab}\left(\mathbf{x}+(1-\xi)Q^{-1}\mathbf{n}_\phi, \mathbf{x}-\xi Q^{-1}\mathbf{n}_\phi, Q\right)\Big],$$

where $a = \alpha_s\beta_0/(4\pi)$, $\mathbb{D}_{ab}(\mathbf{x},\mathbf{y},Q) = \mathrm{tr}\left[T_a\mathbb{S}(\mathbf{x},Q)T_b\mathbb{S}^\dagger(\mathbf{y},Q)\right]/N_c$, $\mathbb{D}^{(F)}_{ab}(\mathbf{x},\mathbf{y},Q) = 2\,\mathrm{tr}\left[t_a\mathbb{V}(\mathbf{x},Q)t_b\mathbb{V}^\dagger(\mathbf{y},Q)\right]$.

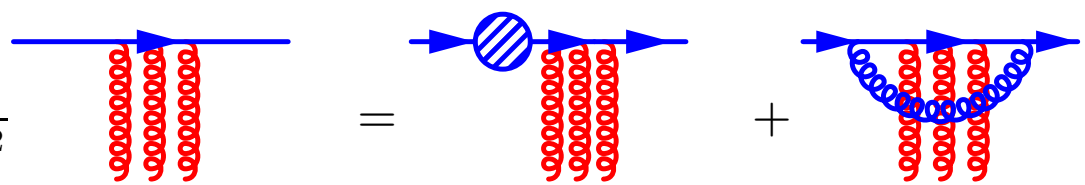
$\frac{\partial}{\partial \ln Q^2}$  $=$  $+$  $+$ 

(anti-)Collinear resummation in JIMWLK (quarks)

In [Kovner,Lublinsky,Skokov 2023] the method to resum large corrections to JIMWLK Hamiltonian, arising when $Q_T \gg Q_P$ was derived. One evolves the "bare" Wilson lines (V^{ij}) at the scale Q_T down to the scale Q_P ($\Rightarrow \mathbb{V}^{ij}_{Q_P}$):

$$\frac{\partial}{\partial \ln Q^2}\mathbb{V}_{ij}(\mathbf{x},Q) = -\frac{3C_F}{2}\frac{2a}{\beta_0}\mathbb{V}_{ij}(\mathbf{x},Q)$$
$$-\frac{2a}{\beta_0}\int\limits_0^1 d\xi \int\limits_0^{2\pi}\frac{d\phi}{2\pi}p_{gq}(\xi)\mathbb{D}^{(FA)}_{ij}\left(\mathbf{x}+(1-\xi)Q^{-1}\mathbf{n}_\phi, \mathbf{x}-\xi Q^{-1}\mathbf{n}_\phi, Q\right),$$

where $a = \alpha_s\beta_0/(4\pi)$, $\mathbb{D}^{(FA)}_{ij}(\mathbf{x},\mathbf{y},Q) = [t_a\mathbb{V}(\mathbf{y},Q)t_b]_{ij}\,\mathbb{S}_{ab}(\mathbf{x},Q)$.

$\frac{\partial}{\partial \ln Q^2}$  $=$ $+$

Resummed JIMWLK Hamiltonian

$$\hat{H}^{(\mathrm{res.})}_{JIMWLK} = \frac{\alpha_s(\mu^2)}{2\pi^2} \int\limits_{\mathbf{x},\mathbf{y},\mathbf{z}} K(\mathbf{x},\mathbf{y},\mathbf{z}) \left[J^a_L(\mathbf{x}) J^a_L(\mathbf{y}) + J^a_R(\mathbf{x}) J^a_R(\mathbf{y}) - \mathbb{S}^{ab}_Q(\mathbf{z}) \left(J^a_L(\mathbf{x}) J^b_R(\mathbf{y}) + J^a_L(\mathbf{y}) J^b_R(\mathbf{x}) \right) \right],$$

with the scale-choice [Kovner, Lublinsky, Skokov 2023] :

$$\boxed{Q^2 = \max((\mathbf{x}-\mathbf{z})^{-2}, (\mathbf{y}-\mathbf{z})^{-2}, Q_0^2)}$$

The task: linearize $\hat{H}^{(\mathrm{res.})}_{JIMWLK} \alpha_a(\mathbf{x}) \alpha_a(\mathbf{y})$, obtain the resummed BFKL equation, do the Fourier transform to momentum space and compute the characteristic function.

Fixed-coupling approximation

(There is no such thing in QCD but it is a useful first step....)

Linearising the resummed JIMWLK

Write an ansatz for $\mathbb{S}$ and $\mathbb{V}$ in terms of Reggeon fields ($\alpha(\mathbf{x})$):

$$\mathbb{S}_{ab}(\mathbf{z},Q) = \delta_{ab} + igT^{c_1}_{ab}\int_{\mathbf{z}_1} R^{(1)}_Q(\mathbf{z}-\mathbf{z}_1)\alpha_{c_1}(\mathbf{z}_1)$$

$$+\frac{(ig)^2}{2}(T^{c_1}T^{c_2})_{ab}\int_{\mathbf{z}_1\mathbf{z}_2} R^{(2,1)}_Q(\mathbf{z}-\mathbf{z}_1,\mathbf{z}-\mathbf{z}_2)\alpha_{c_1}(\mathbf{z}_1)\alpha_{c_2}(\mathbf{z}_2)+\ldots,$$

Other possible colour structures at $O(g^2)$ do not contribute when contracted with $\delta_{c_1c_2}$. Expand the resummed JIMWLK Hamiltonian up to $O(\alpha^2)$:

$$\hat{H}_{\text{lin.},Q} = \frac{\alpha_s}{2\pi^2}\int_{\mathbf{x},\mathbf{y},\mathbf{z}_{(1,2)}} \Bigg\{ f^{[c_1ab_1}f^{c_2]ab_2}\Bigg[-K\alpha^{c_1}(\mathbf{x})\alpha^{c_2}(\mathbf{y}) - K^{(2)}_Q\alpha^{c_1}(\mathbf{z}_1)\alpha^{c_2}(\mathbf{z}_2)$$

$$+K^{(1)}_Q\Big(\alpha^{c_1}(\mathbf{z})\alpha^{c_2}(\mathbf{y})+\alpha^{c_1}(\mathbf{x})\alpha^{c_2}(\mathbf{z})\Big)\Bigg]\frac{\delta^2}{\delta\alpha^{b_1}(\mathbf{x})\delta\alpha^{b_2}(\mathbf{y})}$$

$$+C_A\delta^{(2)}(\mathbf{x}-\mathbf{y})[K\alpha^b(\mathbf{x})-K^{(1)}_Q\alpha^b(\mathbf{z})]\frac{\delta}{\delta\alpha^b(\mathbf{x})}\Bigg\},$$

where $K^{(1)}_Q(\mathbf{x},\mathbf{y},\mathbf{z}) = \int_{\bar{\mathbf{z}}} K(\mathbf{x},\mathbf{y},\bar{\mathbf{z}})R^{(1)}_Q(\bar{\mathbf{z}}-\mathbf{z})$,

$K^{(2)}_Q(\mathbf{x},\mathbf{y},\mathbf{z}_1,\mathbf{z}_2) = \int_{\bar{\mathbf{z}}} K(\mathbf{x},\mathbf{y},\bar{\mathbf{z}})R^{(2,1)}_Q(\bar{\mathbf{z}}-\mathbf{z}_1,\bar{\mathbf{z}}-\mathbf{z}_2)$.

Resummed BFKL kernel

The scale-choice [Kovner, Lublinsky, Skokov, 2023] :
$\boxed{Q^2 = \max((\mathbf{x}-\mathbf{z})^{-2}, (\mathbf{y}-\mathbf{z})^{-2}, Q_0^2)}$. Act $\hat{H}_{\text{lin.},Q}\alpha_c(\mathbf{x})\alpha_c(\mathbf{y})$ and do the Fourier-transform:

$$K^{(\text{res.})}_{\text{BFKL}}(\mathbf{k},\mathbf{q}) = \frac{\alpha_s N_c}{2\pi^2}\frac{\mathbf{k}^4}{\mathbf{q}^4}\Big\{ - \frac{\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2_+}$$

$$-2\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{q})}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}\int\limits_0^\infty \frac{dQ^2}{Q^2}\frac{\partial R_Q^{(1)}(\mathbf{q})}{\partial \ln Q^2} J_0\left(\frac{|\mathbf{k}|}{Q}\right) J_0\left(\frac{|\mathbf{k}-\mathbf{q}|}{Q}\right) \boxed{+\frac{1}{\mathbf{k}^2}\int\limits_0^\infty \frac{dQ^2}{Q^2}\frac{\partial R_Q^{(2,1)}(\mathbf{q})}{\partial \ln Q^2} J_0^2\left(\frac{|\mathbf{k}|}{Q}\right)}$$

$$-\pi\delta^{(2)}(\mathbf{k}-\mathbf{q})\int\limits_0^\infty \frac{dQ^2}{Q^2}\frac{\partial R_Q^{(1)}(\mathbf{q})}{\partial \ln Q^2}\int\limits_0^{Q^2}\frac{dq^2}{q^2} J_0\left(\frac{|\mathbf{k}|}{q}\right) + (\mathbf{q}\to -\mathbf{q})\Big\},$$

where $R_Q^{(1)}(\mathbf{q}) = \int\limits_{\mathbf{z}} R_Q^{(1)}(\mathbf{z}) e^{i\mathbf{q}\mathbf{z}}$, $R_Q^{(2,1)}(\mathbf{q}) = \int\limits_{\mathbf{z}_{1,2}} R_Q^{(2,1)}(\mathbf{z}_1,\mathbf{z}_2) e^{i\mathbf{q}(\mathbf{z}_1-\mathbf{z}_2)}$

and $\int\limits_{\mathbf{q}} \frac{f(\mathbf{q})}{(\mathbf{k}-\mathbf{q})^2_+} = \int\limits_{\mathbf{q}} \frac{f(\mathbf{q})-f(\mathbf{k})\theta(|\mathbf{k}-\mathbf{q}|<|\mathbf{k}|)}{(\mathbf{k}-\mathbf{q})^2}$.

Resummed BFKL kernel, θ-approximation

Let's substitute $J_0(x) \to \theta(1-x)$.

$$K_{\rm BFKL}^{(\text{res.},\theta\text{-approx})}(\mathbf{k},\mathbf{q}) = \frac{\alpha_s N_c}{2\pi^2}\frac{\mathbf{k}^4}{\mathbf{q}^4}\Bigg\{ -\frac{\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2_+}\boxed{+\frac{1}{\mathbf{k}^2}\left(1-R^{(2,1)}_{|\mathbf{k}|}(\mathbf{q})\right)}$$
$$-2\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{q})}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}\left(1-R^{(1)}_{\max(|\mathbf{k}|,|\mathbf{k}-\mathbf{q}|)}(\mathbf{q})\right)$$
$$-\pi\delta^{(2)}(\mathbf{k}-\mathbf{q})\int\limits_{\mathbf{q}^2}^{\infty}\frac{dQ^2}{Q^2}\frac{\partial R^{(1)}_Q(\mathbf{q})}{\partial \ln Q^2}\ln\frac{Q^2}{\mathbf{q}^2} + (\mathbf{q}\to-\mathbf{q})\Bigg\}.$$

The strict LP($\mathbf{k}^2 \ll \mathbf{q}^2$) and LLA: $\left(\alpha_s \ln(\mathbf{q}^2/\mathbf{k}^2)\right)^n/\mathbf{k}^2$ is contained in the first line.

Linearised evolution for $\mathbb{S}_Q$ and $\mathbb{V}_Q$: $R^{(1)}$ and $r^{(1)}$

The quark S-matrix is expanded in a similar way:

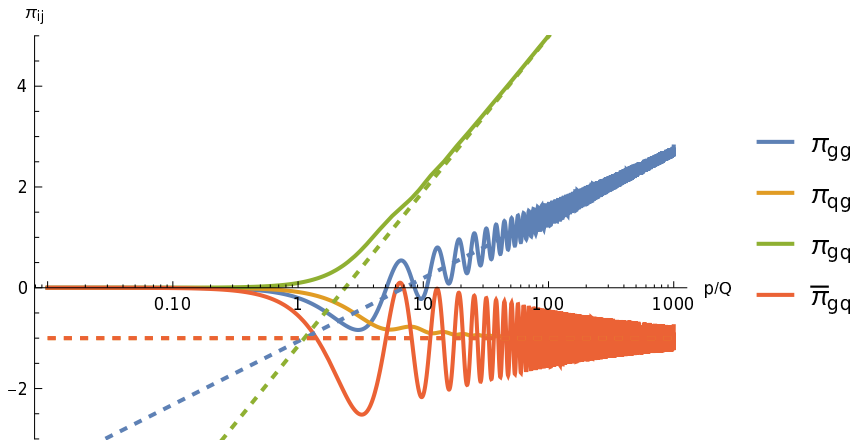
$$\mathbb{V}_{ij}(\mathbf{z}) = \delta_{ab} + igt^{c_1}_{ij} \int_{\mathbf{z}_1} r^{(1)}_Q(\mathbf{z}-\mathbf{z}_1)\alpha_{c_1}(\mathbf{z}_1)$$

$$+\frac{(ig)^2}{2}(t^{c_1}t^{c_2})_{ij} \int_{\mathbf{z}_1\mathbf{z}_2} r^{(2,1)}_Q(\mathbf{z}-\mathbf{z}_1,\mathbf{z}-\mathbf{z}_2)\alpha_{c_1}(\mathbf{z}_1)\alpha_{c_2}(\mathbf{z}_2)+\ldots,$$

Then the evolution for $\mathcal{R}^{(1)}_Q(\mathbf{p}) = \begin{pmatrix} R^{(1)}_Q(\mathbf{p}) \\ r^{(1)}_Q(\mathbf{p}) \end{pmatrix}$ is:

$$\frac{\partial}{\partial \ln Q^2}\mathcal{R}^{(1)}_Q(\mathbf{p}) = -\frac{a}{\beta_0}\Pi_1(|\mathbf{p}|/Q)\mathcal{R}^{(1)}_Q(\mathbf{p}),$$

$$\Pi_1(p) = \begin{pmatrix} -\frac{11N_c}{3}\pi_{gg}(p) & \frac{2n_F}{3}\pi_{qg}(p) \\ -\frac{3}{2}N_c\bar{\pi}_{gq}(p) & \frac{3}{2}\frac{1}{N_c}\pi_{gq}(p) \end{pmatrix},$$



with $\pi_{ij}(p) = \dfrac{\int_0^1 d\xi\, p_{ij}(\xi)\, J_0(p\xi)}{\int_0^1 d\xi\, p_{ij}(\xi)} - 1,$

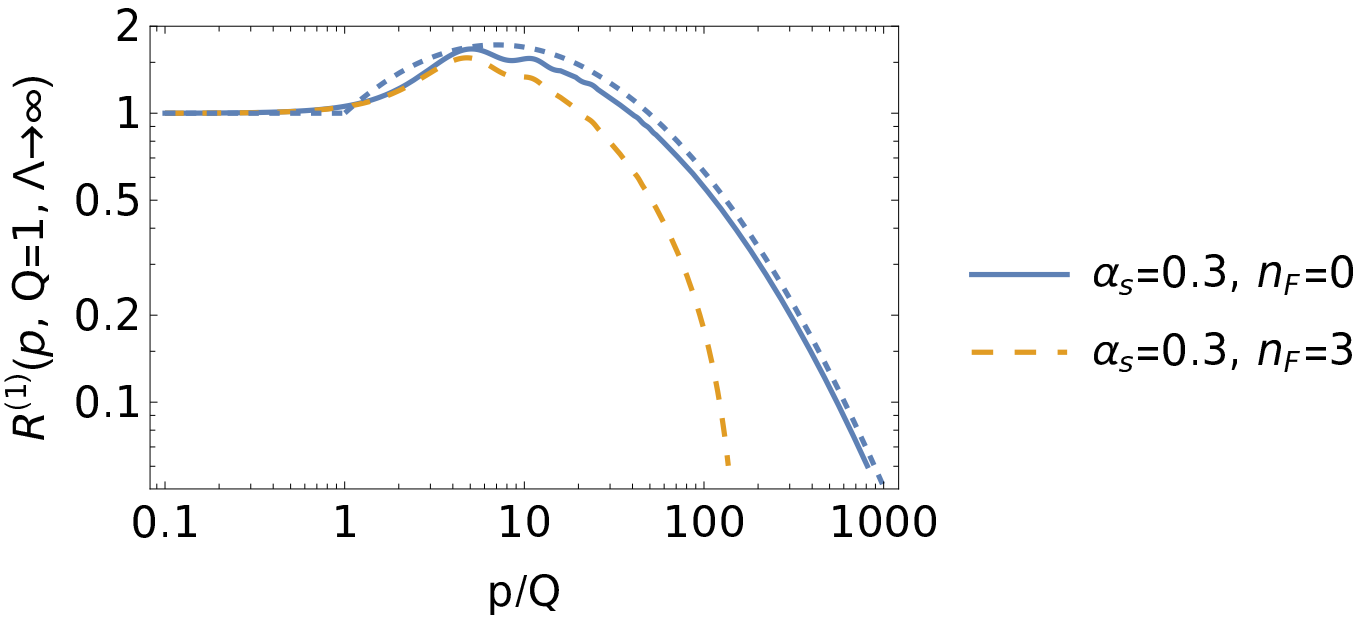
$$\bar{\pi}_{ij}(p) = \frac{\int_0^1 d\xi\, p_{ij}(1-\xi)\, J_0(p\xi)}{\int_0^1 d\xi\, p_{ij}(\xi)} - 1.$$

$$\begin{aligned} \pi_{gg}(p) &\simeq \frac{6}{11}\left(\ln p - \frac{11}{6} - \ln 2 + \gamma_E\right), \\ \pi_{gq}(p) &\simeq \frac{4}{3}\left(\ln p - \frac{3}{4} - \ln 2 + \gamma_E\right), \\ \bar{\pi}_{gq}(p) &\simeq \pi_{qg}(p) \simeq -1. \end{aligned}$$

Sudakov suppression for $R^{(1)}$

Without quarks, the LLA solution for $R^{(1)}$ for $\mathbf{p}^2 \gg Q^2$ looks as follows:

$$R_Q^{(1)}(\mathbf{p}) \simeq \exp\left[-a_s N_c \left(\frac{1}{2} \ln^2 \frac{Q^2}{\mathbf{p}^2} + \left(\frac{11}{3} + 2(\ln 2 - \gamma_E)\right) \ln \frac{Q^2}{\mathbf{p}^2} \right) \right].$$



Linearised evolution for $\mathbb{S}_Q$ and $\mathbb{V}_Q$: $R^{(2,1)}$ and $r^{(2,1)}$

$$\frac{\partial}{\partial \ln Q^2}\mathcal{R}_Q^{(2,1)}(\mathbf{p}) = -\frac{a}{\beta_0}\left[\Pi_2 \mathcal{R}_Q^{(2,1)}(\mathbf{p}) + J_0\left(\frac{|\mathbf{p}|}{Q}\right)\rho_Q^{(1)}(\mathbf{p})\right],$$

$$\rho_Q^{(1)}(\mathbf{p}) = \begin{pmatrix} \frac{11}{3}N_c[R_Q^{(1)}(\mathbf{p})]^2 + \frac{2n_F}{3N_c^2}[r_Q^{(1)}(\mathbf{p})]^2 \\ 3N_c R_Q^{(1)}(\mathbf{p}) r_Q^{(1)}(\mathbf{p}) \end{pmatrix},$$

$$\Pi_2 = \begin{pmatrix} \beta_0 - 2N_c\frac{11}{3} & \frac{4C_F n_F}{3N_c} \\ -3N_c & 0 \end{pmatrix}.$$

We approximate $J_0(x) \to \theta(1-x)$ and then we need only that $R^{(1)}(|\mathbf{p}| < Q) = r^{(1)}(|\mathbf{p}| < Q) = 1$. This way we obtain the LLA solution for the $R_Q^{(2,1)}$ (for $Q_T \to \infty$):

$$R_Q^{(2,1)}(\mathbf{p}) = \frac{3\beta_0 - 11N_c^3 - 11N_c - 3N_c^2\lambda_+}{3N_c^2(\lambda_- - \lambda_+)}\left(\frac{\max(Q^2,\mathbf{p}^2)}{Q^2}\right)^{\frac{a}{\beta_0}\lambda_-} + (\lambda_+ \leftrightarrow \lambda_-),$$

where $\lambda_\pm = \frac{\beta_0}{2} - \frac{11}{3}N_c \pm \sqrt{\left(\frac{\beta_0}{2} - \frac{11}{3}N_c\right)^2 - 4C_F n_F}$.

LLA-resummed BKFL kernel

Only one of the terms contribute in the LLA:

$$K_D^{(\text{res., LLA})}(\mathbf{k},\mathbf{q}) = \frac{\alpha_s N_c}{2\pi^2}\frac{\mathbf{k}^4}{\mathbf{q}^4}\left\{-\frac{\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}+\mathbf{q})_+^2}+\frac{1}{\mathbf{k}^2}\left(1-R^{(2,1)}_{\max(Q_0^2,\mathbf{k}^2)}(\mathbf{q})\right)\right\},$$

with $Q_0^2 = \mathbf{k}^2$. The LLA solution for the $R_Q^{(2,1)}$:

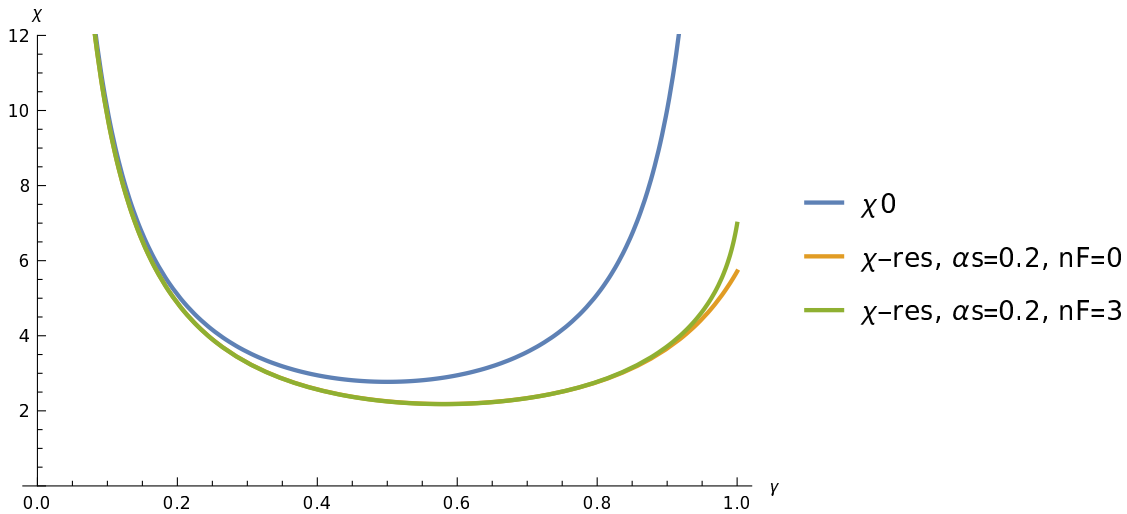
$$R_Q^{(2,1)}(\mathbf{p}) = \frac{3\beta_0 - 11N_c^3 - 11N_c - 3N_c^2\lambda_+}{3N_c^2(\lambda_- - \lambda_+)}\left(\frac{\max(Q^2,\mathbf{p}^2)}{Q^2}\right)^{\frac{a}{\beta_0}\lambda_-} + (\lambda_+ \leftrightarrow \lambda_-),$$

where $\lambda_\pm = \frac{\beta_0}{2} - \frac{11}{3}N_c \pm \sqrt{\left(\frac{\beta_0}{2} - \frac{11}{3}N_c\right)^2 - 4C_F n_F}$.

$$\int\limits_{\mathbf{q}} K(\mathbf{k},\mathbf{q})(\mathbf{q}^2)^\gamma e^{in\phi_{\mathbf{q}}} = \frac{\alpha_s(\cancel{\mathbf{k}^2})N_c}{\pi}\chi(n,\gamma)(\mathbf{k}^2)^\gamma e^{in\phi_{\mathbf{k}}},$$

LLA-resummed characteristic function

$$\chi^{(\mathrm{res.})}(\gamma) = \chi_0(n,\gamma) + \delta_{n,0}\left[\frac{1}{\gamma-1} - \frac{N_c(\gamma-1) - \frac{4}{3}a_s C_F n_F}{N_c\left[(\gamma-1)^2 - a_s(\gamma-1)\left(\frac{22}{3}N_c - \beta_0\right) + 4a_s^2 C_F n_F\right]}\right].$$

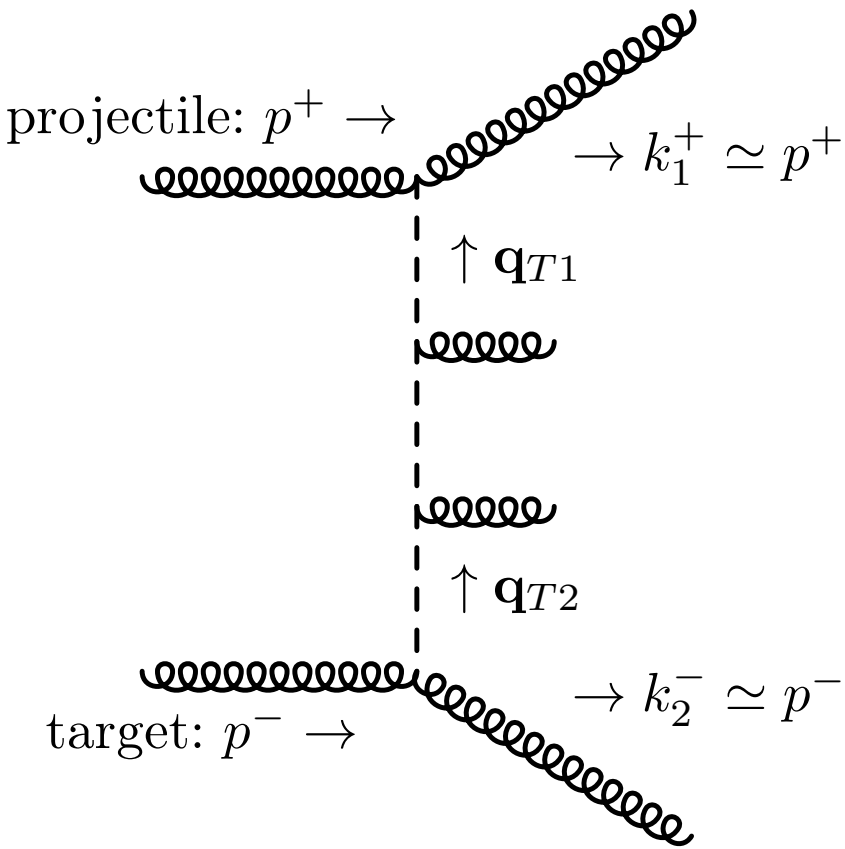


- The anti-collinear pole is removed. Nothing happens to the collinear pole
- The χ-value at the saddle point (*Pomeron intercept*) is decreased, which is good for phenomenology
- The effect of quarks is numerically small

Expanding in α_s we obtain:

$$\chi^{(\mathrm{res.})}(\gamma,n) = \chi_0(\gamma) - \frac{\alpha_s N_c}{4\pi}\frac{11 + 2n_F/N_c^3}{3(1-\gamma)^2}\delta_{n,0} + O(\alpha_s^2),$$

Cheat-sheet on asymmetric schemes in BKFL



$$s = p^+ p^-,\ k_1^- = \frac{\mathbf{q}_{T1}^2}{p^+},\ k_2^+ = \frac{\mathbf{q}_{T2}^2}{p^-},$$

$$L_+ = \ln \frac{k_1^+}{k_2^+} = \ln \frac{s}{\mathbf{q}_{T2}^2},\ \text{(JIMWLK)}$$

$$L_- = \ln \frac{k_2^-}{k_1^-} = \ln \frac{s}{\mathbf{q}_{T1}^2},\ \text{(DIS)}$$

$$Y = \frac{1}{2} \ln \frac{k_1^+}{k_1^-} - \frac{1}{2} \ln \frac{k_2^+}{k_2^-} = \ln \frac{s}{\sqrt{\mathbf{q}_{T1}^2 \mathbf{q}_{T2}^2}}.\ \text{(BFKL)}$$

$$\chi_\mp(\gamma, n) = \chi_0(\gamma, n) \mp \frac{\alpha_s N_c}{4\pi} 2\chi_0'(\gamma, n)\chi_0(\gamma, n) + O(\alpha_s^2),$$

So in $(+)$ scheme the $1/(1-\gamma)^3$-pole cancels and in the $(-)$-scheme the $1/\gamma^3$-pole cancels.

Comparison with NLO BKFL

Pole structure of the NLO characteristic function from [Kotikov, Lipatov, 2000] **(symmetric scheme!)**:

$$\chi(n,\gamma) = \chi_0(n,\gamma) + \frac{\alpha_s(\mathbf{k}^2)}{4\pi}\delta(n,\gamma),$$

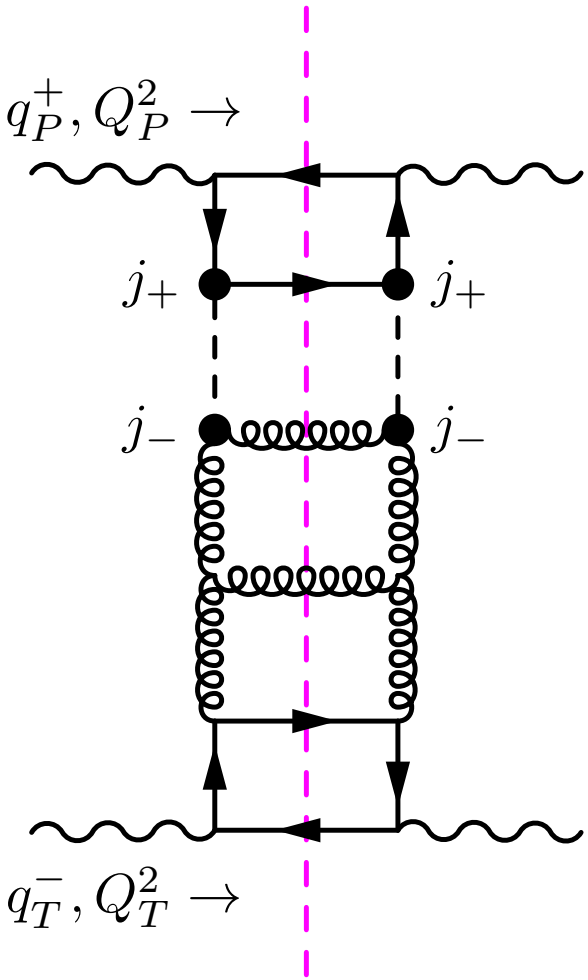
$$\delta(n,\gamma) = -\frac{2}{(\gamma+n/2)^3} - \frac{\delta_{n,0}}{\gamma^2}\left(\frac{11}{3}+\frac{2n_F}{3N_c^3}\right) + 2\frac{\psi(1)-\psi(n+1)}{\left(\gamma+\frac{n}{2}\right)^2}$$

$$-\frac{2}{\left(1-\gamma+\frac{n}{2}\right)^3}\boxed{-\frac{\delta_{n,0}}{(1-\gamma)^2}\left(\frac{11}{3}+\frac{2n_F}{3N_c^3}\right)} + \frac{-\frac{11}{3}+\frac{2n_F}{3N_c}+2\big(\psi(1)-\psi(n+1)\big)}{\left(1-\gamma+\frac{n}{2}\right)^2}$$

$$+\ldots$$

The $1/(1-\gamma+n/2)^3$ pole cancels in the (+)-scheme. The $\delta_{n,0}$-term is reproduced by the resummation. The red term is related with the running-coupling effects. The blue tems are remnants of the leading (cubic) C and AC poles.

$\gamma^*\gamma^*$-scattering and DGLAP



- In the BFKL approach **at fixed coupling**, the amplitude is ($x_B^{(P/T)} = Q^2_{P/T}/s$):

$$\mathrm{Im}\,\mathcal{A}^{(i,j)} \propto \int \frac{d\omega}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{d\gamma}{2\pi i} \left(x_B^{(T)}\right)^{-\omega} \left(\frac{Q_P^2}{Q_T^2}\right)^{\gamma-1} \times \frac{\phi_0^{(i)}(1-\gamma)\phi_0^{(j)}(\gamma)}{\omega - \hat{\alpha}_s \chi_+(0,\gamma)}.$$

- For $Q_T \gg Q_P$ the pole of the denominator dominates(DGLAP-BFKL duality):

$$\boxed{\frac{\alpha_s N_c}{\pi} \chi_+(0, 1-\gamma^{(T)}(\omega)) = \omega.}$$

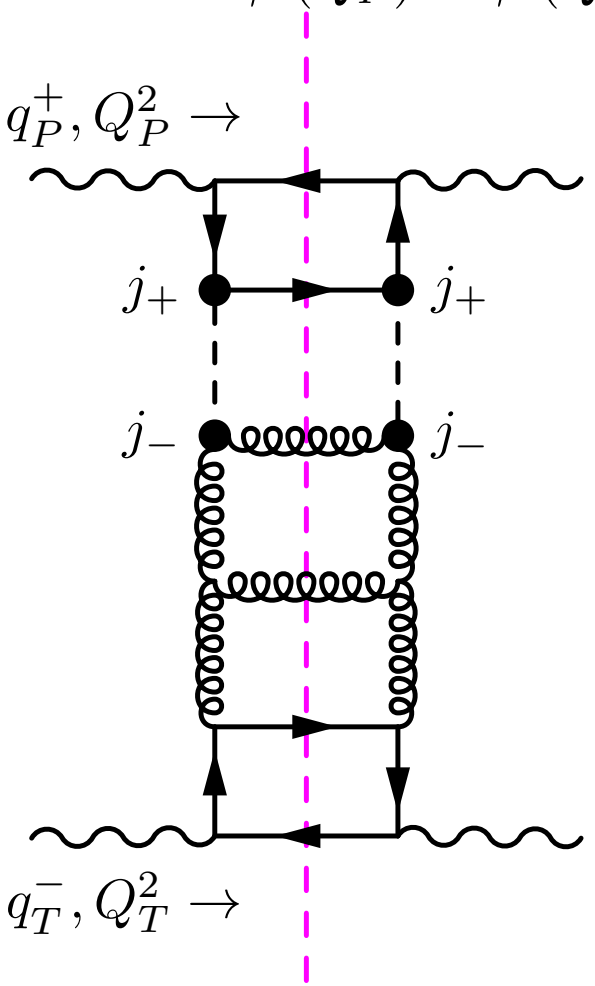
Or for $Q_P \gg Q_T$ (and $\omega \leftrightarrow x_B^{(P)}$):

$$\frac{\alpha_s N_c}{\pi} \chi_-(0, \gamma^{(P)}(\omega)) = \omega.$$

Fixed coupling: $\chi_+(n,\gamma) = \chi_-(n, 1-\gamma)$, so that $\gamma^{(P)}(\omega) = \gamma^{(T)}(\omega)$.

Cross-check: Bjorken limit of the target (Fixed coupling!)

We can cross-check the result by considering the high-energy limit of the $\gamma^*(Q_P) + \gamma^*(Q_T)$-scattering amplitude in the limit $Q_T \gg Q_P$.



- In the BFKL approach, the amplitude $\propto \int\limits_{\omega} (x_B^{(T)})^{\omega} \left(\frac{Q_P^2}{Q_T^2}\right)^{\gamma_\star^{(T)}(\omega)}$, where $x_B^{(T)} = Q_T^2/s$ and the anomalous dimansion $\gamma_\star$ is determined by the equation:

$$\frac{\alpha_s N_c}{\pi} \chi(0, 1 - \gamma_\star^{(T)}(\omega)) = \omega.$$

- The LO solution of this equation:

$$\gamma_\star^{(T)}(\omega) = \frac{\alpha_s}{2\pi}\left[\frac{2N_c}{\omega} + \left(\frac{\beta_0}{2} - \frac{11}{3}N_c + \frac{2n_F}{3N_c}C_F\right) + O(\omega)\right] + \ldots,$$

agrees with the expectation from DGLAP.

$$\frac{\partial}{\partial \ln \mu^2}\begin{pmatrix} f_g(\omega, \mu^2) \\ f_q(\omega, \mu^2) \end{pmatrix} = \frac{\alpha_s}{2\pi}\Gamma \begin{pmatrix} f_g(\omega, \mu^2) \\ f_q(\omega, \mu^2) \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \frac{2N_c}{\omega} + \left(\frac{\beta_0}{2} - \frac{11}{3}N_c\right) & \frac{4n_F C_F}{\omega} - 3n_F C_F \\ \frac{1}{3} & 0 \end{pmatrix} + O(\omega),$$

Comparison with duality at $n_F = 0$

In the approach of [Altarelli, Ball, Forte, 2000] the duality equation

$$\chi_-(0, \gamma_{gg}(\omega)) = \frac{\pi\omega}{\alpha_s N_c} = \chi_+(0, 1 - \gamma_{gg}(\omega)),$$

is assumed to hold exactly **at fixed coupling** in gluodynamics($n_F = 0$). We also know that $\gamma_{gg}(1) = 0$ for $n_F = 0$ exactly, due to momentum sum-rule, so one predicts:

$$\chi_+(0, 1) = \frac{\pi}{\alpha_s N_c}.$$

From our resummation we have:

$$\chi_+^{(\text{anti-coll, LLA})}(0, \gamma \to 1) = \frac{\pi}{\alpha_s N_c} \begin{cases} 4/3 & \text{for } n_F \neq 0, \\ 12/11 & \text{for } n_F = 0. \end{cases}$$

Also if one takes the $\gamma_{gg}^{(LO)} = 2C_A\left(\frac{1}{\omega} - \frac{1}{\omega+1} + \frac{1}{\omega+2} - \frac{1}{\omega+3} - H_{\omega+1}\right) + \frac{\beta_0}{2}$ then one predicts the series:

$\chi_+(\gamma) = \frac{1}{1-\gamma} + c_1 \frac{a_s N_c}{(1-\gamma)^2} + c_2 \frac{(a_s N_c)^2}{(1-\gamma)^3} + c_3 \frac{(a_s N_c)^3}{(1-\gamma)^4} + \ldots$

c_i	Duality	Our resummation ($n_F = 0$)
c_1	$-11/3$	$-11/3$
c_2	$389/9 - \mathbf{16\zeta(2)}$	$121/9$
c_3	$-4493/9 + 176\zeta(2) + 64\zeta(3)$	$-1331/27$

Running-coupling effects

(Ongoing. Many results which need to be understood...)

Running coupling in JIMWLK

The virtual (JSJ) part of the NLO JIMWLK kernel minus the virtual part of the resummation expanded up to NLO:

$$K_{JSJ}^{(\mathrm{NLO})} - K_{JSJ}^{(\text{res. NLO})} = \frac{\alpha_s^2 X\cdot Y}{4\pi^3 X^2 Y^2}\left(\beta_0\left[\frac{1}{2}\ln[X^2\mu^2] + \frac{1}{2}\ln[Y^2\mu^2] - \frac{1}{2}\ln\frac{\mu^2}{Q^2}\right] + \dots\right).$$

If one puts $Q^2 = \max(X^{-2}, Y^{-2})$ then the optimal choice for μ^2 is

$$\boxed{\mu^2 = \min(X^{-2}, Y^{-2}).}$$

When comparing to NLO BFKL results one must understand that BFKL Green's function is a correlator $\langle \rho_T^a(\mathbf{x})\rho_T^a(\mathbf{y})\rangle$ rather than $\langle \alpha^a(\mathbf{x})\alpha^a(\mathbf{y})\rangle$, where

$$\nabla^2\alpha^a(\mathbf{x}) = g\rho(\mathbf{x}) + O(g^2\alpha^2),$$

and the scale-choice for g matters even at LLA. It turns-out that the classic NLO BFKL results are reproduced if one chooses the scales as follows ($\equiv$ [Balitsky, Chirilli, 2008: NLO BK]):

$$\frac{\partial}{\partial Y}\langle \rho_T^a(\mathbf{k})\rho_T^a(-\mathbf{k})\rangle_Y = -\int_{\mathbf{q}} \frac{\mathbf{k}^4\alpha_s(\mathbf{q}^2)}{\mathbf{q}^4\alpha_s(\mathbf{k}^2)} K_{\langle\alpha\alpha\rangle}(\mathbf{k},\mathbf{q})\langle \rho_T^a(\mathbf{q})\rho_T^a(-\mathbf{q})\rangle_Y.$$

LLA-resummed BKFL kernel

"After a tedious but trivial computation..." [Landau, Lifshitz, vol.2]

The kernel in θ-approximation becomes:

$$\Delta K^{(\theta)}_{\langle\rho\rho\rangle}(\mathbf{k},\mathbf{q}) = -\frac{\mathbf{k}^4\alpha_s(\mathbf{q}^2)}{\mathbf{q}^4\alpha_s(\mathbf{k}^2)}\frac{N_c}{2\pi^2}\left[\frac{\alpha_s((\mathbf{q}-\mathbf{k})^2)}{(\mathbf{k}-\mathbf{q})^2_+} + \frac{\alpha_s(\mathbf{k}^2)}{\mathbf{k}^2}\right.$$
$$-\frac{2\mathbf{k}\cdot(\mathbf{k}-\mathbf{q})}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}\alpha_s(\min[\mathbf{k}^2,(\mathbf{k}-\mathbf{q})^2]) + \boxed{\frac{\alpha_s(\mathbf{k}^2)}{\mathbf{k}^2}\left(R^{(2,1)}(\mathbf{q},|\mathbf{k}|)-1\right)}$$
$$\left.-\frac{2\mathbf{k}\cdot(\mathbf{k}-\mathbf{q})}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}\alpha_s(\min[\mathbf{k}^2,(\mathbf{k}-\mathbf{q})^2])\left(R^{(1)}(\mathbf{q},\max[|\mathbf{k}|,|\mathbf{k}-\mathbf{q}|])-1\right)\right]$$
$$+(\text{terms}\propto\delta^{(2)}(\mathbf{q}-\mathbf{k}))+(\mathbf{q}\to-\mathbf{q}).$$

So the LLA-resummed kernel with running coupling in the limit $\mathbf{q}^2 \gg \mathbf{k}^2$ is:

$$\boxed{K^{(\text{AC, LLA})}_{\langle\rho\rho\rangle,+}(\mathbf{k},\mathbf{q}) = \frac{N_c}{\pi^2}\left[\frac{\alpha_s(\mathbf{q}^2)\mathbf{k}^2}{\mathbf{q}^2(\mathbf{k}-\mathbf{q})^2_+} + \frac{\alpha_s(\mathbf{q}^2)\mathbf{k}^2}{\mathbf{q}^4}(R^{(2,1)}_{Q=|\mathbf{k}|}(\mathbf{q})-1)\right].}$$

Solution for $R_Q^{(2,1)}$ with running coupling

$$R^{(2,1)}_{Q,\mathrm{LLA}+\alpha_s}(\mathbf{p}) = \frac{3\beta_0 - 11N_c^3 - 11N_c - 3N_c^2\lambda_+}{3N_c^2(\lambda_- - \lambda_+)} \left(\frac{a_s(Q^2)}{a_s(\max[Q^2,\mathbf{p}^2])}\right)^{\lambda_-/\beta_0} + (\lambda_+ \leftrightarrow \lambda_-).$$

Conversion to the γ-space:

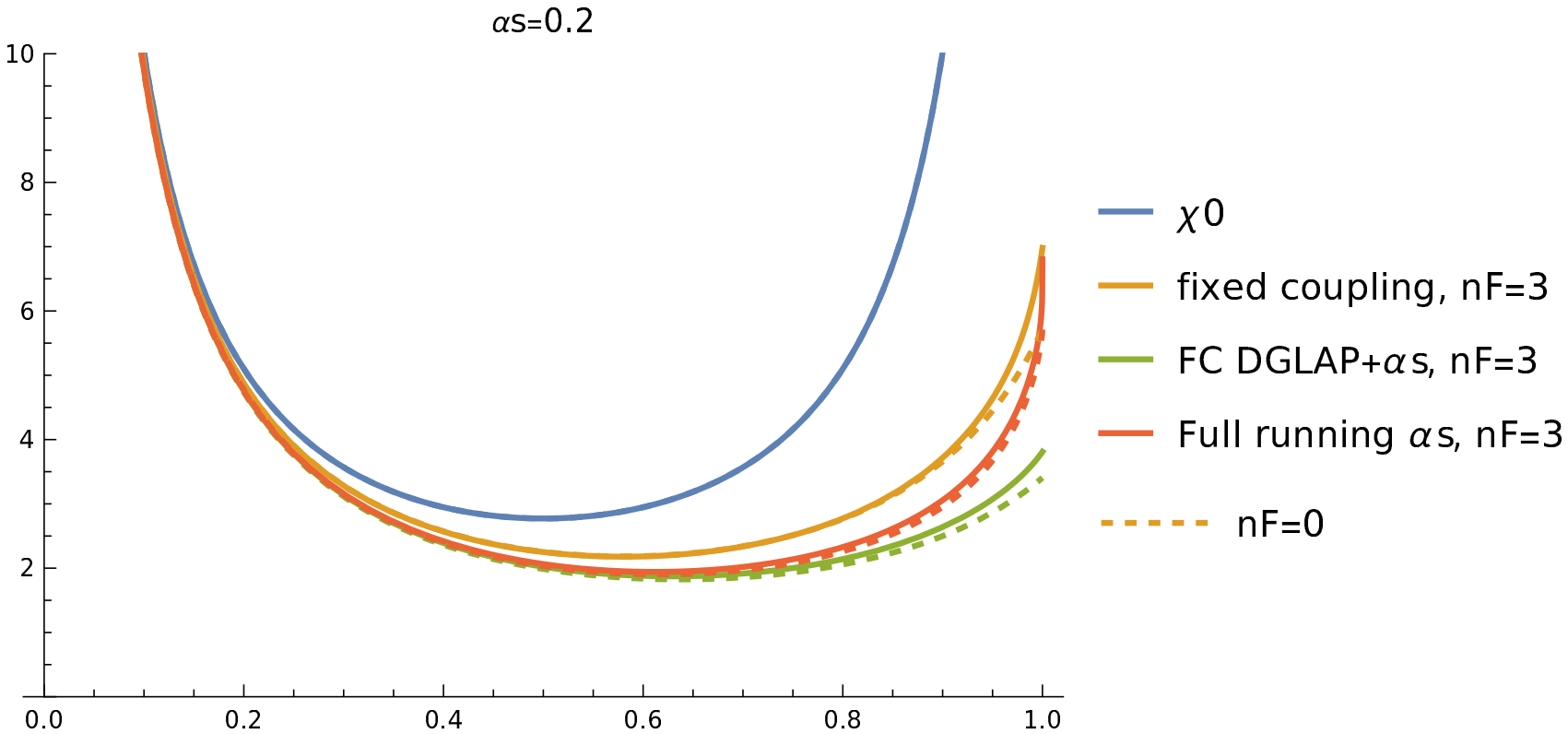
$$\left(\frac{a_s(\mathbf{k}^2)}{a_s(\mathbf{q}^2)}\right)^{\lambda/\beta_0} \to \widetilde{\Gamma}(1+\lambda/\beta_0, 1-\gamma),$$

where $\widetilde{\Gamma}(s,\gamma_0) = \int\limits_0^\infty dL\; e^{-\gamma_0 L}\big(1 + a_s(\mathbf{k}^2)\beta_0 L\big)^{s-1}$.

The following property helps with transition from $\chi_{\langle\alpha\alpha\rangle}$ to $\chi_{\langle\rho\rho\rangle}$:

$$\int\limits_{C_{\bar\gamma}} \frac{d\bar\gamma}{2\pi i}\widetilde{\Gamma}(1+\lambda/\beta_0, 1-\bar\gamma)\widetilde{\Gamma}(0,\bar\gamma-\gamma) = \widetilde{\Gamma}(\lambda/\beta_0, 1-\gamma).$$

Plots of $\chi(0,\gamma)$ with different treatment of RC



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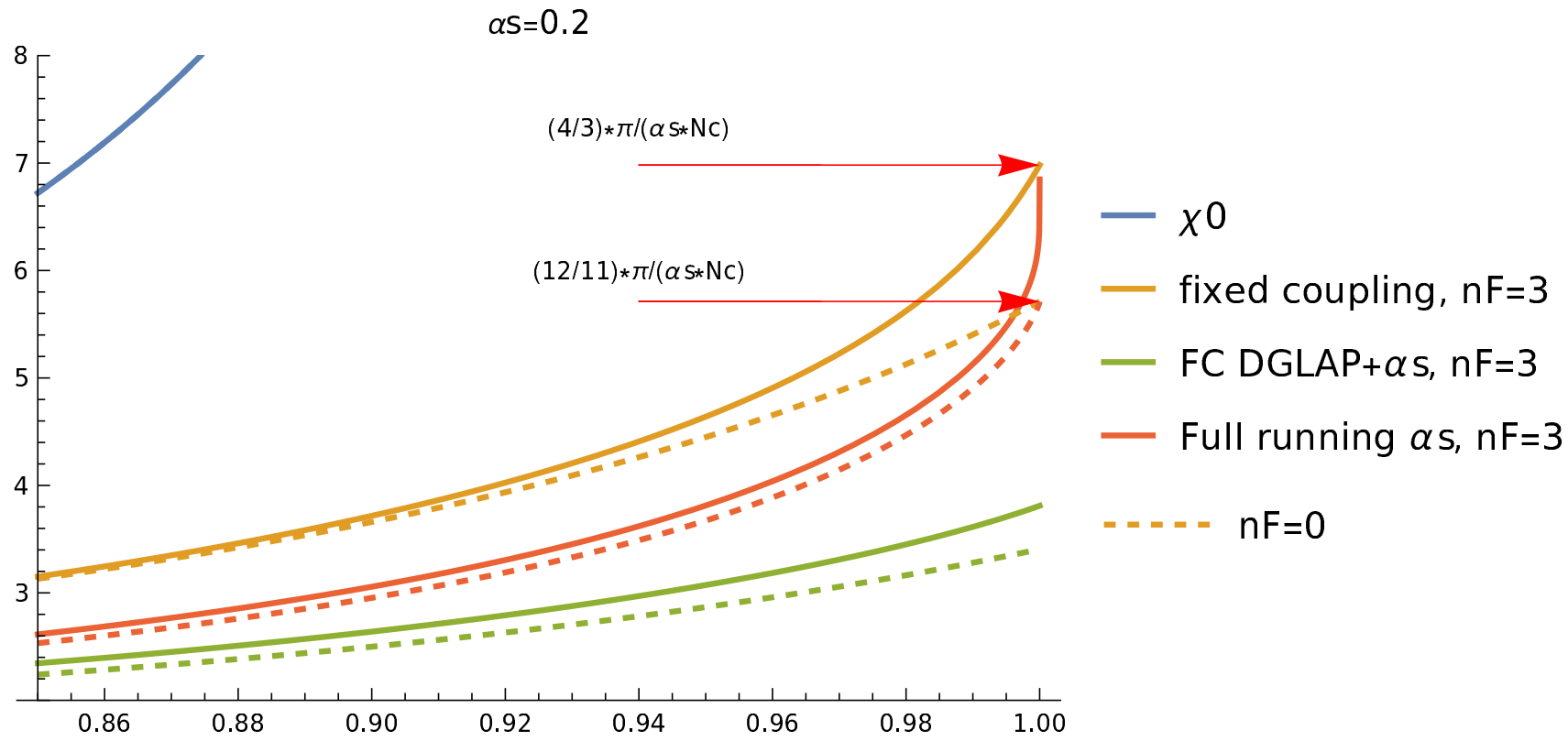


BFKL Green's function with running cupling

What the "running coupling" characteristic function is good for?

$$\int\limits_{\mathbf{q}} K_{\rm BFKL}(\mathbf{k},\mathbf{q})(\mathbf{q}^2)^\gamma e^{in\phi_{\mathbf{q}}} = -\frac{\alpha_s(\mathbf{k}^2)N_c}{\pi}\chi(n,\gamma)(\mathbf{k}^2)^\gamma e^{in\phi_{\mathbf{k}}},$$

note that
$\chi(n,\gamma) = \chi_0(n,\gamma) + a_s(\mathbf{k}^2)N_c\chi_1(n,\gamma) + (a_s(\mathbf{k}^2)N_c)^2\chi_2(n,\gamma) + \ldots$.
However if one neglects the $\mathbf{k}^2$-running in higher orders, then one can write the formal operatorial solution in γ-space:

$$\frac{\partial\phi_Y(\gamma)}{\partial Y} - \frac{\alpha_s(-\partial_\gamma)N_c}{\pi}\chi(0,\gamma)\phi_Y(\gamma) = 0$$

$$\Rightarrow\ \phi_Y(\gamma) = \exp\left[\hat{\alpha}_s Y\big(1 - a_s\beta_0\partial_\gamma\big)^{-1}\chi(\gamma)\right]\phi_0(\gamma),$$

where the 1-loop coupling is $\alpha_s(-\partial_\gamma) = \alpha_s(Q_T)\big(1 - a_s\beta_0\partial_\gamma\big)^{-1}$ and $\hat{\alpha}_s = \alpha_s(Q_T)N_c/\pi$.

(New???) Green's function with RC

For the Green's function one has $\phi_0(\gamma) = 1$ and it is possible to get rid of operators by the price of introducing couple new integrals. One obtains:

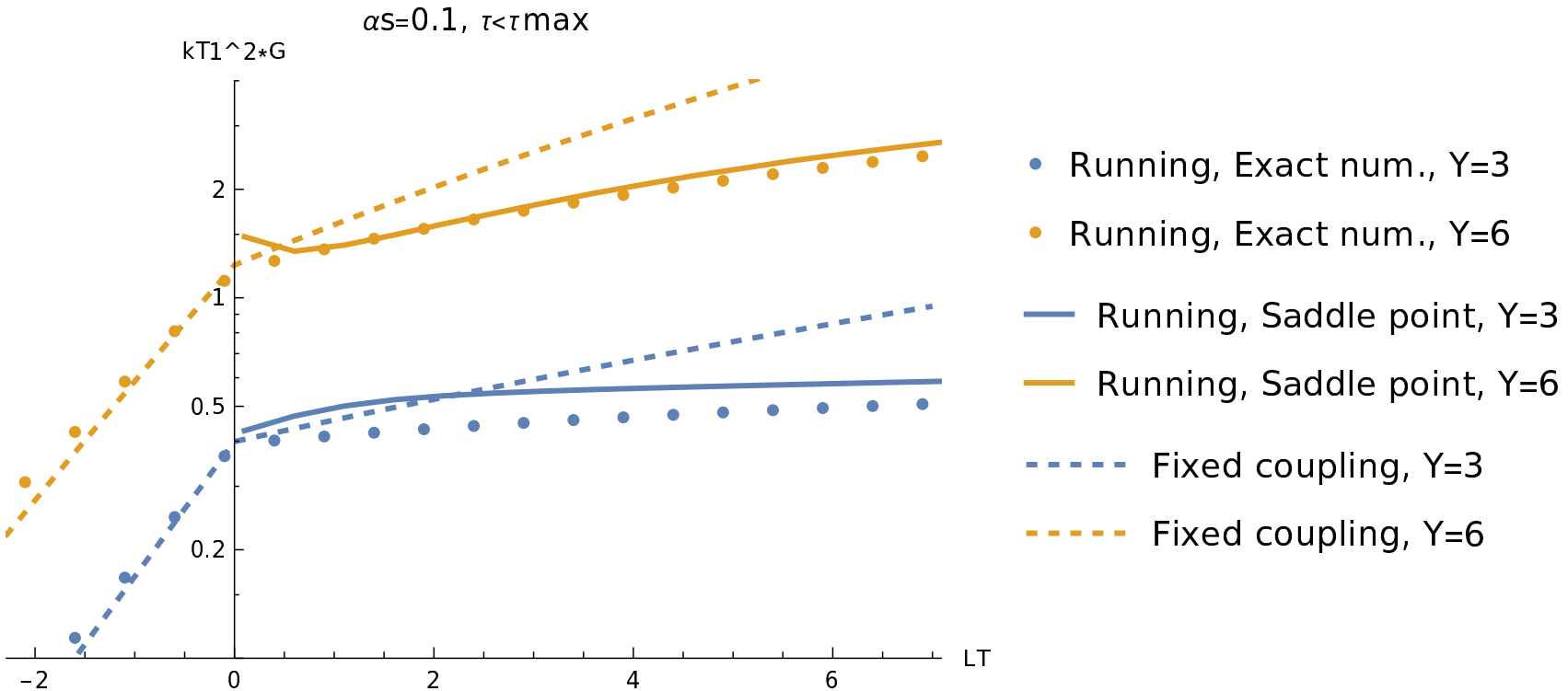
$$G_Y(\gamma) = \int_0^\infty d\tau\, e^{-\tau} I_0\left[2\sqrt{\hat{\alpha}_s Y \int_0^\tau d\tau' \chi(\gamma + a_s\beta_0\tau')}\right].$$

It is an exact solution only if $\chi(\gamma) = \chi_0(\gamma)$, but it is still interesting to substitute the resummed expressions for χ, to understand combined effects of running coupling and resummation on the Green's function. Work in progress...

Note that for real γ the τ-integration runs into the singularity of $\chi(\gamma)$ at $\gamma = 1$. This is a manifestation of Landau pole + diffusion into IR. There are many ways to treat this, the simplest is just to limit the τ-integration before the pole.

Some model studies

So far for the model $\chi_{2P}(\gamma) = 1/\gamma + 1/(1-\gamma)$. Some plots for $G(Y; \mathbf{p}_1^2, \mathbf{p}_2^2)$ with $\alpha_s(\mathbf{p}_2^2)$-fixed. The $L_T = \ln(\mathbf{p}_1^2/\mathbf{p}_2^2)$.



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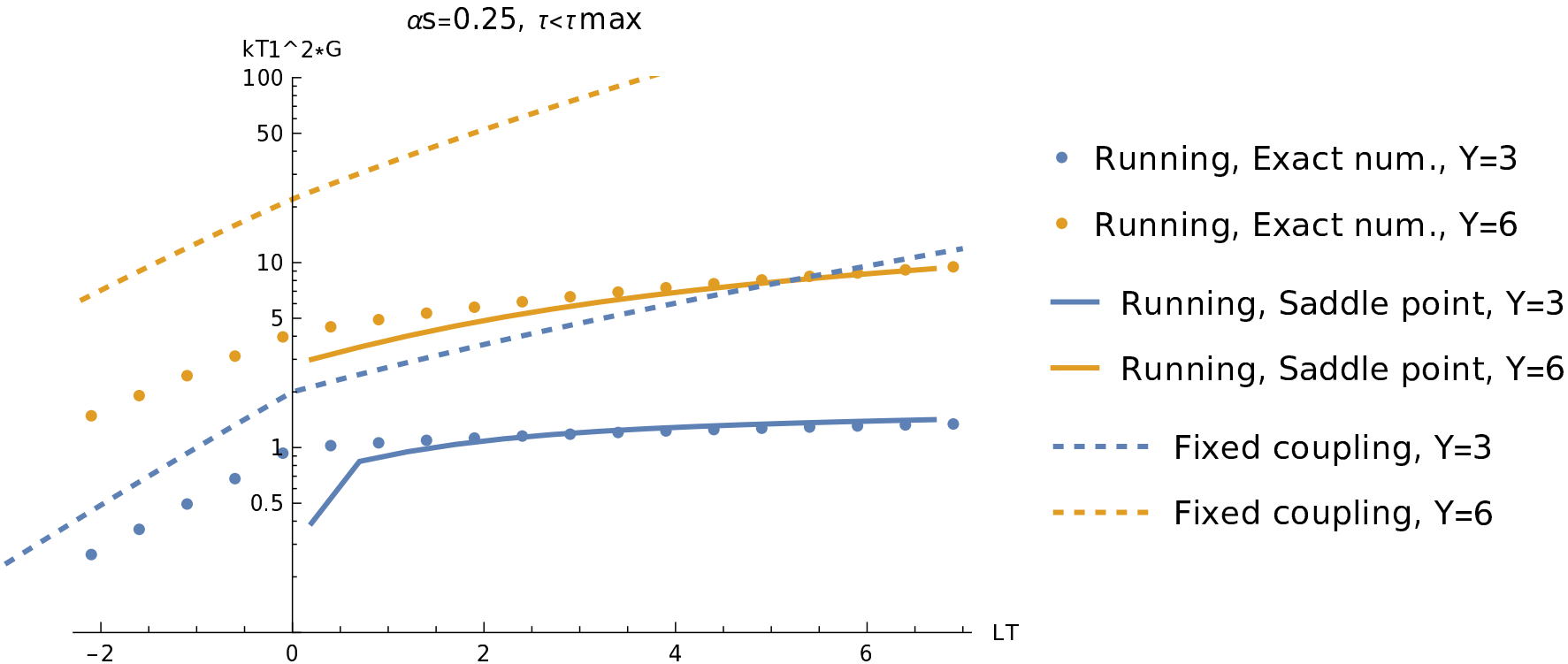
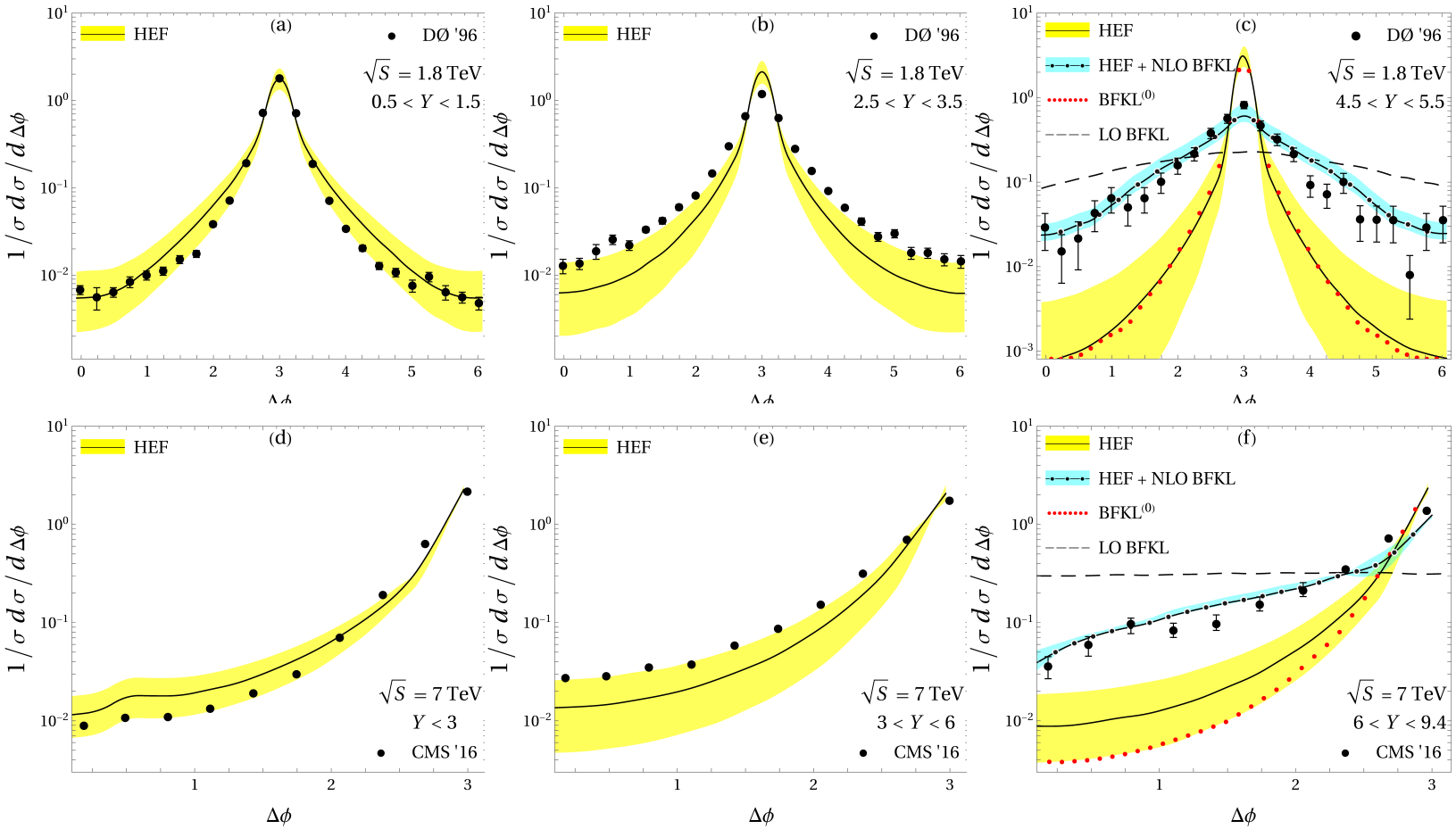


Conclusions and outlook

- The (anti-)collinear resummation for the JIMWLK Hamiltonian [Kovner, Lublinsky, Skokov, 2023] has implications for BFKL evolution.
- The effect for the characteristic function is asymmetric, only anti-collinear pole is affected. The agreement with known NLO BFKL results and with the Bjorken limit of the target, known from DGLAP, has been checked.
- The Pomeron intercept is decreased, the effect of quarks is numerically small
- Running of α_s has been taken into account in the computation of LL resummed characteristic function.
- New (???) solution for the Green's function with running coupling

Thank you for your attention!

Backup: Phenomenology of Mueller-Navelet dijets

[Chernyshev, MN, Saleev, 2025]