TMD factorization in dijets: Relating Large-x and Small-x

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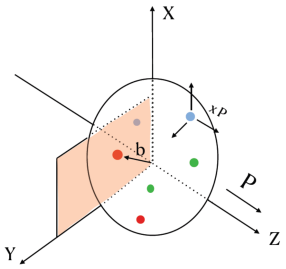
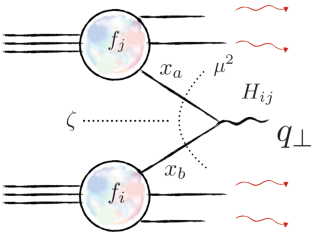
Based on: Mukherjee, Skokov, Tarasov, Yao, SST: 2602.15137 and to appear ..

Motivation

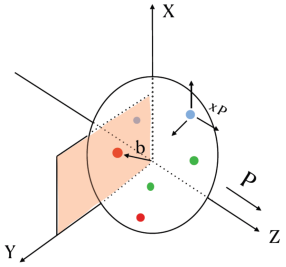


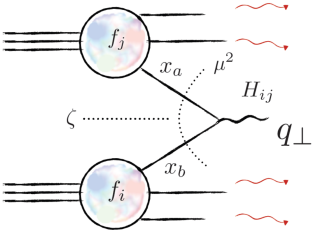


- TMDPDF(x, $b_\perp$) from cross-sections using **TMD factorization**
- At large-x : collinear matching: CSS logarithms
- At small-x : eikonal expansion: BFKL/JIMWLK, CSS
- **Understanding the general structure of TMD factorization at non-zero x beyond collinear matching**

Chiu, Jain, Neill, Rothstein (2013)
Caucal, Salazar, Schenke, Stebel, Venugopalan (2023)

Motivation

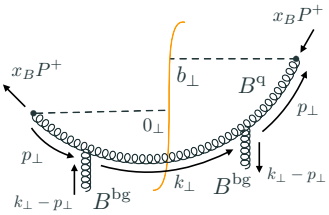


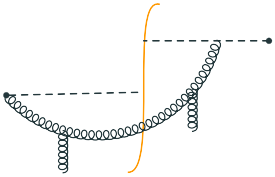


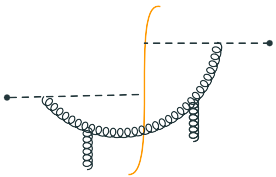
- A first stab: compute one loop correction to TMDPDFs in the presence of background fields
- Assume no strict ordering in transverse momenta
- Assume no strict ordering in longitudinal momenta
- We observed additional rapidity(IR) divergences for gluon TMDPDF

Mukherjee, Skokov, Tarasov, SST (2024,2025,in preparation)

Real corrections: Gluon





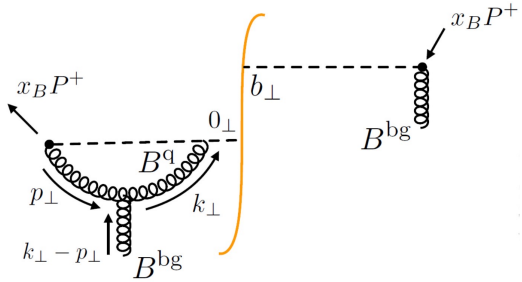


- UV and IR $\mathbf{k}_\perp$ **divergence:**
  Dim reg: Get μ_{UV} and μ_{IR} scales

- **Rapidity UV** ($\mathbf{k}^- \to \infty$) divergence:
  η scheme: Get ν scale

Virtual corrections: Gluon

- UV and IR $\mathbf{k}_\perp$ **divergence:**
  Dim reg: Get μ_{IR}, μ_{UV} scales
- **Rapidity IR ($\mathbf{k}^- \to \mathbf{0}$) divergence:**
  η scheme: Get ρ scale



- Vanishes for collinear matching, where $k_\perp - p_\perp = 0$
  $\Longrightarrow$ **no rapidity IR divergence**

$$B_{ij}^{bg}(x_B, p_\perp) = \int d^2 b_\perp \, e^{-ip_\perp b_\perp} B_{ij}^{bg}(x_B, b_\perp) \xrightarrow[b_\perp = 0]{} \delta^2(p_\perp) \, B_{ij}^{bg}(x_B)$$

Full result: Gluons

$$f_{ij}(x_B,b_\perp,\mu^2_{\rm UV},\zeta) = f_{ij}(x_B,b_\perp,\mu^2_{\rm IR},\rho) + \frac{\alpha_s N_c}{2\pi}\Big(-\frac{1}{2}(L_b^{\mu_{\rm UV}})^2 + L_b^{\mu_{\rm UV}}\ln\frac{\mu^2_{\rm UV}}{\zeta^2} - \frac{\pi^2}{12}\Big) f_{ij}(x_B,b_\perp,\mu^2_{\rm IR},\rho)$$

CSS

$$- \frac{\alpha_s N_c}{\pi} L_b^{\mu_{\rm IR}} \int_0^1 dz\Big[\frac{1}{(1-z)_+} + \frac{1}{z}\Big] f_{ij}\Big(\frac{x_B}{z},b_\perp,\mu^2_{\rm IR},\rho\Big) - \frac{\alpha_s N_c}{2\pi}\int d^2 z_\perp \int \vec{d}^2 p_\perp e^{ip_\perp (b-z)_\perp}$$

Part of DGLAP

$$\times\Big(\ln\frac{\mu^2_{\rm IR}}{p^2_\perp}\ln\frac{\rho}{\zeta}\Big)\frac{g_{il}p_j p_m + p_i p_l g_{mj}}{p^2_\perp} f_{lm}(x_B,z_\perp,\mu^2_{\rm IR},\rho) + \text{finite terms} + O(\alpha_s^2)\,.$$

V. BFKL

Mukherjee, Skokov, Tarasov, SST (2024)

Full result: Quarks

$$\Phi^{[\gamma^+]}_{f;\mathrm{UV}}(x,b_\perp;\mu^2_{UV},\zeta) = \underbrace{\frac{g^2 C_F}{8\pi^2}\Big(-\frac{1}{2}(L_b^{\mu_{UV}})^2+\Big(\frac{3}{2}+\ln\frac{2\mu^2_{UV}}{\zeta}\Big)L_b^{\mu_{UV}}+\frac{1}{2}-\frac{\pi^2}{12}\Big)}_{\text{CSS}}\Phi^{[\gamma^+]}_{f;\mathrm{IR}}(x,b_\perp;\mu^2_{IR},\rho)$$

$$+\text{finite terms}\ +O(\alpha_s^2)$$

Mukherjee, Skokov, Tarasov, SST (2025)

Features of this approach

- A new divergence with virtual part of BFKL logarithms
- Resums all orders in collinear twist expansion (powers of $b_\perp$)
- Reduces to collinear matching formalism under transverse momentum ordering
- Reproduces the full BFKL logarithm under longitudinal momentum ordering
- The procedure:

$$f(\mu_{\rm IR}^0, \rho^0) \xrightarrow[\text{evolution}]{\rho,\, \mu_{\rm IR}} f(\mu_{\rm IR}, \rho) \underset{\text{matching}}{\longrightarrow} f(\mu_{\rm UV}^0, \zeta^0) \xrightarrow[\text{evolution}]{\text{CSS}} f(\mu_{\rm UV}, \zeta)$$

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- A new divergence with virtual part of BFKL logarithms
- Reduces to collinear matching formalism under transverse momentum ordering
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Criticisms

- Matching kernel computed in the dilute approximation (2 gluon exchange)
- Logarithms have not been shown to arise in a process

Aim of this talk

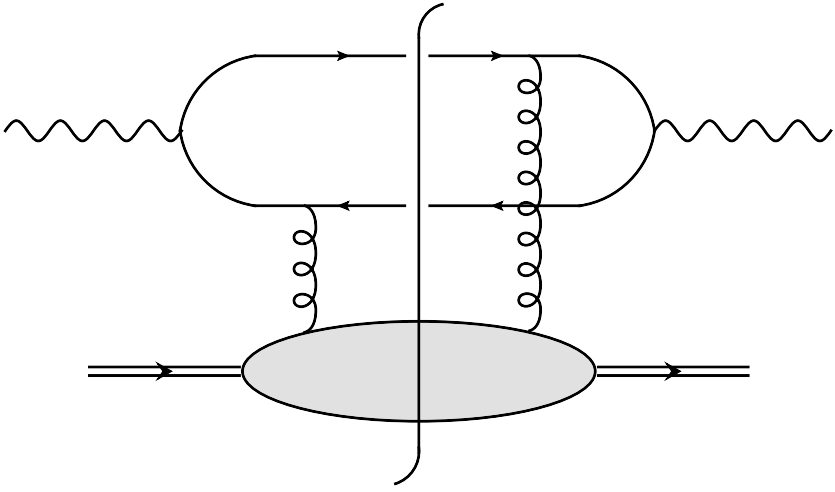
Criticisms

- Matching kernel computed in the dilute approximation (2 gluon exchange)
- Logarithms have not been shown to arise in a process

Aim 1: Develop a formalism to go beyond the 2-gluon (dilute) approximation

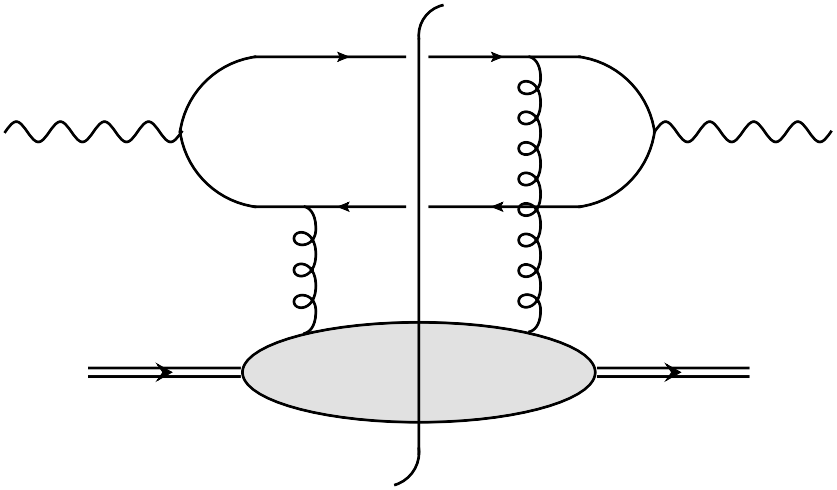
Aim 2: Lay the ground-work for computing dijet production at NLO

Dijet production



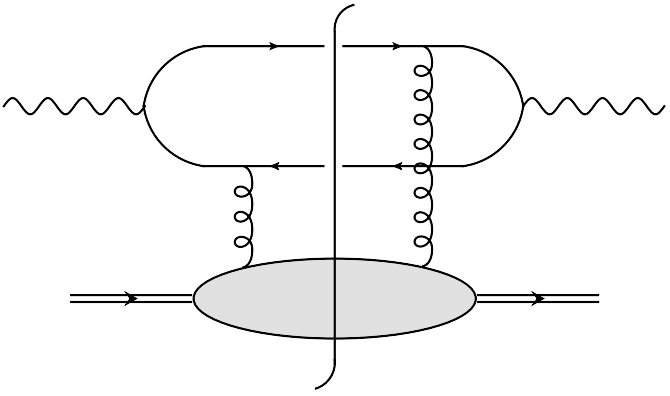
TMD factorization of dijet production in the back to back limit

Dijet production



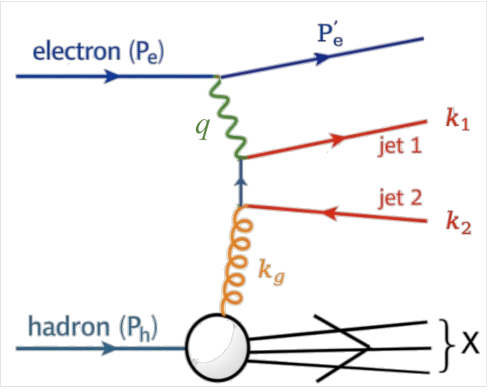
Ultimate goal: show TMD factorization at NLO in an arbitrary background

Dijet production



For this talk: show TMD factorization at leading order for arbitary bjorken x using the background field method

Dijet production

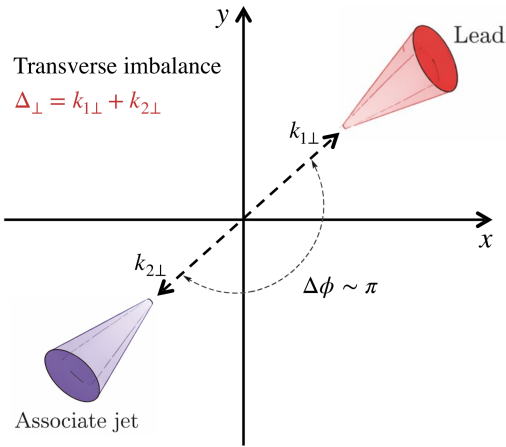


Hard scale: $Q \sim k_{1\perp} \sim k_{2\perp} \sim P_\perp$

Back to back kinematics

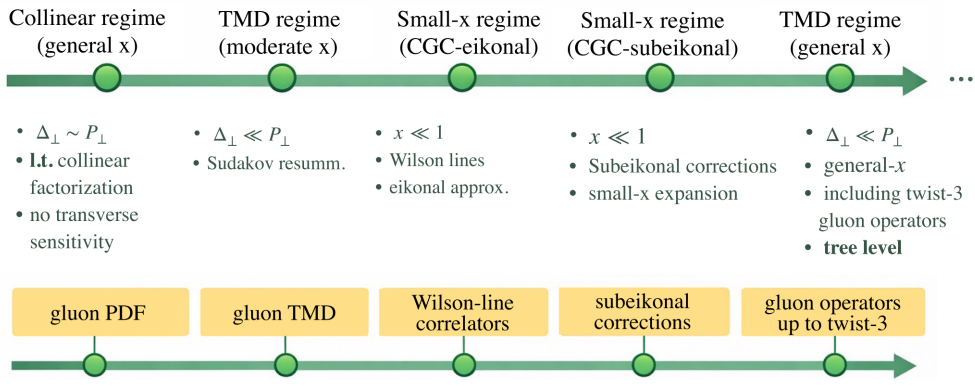
Imbalance: $\Delta_\perp = k_{1\perp} + k_{2\perp}$

Back to back limit: $\Delta_\perp \ll P_\perp$

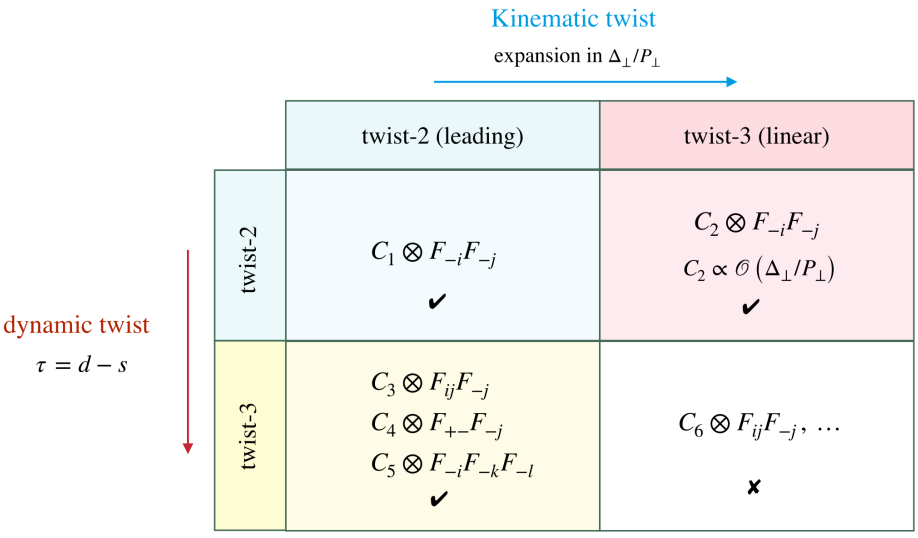


TMD factorization has been shown for dijet production in back to back kinematics

Current status of calculations



What operators are considered?



The procedure summarized

Dijet amplitude

Arbitrary number of insertions ↓ Identify the background propagators

Perform a twist expansion in terms of field strength tensors

Dynamical background ↓ Includes F_{-i}, F_{ij} and F_{+-}

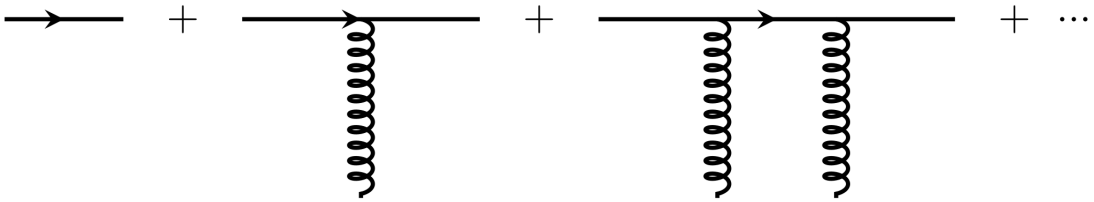
Substitute back into the amplitude

Construct TMD with full phase ↓ Retain terms upto $\mathcal{O}(\frac{\Delta}{P})$

Dijet cross-section upto twist-three accuracy

Mukjerjee, Skokov, Tarasov, Yao, SST (2026)

Quark propagator

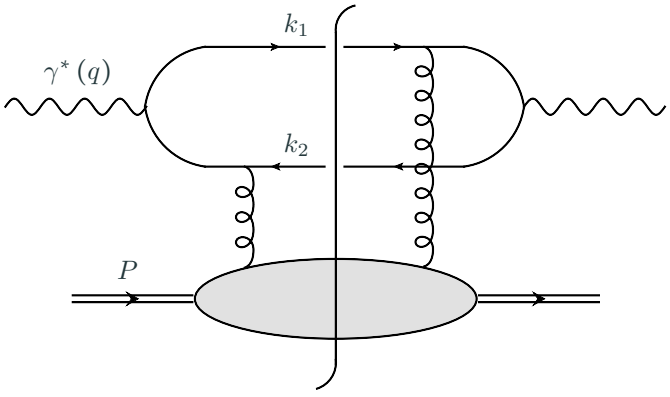


$$(x|\frac{i}{\not{P}}|y) = (x|\frac{i}{\not{p}}|y) + (x|\frac{i}{\not{p}}\not{A}\frac{1}{\not{p}}|y) + (x|\frac{i}{\not{p}}\not{A}\frac{1}{\not{p}}\not{A}\frac{1}{\not{p}}|y) + \ldots$$

$$\boxed{\mathrm{P}^{\mu} = p^{\mu} + A^{\mu}}$$

The dijet amplitude

- $P = (P^+, 0, 0_\perp)$
- $A^+ = 0$



$$i\mathcal{M} = -ie\,\epsilon_\rho(q) \lim_{k_1^2,k_2^2\to 0} \int d^4y\, e^{-iq\cdot y}\, \bar{u}(k_1) \underbrace{(k_1|\not{p}\frac{i}{\not{p}}|y)}_{\text{quark}} \gamma^\rho \underbrace{(y|\frac{i}{\not{p}}\not{p}|-k_2)}_{\text{anti-quark}} v(k_2)$$

The dijet amplitude

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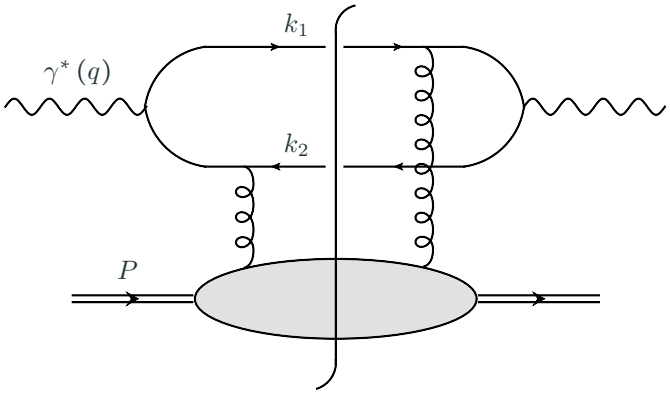

$$\bar{u}(k_1)\,(k_1|\not{p}\,\frac{i}{\not{P}}|y) = \bar{u}(k_1)\,(k_1|\not{p}\not{P}\,\frac{i}{\not{P}\not{P}}|y) = \bar{u}(k_1)k_1^2\,(k_1|\frac{i}{\mathrm{P}^2 + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}|y)\,,$$

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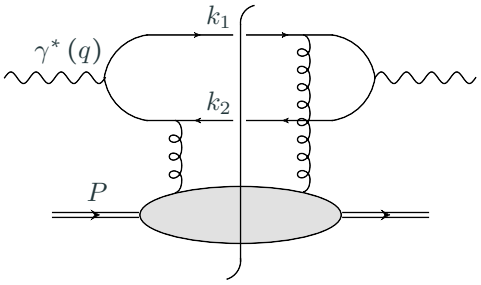


$$i\mathcal{M} = ie\,\epsilon_\rho(q) \int d^4y\, e^{-iq\cdot y} \lim_{k_1^2,k_2^2\to 0} \bar{u}(k_1) \underbrace{k_1^2\,(k_1|\frac{1}{\mathrm{P}^2 + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}|y)}_{\text{quark}} \gamma^\rho \underbrace{(y|\frac{1}{\mathrm{P}^2 + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}|-k_2)\,k_2^2}_{\text{anti-quark}} v(k_2)$$

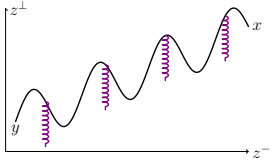
We shall work with these propagators now

The propagator and its expansion

$$(x|\frac{1}{\mathrm{P}^2+\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}+i\epsilon}|y) \;=\; (x|\frac{1}{2p^-\boxed{\mathrm{P}^+}-\boxed{\mathrm{P}_\perp^2}-i\boxed{\partial^- A_-}+\boxed{\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}+i\epsilon}|y)$$

Propagator $\underset{\text{gradient expansion}}{=}$ Wilson line + F insertions + target dynamics + spin-coupling

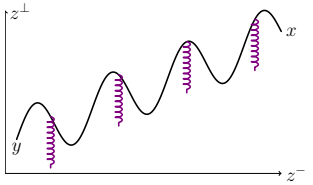
Expanding the scalar propagator



$$(x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}_\perp^2}{2k^-} + i\epsilon}|y) = (x|\Big[\frac{1}{\mathrm{P}^+ + i\epsilon} + \frac{1}{\mathrm{P}^+ + i\epsilon}\frac{\mathrm{P}_\perp^2}{2k^-}\frac{1}{\mathrm{P}^+ + i\epsilon} + \cdots\Big]|y)$$

Let us simplify the eikonal P^+ propagator now

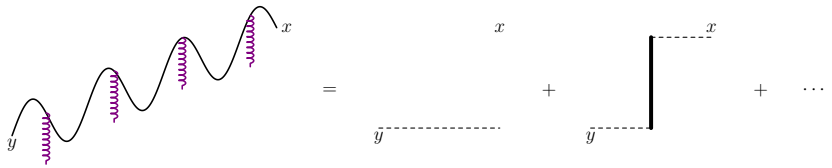
Expanding the scalar propagator: Wilson lines



$$(x^-|\frac{1}{\mathrm{P}^+ + i\epsilon}|y^-) = -i\theta(x^- - y^-)[x^-, y^-]$$

Plugging it back into the propagator

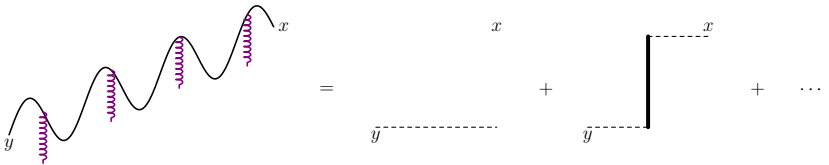
Expanding the scalar propagator



$$(x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}_\perp^2}{2k^-} + i\epsilon}|y) = \delta(x^+ - y^+)\,\theta(x^- - y^-)\,(x_\perp|\Big[-i\,[x^-, y^-] - \int_{y^-}^{x^-} dz^-\,[x^-, z^-]\,\frac{\mathrm{P}_\perp^2(z^-)}{2k^-}\,[z^-, y^-] + \cdots\Big]|\,y_\perp)$$

Let us now generate F insertions

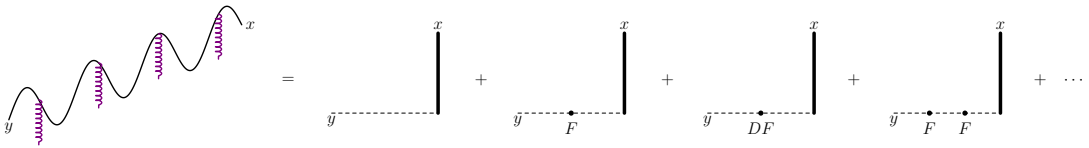
Generating F insertions



$$[x^-, y^-]\,\mathrm{P}_i(y^-) = \mathrm{P}_i(x^-)\,[x^-, y^-] - \int_{y^-}^{x^-} dz^-\,[x^-, z^-]\,F_{-i}(z^-)\,[z^-, y^-]$$

Let us now generate F insertions

Intermediate form of the scalar propagator

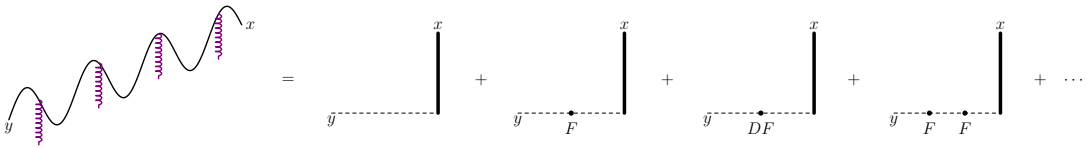


$$(x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}^2_\perp}{2k^-} + i\epsilon}|y) = \delta(x^+ - y^+)\,\theta(x^- - y^-)\,(x_\perp|\Big[-i\,[x^-, y^-] - (x^- - y^-)\frac{\mathrm{P}^2_\perp(x^-)}{2k^-}[x^-, y^-]$$

$$+ \int_{y^-}^{x^-} dz_1^-\,(z_1^- - y^-)\frac{2\mathrm{P}_i(x^-)}{2k^-}[x^-, z_1^-]F_{-i}(z_1^-)[z_1^-, y^-] + \cdots\Big]|\,y_\perp)$$

We can resum the exponential at the boundary: phase ignored in small-x/CGC

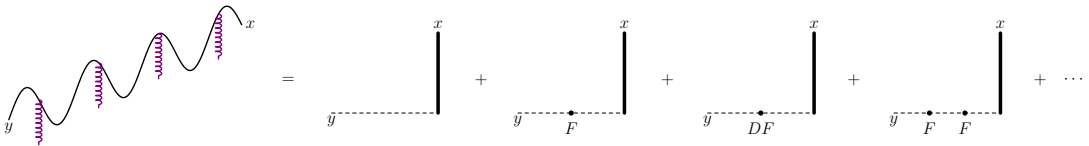
Final form of the scalar propagator



$$(x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}_\perp^2}{2k^-} + i\epsilon}|y) = \delta(x^+ - y^+)\theta(x^- - y^-)(x_\perp|e^{-i(x^- - y^-)\frac{\mathrm{P}_\perp^2(x^-)}{2k^-}}\Bigg[- i\,[x^-, y^-]$$
$$+ \frac{1}{2k^-}\int_{y^-}^{x^-} dz^-(z^- - y^-)\Big(2\mathrm{P}_k(x^-)\,[x^-, z^-]\,F_{-k}(z^-)\Big)[z^-, y^-]\Bigg]|y_\perp)\,,$$

We can resum the exponential at the boundary: phase ignored in small-x/CGC

Going back to quark propagator

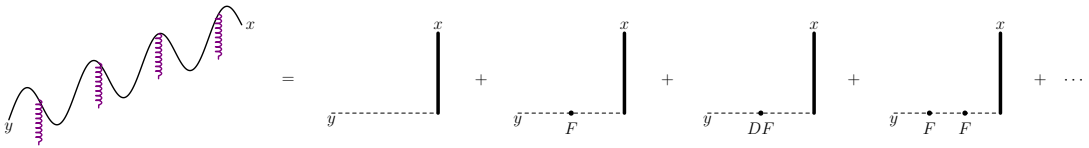


$$(x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}_\perp^2}{2k^-} + \frac{\sigma F}{4k^-} + i\epsilon}|y) \;=\; (x|\frac{1}{\mathrm{P}^+ - \frac{\mathrm{P}_\perp^2}{2k^-} + i\epsilon}|y) \;+\; \delta(x^+ - y^+)\,\theta(x^- - y^-)$$

$$(x_\perp|e^{-i(x^- - y^-)\frac{\mathrm{P}_\perp^2(x^-)}{2k^-}}\left[\frac{1}{4k^-}\int_{y^-}^{x^-} dz^-[x^-, z^-]\,\sigma F(z^-)\,[z^-, y^-] + \ldots\right]|y_\perp)$$

Carrying out amputation (LSZ)

Quark propagator amputated



$$\lim_{k^2\to 0} k^2 (k|\frac{1}{P^2+\frac{1}{2}\sigma F+i\epsilon}|y)$$

$$= ie^{iky}\left\{-i[\infty,y^-]_{y_\perp}+\int_{y^-}^{\infty} dz^-\left[\frac{k_k}{k^-}(z^--y^-)\bar{F}_{-k}(z^-,y_\perp)+\frac{1}{4k^-}\sigma\bar{F}(z^-,y_\perp)\right][\infty,y^-]_{y_\perp}\right\}$$

where $\bar{F}(z^-) = [\infty,z^-]F(z^-)[z^-,\infty]$

Similar procedure for target dynamics

Summary of different terms

Four layers of gradient expansion

(i) Longitudinal propagation — Wilson line

- Resumption of multiple A^- interactions
- Defines the gauge-invariant backbone

(ii) dynamics in target — F_{+-}

- Encodes longitudinal field gradients
- Generate twist-2 F_{+-} insertions
- Absent in strict eikonal limit

(iii) Field-strength insertions — F_{-i}

- merges from transverse momentum expansion
- Generates twist-1 F_{-i}, two-gluon body $F_{-k}F_{-l}$, and kinematic twist-2 from $D_i F_{-j}$

(iv) Spin-field coupling — $\sigma^{\mu\nu}F_{\mu\nu}$

- Pauli spin–field coupling
- Produces additional transverse twist-2 structures F_{ij}
- Absent for longitudinal photon

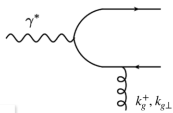
Final propagators

Quark propagator: $$\lim_{k^2\to 0} k^2 (k|\frac{1}{\mathrm{P}^2+\frac{1}{2}\sigma F+i\epsilon}|y) = ie^{iky}\Big\{-i[\infty,y^-]_{y_\perp} + c_1(k)\,F + c_2(k)\,\sigma F + c_3(k)\,DF + c_4(k)\,D(\sigma F) + c_5(k)\,FF + c_6(k)\,(\sigma F)F + \ldots\Big\}$$

Antiquark propagator: $$\lim_{k^2\to 0} k^2 (y|\frac{1}{\mathrm{P}^2+\frac{1}{2}\sigma F+i\epsilon}|-k) = -ie^{iky}\Big\{i[y^-,\infty]_{y_\perp} + c_1^*(-k)\,F + c_2^*(-k)\,\sigma F + c_3^*(-k)\,DF + c_4^*(-k)\,D(\sigma F) + c_5^*(-k)\,FF + c_6^*(-k)\,(\sigma F)F + \ldots\Big\}$$

Dijet amplitude

$$i\mathcal{M} = ie\,\epsilon_\rho(q)\int d^4y\, e^{-iq\cdot y}\lim_{k_1^2,k_2^2\to 0}\bar{u}(k_1)\,k_1^2\,(k_1|\underbrace{\frac{1}{\mathrm{P}^2+\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}}_{\text{quark}}|y)\gamma^\rho(y|\underbrace{\frac{1}{\mathrm{P}^2+\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}}_{\text{antiquark}}|-k_2)\,k_2^2\,v(k_2)\,.$$



$$\begin{aligned}
i\mathcal{M} = &-e\,\epsilon_\rho(q)\int\frac{dx^-d^2x_\perp}{2q^-}e^{i(k_g^+x^- - \Delta_\perp x_\perp)}\bar{u}(k_1)\Big\{\mathbb{C}_1^i\gamma^\rho\overline{F}_{-i}(x^-,x_\perp)+i\,(\mathbb{S}_1\sigma^{\mu\nu}\gamma^\rho+\mathbb{S}_2\gamma^\rho\sigma^{\mu\nu})\,\overline{F}_{\mu\nu}(x^-,x_\perp)\\
&+\mathbb{C}_2^+\,\gamma^\rho\,\overline{F}_{+-}(x^-,x_\perp)+\Big[\left(\mathbb{T}_1^{kl}\,\gamma^\rho+i\,\mathbb{C}_3^l\,\sigma^{-k}\,\gamma^\rho+i\,\mathbb{C}_4^k\,\sigma^{-l}\gamma^\rho+i\,\mathbb{C}_5^k\,\gamma^\rho\,\sigma^{-l}\right)\mathbb{D}_{kl}(\,l^+,x^-,x_\perp)\\
&+\left(\mathbb{T}_2^{kl}\,\gamma^\rho+i\,\mathbb{C}_6^l\,\sigma^{-k}\,\gamma^\rho+i\,\mathbb{C}_7^l\,\gamma^\rho\,\sigma^{-k}+i\,\mathbb{C}_8^k\,\gamma^\rho\,\sigma^{-l}\right)\mathbb{D}_{kl}^\dagger(\,-l^+,x^-,x_\perp)\Big]_{l^+=0}\Big\}v(k_2)
\end{aligned}$$

Two-body (twist-2) operator:

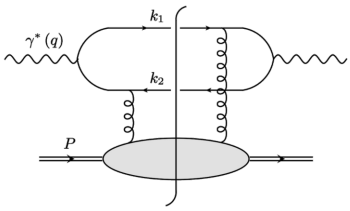
$$\mathbb{D}_{kl}(\,l^+,x^-,x_\perp) = i\int\frac{dk^+}{(2\pi)}\frac{1}{k^+-l^++i\epsilon}\int dx'^-e^{ik^+(x^--x'^-)}[\infty,x'^-]_{x_\perp}\,F_{-k}(x'^-,x_\perp)\,[x'^-,x^-]_{x_\perp}\,F_{-l}(x^-,x_\perp)\,[x^-,\infty]_{x_\perp}$$

Cross-section

Differential cross-section

$$k_1^- k_2^- \frac{d\sigma}{đ k_1^- đ^2 k_{1\perp} đ k_2^- đ^2 k_{2\perp}} = \pi q^- \delta(k_1^- + k_2^- - q^-) |i\mathcal{M}|^2$$

Longtudinal polarized	Transverse polarized
$\epsilon_-(q)\epsilon_-^*(q) = \frac{Q^2}{q^{-2}}$	$\frac{1}{2}\sum_{\lambda=\pm1} \epsilon_k^\lambda(q)\epsilon_l^{\lambda *}(q) = -\frac{1}{2} g_{kl}$



Leading-twist contribution to the cross section.

Longitudinal photon cross-section

complete phase

$$|i\mathcal{M}|_L^2 \approx e^2 \int dx^- d^2x_\perp dy^- d^2y_\perp \, e^{ik_g^+(x^- - y^-) - i\Delta_\perp(x_\perp - y_\perp)} \frac{16\, z^2\bar{z}^2\, \epsilon_f^2}{(P_\perp^2 + \epsilon_f^2)^4} \mathrm{Tr}\Big[O_1 + O_2 + O_3 + O_4\Big] \qquad \epsilon_f^2 = z\bar{z}Q^2$$

Kinematic twist-3

$$O_1 = \Big[2P^iP^j - (\bar{z}-z)(P^i\Delta^j + P^j\Delta^i) + 8(\bar{z}-z)\frac{(\Delta\cdot P)P^iP^j}{P_\perp^2+\epsilon_f^2}\Big]\overline{F}_{-i}(x^-, x_\perp)\,\overline{F}_{-j}(y^-, y_\perp), \quad \text{dynamic leading-twist}$$

$$O_2 = \frac{(z-\bar{z})}{z\bar{z}\,q^-} P^i P_\perp^2 \Big[\overline{F}_{-i}(x^-, x_\perp)\,\overline{F}_{+-}(y^-, y_\perp) + \overline{F}_{+-}(x^-, x_\perp)\,\overline{F}_{-i}(y^-, y_\perp)\Big], \quad \text{dynamic twist-3}$$

$$O_3 = P^i\Big[\mathcal{H}_{kl}(z)\,\overline{F}_{-i}(x^-, x_\perp)\,\mathbb{D}_{kl}{}^\dagger(l^+, y^-, y_\perp) + \mathcal{H}_{kl}(z)\,\mathbb{D}_{kl}(l^+, x^-, x_\perp)\,\overline{F}_{-i}(y^-, y_\perp)\Big]_{l^+=0},$$

$$O_4 = P^i\Big[\mathcal{H}_{kl}(\bar{z})\,\overline{F}_{-i}(x^-, x_\perp)\,\mathbb{D}_{kl}(-l^+, y^-, y_\perp) + \mathcal{H}_{kl}(\bar{z})\,\mathbb{D}_{kl}^\dagger(-l^+, x^-, x_\perp)\,\overline{F}_{-i}(y^-, y_\perp)\Big]_{l^+=0}, \quad \text{dynamic twist-3}$$

where we defined

$$\mathcal{H}_{kl}(z) \equiv \delta_{kl} - \frac{4P^kP^l}{P_\perp^2+\epsilon_f^2} + \frac{2P^kP^l}{zq^-}\frac{\partial}{\partial l^+}.$$

Transverse photon cross-section

complete phase

$$
\begin{aligned}
|i\mathcal{M}|_T^2 = 4\,e^2 \int \frac{dx^- d^2x_\perp dy^- d^2y_\perp}{(2q^-)^2}\, e^{ik_g^+(x^- - y^-) - i\Delta_\perp (x_\perp - y_\perp)}\, \mathrm{Tr}\Big[& A^{ij}\, \overline{F}_{-i}(x^-, x_\perp)\, \overline{F}_{-j}(y^-, y_\perp) \\
& + B^i \Big(\overline{F}_{-i}(x^-, x_\perp)\, \overline{F}_{+-}(y^-, y_\perp) + \overline{F}_{+-}(x^-, x_\perp)\, \overline{F}_{-i}(y^-, y_\perp) \Big) \\
& + C^i \Big(\overline{F}_{ij}(x^-, x_\perp) \overline{F}_{-j}(y^-, y_\perp) + \overline{F}_{-j}(x^-, x_\perp) \overline{F}_{ij}(y^-, y_\perp) \Big) \qquad \text{dynamic twist-3} \\
& + \Big(E^{ikl}\, \overline{F}_{-i}(x^-, x_\perp)\, \mathbb{D}^\dagger_{kl}(l^+, y^-, y_\perp) + E^{ikl}\, \mathbb{D}_{kl}(l^+, y^-, y_\perp)\, \overline{F}_{-i}(y^-, y_\perp) \Big)_{l^+=0} \\
& + \Big(G^{ikl}\, \overline{F}_{-i}(x^-, x_\perp)\, \mathbb{D}_{kl}(-l^+, y^-, y_\perp) + G^{ikl}\, \mathbb{D}^\dagger_{kl}(-l^+, x^-, x_\perp)\, \overline{F}_{-i}(y^-, y_\perp) \Big)_{l^+=0} \Big]
\end{aligned}
$$

Bianchi identity → operator $F_{ij}F_{-j}$ reduction: kinematic twist-3 + three-body operator

TMD factorization at small-x

- Full longitudinal phase (**general-x** TMD framework):

$$e^{i\,x\,k_g^+\,(x^- - y^-)}$$

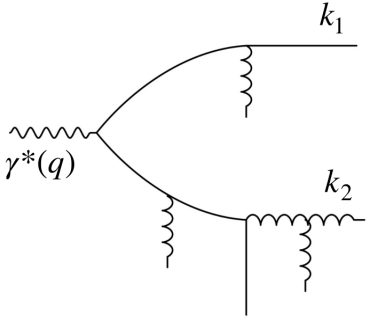
- Small-x limit ($x \to 0$), taking phase expansion:

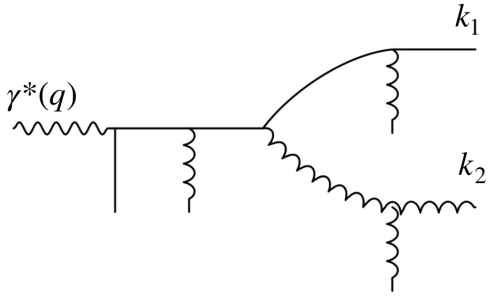
$$e^{ixP^+(x^- - y^-)} \approx 1 + i\,x\,k_g^+\,(x^- - y^-)$$

- Operator reduction:
  - $F_{-i}F_{-j} \to$ eikonal, subeikonal
  - $F_{-i}F_{-j}F_{-k} \to$ eikonal, subeikonal
  - $F_{+-}F_{-j}, F_{ij}\,F_{-j} \to$ subeikonal
- Reproduces eikonal and sub-eikonal CGC expressions in Ref. [Altinoluk, Beuf, Czajka & Marquet, PRD 111 (2025)]

What about quarks?

At large-x we also need to consider dijet in quark background





Amplitude

$$i\mathcal{M}^1 = -\,e \lim_{k^2,\,k_1^2 \to 0} \int d^4z\, d^4y\, e^{-iq.y} \epsilon_i^{\lambda}(k)\, \epsilon_\mu(q) \underbrace{k^2\, (k| \frac{1}{P^2 g_{\lambda\nu} + iF_{\lambda\nu} + i\epsilon} |z)^{ij}}_{\text{gluon}} \bar{\psi}^c(z)\, t^j_{cb}\, \gamma^\nu$$

$$\underbrace{(z| \frac{i}{P^2 + \frac{1}{2}\sigma F + i\epsilon} \not{P} |y)^{ba}}_{\text{intermediate anti-quark}} \gamma^\mu \underbrace{(y| \frac{i}{P^2 + \frac{1}{2}\sigma F + i\epsilon} |k_1)^{ad}\, k_1^2}_{\text{quark}} v^d(k_1)$$

$$i\mathcal{M}^2 = -\,ie \lim_{k^2,\,k_1^2 \to 0} \int d^4y\, d^4z\, e^{-iq.y}\, \bar{u}^d(k_1)\, k_1^2 (k_1| \frac{i}{P^2 + \frac{\sigma F}{2} + i\epsilon} |z)^{de} \gamma^\nu t^i_{ef}\, (z| \frac{i}{P^2 + \frac{\sigma F}{2} + i\epsilon} \not{P} |y)^{fg}$$

$$\gamma^\mu\, \psi^g(y)\, \epsilon_j^{\lambda}(k)\, \epsilon_\mu(q)\, (z| \frac{-i}{P^2 g_{\nu\lambda} + iF_{\nu\lambda} + i\epsilon} |k)^{ij}$$

Propagators

$$\lim_{k^2\to 0} k^2 (k|\frac{1}{P^2 g_{\nu\lambda} + iF_{\nu\lambda} + i\epsilon}|z)^{ij} = ie^{ik^+z^- + ik^-z^+}(k_\perp| - i\, g_{\nu\lambda}[\infty, z^-]_A + \frac{1}{2k^-}\int_{z^-}^{\infty} dz'^-$$
$$(z'^- - z^-)\,[\infty, z'^-]_A \left[g_{\nu\lambda}\, 2p_k F_{-k}(z'^-) + \frac{F_{\nu\lambda}(z'^-)}{(z'^- - z^-)} \right] [z'^-, z^-]_A |z_\perp)^{ij}$$

$$(z|\frac{i}{\not{P}}|y)^{ba}_{\beta\gamma} = \int_0^\infty đp^-\, e^{-ip^-(z^+ - y^+)}\, (z_\perp|\gamma^\mu \frac{P_\mu(z^-)}{2p^-} e^{-i(z^- - y^-)\frac{P_\perp^2(z^-)}{2p^-}} \Big[- i[z^-, y^-] + \frac{1}{2p^-}\int_{y^-}^{z^-} dz'^-$$
$$(z'^- - y^-)[z^-, z'^-] \Big(\{\mathrm{P}_k, F_{-k}\}(z'^-) + \frac{\sigma F(z'^-)}{2(z' - y^-)} - i\frac{\delta_-^\mu\, p^-\, \sigma F(z^-)}{(z'^- - y^-)(z^- - y^-)} \Big) [z'^-, y^-] \Big] |y_\perp)^{ba}_{\beta\gamma}$$

Longitudinal cross-section at leading twist

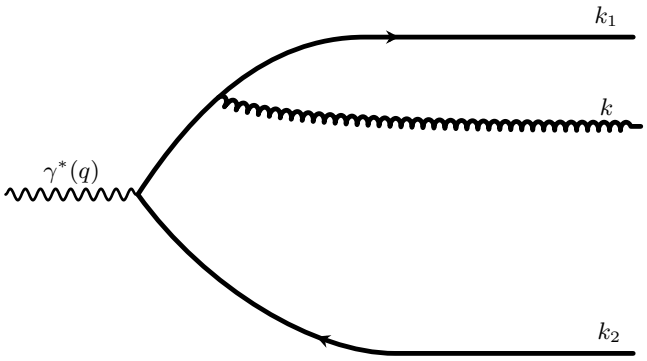
$$d\sigma^L = -8e^2 \left(\frac{\bar{z} z^2 q^- \epsilon_f^2}{(P_\perp^2 + \epsilon_f^2)^2} \right) \int dy^- dz^- d^2 y_\perp d^2 z_\perp \, e^{-i(y^- - z^-)k_g^+ + i(y_\perp - z_\perp)k_{g\perp}}$$
$$\bar{\psi}(y) \, [y^-, \infty]_{z_\perp} \, \gamma^+ \, [\infty, z^-]_{z_\perp} \psi^c(z)$$

- Matches with sub-eikonal results at small-x
- Can be used to systematically go beyond leading twist ($\bar{\psi}\gamma^i\psi$, $\bar{\psi}\gamma^+ F\psi$)
- Similar procedure for transverse cross-section

Altinoluk, Armesto, Beuf (2023)
Mukherjee, Skokov, Tarasov, Yao, SST..to appear

Dijet NLO

Work in progress



$$\lim_{k^2\to 0} k^2 (k|\frac{1}{P^2 g_{\nu\lambda} + iF_{\nu\lambda} + i\epsilon}|z)^{ij}$$

$$(z|\frac{i}{P^2 + \frac{1}{2}\sigma F + i\epsilon}\not{P}|y)^{ba}$$

$$(k_1|\frac{1}{\mathrm{P}^2 + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}}|y)$$

We shall report on it in the future

Conclusions and outlook

- TMD factorization of dijet cross-section at LO upto twist 3 accuracy
- The methodology goes beyond dilute (2-gluon) limit
- Systematic procedure for higher twist calculations
- Recover the full TMD phase and match with subeikonal/CGC calculations
- Can be used to compute quark background and NLO dijet

Thank you for your attention!

Full result: Gluons

$$f_{ij}(x_B,b_\perp,\mu^2_{\rm UV},\zeta) = f_{ij}(x_B,b_\perp,\mu^2_{\rm IR},\rho) - 4\alpha_s N_c \int \vec{d}^2 p_\perp e^{ip_\perp b_\perp} \int_0^1 \frac{dz}{z(1-z)} \int \vec{d}^2 k_\perp \Big[\mathcal{R}^a_{ij;lm}(z,p_\perp,k_\perp)$$

$$+\mathcal{R}^b_{ij;lm}(z,p_\perp,k_\perp)\Big] \int d^2 z_\perp e^{-i(p_\perp-k_\perp)z_\perp} f_{lm}\Big(\frac{x_B}{z},z_\perp,\mu^2_{\rm IR},\rho\Big) + \boxed{\frac{\alpha_s N_c}{2\pi}\Big(-\frac{1}{2}(L_b^{\mu_{\rm UV}})^2 + L_b^{\mu_{\rm UV}} \ln\frac{\mu^2_{\rm UV}}{\zeta^2} - \frac{\pi^2}{12}\Big)}$$

CSS

$$\times f_{ij}(x_B,b_\perp,\mu^2_{\rm IR},\rho) - \boxed{\frac{\alpha_s N_c}{\pi} L_b^{\mu_{\rm IR}} \int_0^1 dz\Big[\frac{1}{(1-z)_+} + \frac{1}{z}\Big]} f_{ij}\Big(\frac{x_B}{z},b_\perp,\mu^2_{\rm IR},\rho\Big) - \frac{\alpha_s N_c}{2\pi} \int d^2 z_\perp \int \vec{d}^2 p_\perp e^{ip_\perp (b-z)_\perp}$$

Part of the DGLAP

$$\times\Big(\frac{1}{2}\ln^2\frac{\mu^2_{\rm IR}}{p^2_\perp} + \boxed{\ln\frac{\mu^2_{\rm IR}}{p^2_\perp}\ln\frac{\rho}{\zeta}} - \frac{\pi^2}{12}\Big)\frac{g_{il}p_j p_m + p_i p_l g_{mj}}{p^2_\perp} f_{lm}(x_B,z_\perp,\mu^2_{\rm IR},\rho)$$

Part of BFKL

$$+\frac{\alpha_s N_c}{2\pi} \int d^2 z_\perp \int \vec{d}^2 p_\perp e^{ip_\perp (b-z)_\perp}\Big(\frac{\beta_0}{2N_c}\ln\frac{\mu^2_{\rm UV}}{p^2_\perp} + \frac{67}{18} - \frac{5N_f}{9N_c}\Big) f_{ij}(x_B,z_\perp,\mu^2_{\rm IR},\rho) + O(\alpha_s^2)\,.$$

Full result for quarks

$$\Phi^{[\gamma^+]}_{f;\mathrm{UV}}(x,b_\perp;\mu^2_{UV},\zeta) = \frac{g^2C_F}{8\pi^2}\Big(-\frac{1}{2}(L_b^{\mu_{UV}})^2+\Big(\frac{3}{2}+\ln\frac{2\mu^2_{UV}}{\zeta}\Big)L_b^{\mu_{UV}}+\frac{1}{2}-\frac{\pi^2}{12}\Big)\Phi^{[\gamma^+]}_{f;\mathrm{IR}}(x,b_\perp;\mu^2_{IR},\rho)$$

CSS

$$+\frac{g^2C_F}{8\pi^2}\int \vec{d}^2p_\perp e^{ip_\perp b_\perp}\Big(-\frac{1}{2}\ln\frac{4e^{-2\gamma_E}}{p_\perp^2b_\perp^2}-1\Big)\int d^2z_\perp e^{-ip_\perp z_\perp}\Phi^{[\gamma^+]}_{f;\mathrm{IR}}(x,z_\perp;\mu^2_{IR},\rho)$$

$$+\frac{g^2C_F}{\pi}\int \vec{d}^2p_\perp e^{ip_\perp b_\perp}\int_0^1\frac{du}{1-u}\int\vec{d}^2k_\perp\frac{1}{(1-u)p_\perp^2+uk_\perp^2}\int d^2z_\perp e^{i(k_\perp-p_\perp)z_\perp}\Big(\Phi^{[\gamma^+]}_{f;\mathrm{IR}}(\frac{x}{u},z_\perp;\mu^2_{IR},\rho)$$

$$-u\Phi^{[\gamma^+]}_{f;\mathrm{bg}}(x,z_\perp;\mu^2_{IR},\rho)\Big)+\frac{g^2C_F}{\pi}\int d^2z_\perp\int\vec{d}^2p_\perp e^{ip_\perp b_\perp}\int\frac{\vec{d}^2k_\perp}{k_\perp^2}\Big(1-e^{-ik_\perp b_\perp}\Big)e^{ik_\perp z_\perp}$$

$$\times\Big(1+\frac{1}{1-k_\perp^2/p_\perp^2}\ln\frac{k_\perp^2}{p_\perp^2}\Big)e^{-ip_\perp z_\perp}\Phi^{[\gamma^+]}_{f;\mathrm{IR}}(x,z_\perp;\mu^2_{IR},\rho)+\frac{ig^2C_F}{2\pi}\int d^2z_\perp\int\vec{d}^2p_\perp e^{ip_\perp b_\perp}\int\vec{d}^2k_\perp e^{i(k_\perp-p_\perp)z_\perp}$$

$$\times\int_0^1\frac{du}{u}\frac{1}{(p_\perp^2(1-u)+uk_\perp^2)^2}\Big[\frac{2ig^{\mu-}}{(2z^-)^2}\Big(\delta(1-u)-\frac{4p_\perp^2}{p_\perp^2(1-u)+uk_\perp^2}+\frac{6(1-u)p_\perp^4}{(p_\perp^2(1-u)+uk_\perp^2)^2}\Big)$$

$$+\frac{g^{\mu k}p_k}{2z^-}\Big(2-\frac{4(1-u)p_\perp^2}{p_\perp^2(1-u)+uk_\perp^2}\Big)-ig^{\mu+}(1-u)p_\perp^2\Big]\Phi^{[\gamma_\mu]}_{f;\mathrm{IR}}(\frac{x}{u},z_\perp;\mu^2_{IR},\rho)\,.$$