BRIDGING TMD FRAMEWORKS: INTERSECTIONS, TENSIONS, AND APPLICATIONS

09-13 March 2026

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Building bridges from TMD phenomenology to the wider QCD landscape

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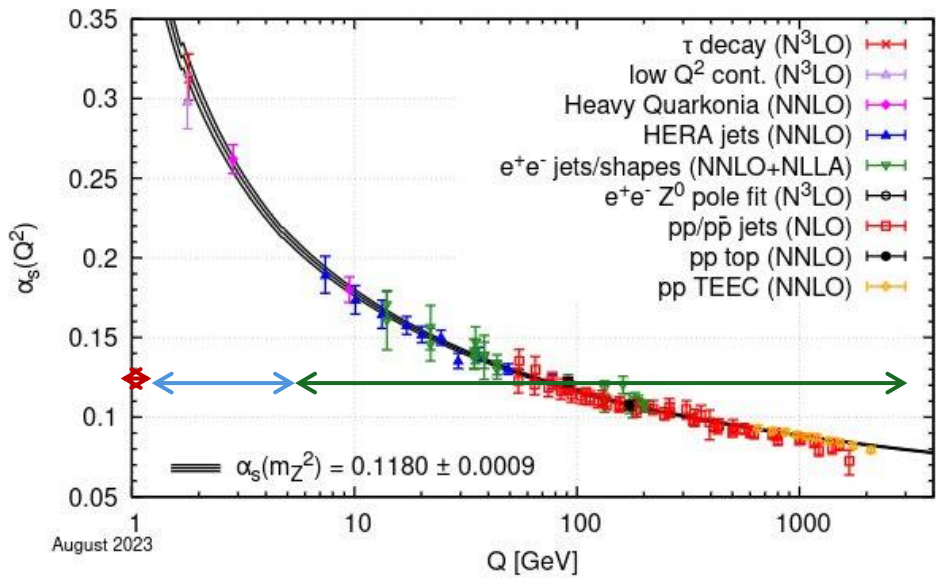
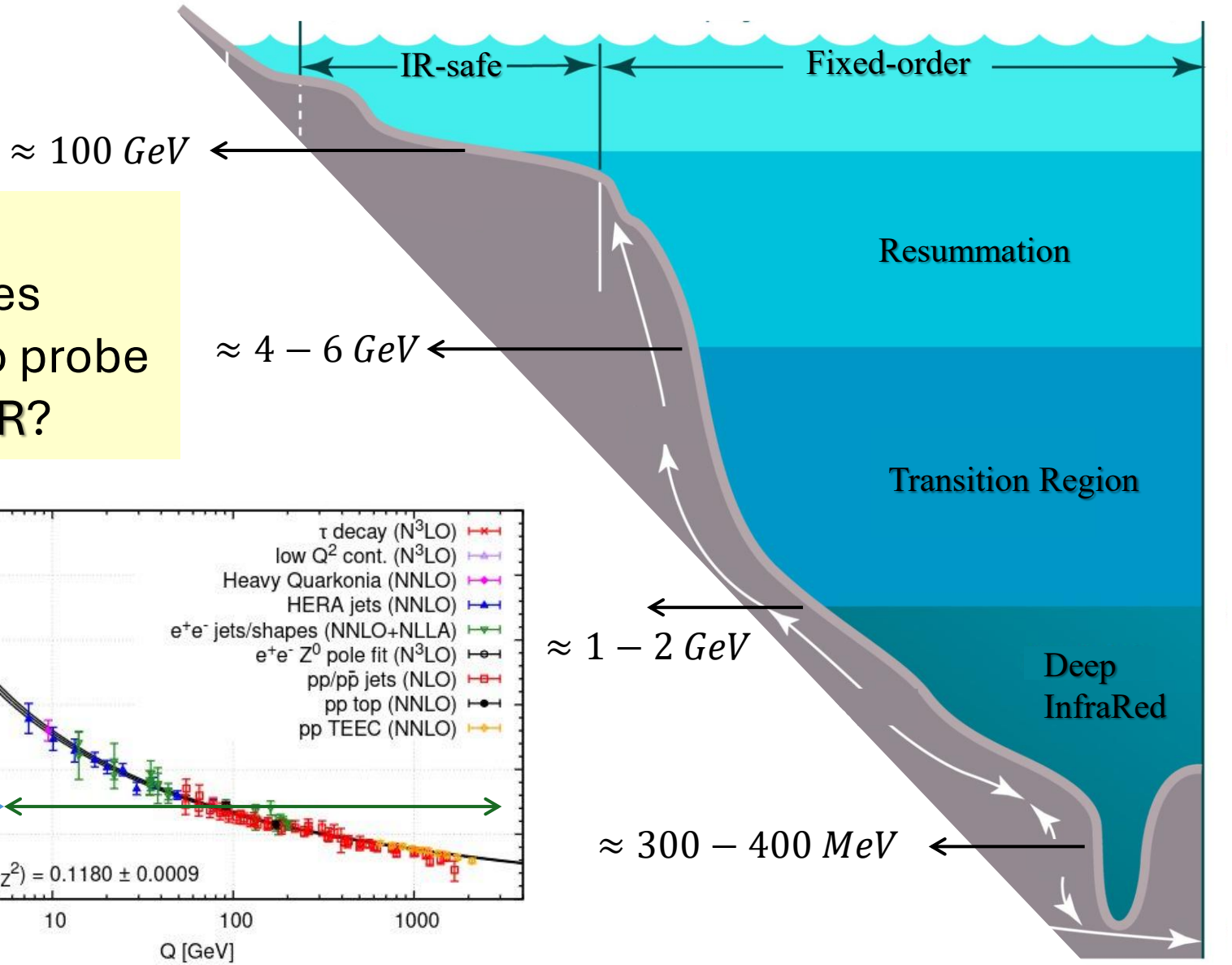
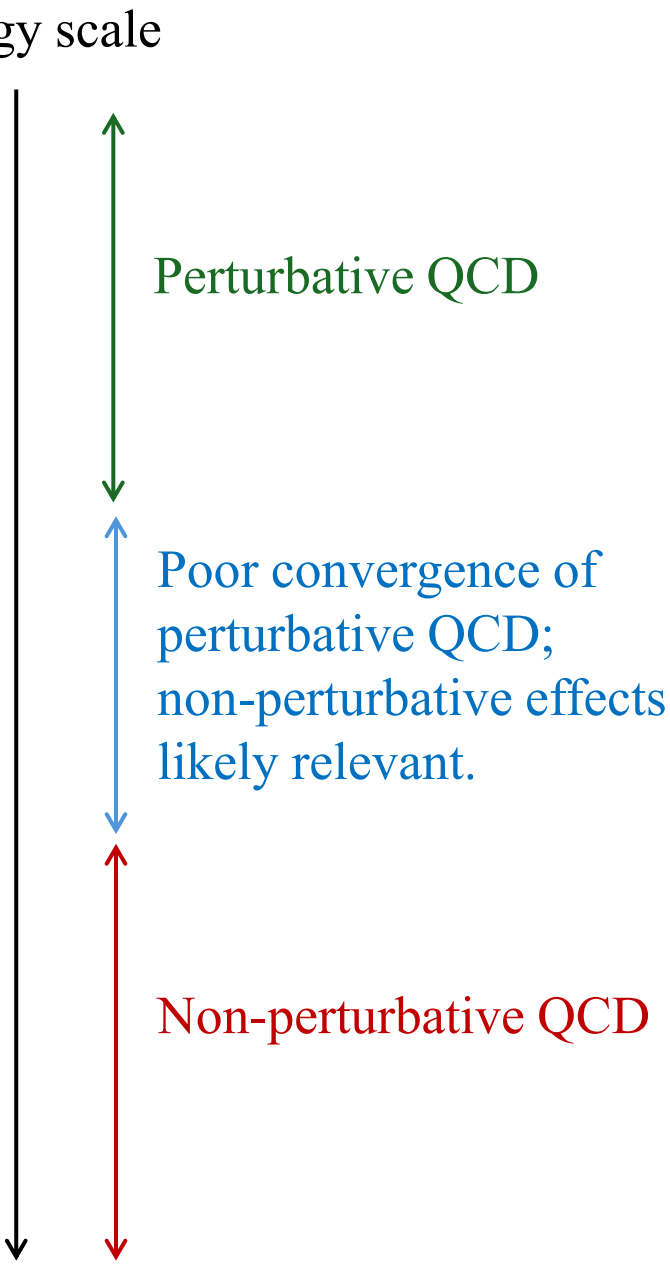
How deep can we probe QCD?

Which observables allow us to probe the Deep IR?







How deep can we probe QCD?

Deep Inelastic Scattering

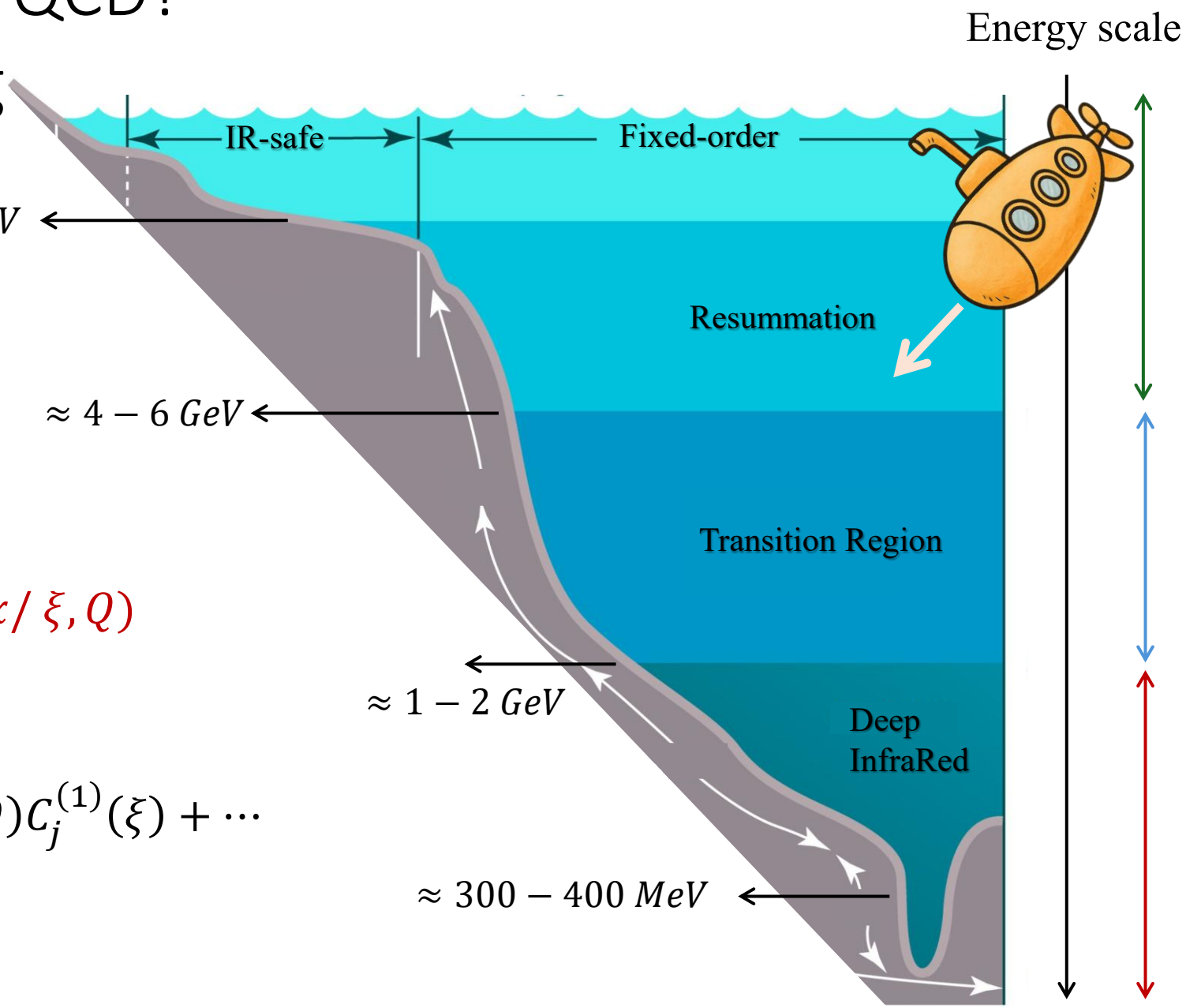
(and in general, collinear factorization)

Proton (longitudinal)

Structure probed at scale Q

$$\frac{d\,\sigma}{d\,x\,d\,Q^2} = \sum_j \int_x^1 \frac{d\,\xi}{\xi} \frac{d\,\hat{\sigma}_j}{d\,\xi\,d\,Q^2} f_{j/\,h}(x/\,\xi, Q)$$

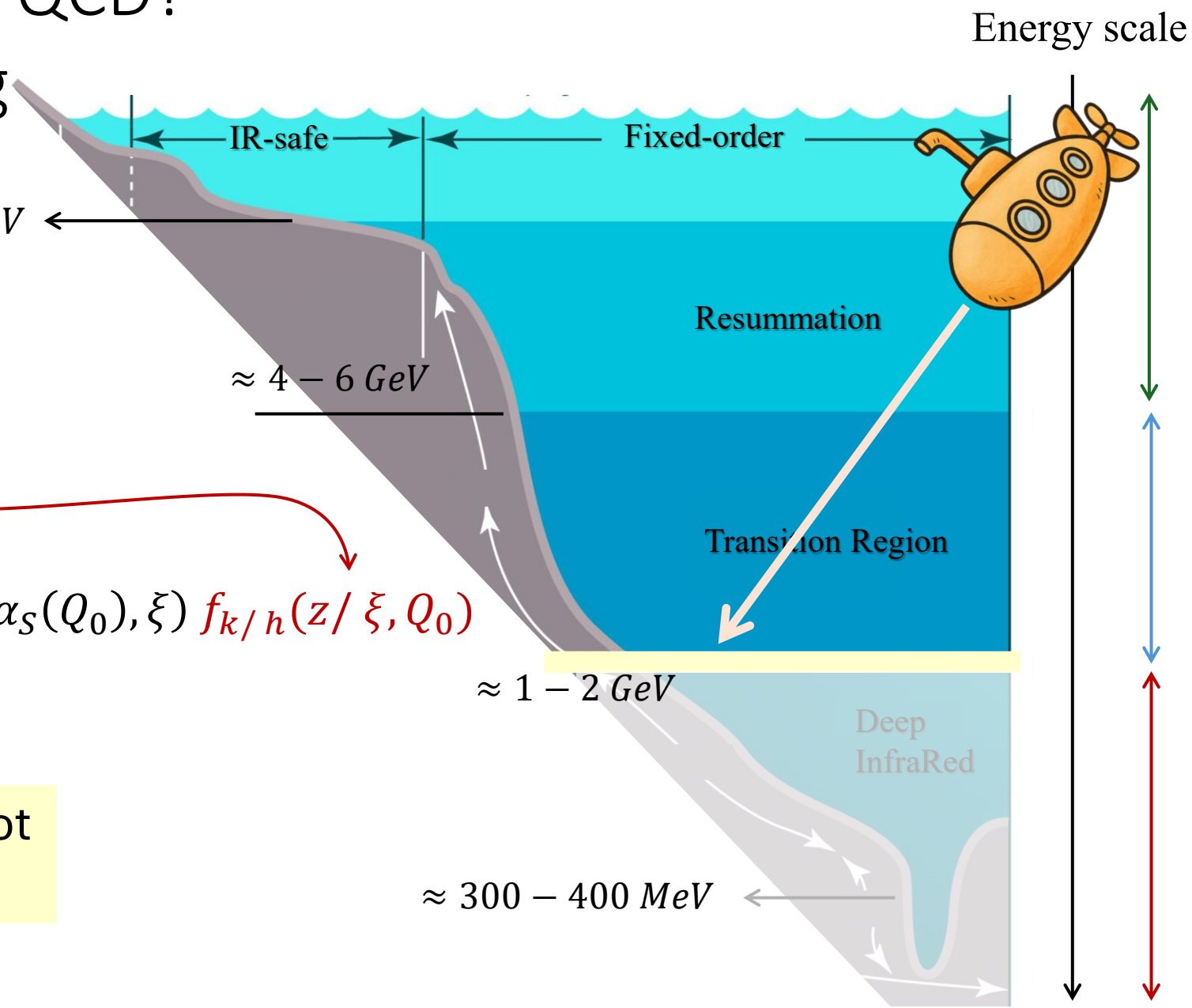
$$= C_j^{(0)}(\xi) + \alpha_S(Q) C_j^{(1)}(\xi) + \cdots$$



How deep can we probe QCD?

Deep Inelastic Scattering

(and in general, collinear factorization)



Proton (longitudinal)
Structure probed up to scale Q_0

$$f_{j/h}(z,Q) = \sum_k \int_z^1 \frac{d\,\xi}{\xi} E_{j/k}^{DGLAP}(\alpha_S(Q), \alpha_S(Q_0), \xi)\, f_{k/h}(z/\xi, Q_0)$$

As low as Q_0 is large enough to not violate collinear factorization

How deep can we probe QCD?

Thrust distribution

(and in general, event-shape observables)

$\left[\text{saddle point } |u| \approx 1/\tau\right]$

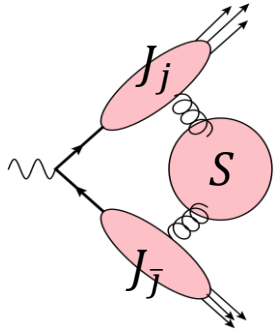
$$\frac{d\,\sigma}{d\,\tau\,d\,Q^2} = \sum_j H_{j\,\bar{j}}\left(\alpha_S(Q)\right) \int \frac{d\,u}{2\,\pi\,i} e^{u\,\tau}$$

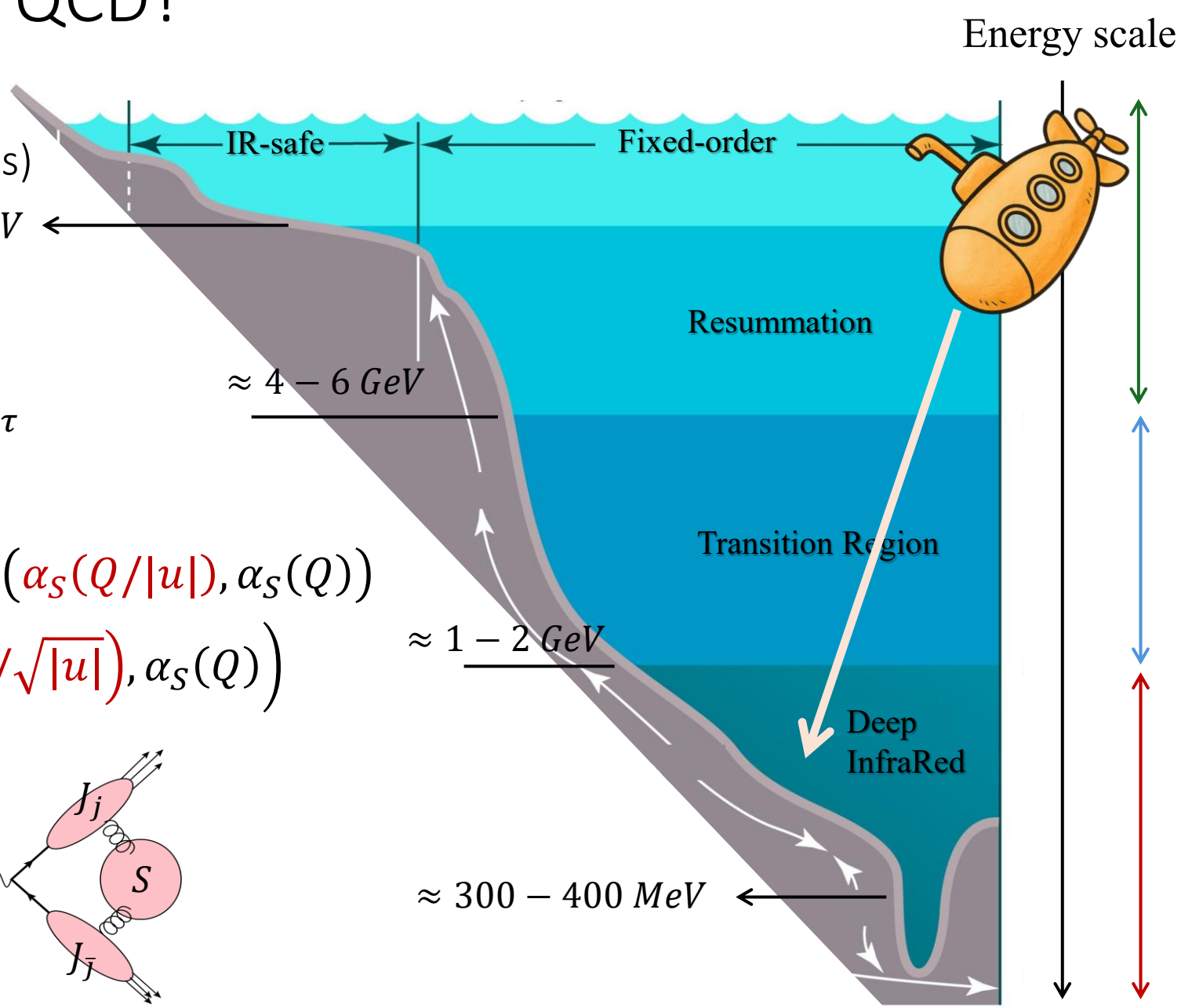
$$\times J_j\left(\alpha_S\left(Q/\sqrt{|u|}\right), \alpha_S(Q)\right) S\left(\alpha_S(Q/|u|), \alpha_S(Q)\right)$$

$$\times J_{\bar{j}}\left(\alpha_S\left(Q/\sqrt{|u|}\right), \alpha_S(Q)\right)$$

Strong coupling probed in the Deep InfraRed

As low as τ goes!

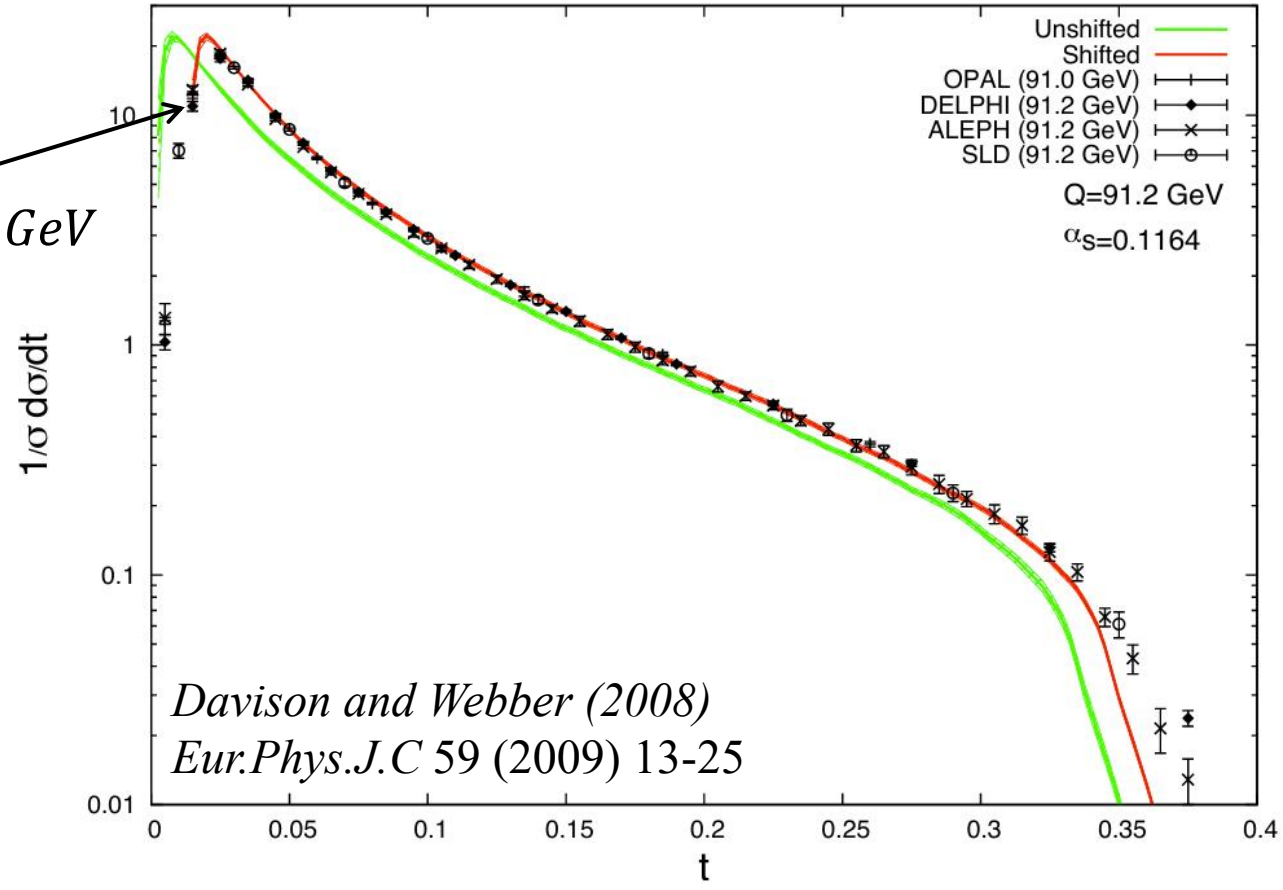




How deep can we probe QCD?
Thrust distribution
(and in general, event-shape observables)

Strong coupling probed up to $\approx 1.8\ GeV$

Davison and Webber used an *extremely simple model* (1 param.) for an effective coupling α_{eff} which extrapolates below $Q_0 = 2\ GeV$ results in a $1/Q$-dependent shift in the distribution

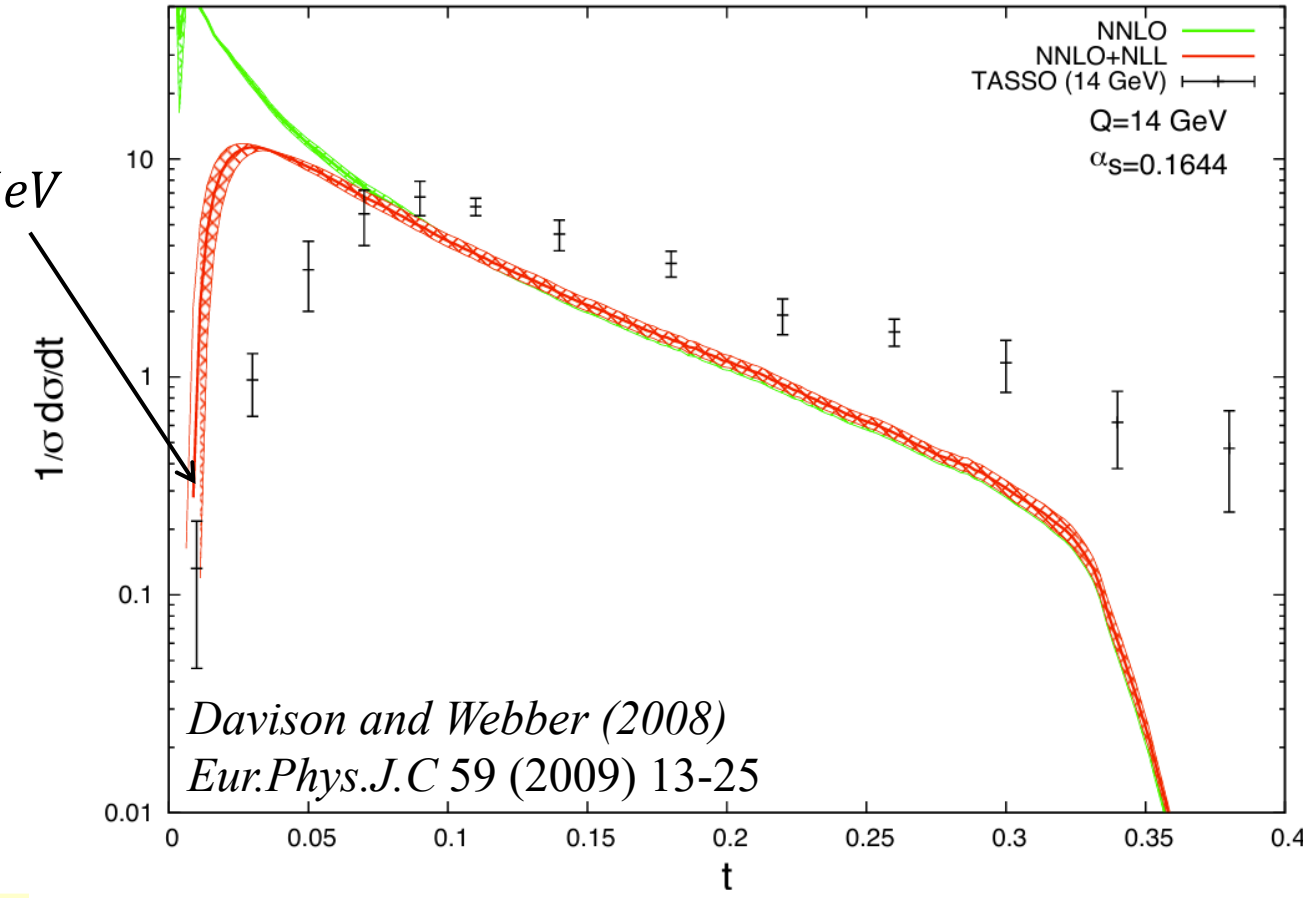


How deep can we probe QCD?
Thrust distribution

(and in general, event-shape observables)

Strong coupling probed up to $\approx 200\ MeV$

Davison and Webber used an *extremely simple model* (1 param.) for an effective coupling α_{eff} which extrapolates below $Q_0 = 2\ GeV$ results in a $1/Q$-dependent shift in the distribution



Too simple model? Wrong philosophy?

How deep can we probe QCD?

Thrust distribution

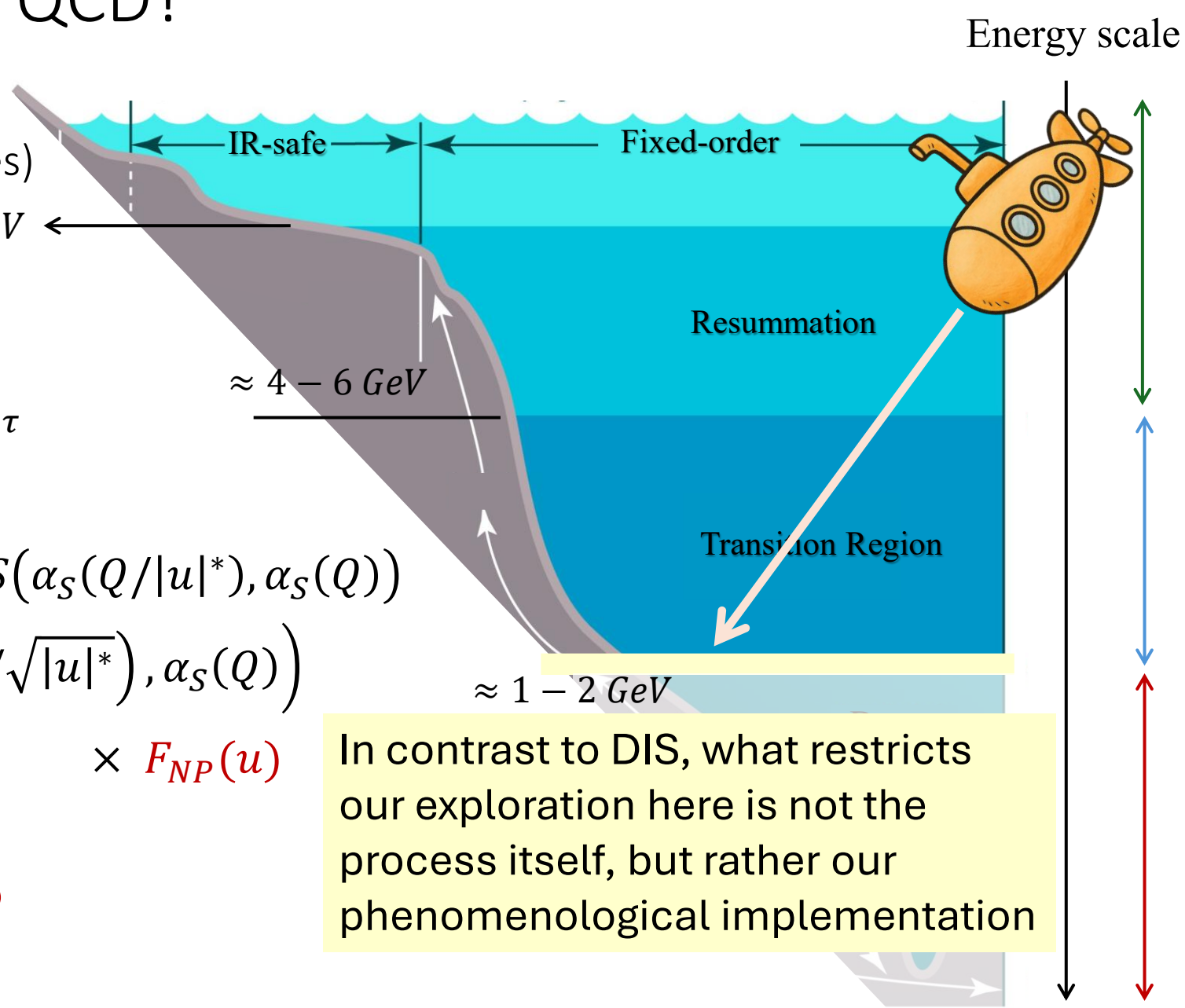
(and in general, event-shape observables)

$\left[\text{saddle point } |u| \approx 1/\tau\right]$

$$\frac{d\,\sigma}{d\,\tau\,d\,Q^2} = \sum_j H_{j\,\bar{J}}\left(\alpha_S(Q)\right) \int \frac{d\,u}{2\,\pi\,i} e^{u\,\tau} \times J_j\left(\alpha_S\left(Q/\sqrt{|u|^*}\right), \alpha_S(Q)\right) S\left(\alpha_S(Q/|u|^*), \alpha_S(Q)\right) \times J_{\bar{J}}\left(\alpha_S\left(Q/\sqrt{|u|^*}\right), \alpha_S(Q)\right) \times F_{NP}(u)$$

Strong coupling probed up to
$Q_0 = Q/|u_{max}| \approx 1 - 2\ GeV$

QCD Deep IR behavior is encoded into a non-perturbative black-box F_{NP}

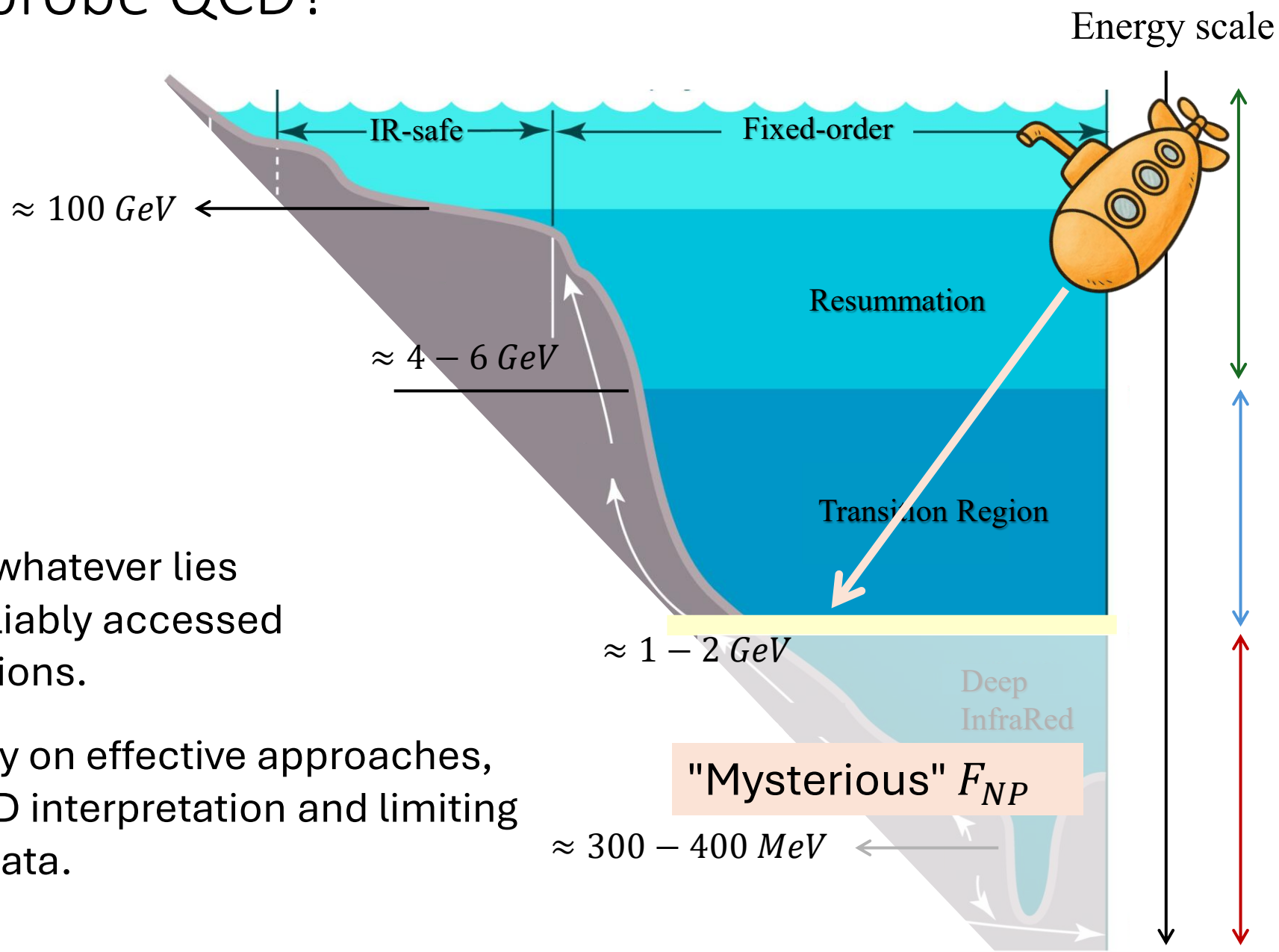


In contrast to DIS, what restricts our exploration here is not the process itself, but rather our phenomenological implementation

How deep can we probe QCD?

Accepting this **philosophy**, whatever lies below 1–2 GeV cannot be reliably accessed through hadronic cross sections.

In this region we can only rely on effective approaches, giving up a fundamental QCD interpretation and limiting ourselves to modelling the data.

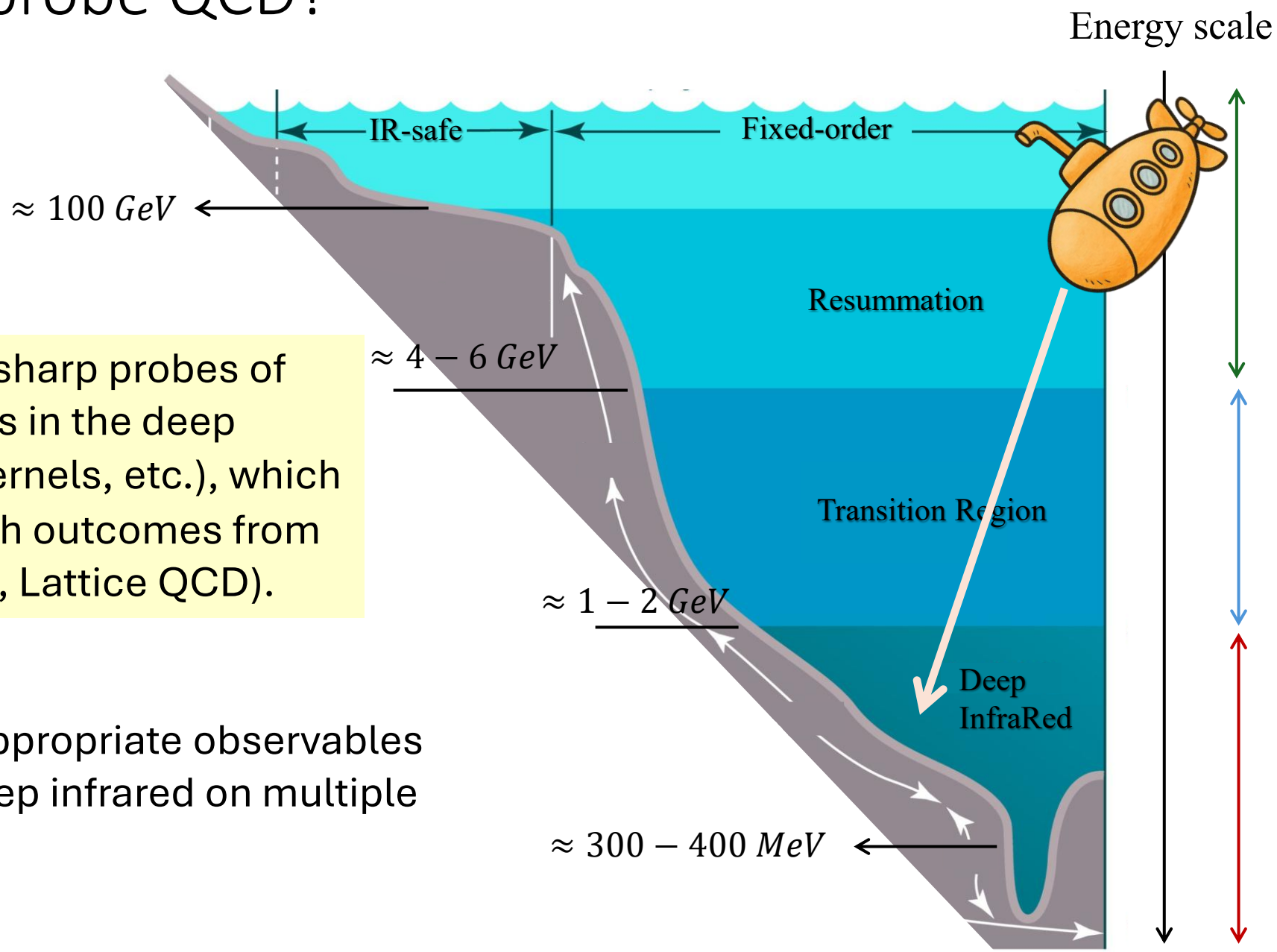


How deep can we probe QCD?

But I believe it is worth at least giving a chance to the **alternative perspective**:

Hadronic cross sections are sharp probes of fundamental QCD ingredients in the deep infrared (e.g., β_{QCD}, DGLAP kernels, etc.), which can be directly compared with outcomes from other channels (e.g., NRQCD, Lattice QCD).

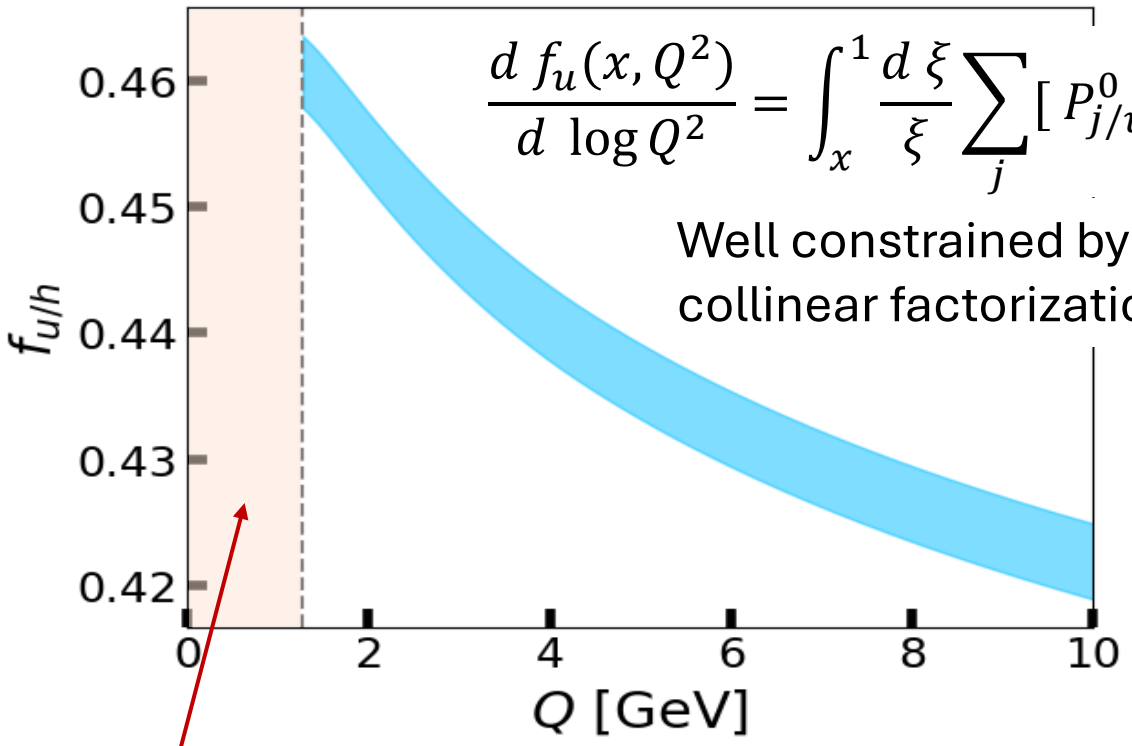
We simply need to identify appropriate observables that allow us to probe the deep infrared on multiple fronts (coupling, PDFs, etc.).



How deep can we probe QCD?

$$\frac{d\, f_u(x,Q^2)}{d\, \log Q^2} = \int_x^1 \frac{d\,\xi}{\xi} \sum_j [\, P^0_{j/u}(\xi)\alpha_S\,(Q) + P^1_{j/u}(\xi)\alpha_S^2(Q) + \cdots]\, f_j(x/\xi, Q^2)$$

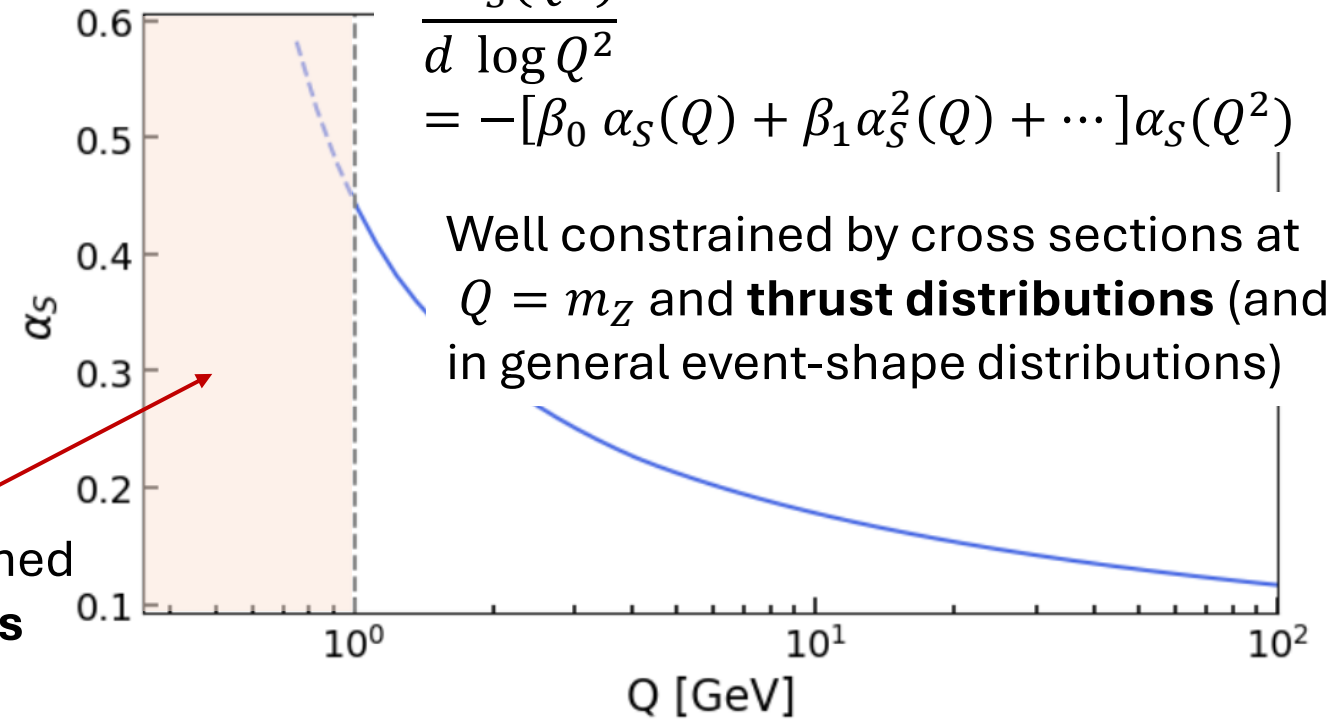
Well constrained by **DIS** (and in general collinear factorization theorems)



An actual mystery

$$\frac{d\alpha_S(Q^2)}{d\, \log Q^2} = -[\beta_0\, \alpha_S(Q) + \beta_1 \alpha_S^2(Q) + \cdots]\alpha_S(Q^2)$$

Well constrained by cross sections at $Q = m_Z$ and **thrust distributions** (and in general event-shape distributions)



Hard to control within phenomenological analyses, yet constrained by **thrust distributions**

Transverse Momentum Dependent cross sections

Factorization theorems typical of processes sensitive to the transverse momentum $\boldsymbol{q}_T$ of a final state relative to a specified axis, in the kinematic limit where $|\boldsymbol{q}_T| \ll Q$

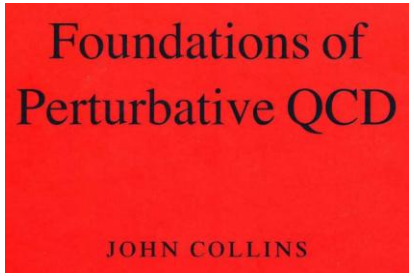
Standard cases are:

- **Drell-Yan** at small transverse momentum of the lepton pair
- **Semi-Inclusive DIS** at small transverse momentum of the detected hadron w.r.t. the target-γ^* plane
- **Double-inclusive $\boldsymbol{e^+e^-}$ annihilation** into two back-to-back hadrons

In all these cases

$$\frac{d\,\sigma}{d\boldsymbol{q}_T d\,Q^2} = H(\alpha_S(\mu), Q/\mu) \int \frac{d^2\boldsymbol{b}_T}{(2\pi)^2} e^{i\,\boldsymbol{q}_T\cdot\boldsymbol{b}_T}\, F_1(\mu, x_1, |\boldsymbol{b}_T|, y_n) F_2(\mu, x_2, |\boldsymbol{b}_T|\, y_n) + pol.terms$$

Transverse-Momentum-Dependent (TMD) factorization [Collins, 2011]

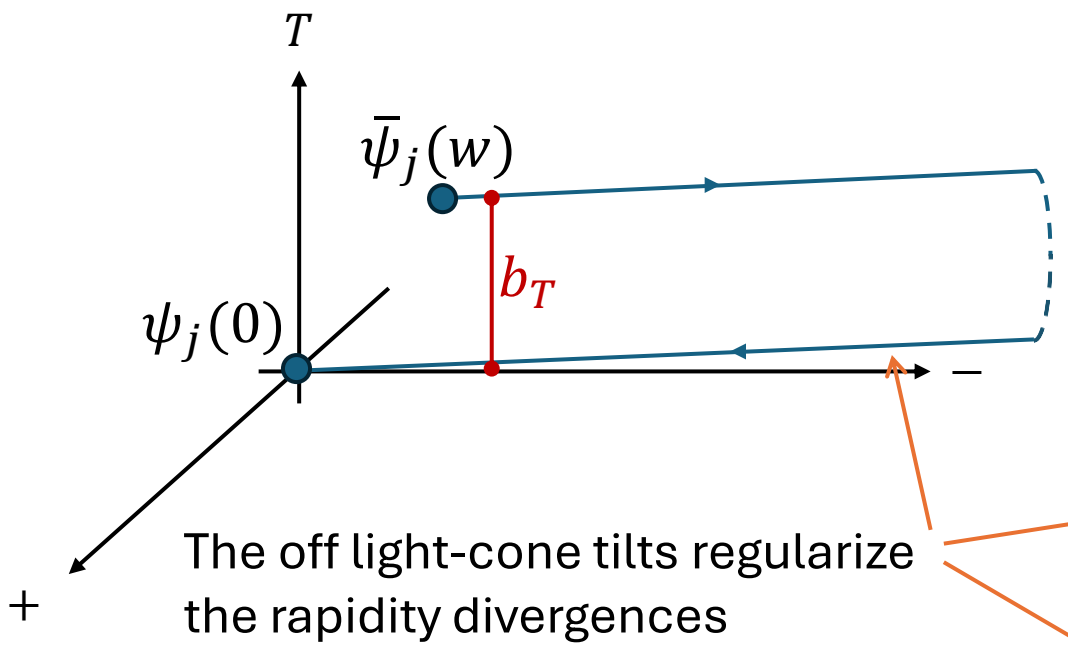


TMD parton densities

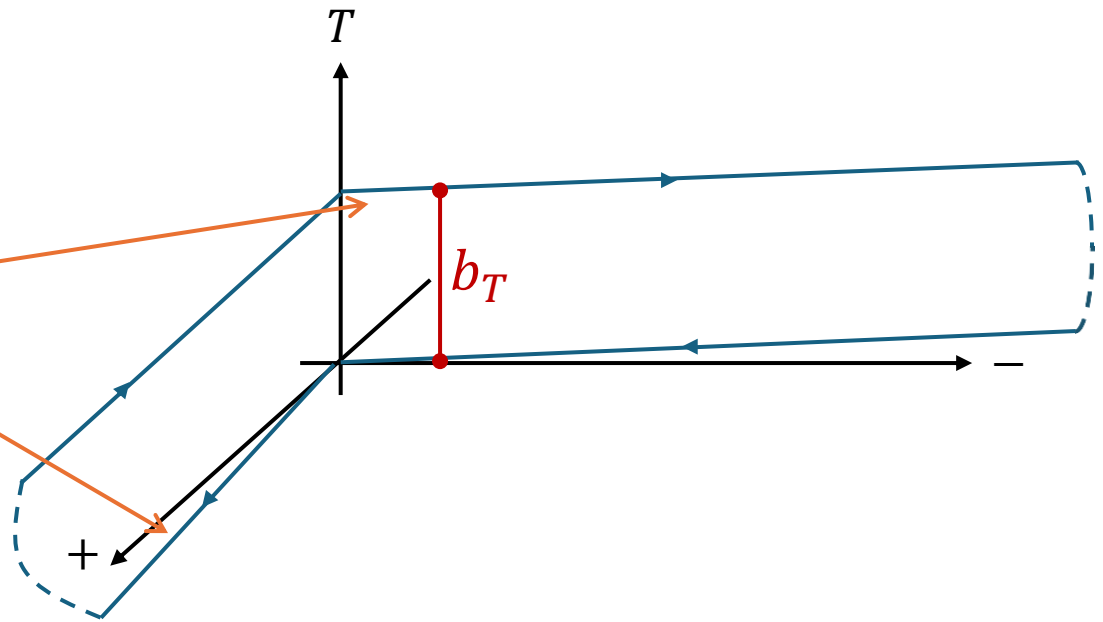
$$F_j(x, b_T, y_n) = \int \frac{d\, w^-}{2\,\pi} e^{-i\, x\, P^+\, w^-} Tr\, \langle P, S | \bar{\psi}_j(w) \gamma^+ W_{0\to w;n} \psi_j(0) | P, S \rangle \times [S(b_T, y_n)]$$

$w = (0, w^-, \boldsymbol{b}_T)$

Required to avoid double-counting of soft-collinear contributions and to handle the regularization of rapidity divergences



The off light-cone tilts regularize the rapidity divergences



Transverse Momentum Dependent parton densities

Evolution equations:

- Renormalization group μ
- Light-cone tilt (or rapidity regulator) y_n

$$\begin{cases} \dfrac{\partial \log F_j(x, b_T, \mu, y_n)}{\partial \log \mu} = \gamma_F(a_S(\mu), \log(Q/\mu) \pm y_n) \\ \dfrac{\partial \log F_j(x, b_T, \mu, y_n)}{\partial y_n} = \pm K(a_S(\mu), \log(b_T\ \mu/c_1)) \end{cases}$$

Collins-Soper kernel or
rapidity anomalous dimension (RAD)

$$\frac{dK(a_S(\mu), \log(b_T\ \mu/c_1))}{d\ \log \mu} = -\gamma_K(a_S(\mu))$$

Associated with recoil of soft gluons

Setting the reference scale to the *natural* choice $\mu_b = c_1/b_T$

$$\mathrm{F_j}(\mathrm{x}, \mathrm{b_T}, \mu, y_n) = \mathrm{F_j}(\mathrm{x}, \mathrm{b_T}, \mu_b, \mu_b^2)\, exp\left\{\log\left(\frac{\mu}{\mu_b}\right) K(a_S(\mu_b), 0) + \int_{\mu_b}^{\mu} \frac{d\ \mu'}{\mu'} \gamma_F(a_S(\mu), \log(\mu/\mu'))\right\}$$

$$\times\, exp\left\{\left[\log\left(\frac{Q}{\mu}\right) \pm y_n\right] K(a_S(\mu), \log(b_T\ \mu/c_1))\right\}$$

Transverse Momentum Dependent parton densities

Evolution equations:

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Collins-Soper kernel or rapidity anomalous dimension (RAD)

$$\frac{dK(a_S(\mu), \log(b_T\, \mu/c_1))}{d\ \log \mu} = -\gamma_K(a_S(\mu))$$

Associated with recoil of soft gluons

Setting the reference scale to the *natural* choice $\mu_b = c_1/b_T$

$$\mathrm{F_j}(\mathrm{x}, \mathrm{b_T}, \mu, y_n) = \mathrm{F_j}(\mathrm{x}, \mathrm{b_T}, \mu_b, \mu_b^2)\, exp\left\{\log\left(\frac{\mu}{\mu_b}\right) K(a_S(\mu_b), 0) + \int_{\mu_b}^{\mu} \frac{d\,\mu'}{\mu'} \gamma_F(a_S(\mu), \log(\mu/\mu'))\right\}$$

The dependence on y_n disappears in the cross section, so it can be fixed to any value we choose

- MAP: $y_n = 0$
- ART: $y_n = y_n^*$ (zeta prescription)

$$\times\, exp\left\{\left[\log\left(\frac{Q}{\mu}\right) \pm y_n\right] K(a_S(\mu), \log(b_T\, \mu/c_1))\right\}$$

How far in b_T can we push perturbative QCD?

$$\mathrm{F_j}(\mathrm{x}, \mathrm{b_T}, Q, 0)$$

$$= \mathrm{F_j}(\mathrm{x}, \mathrm{b_T}, \mu_b, \mu_b^2)\, exp\left\{\log\left(\frac{Q}{\mu_b}\right) K(a_S(\mu_b), 0) + \int_{a_S(\mu_b)}^{a_S(Q)} \frac{d\,a}{2\,\beta_{QCD}(a)}\left[\gamma_f(a) - \gamma_k(a)\int_a^{a_S(Q)} \frac{d\,a'}{2\,\beta_{QCD}(a')}\right]\right\}$$

$$a_S(\mu_b)^2\,\gamma_r^1 + \cdots$$

It can be integrated *analytically*, provided that the integrand can be expanded in powers of a, which in turn is valid as long as $\alpha_S(\mu_b)$ is perturbative

$$f_j(x, \mu_b) + a_S(\mu_b)\int_x^1 \frac{d\,\xi}{\xi} C_{j/k}^1(\xi) f_k\left(\frac{x}{\xi}, \mu_b\right) + \cdots$$

Valid approximations as long as $a_S(\mu_b)$ is perturbative

How far in b_T can we push perturbative QCD?

$$\mathrm{F_j}(\mathrm{x}, \mathrm{b_T}, Q, 0)$$

$$= \mathrm{F_j}\left(\mathrm{x}, \mathrm{b_T}, \mu_b, \mu_b^2\right) exp\left\{\log\left(\frac{Q}{\mu_b}\right) K(a_S(\mu_b), 0) + \int_{a_S(\mu_b)}^{a_S(Q)} \frac{d\,a}{2\,\beta_{QCD}(a)}\left[\gamma_f(a) - \gamma_k(a)\int_a^{a_S(Q)} \frac{d\,a'}{2\,\beta_{QCD}(a')}\right]\right\}$$

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Valid approximations as long as $a_S(\mu_b)$ is perturbative

$\mu_b \gtrsim 1\ GeV \quad \longleftrightarrow \quad \mathrm{b_T} \lesssim 1\ GeV^{-1}$

We *know* the TMD up to around $1\ GeV^{-1}$

How far in b_T can we push perturbative QCD?

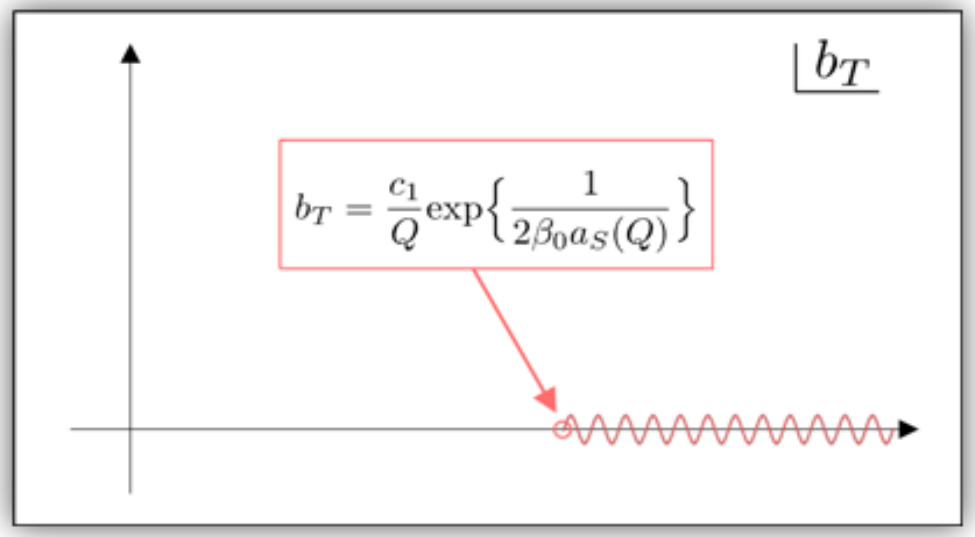
The perturbative control can be made explicit by expanding the integrals in descending powers of $L_b = \log(Q/\mu_b)$, after trading $a_S(\mu_b)$ with $a_S(Q) \times L_b$

$$F_j(x, b_T, Q, 0) = f_j(x, \mu_b)\, exp\left\{ \underbrace{L_b\, g_1(\lambda_b)}_{\text{LL}} + \underbrace{g_2(\lambda_b)}_{\text{NLL}} + \underbrace{\frac{1}{L_b} g_3(\lambda_b) + \cdots}_{\text{NNLL}} \right\}$$

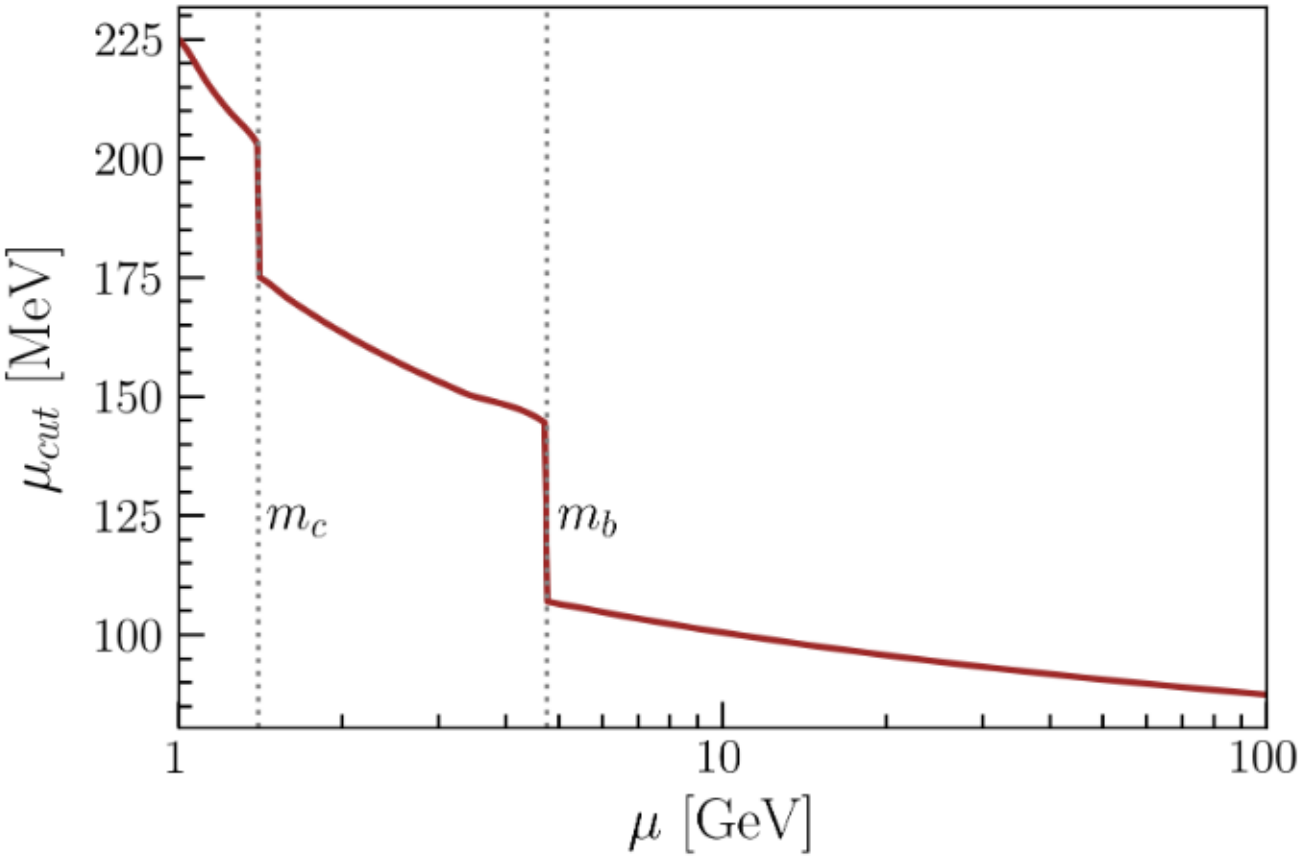
$$\lambda_b = \beta_0 a_S(Q) \log(Q^2/\mu_b^2)$$

What do we do for $b_T \gtrsim 1\ GeV^{-1}$?
The scale μ_b is not perturbative anymore,
but can how far can we stretch the expression above?

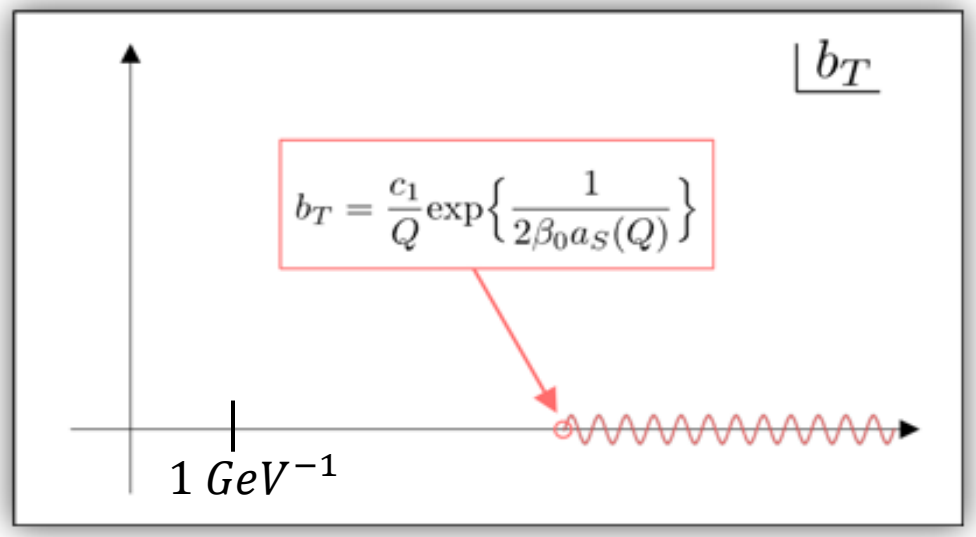
The function loses *analyticity* (manifestation of Landau pole)
and has a **cut** in $\lambda_b = 1$



How far in b_T can we push perturbative QCD?



Note that the cut is very far in b_T, very deep in the infrared (order of 100-200 MeV).



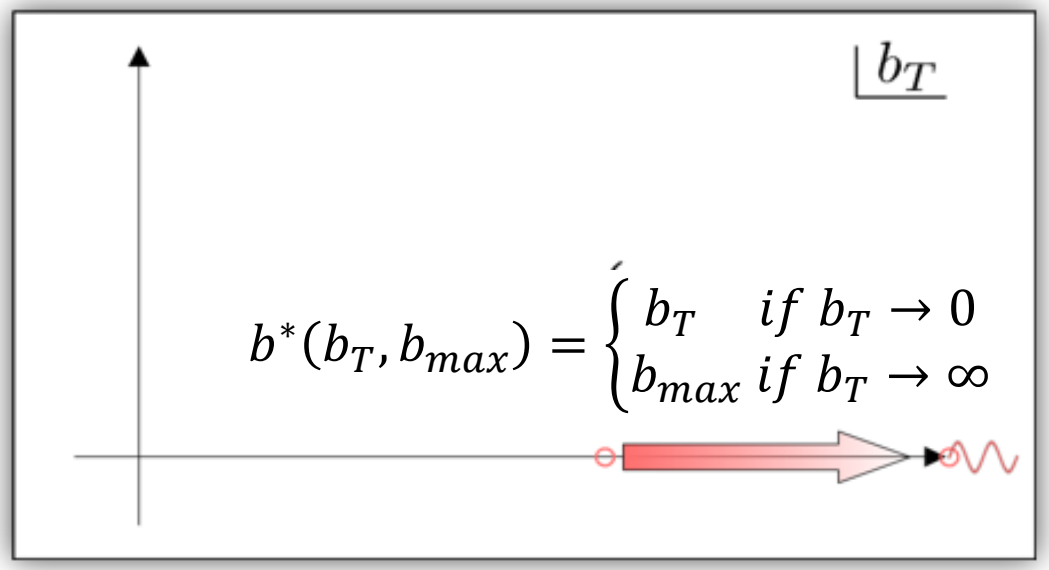
How far in b_T can we push perturbative QCD?

Having a branch cut for real values of b_T , although large, leads to an even more problematic issue.
The inverse Fourier transform becomes ambiguous:
which value should be kept, the one above the cut, or the one below it?

A prescription must be introduced. By far the most famous and popular is the **b^*-prescription**.

A possible (simple) parametrization:

$$b^*(b_T, b_{max}) = \frac{bT}{\sqrt{1 + \frac{b_T^2}{b_{max}^2}}}$$

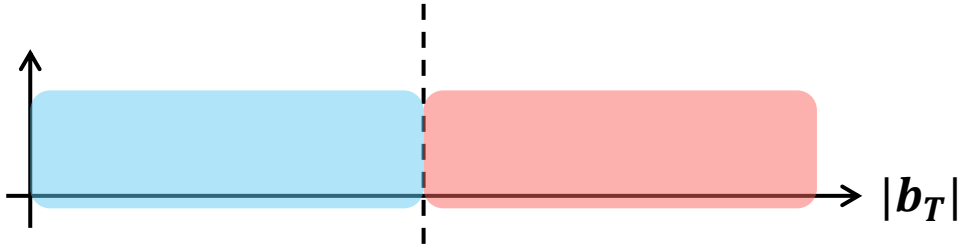


The *intrinsic arbitrariness* is in
- The choice of the saturation point b_{max}
- The functional form describing the saturation

If $b_{\max} \approx 1\ GeV^{-1}$ everything is evaluated in the perturbative region

Standard approach

$$F_j(x, |\boldsymbol{b_T}|, \mu, y_n) = F_j(x, |\boldsymbol{b_T}|^*, \mu, y_n) \otimes F_{NP}(x, |\boldsymbol{b_T}|, j) e^{-g_K(|\boldsymbol{b_T}|)\log\left(\frac{Q^2 e^{-y_n}}{\mu}\right)}$$



All efforts are dedicated to model and parametrize them

For instance:

$$|\boldsymbol{b_T}|^* = \frac{|\boldsymbol{b_T}|}{\sqrt{1 + \frac{|\boldsymbol{b_T}|^2}{b_{max}^2}}}$$

Completely determined by perturbative QCD

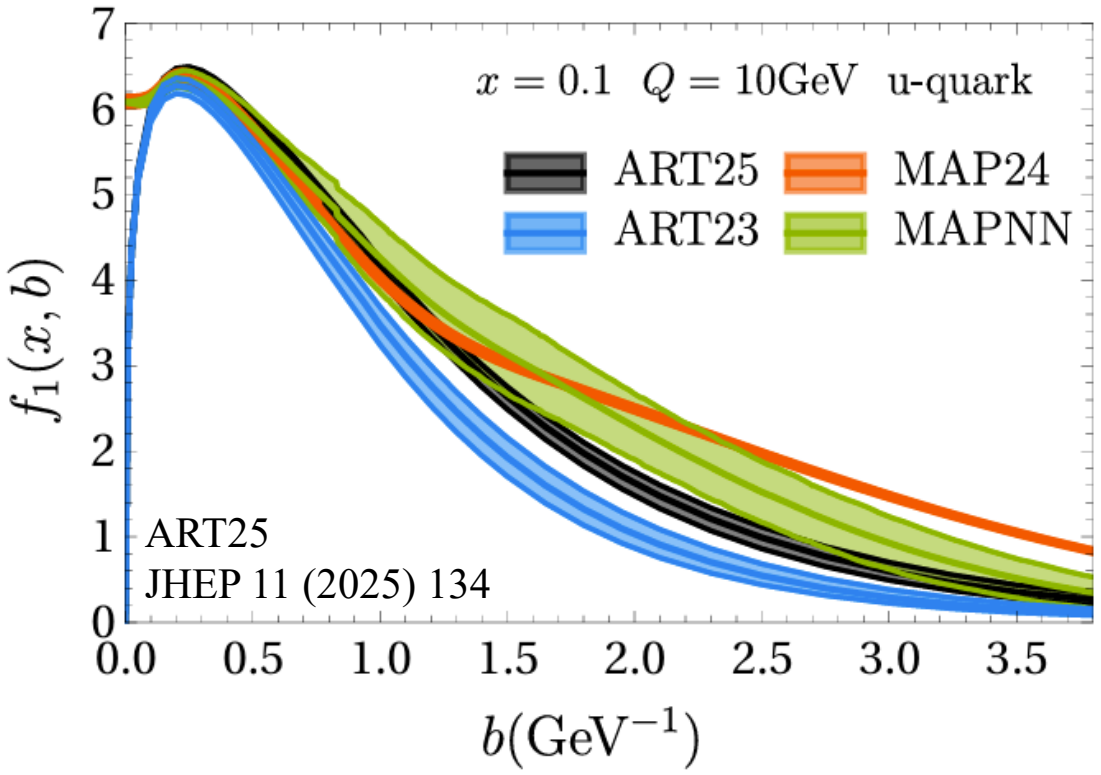
Black boxes that have to be determined by imposing agreement with data

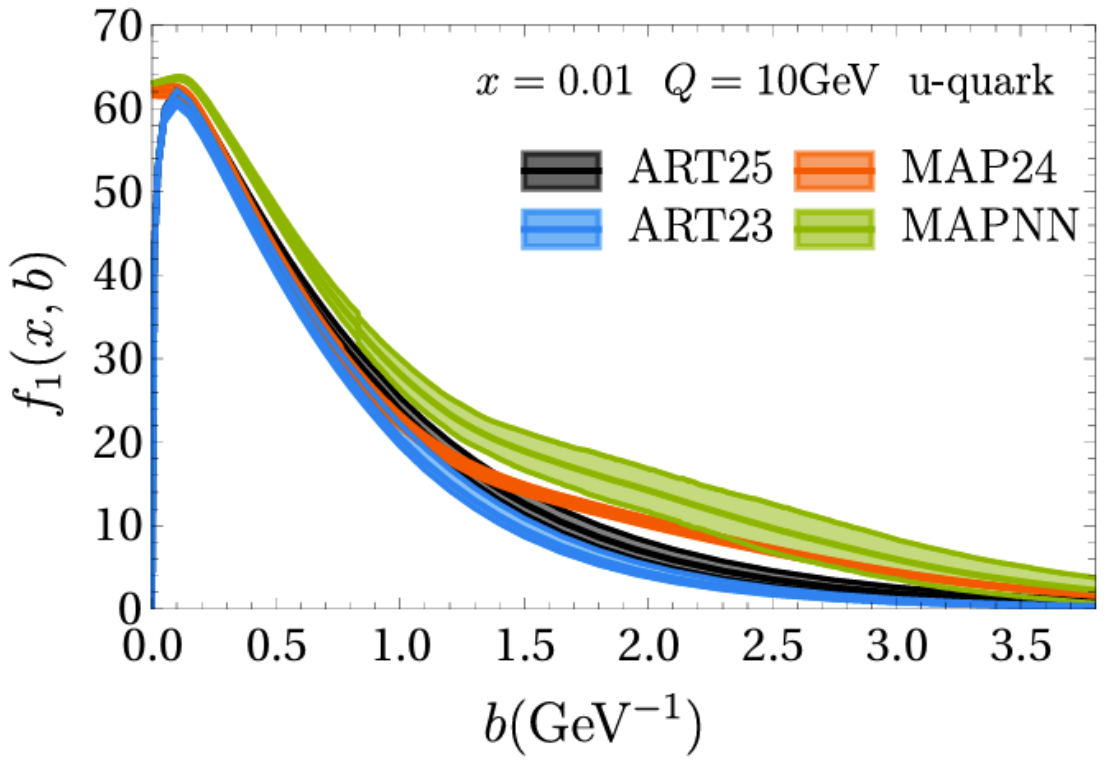
$\approx 1\ GeV^{-1}$

Standard hypothesis: the way the transition is implemented (how and where) is *unphysical* and should not affect the predictions.

The b* prescription (functional form and b_{max} value) is fixed, not fitted

Something is not working properly





Is it possible to test the standard hypothesis in a controlled set-up?

Impact of prescriptions

The impact of prescriptions in phenomenological extractions of Transverse Momentum Dependent distributions

Matteo Cerutti[1,*] and Andrea Simonelli[2,†]

Coming very soon!

Data from E288 (130 pts) and E605 (50 pts) experiments:

$4\ GeV \leq Q \leq 13.5\ GeV$ ⟶ Energy is low but "still perturbative".

TMD extraction performed at N2LL and with a simple NP ansatz:

$$g_K = \frac{g_2}{2} b_T^2 \qquad g_{j/h} = \frac{g_1}{2} b_T^2$$

We consider different b^*-prescriptions while keeping *the same functional form* for NP models.

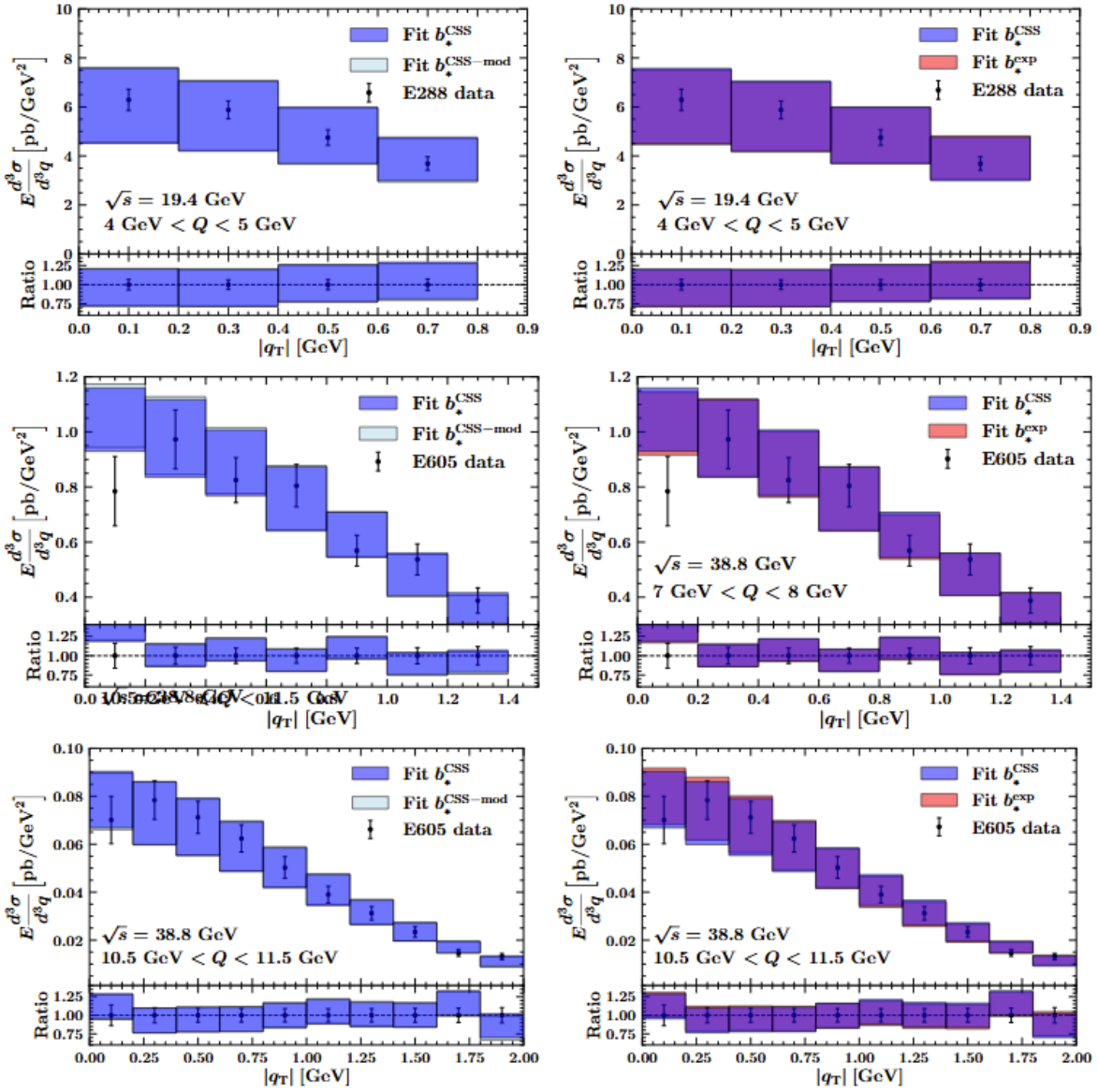
- Different functional form:

$$b_*^{\mathrm{CSS}}(\boldsymbol{b}_T^2) = \sqrt{\frac{\boldsymbol{b}_T^2}{1 + \frac{\boldsymbol{b}_T^2}{b_{\mathrm{max}}^2}}} \qquad b_*^{\mathrm{exp}}(\boldsymbol{b}_T^2) = b_{\mathrm{max}}\left[1 - \exp\left(-\boldsymbol{b}_T^4 / b_{\mathrm{max}}^4\right)\right]^{\frac{1}{4}}$$

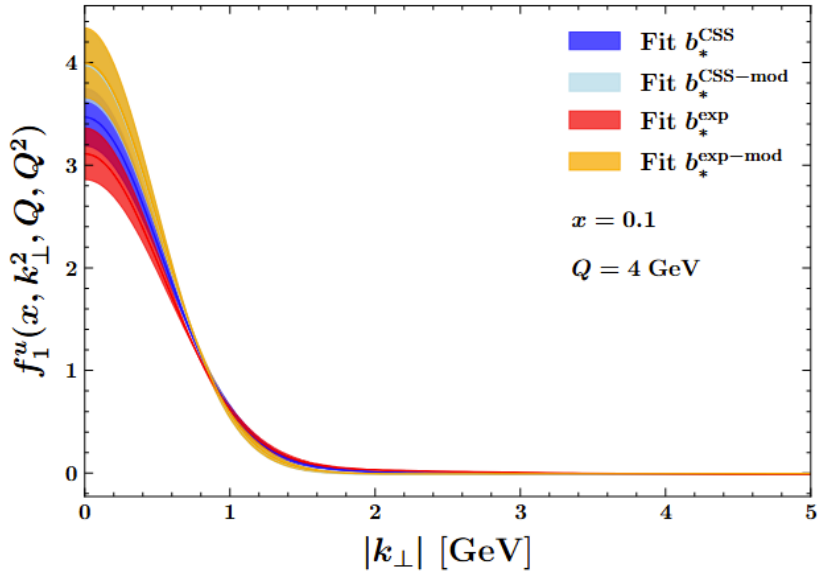
- Different b_{max}:

$$b_{\mathrm{max}} = c_1 \approx 1.12\ \mathrm{GeV}^{-1} \qquad b_{\mathrm{max}} = c_1/2$$

Impact of prescriptions

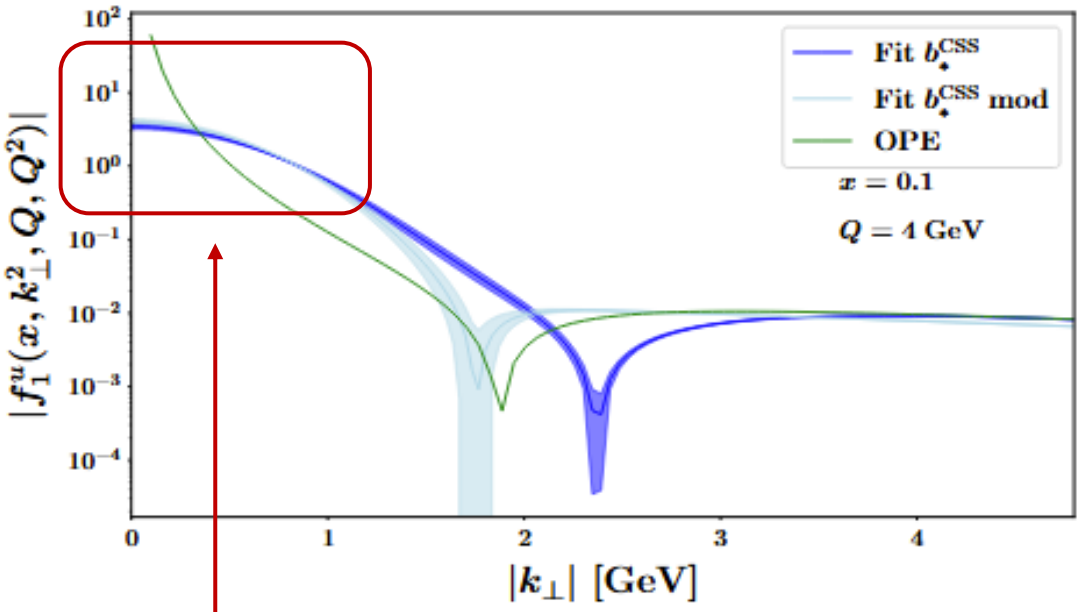


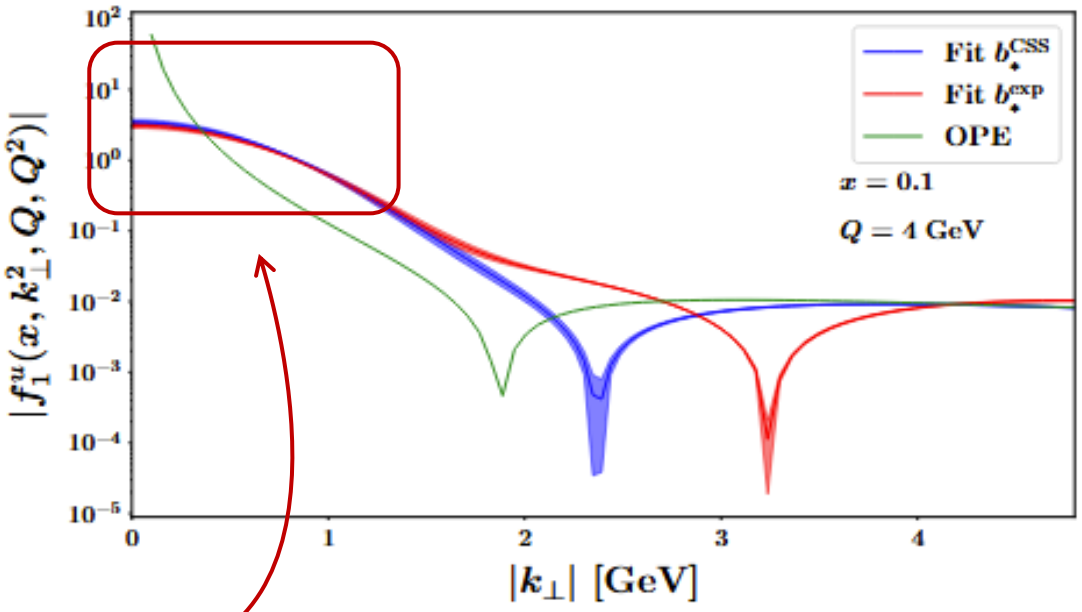
		$\chi^2/N_{\rm dat}$			
Dataset	$N_{\rm dat}$	$b_*^{\rm CSS}$	$b_*^{\rm CSS-mod}$	$b_*^{\rm exp}$	$b_*^{\rm exp-mod}$
E288	130	0.99	0.83	1.00	0.96
E605	50	1.66	1.78	1.40	1.85
Total	180	1.18	1.09	1.11	1.21



It seems that the arbitrariness of our choice just increases the uncertainty bands of the TMDs.

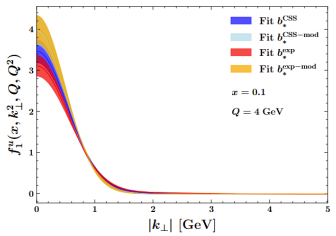
Ambiguity at intermediate k_T



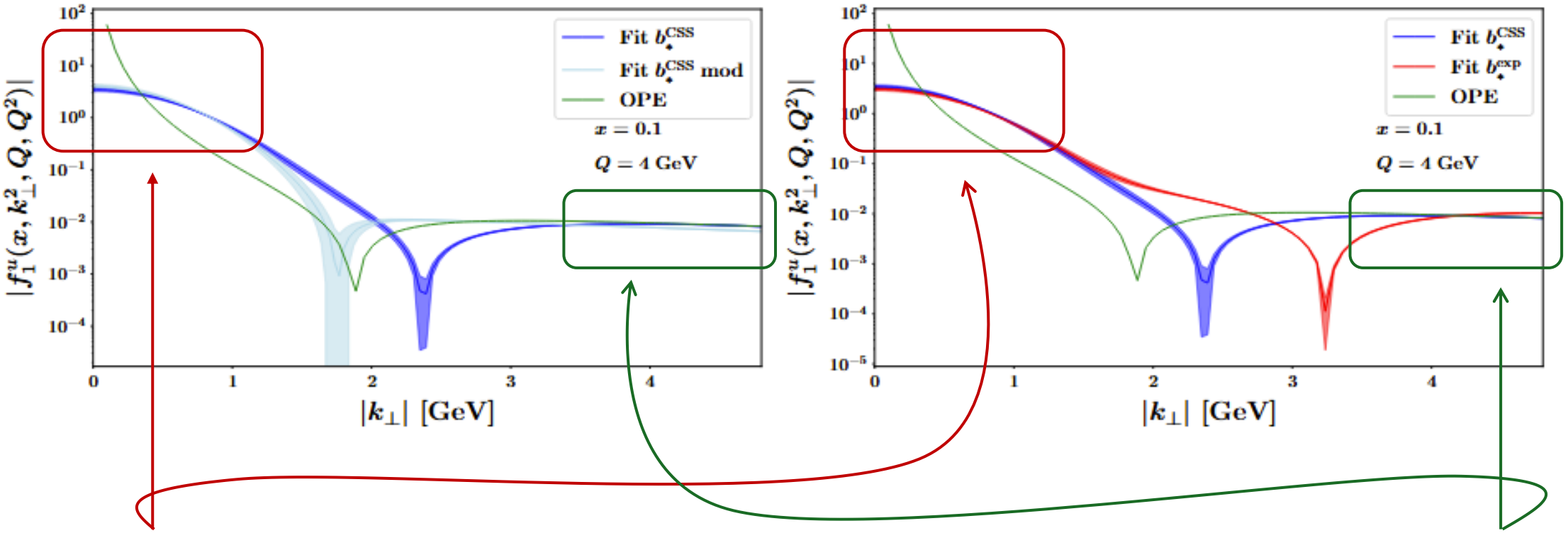


Constrained by data (fit)

See previous plot

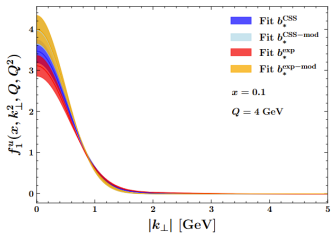


Ambiguity at intermediate k_T



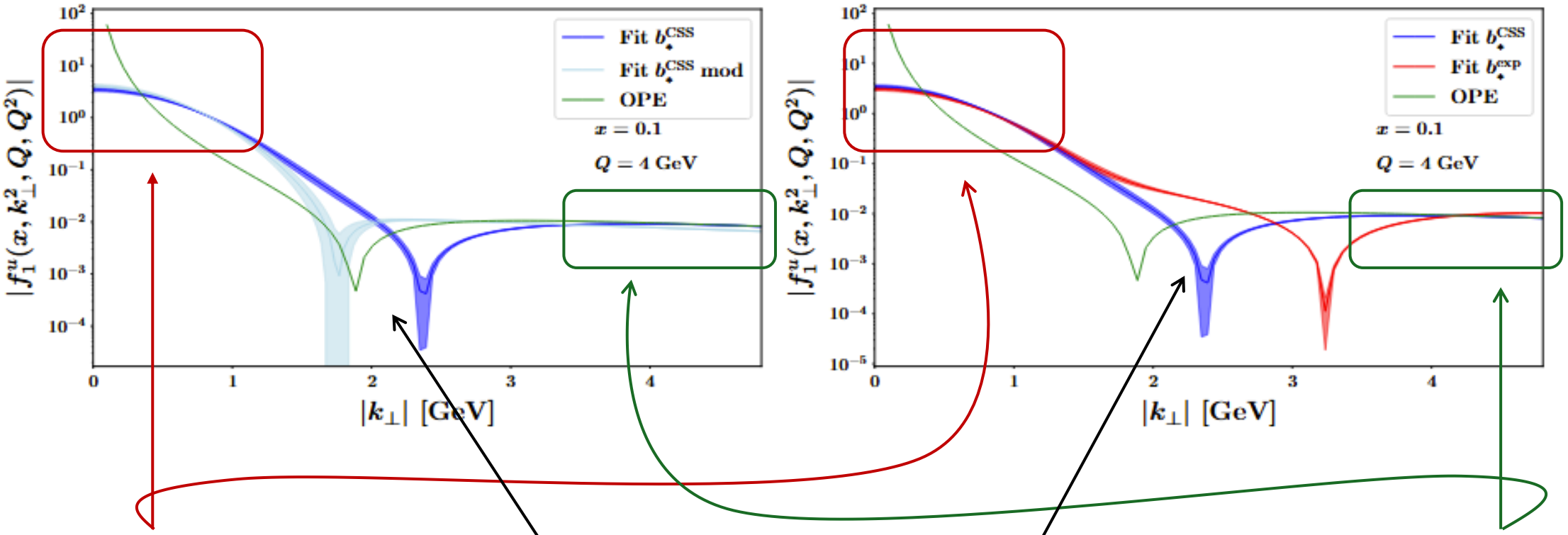
Constrained by data (fit)

See previous plot



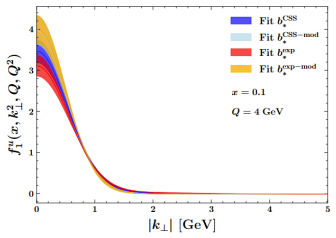
Constrained by fixed-order (OPE)

Ambiguity at intermediate k_T



Constrained by data (fit)

See previous plot

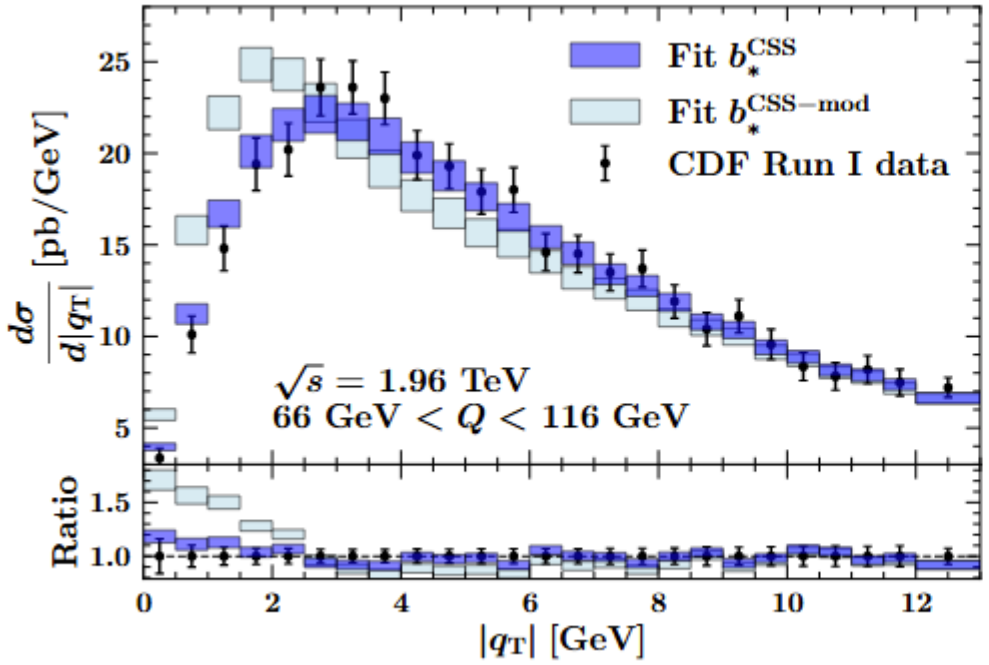


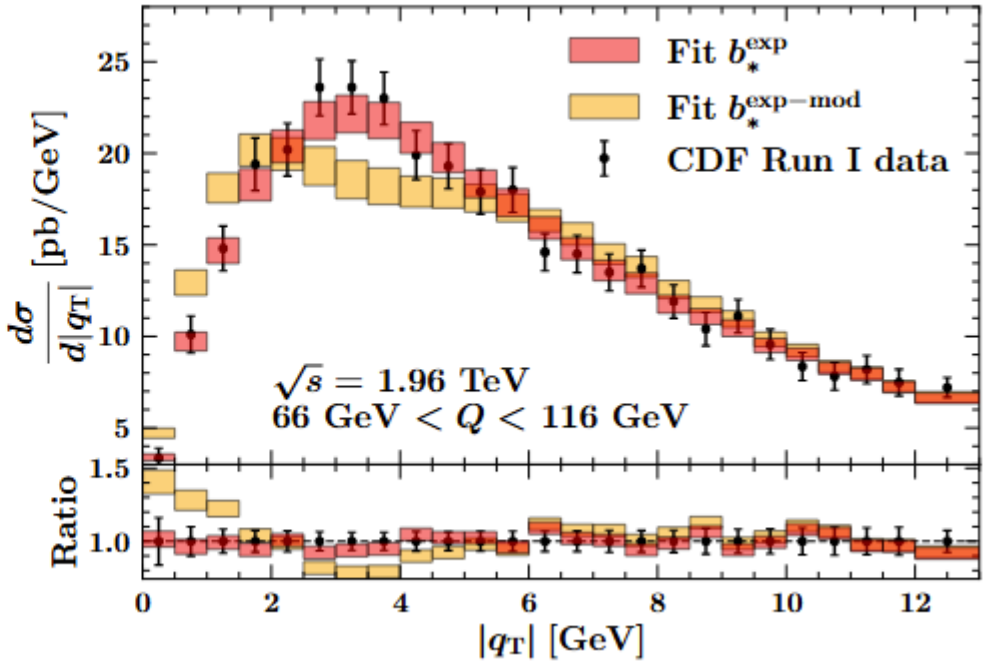
Constrained by fixed-order (OPE)

Completely unconstrained in standard approach.

What constrains it?

Postdicting high-energy data





		χ^2/N_{dat}			
Dataset	N_{dat}	b_*^{CSS}	$b_*^{\mathrm{CSS-mod}}$	b_*^{exp}	$b_*^{\mathrm{exp-mod}}$
CDF I	25	0.83	6.16	0.57	2.87

Postdicting high-energy data

Global fits *mitigate* but *not solve* the issue

		$\chi^2/N_{\rm dat}$			
Dataset	$N_{\rm dat}$	$b_*^{\rm CSS}$	$b_*^{\rm CSS-mod}$	$b_*^{\rm exp}$	$b_*^{\rm exp-mod}$
CDF I	25	0.83	6.16	0.57	2.87

		$\chi^2/N_{\rm dat}$			
Dataset	$N_{\rm dat}$	$b_*^{\rm CSS}$	$b_*^{\rm CSS-mod}$	$b_*^{\rm exp}$	$b_*^{\rm exp-mod}$
E288	130	0.98	0.89	1.01	0.96
E605	50	1.72	2.12	1.38	1.96
CDF I	25	0.74	3.38	0.56	2.19
Total	205	1.13	1.49	1.04	1.36

The standard approach *requires* **global fits**:

- The low-energy data mainly constraint the NP region at very low k_T
- The high-energy data provide information about the intermediate region, helping to select the "proper prescription" at a given NP model.

Note that the global fit selects **two** “good” prescriptions.

Could it be that the philosophy needs to be revised?

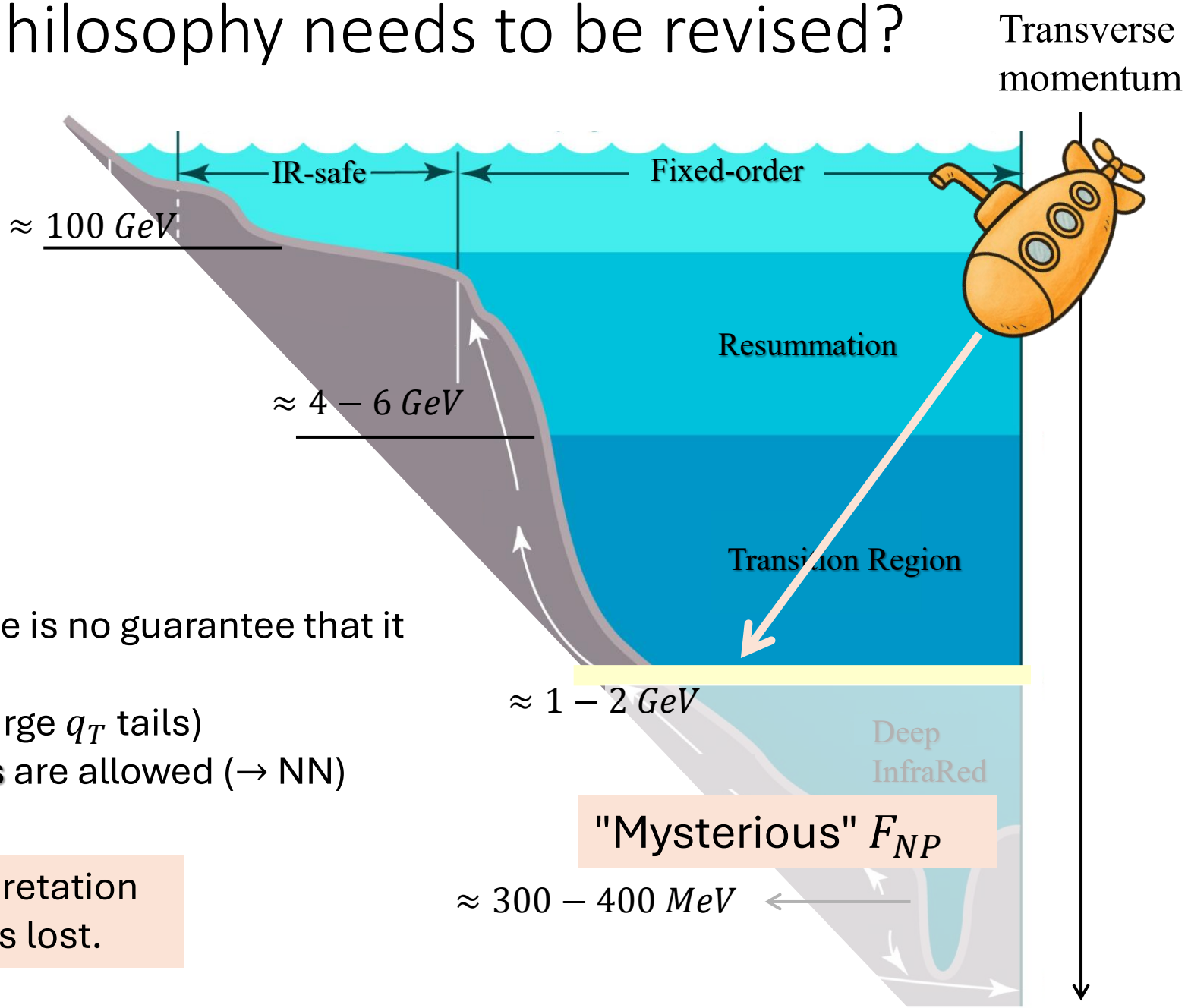
The standard approach does not give a unique solution for the distributions, despite an excellent agreement with data.

It is the combination between
- the chosen prescription,
- the non-perturbative model F_{NP},
- the constraints from data,

that fix the TMDs.

The strategy improves (although there is no guarantee that it will ever fully work) if:
- More data are added (including large q_T tails)
- Extreme flexible parametrizations are allowed (→ NN)

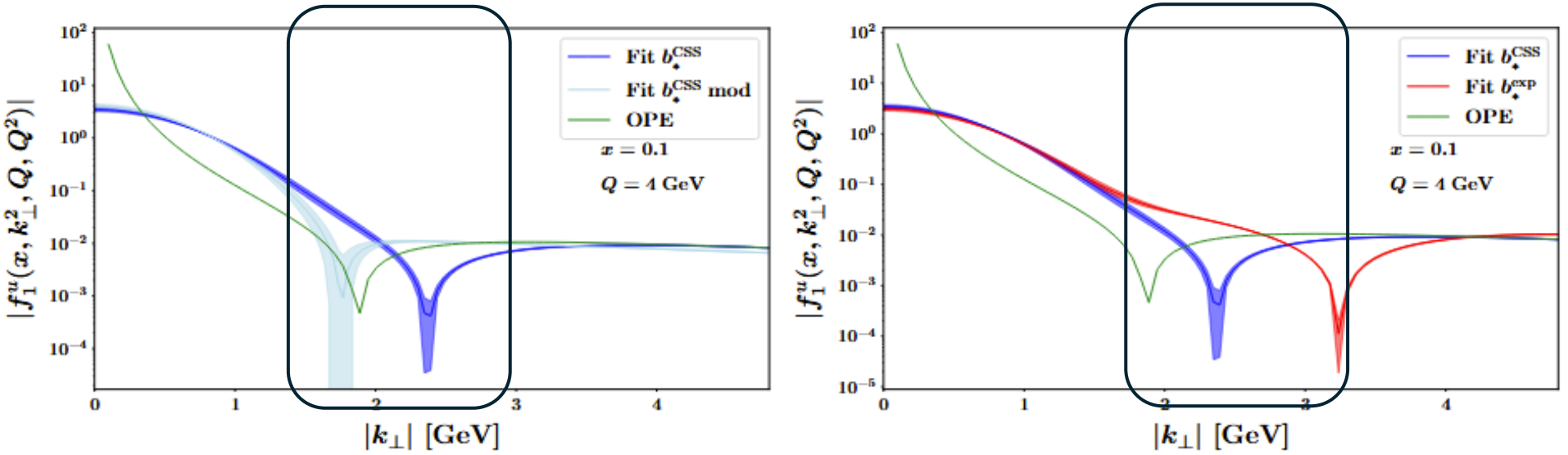
But the fundamental physical interpretation in terms of intrinsic QCD elements is lost.



Avoiding prescriptions

Is there a way to map the TMD from b_T-space to k_T-space, before introducing any prescription or model, *while preserving logarithmic accuracy*?

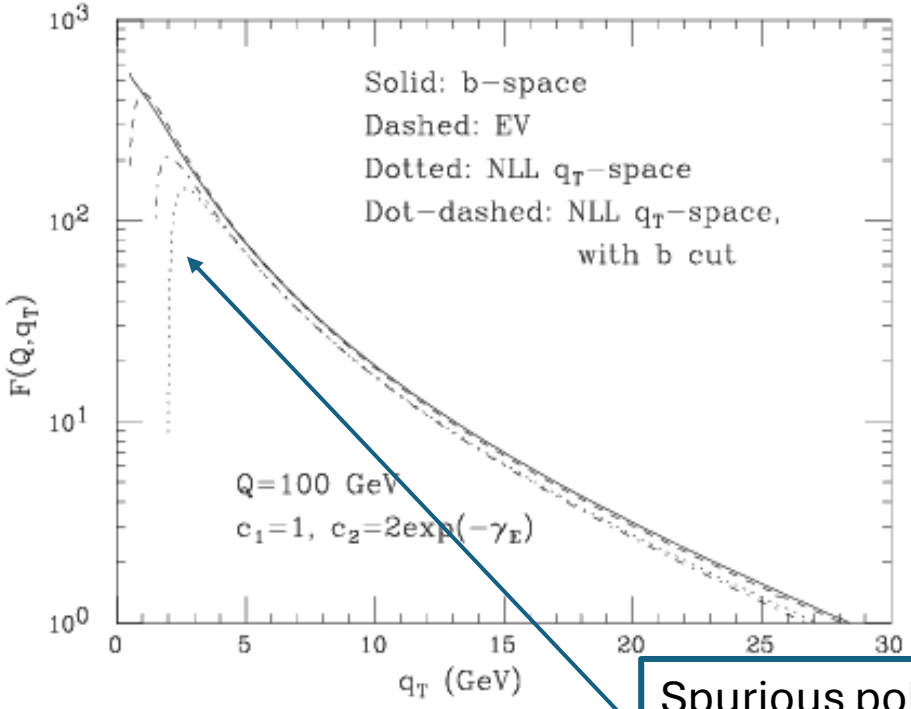
$$\mathrm{F_j(x, b_T}, Q, 0) = f_j(x, \mu_b)\, exp\left\{L_b\, g_1(\lambda_b) + g_2(\lambda_b) + \frac{1}{L_b} g_3(\lambda_b) + \cdots\right\} \quad \overset{?}{\Rightarrow} \quad F_j(x, k_T, Q)$$



Analytic resummation in k_T space

Previous attempts only worked out at leading-log (LL) (see Ellis-Veseli ('97), Kulesza-Stirling ('99)).

Extensions at NLL are dangerously problematic. Beyond the trivial order, a series of "spurious poles" at low k_T appears (see Frixione-Nason-Ridolfi ('98)), making the predictions in transverse momentum space completely non-sense.



to the fact that it is approaching the singularity $h \to -2$. As an illustration, we also plot a modification of the NLL q_T-space formula, obtained by letting

$$\mathcal{S}_{\rm NLL} \to \theta(b - 1/Q)\mathcal{S}_{\rm NLL} = \theta(\hat{b} - q_T/Q)\mathcal{S}_{\rm NLL} \qquad (3.5)$$

before performing the $\hat{b}$ integration (the relevant formulae are given in Appendix B). Modified in this way, we see that the NLL result is stabilized down to smaller values of q_T.

"Theory" justification for introducing a b_{min} cut-off

Spurious pole(s) due to high energetic gluons, absorbed in the UV region at very small b_T

Analytic resummation in k_T space

The problem is in the low-b_T region of the integral to get to k_T-space, starting from NLL and beyond:

$$F(k_T) = \int_0^{c_1} db_T \ldots F(b_T) + \int_{c_1}^{\infty} db_T \ldots F(b_T)$$

Saddle point approximation in the Sudakov is a terrible approximation.
Better expand the Bessel function.

Saddle point approximation in the Sudakov (→ Catani) is a great approximation

$$k_T \approx c_1/b_T$$

If one persists into forcing the saddle point approximation for the low- b_T term, then the spurious poles arise. Instead, this part can be solved *analytically*, in terms of a series that is very fast converging.

$$\Phi(\lambda, L) = \frac{c_1^2}{2} \sum_{k\geq 0} \left(-\frac{c_1^2}{4}\right)^k \frac{1}{k!\,(2)_k} \frac{1}{c+2(k+1)} e^{-\frac{cL}{h_1(\lambda)}(c+2(k+1))}$$
$$\times \left(-\frac{cL}{h_1(\lambda)}(c+2(k+1))\right)^{-\frac{cL}{\lambda}} \Gamma_{1+\frac{cL}{\lambda}}\left(-\frac{cL}{h_1(\lambda)}(c+2(k+1))\right)$$

Where now:

$$\lambda = \beta_0\, a_S(Q) \log(Q^2/k_T^2)$$

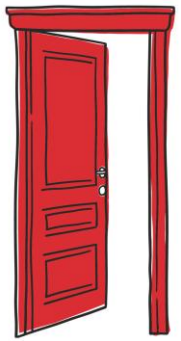
Analytic resummation in k_T space

The TMD in k_T-space, at NLL reads as:

$$F_j^{(NLL)}(x,k_T) = \frac{1}{2\pi k_T^2} e^{Lg_1(\lambda)+g_2(\lambda)}$$
$$\times \left\{ \left[-\left(h_1(\lambda) + \frac{1}{L} h_2(\lambda)\right)\Psi(\lambda,L) - \frac{d\Psi(\lambda,L)}{dL} \right] f_j(x;k_T) + \Psi(\lambda,L)\,\frac{df_j(x;k_T)}{d\log k_T} \right\}$$

Notice that *everything*, except for the collinear PDFs, is perturbative.
Were can we trust this expression?

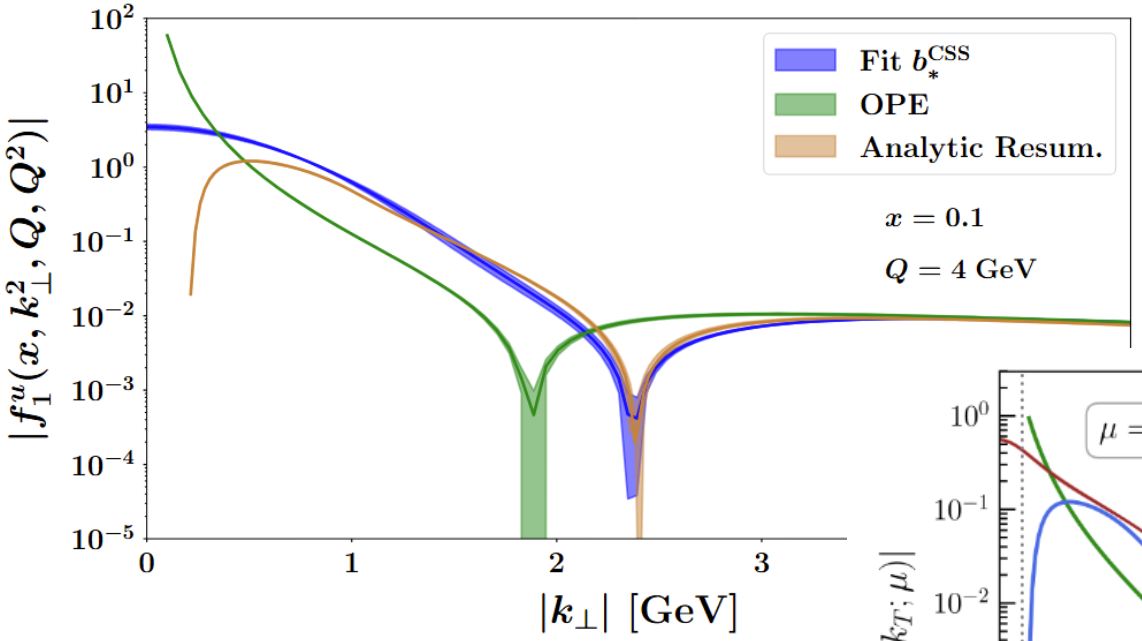
$$k_T \gtrsim Q_0 \approx 1 \text{ GeV}$$



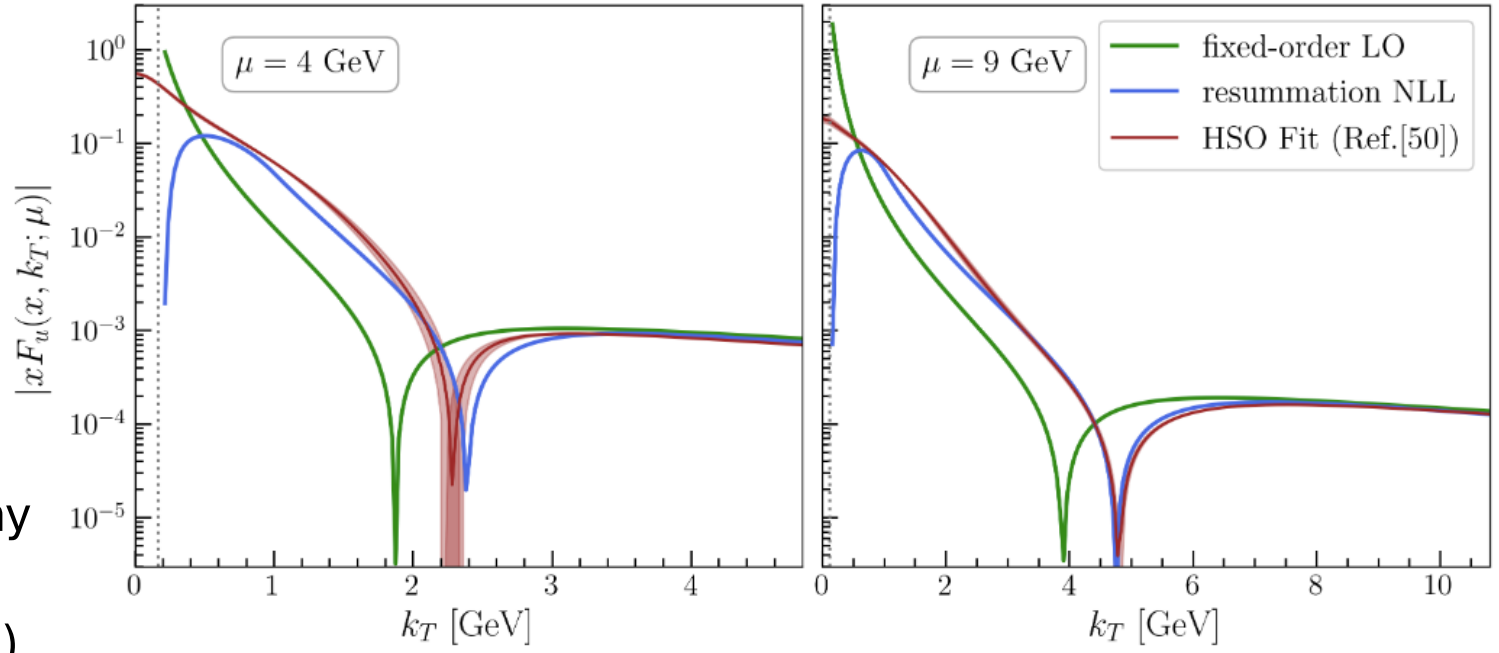
A. Simonelli "*Transverse Momentum Resummation and Analytic continuation into the Deep Infrared*"
hep-ph: 2509.06843 (now accepted in EpJC)

Analytic resummation in k_T space

This expression reproduces one of the best curve of the DY stress test in the intermediate region



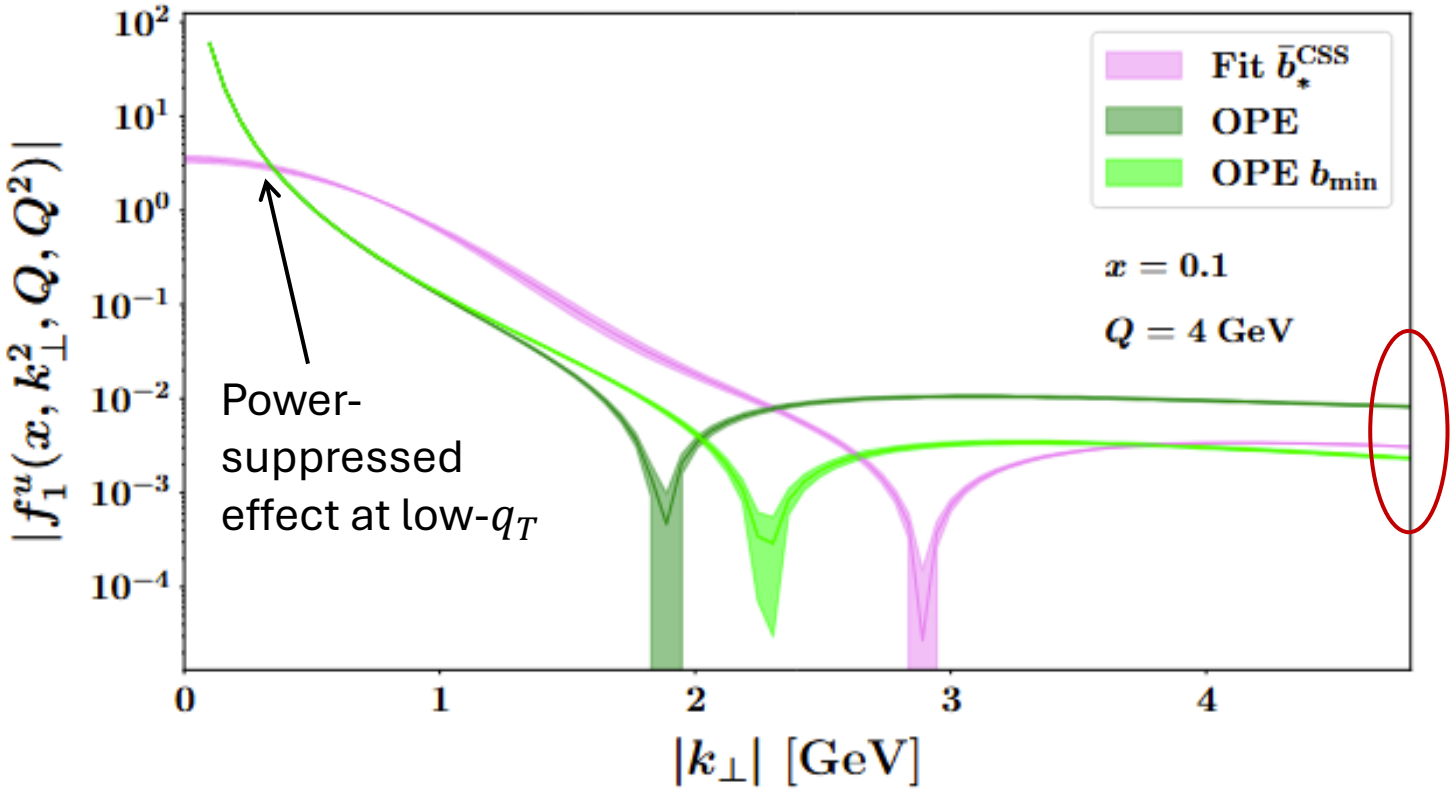
As well as the result from an actual fit Phys.Rev.D 110 (2024) 7, 074016



With the significative advantage of being completely perturbative and free from any contamination due to prescriptions and NP models (look at the size of the bands!)

The unnecessariness of $b_{\min}$

Since the spurious poles motivating the introduction of $b_{\min}$ are not really there, do we really need to use $b_{\min}$?

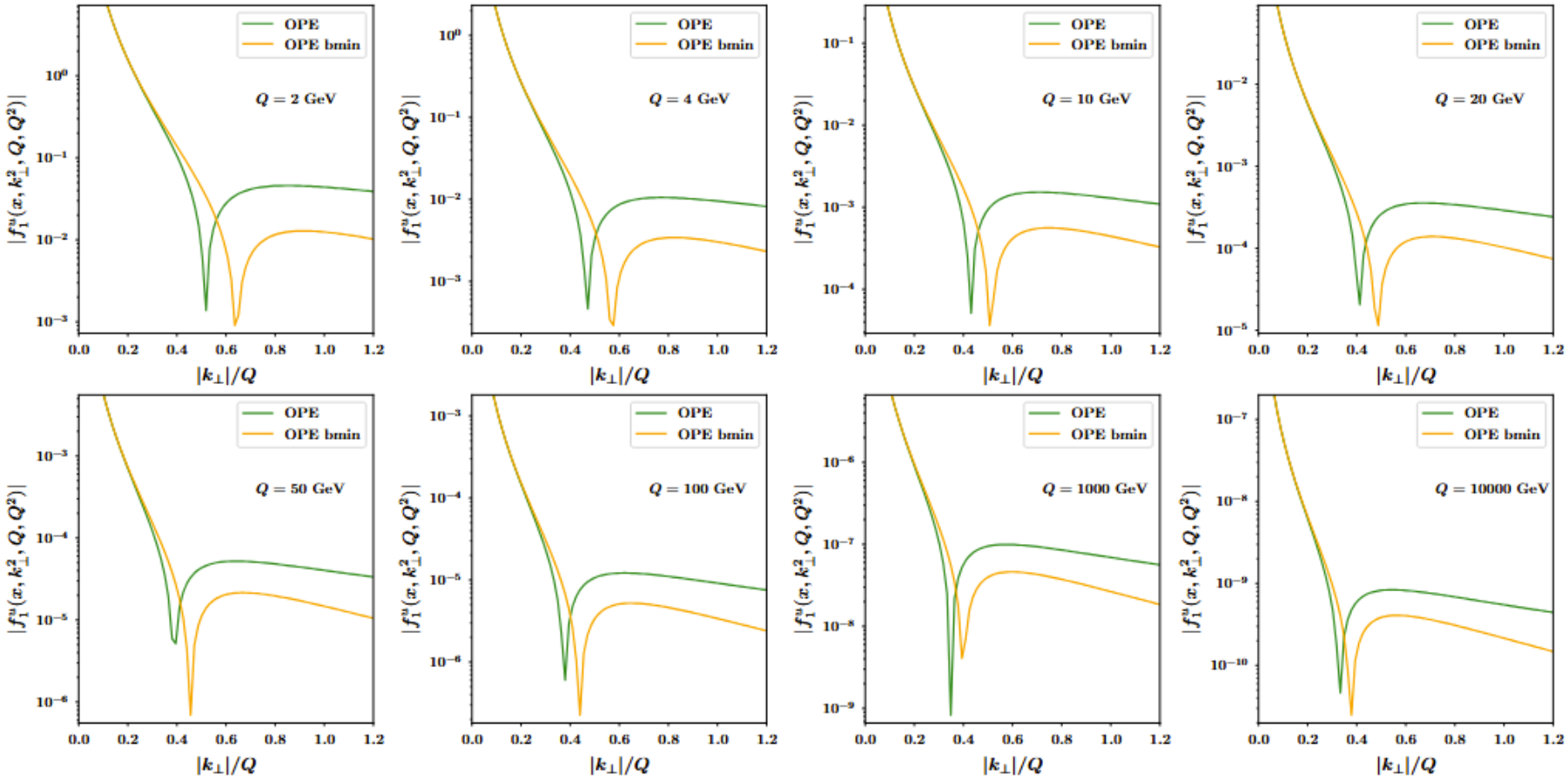


Drastic side effect on the OPE
(→ matching at cross-section level)

The unnecessariness of $b_{\min}$

Since the spurious poles motivating the introduction of $b_{\min}$ are not really there, do we really need to use $b_{\min}$?

The side effect does not vanish as the scale increases (counterintuitive!)



The unnecessariness of $b_{\min}$

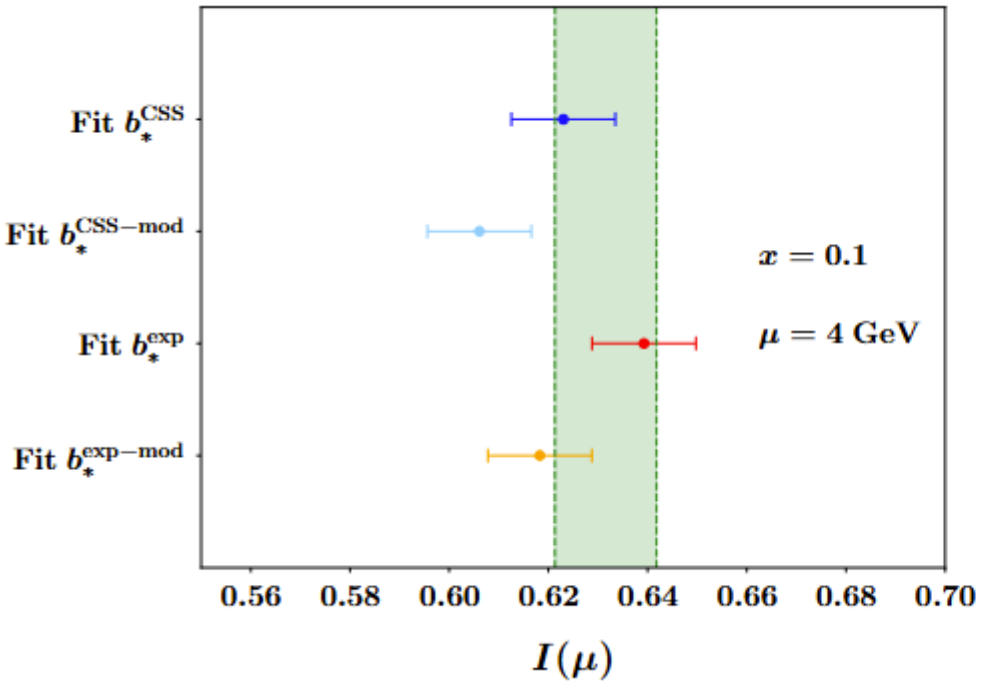
The only concrete advantage is the integrals

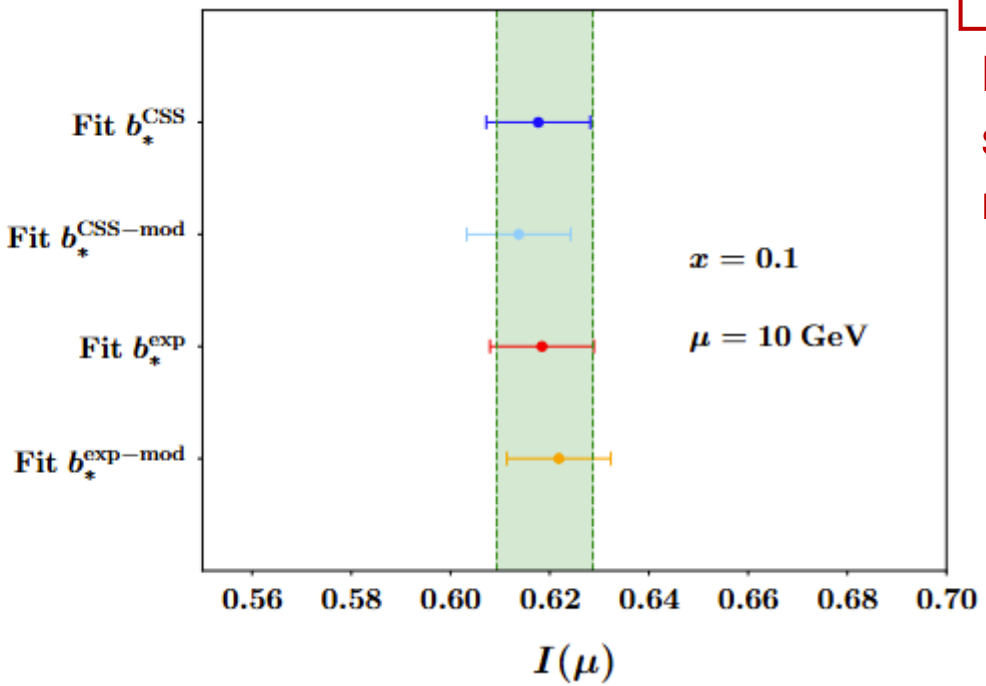
$$\int d\vec{k}_T F_{q/h}\left(x, |\boldsymbol{k}_\perp|, \mu, 1; b_{\min}\right) = f_{q/h}(x,\mu) + a_S(\mu)\left[\Delta^{[1]}_{q/j} \otimes f_{j/h}\right](x,\mu) + \mathcal{O}(a_S(\mu)^2)$$

vs

$$\int^{\mu} d\vec{k}_T F_{q/h}\left(x, |\boldsymbol{k}_\perp|, \mu, 1\right) = f_{q/h}(x,\mu) + a_S(\mu)\left[\Delta^{[1]}_{q/j} \otimes f_{j/h}\right](x,\mu) + \mathcal{O}(a_S(\mu)^2) + \mathcal{O}\left(\frac{1}{\mu b_{\max}}\right)$$

Non-perturbative suppressed remnant





Analytic resummation

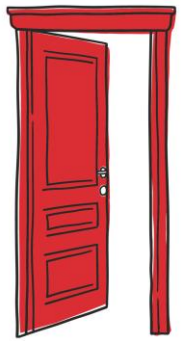
Following the same logic, the Drell-Yan cross section at NLL is

$$\frac{d\sigma^{(NLL)}}{dQ^2 dq_T^2 dy} = H^{[0]} \frac{1}{2\pi q_T^2} e^{2\left(Lg_1(\lambda)+g_2(\lambda)\right)}$$
$$\times \left\{ \left[-2\left(h_1(\lambda) + \frac{1}{L} h_2(\lambda)\right)\Psi'(\lambda, L) - \frac{d\Psi'(\lambda, L)}{dL} \right] \mathrm{f}(q_T; \tau, y) + \Psi'(\lambda, L) \frac{d\mathrm{f}(q_T; \tau, y)}{d\log q_T} \right\}$$

Where $\mathrm{f}(q_T, \tau, y) = \sum_q e_q^2 f_q\left(\sqrt{\tau} e^y, q_T\right) f_{\bar{q}}\left(\sqrt{\tau} e^{-y}, q_T\right)$

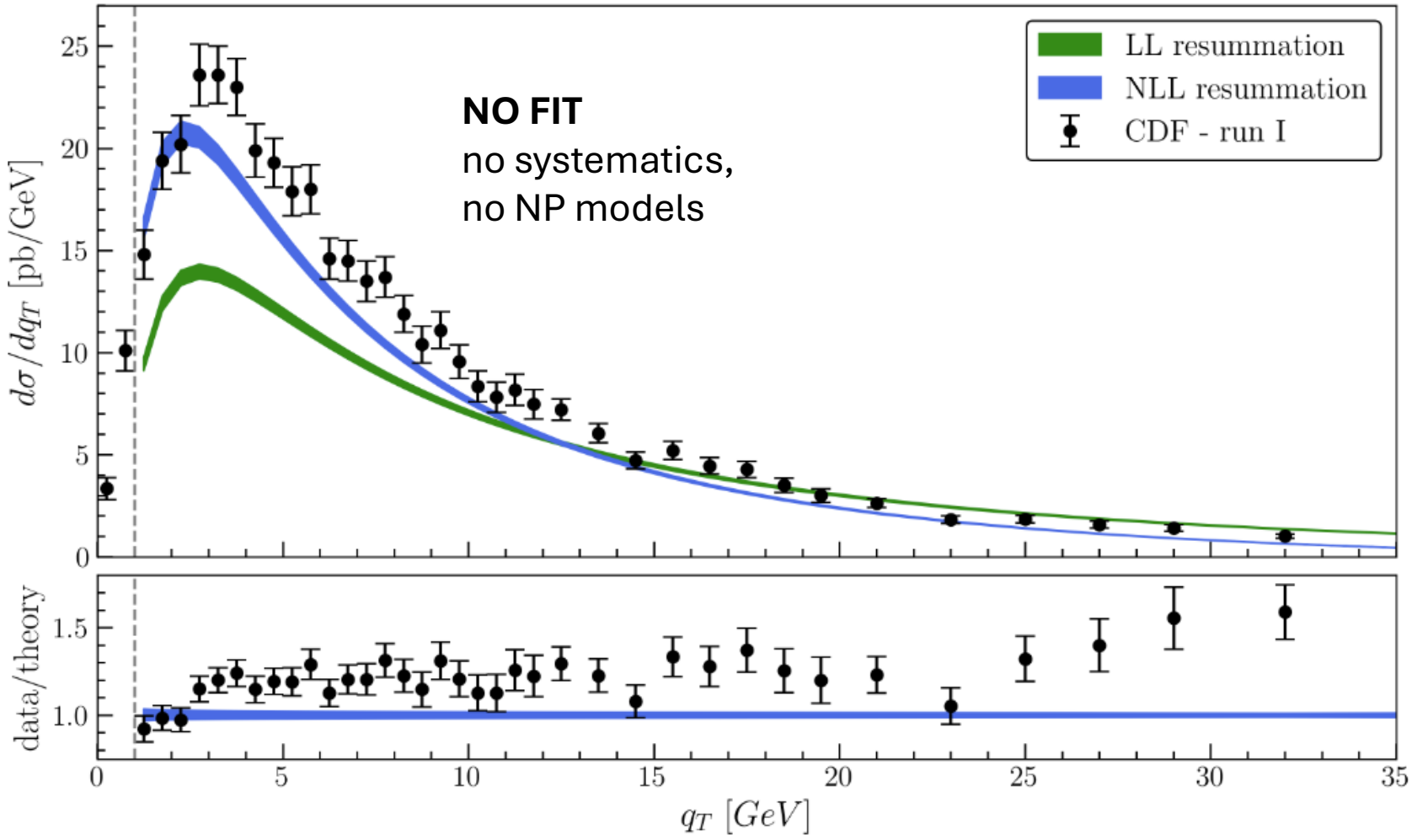
Again, *everything*, except for the collinear PDFs, is perturbative.
Were can we trust this expression?

$$q_T \gtrsim Q_0 \approx 1 \text{ GeV}$$



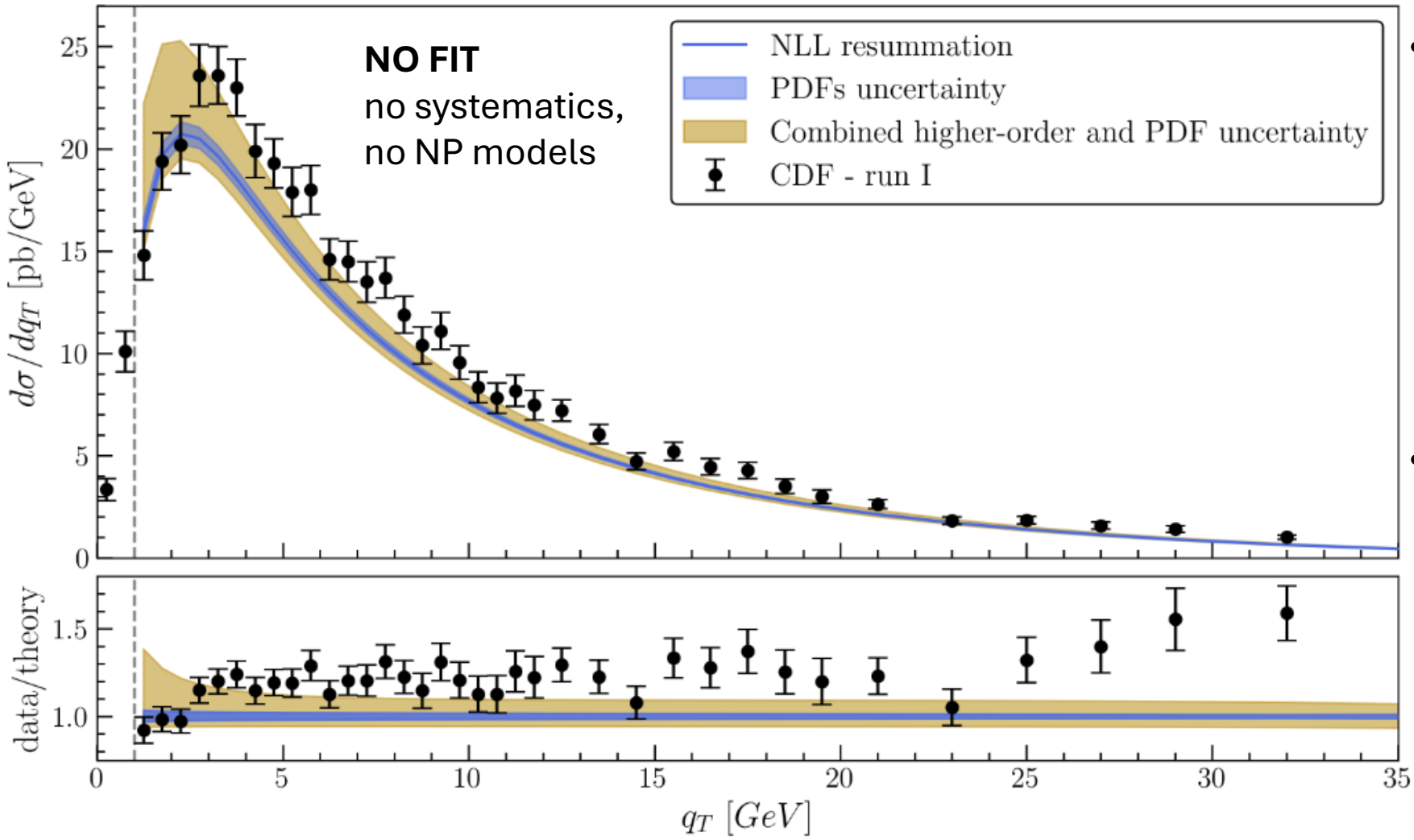
High energy data are the perfect environment where to test it

Analytic resummation



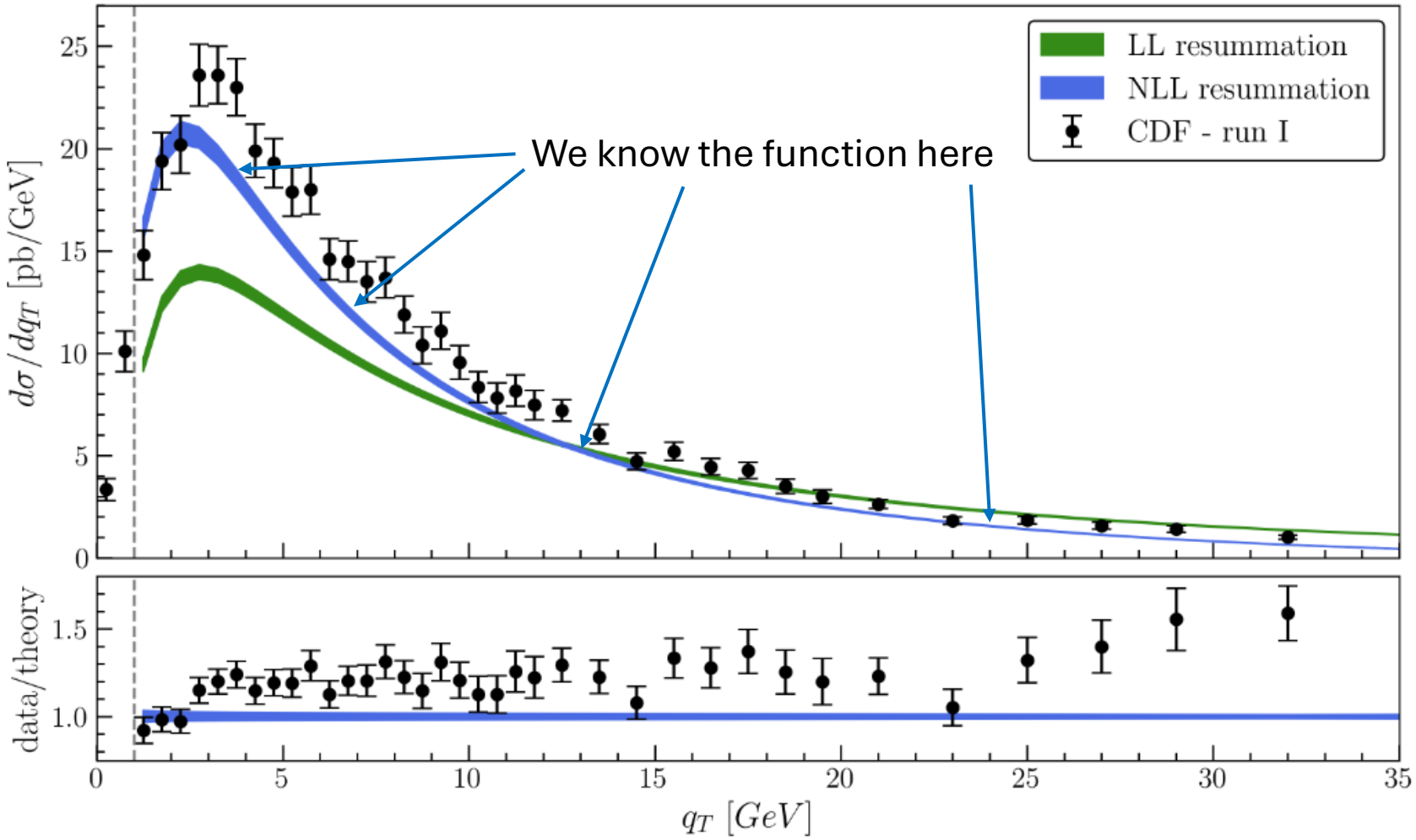
- Clear improvement from LL to NLL precision
- Below $q_T \approx 1$ GeV the full-perturbative cross-section is no longer applicable as it stands.
- Above $q_T \approx 25$ GeV the agreement with data deteriorates. This indicates the region *where* the matching to fixed-order collinear factorization becomes relevant.

Analytic resummation extra bonuses



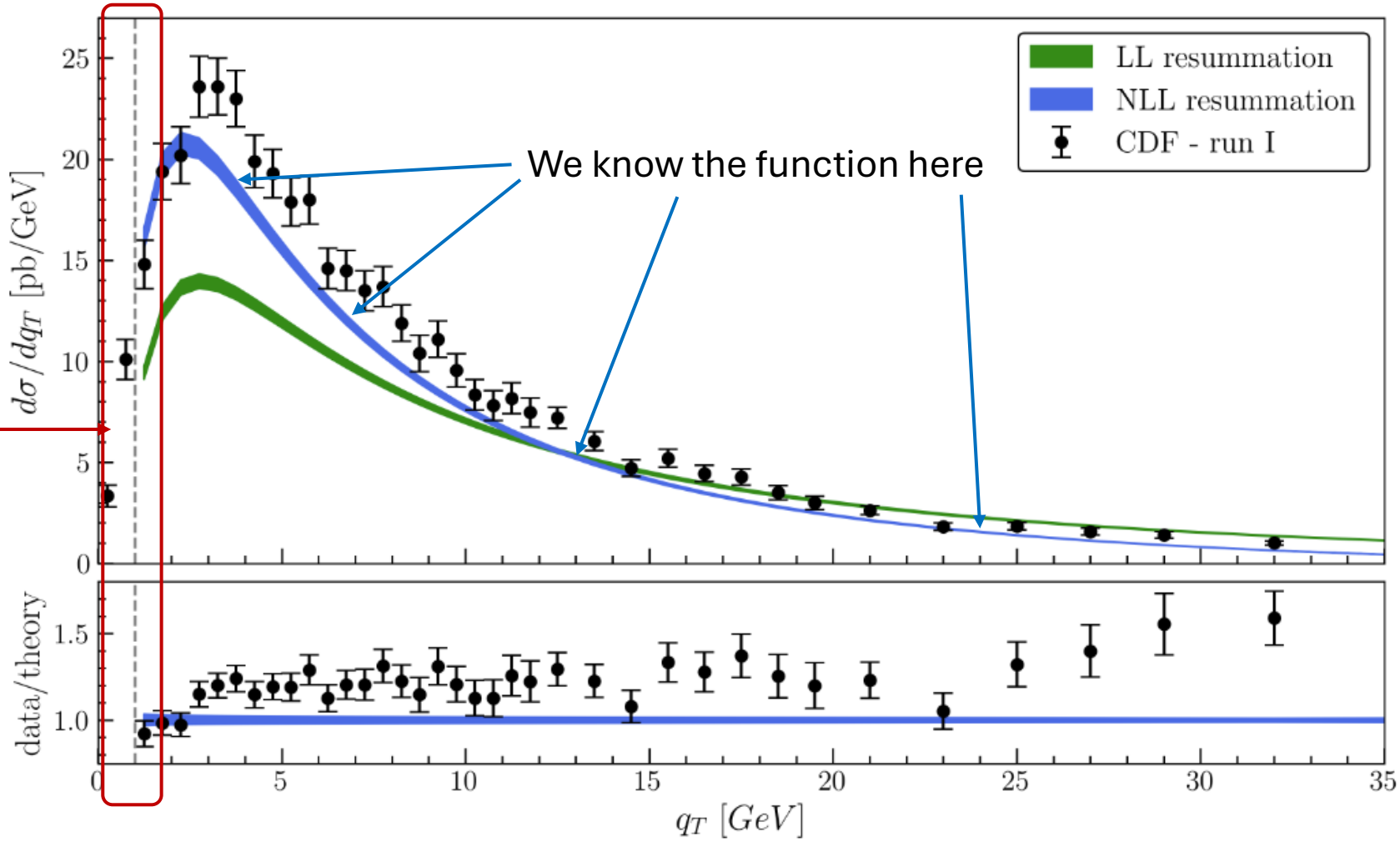
- Simple estimate of missing higher-order corrections based on scale variation. This is a new result, and it would be difficult to obtain within the standard approach.
- All data above $q_T \approx Q_0$ can be used to extract the usual PDFs, which are thus elevated from simple inputs to active ingredients of the phenomenology

Analytic continuation into the Deep Infrared



Analytic continuation into the Deep Infrared

But what can we say here?



The region of q_T below Q_0 becomes larger and larger as the energy decreases, making its proper description essential.

From where do non-perturbative effects come?

Ok, but what do we do *below* this threshold?
We need to pin down the sources of non-perturbative effects!

Two sources of non-perturbative effects:

1. The strong coupling entering the deep IR region

$$\log\left(\frac{Q}{q_T}\right) \to \log\left(\frac{Q}{q_T^{Deep\ IR}(Q_0, pars)}\right)$$

Recall we traded $\alpha_S(q_T)$ for the logarithms $\log(Q/q_T)$

2. The collinear PDFs entering the deep IR region

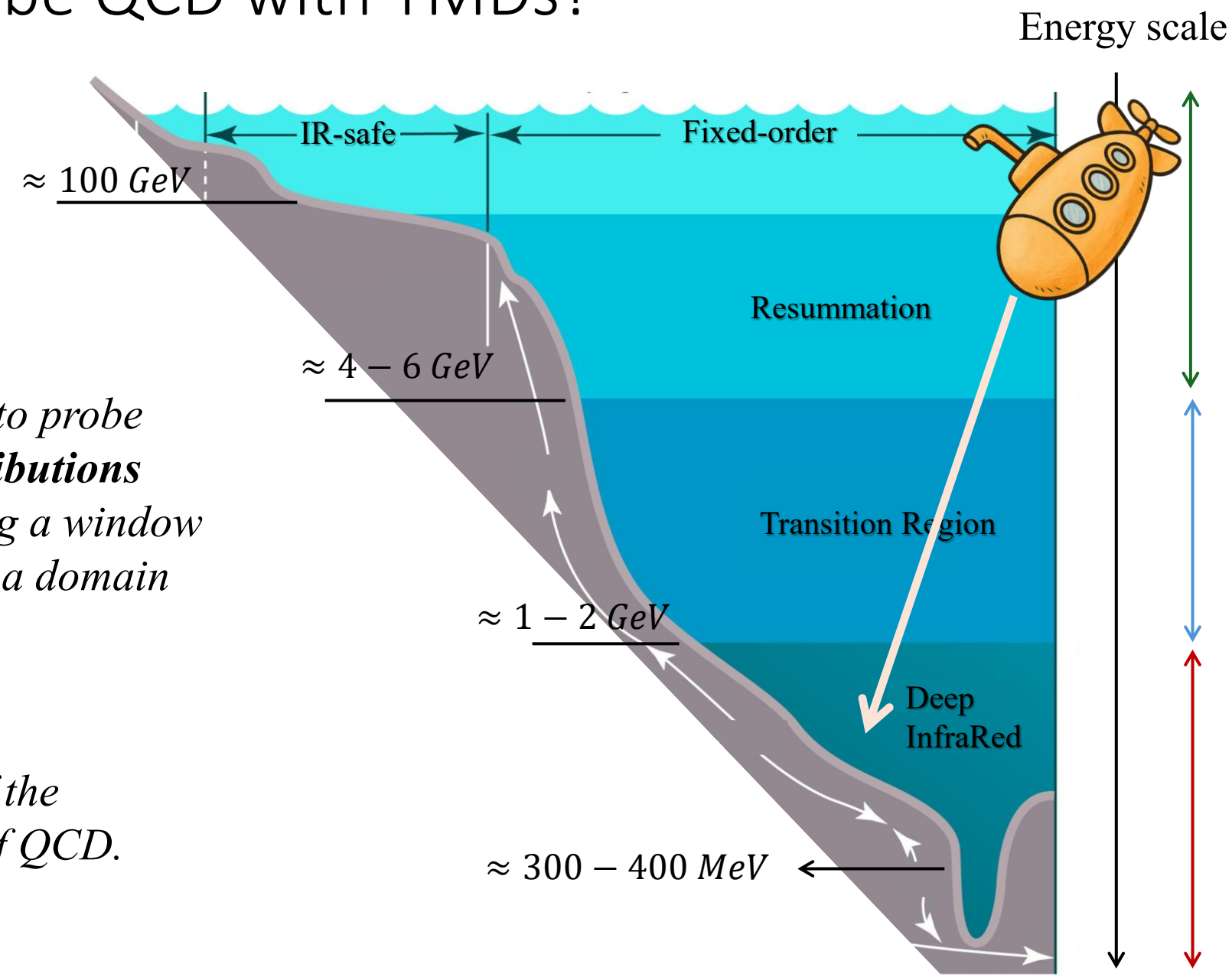
$$\frac{d\ f_j(x, q_T)}{d\ \log q_T^2} \to P_j^{Deep\ IR}(q_T, Q_0, pars) \otimes f_k(x, q_T)$$

How deep can we probe QCD with TMDs?

As low as q_T goes!

*TMD phenomenology can be used to probe the Deep IR regime of **parton distributions** and of the **strong coupling**, opening a window onto the inner structure of QCD in a domain that is typically inaccessible.*

*In this perspective, the structure of the hadrons is an **emergent property** of QCD.*

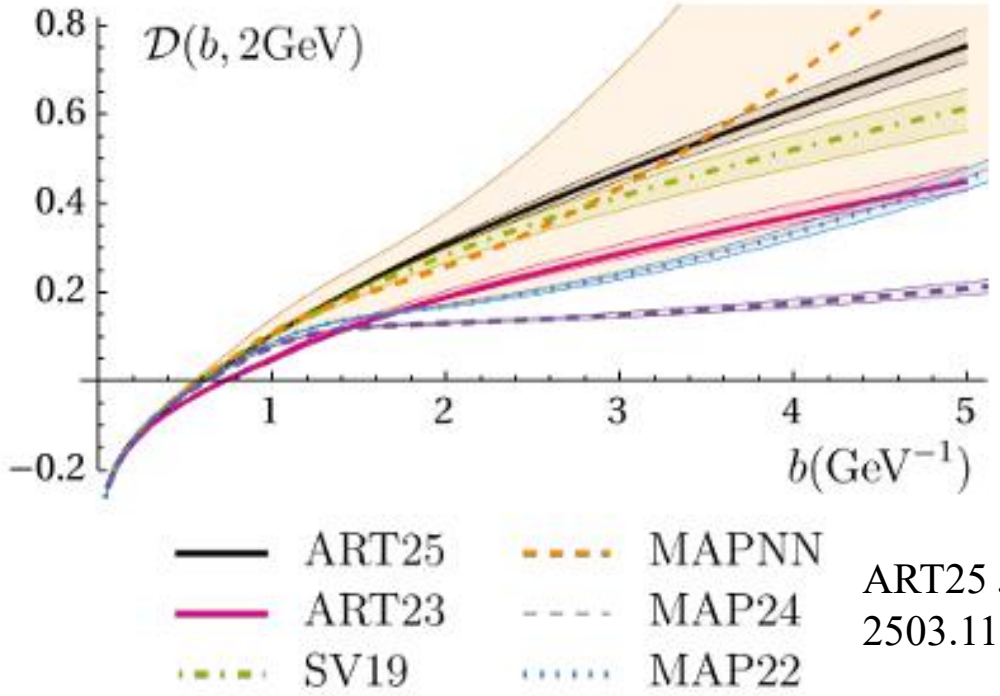


What about the Collins-Soper kernel?

In this framework, it is not manifest: the analytic solution in k_T-space does not resolve the two TMDs separately.

However, the structure is still present, but its long-distance behavior is entirely dictated by the deep-IR behavior of the strong coupling.

$$K(\alpha_S, \mu\, b_T/c_1) = K(\alpha_b, 1) - \int_{\alpha_b}^{\alpha_S} \frac{d\,\alpha}{2\beta(\alpha)} \gamma_K(\alpha)$$



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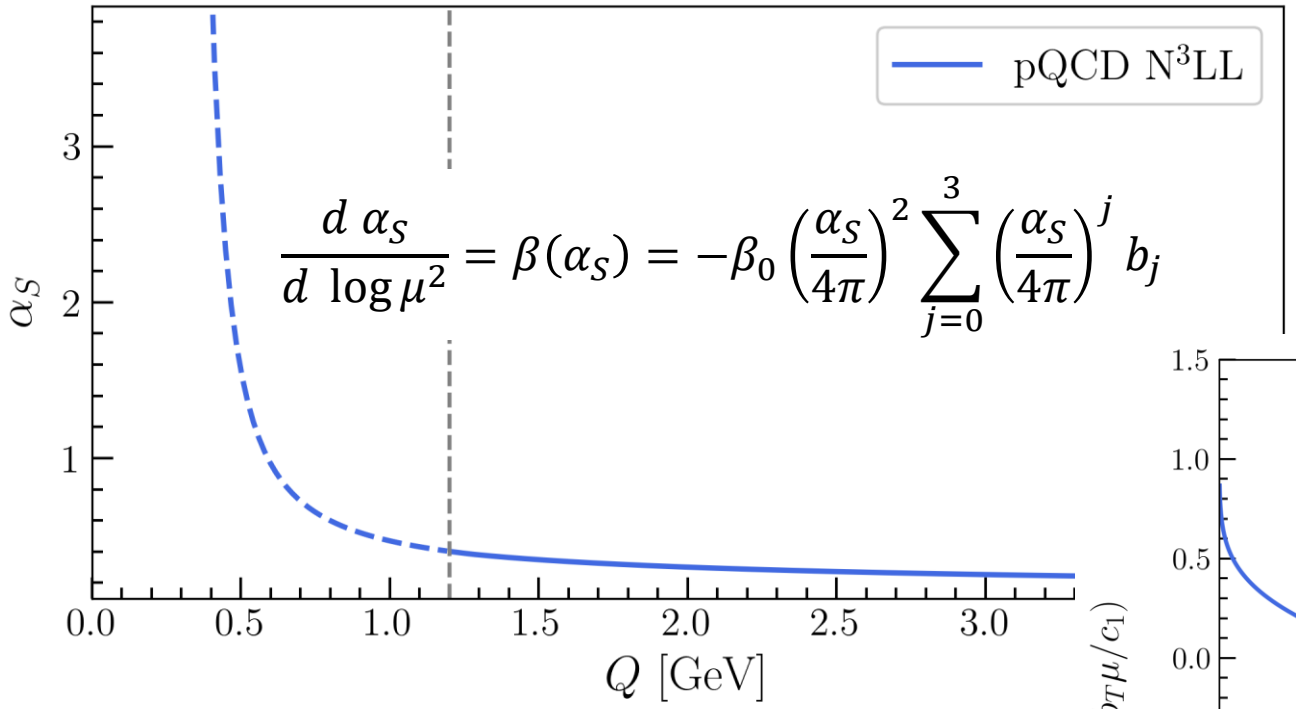
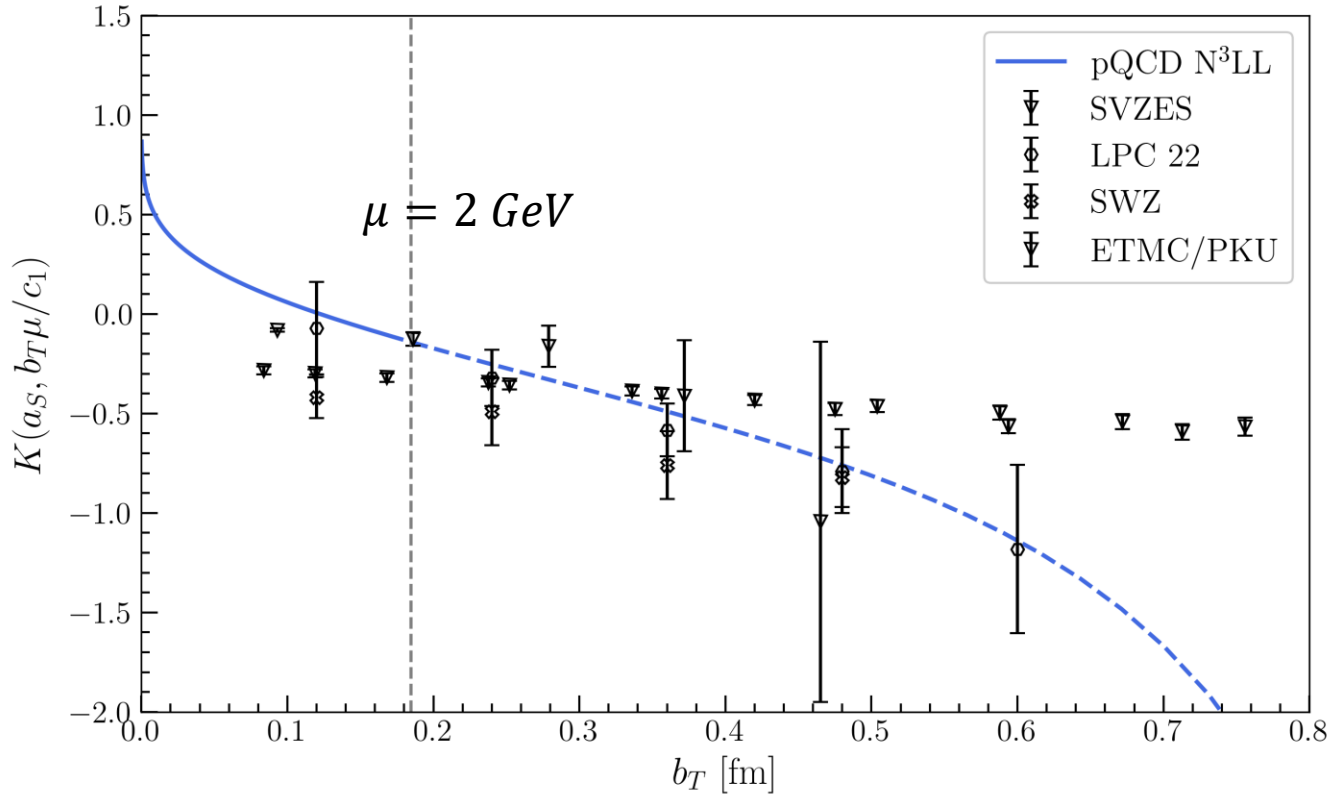
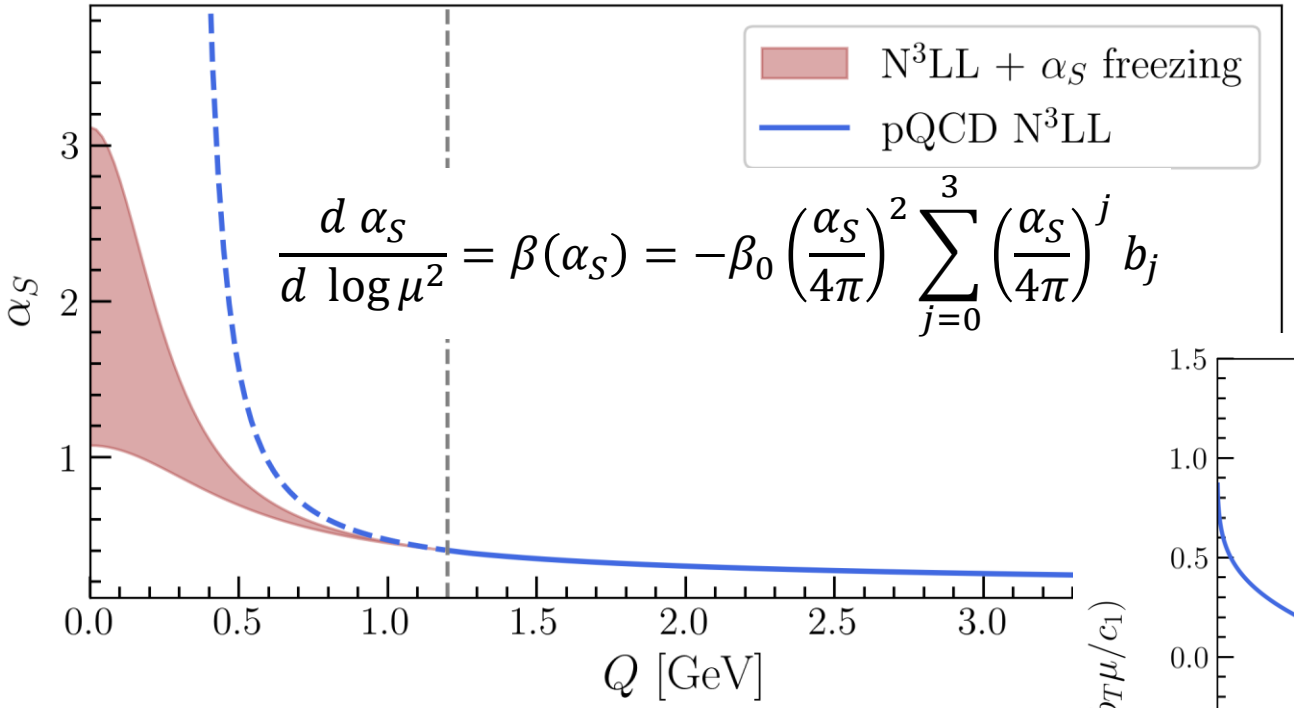
2503.11201 [hep-ph]

What about the Collins-Soper kernel?





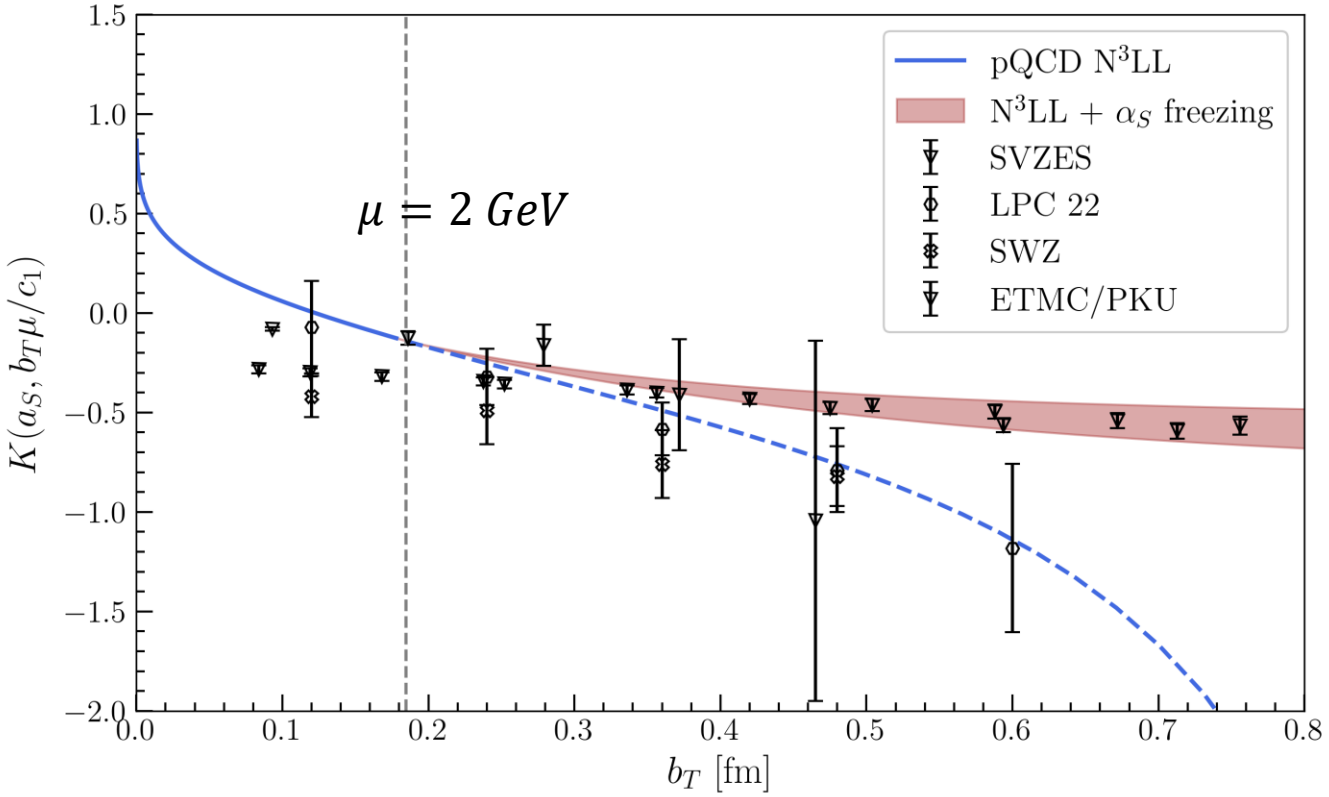
What about the Collins-Soper kernel?



$$\frac{d\,\alpha_S}{d\,\log\mu^2} = \beta(\alpha_S) = -\beta_0\left(\frac{\alpha_S}{4\pi}\right)^2 \sum_{j=0}^{3}\left(\frac{\alpha_S}{4\pi}\right)^j b_j$$

Here we assumed the deep IR model:

- $Q_0 = 1.2\ GeV$
- $450\ MeV \leq m_g \leq 650\ MeV$

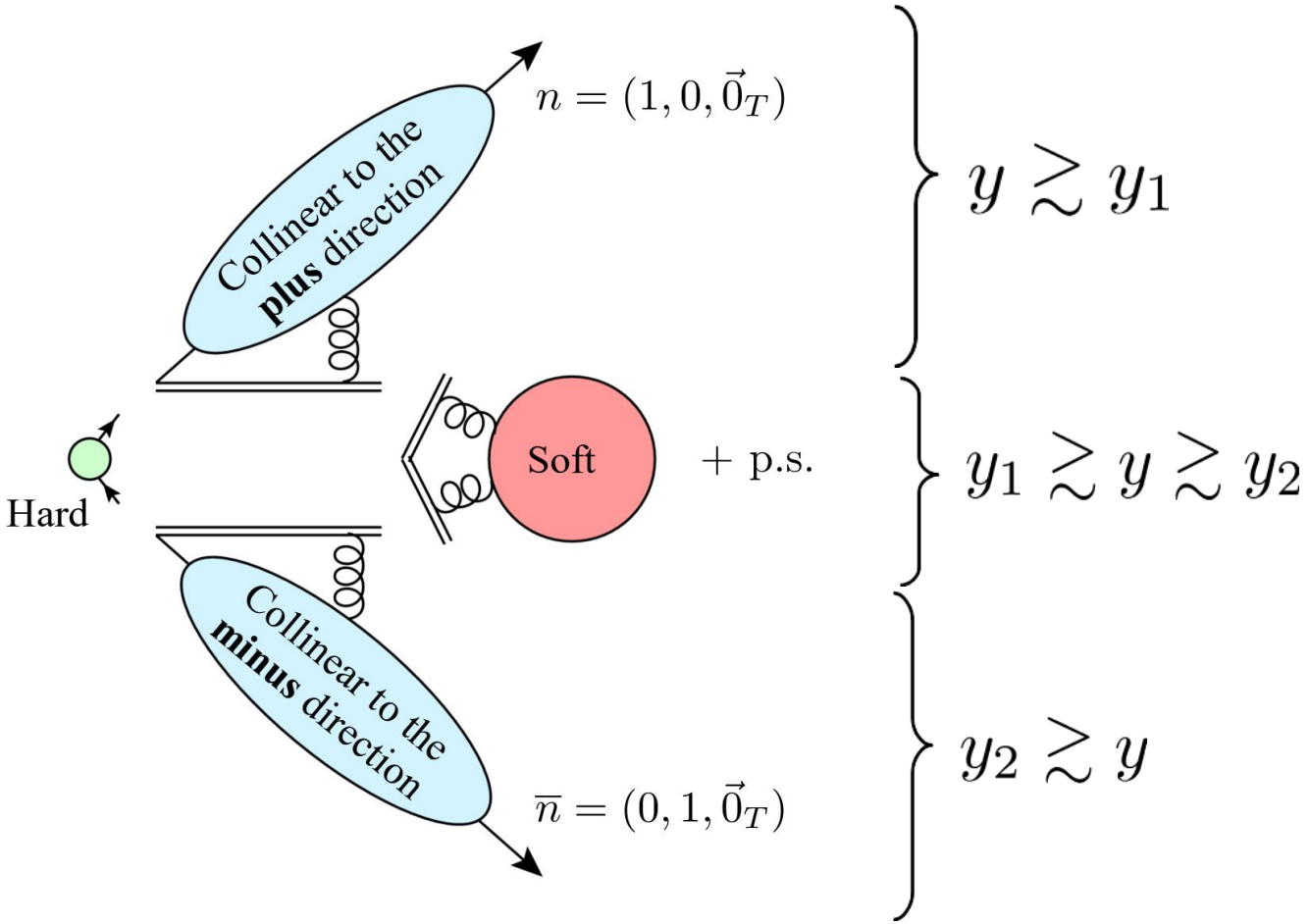


The freezing of alpha strong implies the large b_T behavior:

$$K(\alpha_S, b_T\mu/c_1) = K_0^{\infty} + \frac{1}{b_T^2}K_1^{\infty} + \cdots$$

Geometric Universality

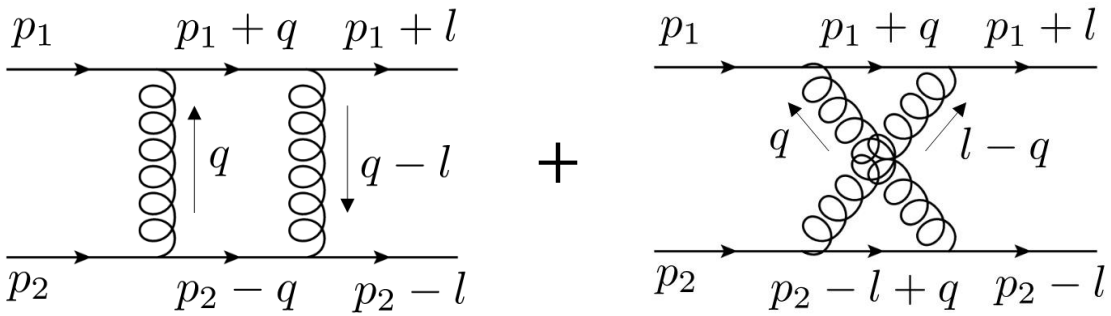
$$H \cdot C_+(y_1) \cdot S(y_1, y_2) \cdot C_-(y_2)$$



There is a clear and transparent separation in rapidity

Gluon Regge trajectory

At NLO two graphs contribute:



Rapidity divergence regularized

$$= M^{tree}_{ij\to ij}\left[(y_1-y_2)\,\frac{\alpha_S}{4\pi}\omega_G^{[1]}(\mu^2/t)+O\left(e^{-2(y_1-y_2)}\right)\right]$$

The two gluons are soft, but the sum of their momenta is Glauber.

The structure is reminiscent of the NLO K-term of the TMD soft factor, in fact:

$$\omega_G^{[1]}(\mu^2/|t|) = -\frac{1}{2}K_G^{[1]}\left(\mu/\sqrt{|t|}\right) \longleftarrow$$

NLO CS-kernel in adjoint representation under replacing: $\frac{c_1^2}{b_T^2} \leftrightarrow t$

What can we say beyond NLO?

Gluon Regge trajectory

There is a conjecture verified up to N^{3}LO :

$$\frac{d\,\omega_G(\alpha_S,\mu^2/|t|\,)}{d\,\log\mu}=\frac{1}{2}\gamma_K^{(8)}(\alpha_S)$$

The soft factor of $2\to 2$ forward scattering amplitude admits a KP-decomposition where the K-term is the Regge trajectory:

$$S_{2\to 2\ forward}=exp\{(y_1-y_2)\omega_G+\cdots\}$$

Suppose it holds true.
Then we can define the *RG-invariant* discrepancy operator:

$$\Delta_G(\alpha_S,\mu^2/|t|)=\ \omega_G(\alpha_S,\mu^2/|t|)+\frac{1}{2}K_G(\alpha_S,\mu^2/|t|)$$

$$=\left(\frac{\alpha_t}{4\pi}\right)^2\Delta_G^{[2]}+\left(\frac{\alpha_t}{4\pi}\right)^3\Delta_G^{[3]}+\cdots$$

Exploiting the RG-invariance we set $\mu=|t|$ and also $\alpha_S(|t|)\equiv a_t$.

Gluon Regge trajectory

The discrepancy operator Δ is the results of *several delicate cancellations*:

$$\omega_G^{[2]}(\mu^2 = t) = N^2\left(\frac{404}{27} - 2\zeta_3\right) - N\, n_f \frac{56}{27}$$

$$-\frac{1}{2}K_G^{[2]}(\mu^2 = t) = N^2\left(\frac{404}{27} - 14\zeta_3\right) - N\, n_f \frac{56}{27}$$

$$\longrightarrow \quad \Delta_G^{[2]} = N^2 12\,\zeta_3$$

- Large-N (planar contributions)
- Max trascendental degree at 2-loops (rational terms cancel out)

Beyond* the NNLO the trend keeps going:

$$\Delta_G^{[3]}(\alpha_S, |t|) = -N^3\left[\frac{4}{27}\left(125 - 304\frac{n_f}{N} + 44\left(\frac{n_f}{N}\right)^2\right)\zeta_3 + 16\,\zeta_2\zeta_3 + 80\,\zeta_5\right]$$

Gluon Regge trajectory

After even subtler cancellations :

$$\omega_G^{[3]}(\mu^2 = t\,)$$
$$= N^3\left(16\,\zeta_5 + \frac{40}{3}\zeta_2\zeta_3 - \frac{77}{3}\zeta_4 - \frac{6664}{27}\zeta_3 - \frac{3196}{81}\zeta_2 + \frac{297029}{1458}\right)$$
$$+ N^2\,n_f\left(\frac{412}{81}\zeta_2 + \frac{2}{3}\zeta_4 + \frac{632}{9}\zeta_3 - \frac{171449}{2916}\right) + N\,n_f^2\left(\frac{928}{729} - \frac{128}{27}\zeta_3\right)$$
$$+ n_f\left(-4\zeta_4 - \frac{76}{9}\zeta_3 + \frac{1711}{108}\right)$$

$$-\frac{1}{2}K_G^{[3]}(\mu^2 = t)$$
$$= N^3\left(96\,\zeta_5 + \frac{88}{3}\zeta_2\zeta_3 - \frac{77}{3}\zeta_4 - \frac{6164}{27}\zeta_3 - \frac{3196}{81}\zeta_2 + \frac{297029}{1458}\right)$$
$$+ N^2\,n_f\left(\frac{412}{81}\zeta_2 + \frac{2}{3}\zeta_4 + \frac{680}{27}\zeta_3 - \frac{171449}{2916}\right) + N\,n_f^2\left(\frac{928}{729} - \frac{16}{27}\zeta_3\right)$$
$$+ n_f\left(-4\zeta_4 - \frac{76}{9}\zeta_3 + \frac{1711}{108}\right)$$

Gluon Regge trajectory

Is there a simpler way to get the Regge trajectory?

$$\omega_G(\alpha_S, \mu^2/|t|) = -\frac{1}{2} K_G(\alpha_S, \mu^2/|t|) + \Delta_G(\alpha_S, \mu^2/|t|)$$

Easier than Regge trajectory to compute in full QCD (1 loop less)

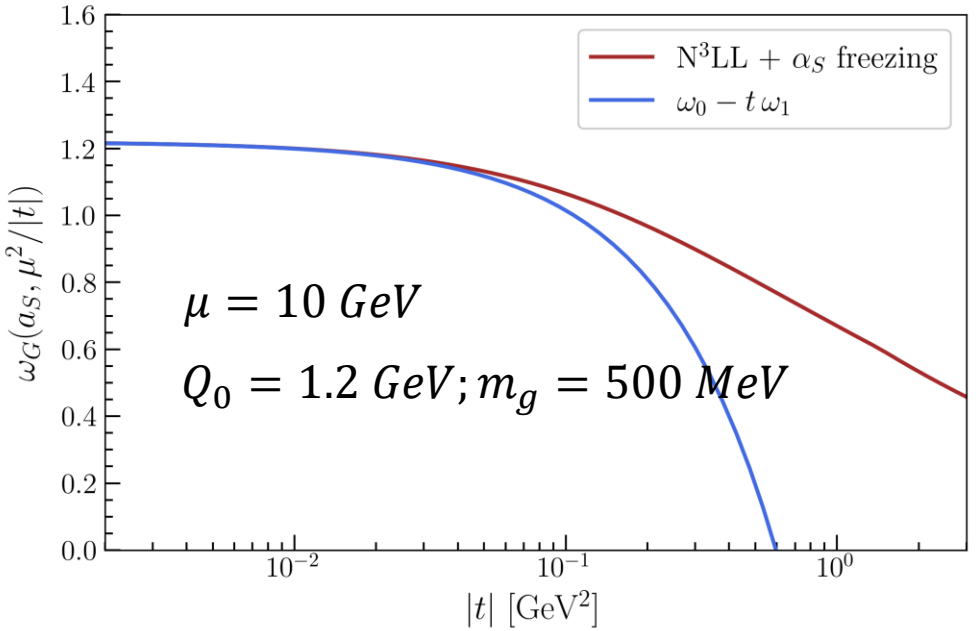
Can it be obtained in some simpler theory? Hints from SYM N=4 in the planar limit?

The two operators starts to be different at N²LL

The IR behavior of the CS-kernel reflects onto the IR behavior of the gluon Regge trajectory

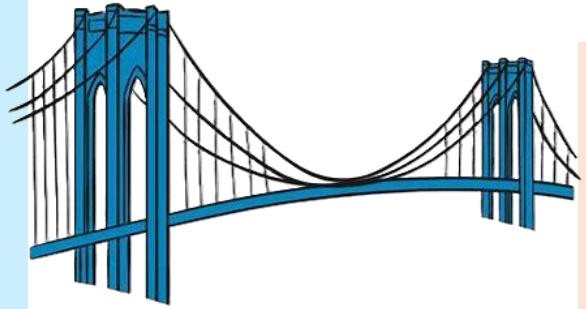
$$\omega_G(\alpha_S, \mu^2/t) = \omega_0 - t\,\omega_1 + \cdots$$

if alpha strong freezes.



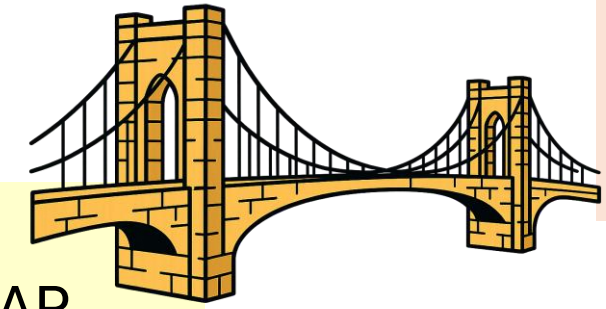
Summary: building bridges

Collinear world
Collinear functions (PDFs, FFs) can be extracted directly from q_T-data

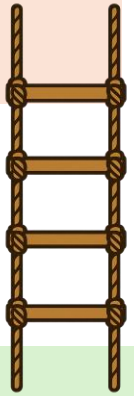


TMD world

- TMD observables as sharp probes of QCD Deep IR.
- TMDs as emergent properties of QCD.



Non-Perturbative world
The strong coupling and the DGLAP evolution (etc..) can be probed in the Deep IR and compared with e.g. NRQCD, Lattice QCD



Broader picture
Geometric universality and link with high-energy factorization