Rapidity-only TMD factorization

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QCD Evolution 2026

Outline

1. Part 1: Rapidity-only TMD factorization and power corrections
   - TMD factorization for particle production in hadron collisions.
   - Leading-N_c power corrections for DY hadronic tensor.
   - Results for Z-boson production: angular asymmetries.
2. Part 2: Rapidity factorization and rapidity evolution of TMDs:
   - Rapidity-only cutoff *vs* UV+rapidity regularization
   - Rapidity evolution of TMDs in the Sudakov region.
   - Argument of coupling constant by BLM.
   - Rapidity-only factorization at one loop.
3. Conclusions

Part 1: Rapidity-only TMD factorization and power corrections

TMD factorization

TMD factorization formula for particle production in hadron-hadron scattering looks like

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp)\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)C(q, k_\perp)$$
$$+ \text{power corrections} + \text{“Y} - \text{terms}''$$

- $\mathcal{D}_{f/A}(x_A, k_\perp)$ is the TMD density of a parton f in hadron A with fraction of momentum x_A and transverse momentum $k_\perp$,
- $\mathcal{D}_{f/B}(x, q_\perp - k_\perp)$ is a similar quantity for hadron B,
- $C_i(q, k)$ are determined by the cross section $\sigma(ff \to \mu^+\mu^-)$ of production of DY pair of invariant mass q^2 in the scattering of two partons.

Examples: Drell-Yan process with Q being the mass of DY pair and Higgs production by gluon-gluon fusion

TMD approach is relevant when the transverse momentum $q_\perp \ll Q$

Long-term project: TMD factorization valid at small and moderate x

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp)\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)C(q, k_\perp)$$
$$+ \text{power corrections} + \text{“Y} - \text{terms}''$$

The quantities $\mathcal{D}_{f/A}(x_A, k_\perp)$, $\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)$, and $C(q, k_\perp)$ are defined with cutoffs. The dependence on the cutoffs cancels in their product order by order in α_s.

At moderate x_A, x_B: CSS/SCET approach. The TMDs $\mathcal{D}_{f/A}(x_A, k_\perp)$ are defined with a combination of UV and rapidity cutoffs.

At $x_A, x_B \ll 1$: k_T-factorization approach. The TMDs are defined with rapidity-only cutoffs.

It is impossible to extend CSS to small x. (Recently: LO BFKL from SCET)

It *is* possible to study TMD factorization at moderate x using small-x methods (rapidity-only factorization etc.) (A. Tarasov, G. Chirilli, I.B, 2015-2023)

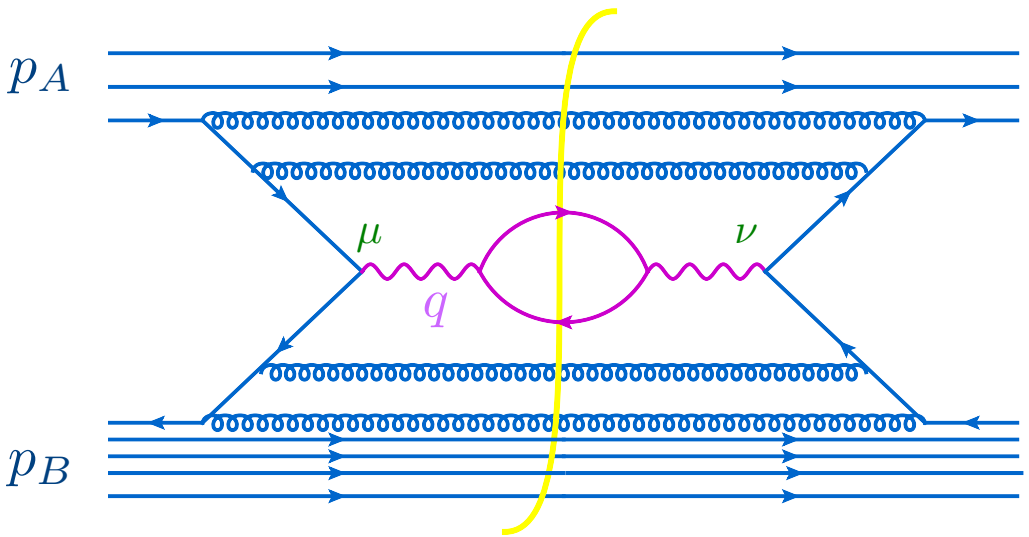
Example: full list of power corrections $\sim \frac{1}{Q^2}$ for DY hadronic tensor.

They are not obtained (yet?) by CSS or SCET

Classical example: DY hadronic tensor

DY cross section is given by the product of leptonic tensor and hadronic tensor. The hadronic tensor $W_{\mu\nu}$ is defined as

$$W_{\mu\nu}(p_A, p_B, q) = \frac{1}{(2\pi)^4} \int d^4x \, e^{-iqx} \langle p_A, p_B | J_\mu(x) J_\nu(0) | p_A, p_B \rangle$$



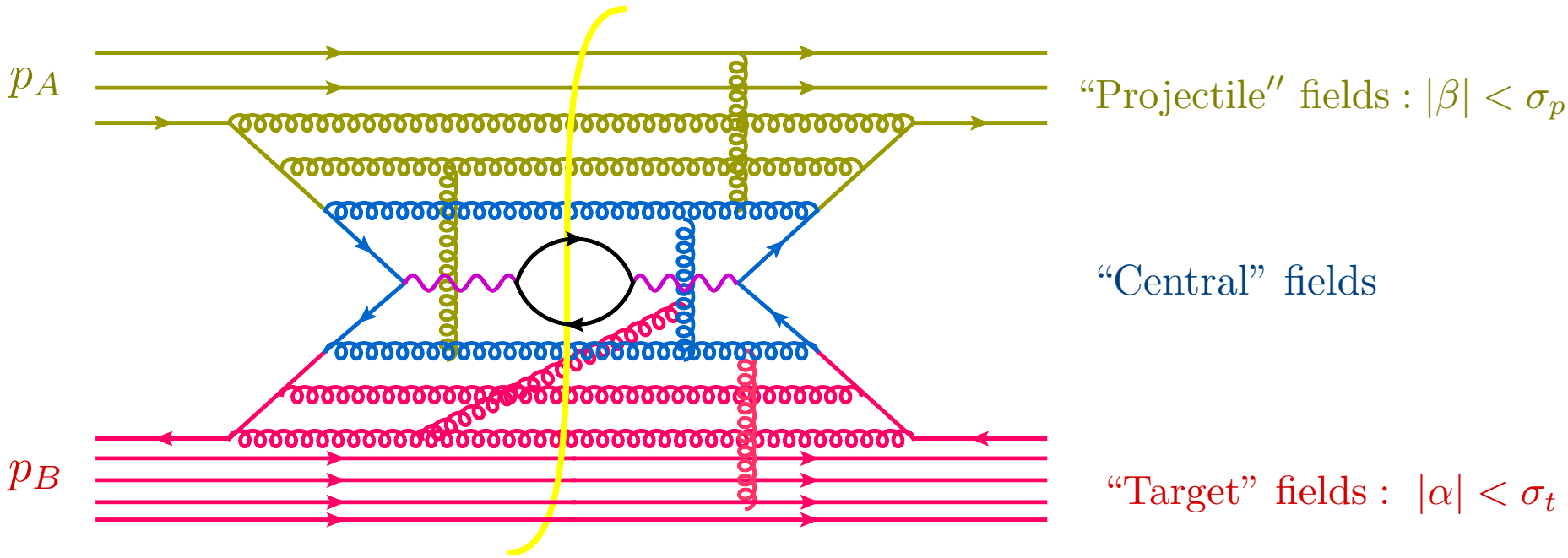
p_A, p_B = hadron momenta, q = the momentum of DY pair, and J_μ is the electromagnetic or Z-boson current.

There are four tensor structures W_T, W_L, W_Δ, $W_{\Delta\Delta}$

TMD factorization from rapidity factorization (A. Tarasov and I.B.)

Sudakov variables:

$$p = \alpha p_1 + \beta p_2 + p_\perp, \qquad p_1 \simeq p_A, \quad p_2 \simeq p_B, \quad p_1^2 = p_2^2 = 0$$



The result of the integration over "central" fields in the background of projectile and target fields is a series of TMD operators made from projectile (or target) fields multiplied by powers of $\frac{1}{Q^2} \Rightarrow$ power corrections

TMD representation for W_i

The hadronic tensor in the Sudakov region $q^2 \equiv Q^2 \gg q_\perp^2$ can be studied by TMD factorization. For example, functions W_T and $W_{\Delta\Delta}$ can be represented as

$$W_i = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}^{(i)}_{f/A}(x_A, k_\perp)\mathcal{D}^{(i)}_{f/B}(x_B, q_\perp - k_\perp)C_i(q, k_\perp)$$
$$+ \text{power corrections} + \text{Y} - \text{terms}$$

There is, however, a problem with this equation for the functions W_L and W_Δ.

W_T and $W_{\Delta\Delta}$ are determined by leading-twist quark TMDs,
but W_Δ and W_L start from terms $\sim \frac{q_\perp}{Q}$ and $\sim \frac{q_\perp^2}{Q^2}$ determined by quark-quark-gluon TMDs.

The power corrections $\sim \frac{q_\perp}{Q}$ were found more than two decades ago but there was no calculation of power corrections $\sim \frac{q_\perp^2}{Q^2}$ until recently.

Goal: TMD factorization formula (with power corrections)

TMD factorization formula structure :

$$\langle p'_A, p'_B | J(x_1) J(x_2) | p_A, p_B \rangle = \sum_{\text{TMD operators}} \int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_i(x_1, x_2; z_i^-, w_i^+; \sigma_p, \sigma_t)$$
$$\times \langle p'_A | \hat{\mathcal{O}}_i^{\sigma_p}(z_2^-, x_{2\perp}; z_1^-, x_{1\perp}) | p_B \rangle \langle p'_B | \hat{\mathcal{O}}_i^{\sigma_t}(z_2^+, x_{2\perp}; z_1^+, x_{1\perp}) | p'_B \rangle$$

$q_\perp^2 \ll Q^2 \Rightarrow$ no dynamics in the transverse space (to be demonstrated below)

$\hat{\mathcal{O}}_i^{\sigma_p}$ - "projectile" TMD operators with $\beta < \sigma_p$ cutoff, e.g

$$\mathcal{O}(z_{1-}, z_{1\perp}, z_{2-}, z_{2\perp}) \equiv \bar{\psi}(z_{1-}, z_{1\perp})[z_{1-}, -\infty]_{z_{1\perp}} \Gamma[-\infty, z_{2+}]_{z_2} \psi(z_{2+}, z_{2\perp})$$

$\hat{\mathcal{O}}_i^{\sigma_p}$ - "target" TMD operators with $\alpha < \sigma_t$ cutoff, e.g

$$\mathcal{O}(z_{1+}, z_{1\perp}, z_{2+}, z_{2\perp}) \equiv \bar{\psi}(z_{1+}, z_{1\perp})[z_{1+}, -\infty]_{z_{1\perp}} \Gamma[-\infty, z_{2+}]_{z_{2\perp}} \psi(z_{2+}, z_{2\perp}).$$

Standard notation for straight-line gauge link

$$[x, y] \equiv \mathrm{P} e^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)} - \text{gauge link}$$

Convenient notations

$$[x_+, y_+]_{z_\perp} \equiv [x_+, 0_-, z_\perp; y_+, 0_-, z_\perp], \quad [x_-, y_-]_{z_\perp} \equiv [x_-, 0_+, z_\perp; y_-, 0_+, z_\perp]$$

Means: "double operator expansion"

Intermediate step: double operator expansion

$$\hat{J}(x_1)\hat{J}(x_2) = \sum_{I,J}\int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_{IJ}(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t) \times \hat{\mathcal{O}}_I^{\sigma_p}(z_2^-,x_{2\perp};z_1^-,x_{1\perp})\hat{\mathcal{O}}_J^{\sigma_t}(z_2^+,x_{2\perp};z_1^+,x_{1\perp})$$

To find relevant operators and coefficients, it is convenient to consider "matrix" elements of the l.h.s. and r.h.s. in suitable background field

Suitable field $\mathbb{A}$: solution of classical YM equations with boundary condition that at the remote past the field is a sum of projectile and target fields

$$\langle\hat{J}(x_1)\hat{J}(x_2)\rangle_{\mathbb{A}} = \sum_{I,J}\int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_{IJ}(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t) \times \langle\hat{\mathcal{O}}_I^{\sigma_p}(z_2^-,x_{2\perp};z_1^-,x_{1\perp})\hat{\mathcal{O}}_J^{\sigma_t}(z_2^+,x_{2\perp};z_1^+,x_{1\perp})\rangle_{\mathbb{A}}$$

Method of solution:

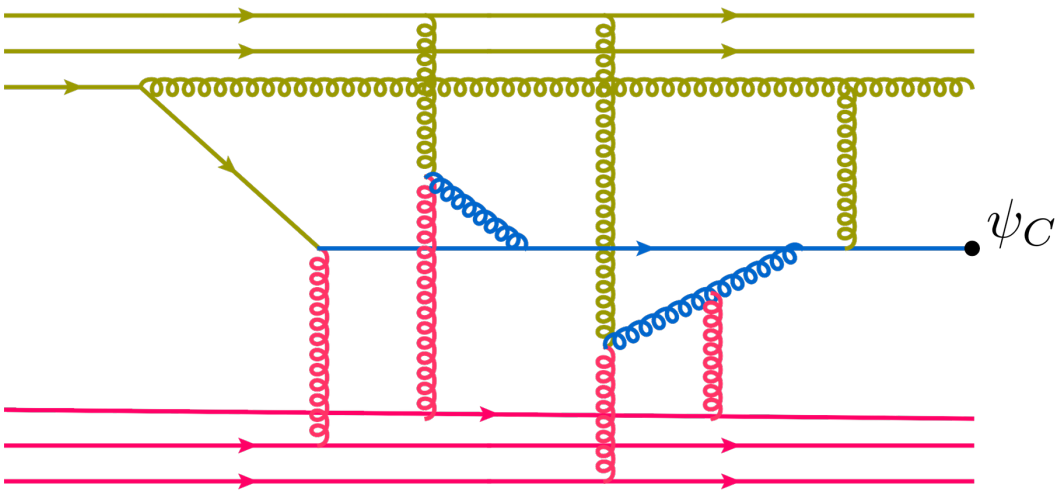
- Start with $\Psi_{\text{trial}} = \psi_A + \psi_B$ and $\mathbb{A}_{\text{trial}} = \bar{A}_\mu + \bar{B}_\mu$ in the gauge $A^+ = 0$, $A^- = 0$
- Correct by computing Feynman diagrams (with retarded propagators) with sources $(\not{P} + m)(\psi_A + \psi_B)$ and $J_\nu = D^\mu F^{\mu\nu}(A+B)$

ψ_C in the tree approximation

It is convenient to choose projectile/target fields as

Projectile fields: $\beta = 0 \Rightarrow A(x^-, x_\perp),\ \psi_A(x^-, x_\perp)$

Target fields: $\alpha = 0 \Rightarrow B(x^+, x_\perp),\ \psi_B(x^-, x_\perp)$



Classical background fields: Ψ , $\mathbb{A}_\mu$

ψ_C = sum of tree diagrams in external A, ψ_A and B, ψ_B fields with sources

$J_\psi = (\not{P} + m)(\psi_A + \psi_B), \qquad J_\nu = D^\mu F^{\mu\nu}(A + B)$

Classical fields in the leading order in $p_\perp^2/p_\parallel^2 \sim q_\perp^2/Q^2$

The solution of such YM equations in general case is yet unsolved problem (goes under the name "glasma" $\Leftrightarrow$ scattering of two "color glass condensates").

Fortunately, for our case of particle production with $\frac{q_\perp}{Q} \ll 1$ we can use this small parameter and construct the approximate solution.

At the tree level transverse momenta are $\sim q_\perp^2$ and longitudinal are $\sim Q^2 \Rightarrow$

$$\Psi, \mathbb{A} \;=\; \text{series in } \frac{q_\perp}{Q}: \qquad \Psi = \psi^{(0)} + \psi^{(1)} + ..., \qquad \mathbb{A} = A^{(0)} + A^{(1)} + ...$$

NB: After the expansion

$$\frac{1}{p^2 + i\epsilon p_0} = \frac{1}{p_\parallel^2 - p_\perp^2 + i\epsilon p_0} = \frac{1}{p_\parallel^2} - \frac{1}{p_\parallel^2 + i\epsilon p_0} p_\perp^2 \frac{1}{p_\parallel^2 + i\epsilon p_0} + ...$$

the dynamics in transverse space is trivial.

Fields are either at the point $x_\perp$ or at the point $0_\perp \Rightarrow$ TMDs

Leading-N_c power corrections

Power corrections are $\sim$ leading twist $\times\left(\frac{q_\perp}{Q}\text{ or }\frac{q_\perp^2}{Q^2}\right)\times\left(1+\frac{1}{N_c}+\frac{1}{N_c^2}\right)$.

NB: almost all $\bar{q}Gq$ TMDs not suppressed by $\frac{1}{N_c}$ are determined by the $\bar{q}q$ TMDs due to QCD equations of motion

Leading twist (for the projectile nucleon): $\varrho\equiv\sqrt{s/2}$

$$\frac{1}{8\pi^3 s}\int dx^- d^2x_\perp\, e^{-i\alpha\varrho x^- + i(k,x)_\perp}\,\langle N|\hat{\bar{\psi}}(x^-,x_\perp)\,\not{p}_2\hat{\psi}(0)|N\rangle \;=\; f_1(\alpha,k_\perp^2)$$

Power correction:

$$\frac{1}{8\pi^3 s}\int dx^- dx_\perp\, e^{-i\alpha\varrho x^- + i(k,x)_\perp}$$

$$\times\,\langle N|\hat{\bar{\psi}}(x^-,x_\perp)\hat{\not{A}}(x^-,x_\perp)\not{p}_2\gamma_i\hat{\psi}(0)|N\rangle \;=\; k_i\tilde{f}_1(\alpha,k_\perp)$$

$$=\; k_i f_1(\alpha,k_\perp) - \alpha k_i\left[f_\perp(\alpha,k_\perp)+ig^\perp(\alpha,k_\perp)\right],$$

(Mulders & Tangerman, 1996)

Result for $W_{\mu\nu}$ for unpolarized hadrons

Result:

$$W_{\mu\nu}(q) \;=\; W^{1}_{\mu\nu}(q) + W^{2}_{\mu\nu}(q) + W^{3}_{\mu\nu}(q)$$

The first, gauge-invariant, part is a "gauge completion" of leading-twist result

$$\begin{aligned}
W^{1}_{\mu\nu}(q) \;&=\; W^{1F}_{\mu\nu}(q) + W^{1H}_{\mu\nu}(q),\\
W^{1F}_{\mu\nu}(q) \;&=\; \sum_f e_f^2 W^{fF}_{\mu\nu}(q), \quad W^{fF}_{\mu\nu}(q) \;=\; \frac{1}{N_c}\int d^2k_\perp F^f(q,k_\perp)\mathcal{W}^{F}_{\mu\nu}(q,k_\perp),\\
W^{1H}_{\mu\nu}(q) \;&=\; \sum_f e_f^2 W^{fH}_{\mu\nu}(q), \quad W^{fH}_{\mu\nu}(q) \;=\; \frac{1}{N_c}\int d^2k_\perp H^f(q,k_\perp)\mathcal{W}^{H}_{\mu\nu}(q,k_\perp)
\end{aligned}$$

where F^f and H^f are ($\alpha_q \equiv x_A$, $\beta_q \equiv x_B$)

$$\begin{aligned}
F(q,k_\perp) \;&=\; f_1\alpha_q, k_\perp)\bar{f}_1\beta_q, (q-k)_\perp) \;+\; f_1 \leftrightarrow \bar{f}_1\\
H(q,k_\perp) \;&=\; h_1^\perp(\alpha_q, k_\perp)\bar{h}_1^\perp(\beta_q, (q-k)_\perp) \;+\; h_1^\perp \leftrightarrow \bar{h}_1^\perp
\end{aligned}$$

Gauge-invariant structures $q^\mu W^F_{\mu\nu} = q^\mu W^H_{\mu\nu} = 0$

$$\mathcal{W}^F_{\mu\nu}(q,k_\perp) \;=\; \mathrm{LT} + \text{“gauge}-\text{invariant completion”}$$

$$= \; -g^\perp_{\mu\nu} + \frac{1}{Q^2}(q^\parallel_\mu q^\perp_\nu + q^\parallel_\nu q^\perp_\mu) + \frac{q^2_\perp}{Q^4} q^\parallel_\mu q^\parallel_\nu + \frac{\tilde{q}_\mu \tilde{q}_\nu}{Q^2}[q^2_\perp - 4(k,q-k)_\perp]$$

$$- \left[\frac{\tilde{q}_\mu}{Q^2}\left(g^\perp_{\nu i} - \frac{q^\parallel_\nu q_i}{Q^2}\right)(q-2k)^i_\perp + \mu \leftrightarrow \nu\right] \qquad \tilde{q} \equiv \alpha_q p_1 - \beta_q p_2$$

Non-trivial observation: take $\tilde{f}_1 = f_1 - \alpha(f^\perp - ig^\perp) \simeq f_1 \Rightarrow$ gauge invariance.
NB: you need both $\frac{1}{Q}$ and $\frac{1}{Q^2}$ corrections proportional to $\tilde{f}_1$.

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$$- \left[\frac{\tilde q_\mu}{Q^2}\left(g^\perp_{\nu i} - \frac{q^\parallel_\nu q_i}{Q^2}\right)(q-2k)^i_\perp + \mu \leftrightarrow \nu\right] \qquad \tilde q \equiv \alpha_q p_1 - \beta_q p_2$$

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NB: you need both $\frac{1}{Q}$ and $\frac{1}{Q^2}$ corrections proportional to $\tilde f_1$.

$$m^2 \mathcal{W}^H_{\mu\nu}(q,k_\perp)$$

$$= -\left[k^\perp_\mu (q-k)^\perp_\nu + k^\perp_\nu (q-k)^\perp_\mu + g^\perp_{\mu\nu}(k,q-k)_\perp\right] + 2\frac{\tilde q_\mu \tilde q_\nu - q^\parallel_\mu q^\parallel_\nu}{Q^4} k^2_\perp (q-k)^2_\perp$$

$$- \left(\frac{q^\parallel_\mu}{Q^2}\left[k^2_\perp (q-k)^\perp_\nu + k^\perp_\nu (q-k)^2_\perp\right] + \frac{\tilde q_\mu}{Q^2}\left[k^2_\perp (q-k)^\perp_\nu - k^\perp_\nu (q-k)^2_\perp\right] + \mu \leftrightarrow \nu\right)$$

$$- \frac{\tilde q_\mu \tilde q_\nu + q^\parallel_\mu q^\parallel_\nu}{Q^4}\left[q^2_\perp - 2(k,q-k)_\perp\right](k,q-k)_\perp - \frac{q^\parallel_\mu \tilde q_\nu + \tilde q_\mu q^\parallel_\nu}{Q^4}(2k-q,q)_\perp (k,q-k)_\perp$$

Second gauge-invariant part

$$
\begin{aligned}
W^2_{\mu\nu}(q) \;=\; & \frac{2}{N_c Q^2}\int d^2k_\perp \\
\times & \Big\{\Big[\tilde{q}_\mu (q-k)_\nu + \frac{2}{\beta_q s}\tilde{q}_\mu p_{1\nu}(k,q-k)_\perp + \frac{2}{\alpha_q s}\tilde{q}_\mu p_{2\nu}(q-k)^2_\perp \;+\; \mu \leftrightarrow \nu\Big] \\
& \qquad \times \Big(\beta_q\{f_1\bar{f}_\perp + \bar{f}_1 f_\perp\} - \alpha_q\{h\bar{h}_1^\perp + \bar{h}h_1^\perp\}\Big) \\
+ & \Big[\tilde{q}_\mu k_\nu^\perp + \frac{2}{s\beta_q}k^2_\perp \tilde{q}_\mu p_{1\nu} + \frac{2}{s\alpha_q}(k,q-k)_\perp \tilde{q}_\mu p_{2\nu} \;+\; \mu \leftrightarrow \nu\Big] \\
& \qquad \times \Big(-\alpha_q\{f_\perp \bar{f}_1 + \bar{f}_\perp f_1\} + \beta_q\{h_1^\perp \bar{h} + \bar{h}_1^\perp h\}\Big) \\
+ & \frac{4\tilde{q}_\mu \tilde{q}_\nu}{Q^2}\Big[m^2\Big(\alpha_q^2\{f_3\bar{f}_1 + \bar{f}_3 f_1\} + \beta_q^2\{f_1\bar{f}_3 + \bar{f}_1 f_3\} + \alpha_q\beta_q\big[\{e\bar{e}+\bar{e}e\} + \{h\bar{h}+\bar{h}h\}\big]\Big) \\
& \qquad + (k,q-k)_\perp\Big(-\alpha_q\beta_q\big[\{f_\perp\bar{f}_\perp + \bar{f}_\perp f_\perp\} + \{g_\perp\bar{g}_\perp + \bar{g}_\perp g_\perp\}\big] \\
& \qquad + \beta_q^2\{h_1^\perp \bar{h}_3^\perp + \bar{h}_1^\perp h_3^\perp\} + \alpha_q^2\{h_3^\perp \bar{h}_1^\perp + \bar{h}_3^\perp h_1^\perp\}\Big)\Big]
\end{aligned}
$$

$f_\perp, f_3, h, h_3^\perp, g_\perp, e$ are the quark-antiquark TMDs of a non-leading twist.

Third part

$$W^3_{\mu\nu}(q) \;=\; \frac{2}{N_c Q^2}\int d^2k_\perp$$

$$\times\Big\{g^\perp_{\mu\nu}\times\big[\text{a bunch of quark–antiquark and quark–antiquark–gluon TMDs}\big]$$

$$+\;\big[k^\perp_\mu(q-k)^\perp_\nu + k^\perp_\nu(q-k)^\perp_\mu + g^\perp_{\mu\nu}(k,q-k)_\perp\big]\times\text{same}\Big\}$$

Similarly to LT contribution, is not EM gauge invariant $\Rightarrow$ needs "gauge completion" by $\frac{1}{Q^3}$ and $\frac{1}{Q^4}$ power corrections, for example

$$g^\perp_{\mu\nu}\to\mathcal{W}^F_{\mu\nu},\quad k^\perp_\mu(q-k)^\perp_\nu + k^\perp_\nu(q-k)^\perp_\mu + g^\perp_{\mu\nu}(k,q-k)_\perp\to\mathcal{W}^H_{\mu\nu}$$

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NB: basis of operators for $\frac{1}{Q^2}$ corrections ($\varrho\equiv\sqrt{s/2}$)

$$EOM:\qquad \begin{aligned} \not{A}_\perp(x)\psi(x) &= -i\not{\partial}_\perp\psi(x) - i\tfrac{1}{\varrho}\not{p}_1\partial_+\psi(x) - i\tfrac{1}{\varrho}\not{p}_2 D_-\psi(x)\\ \not{B}_\perp(x)\psi(x) &= -i\not{\partial}_\perp\psi(x) - i\tfrac{1}{\varrho}\not{p}_2\partial_-\psi(x) - i\tfrac{1}{\varrho}\not{p}_1 D_+\psi(x)\end{aligned}$$

LHS = Vladimirov, Scimemi, *et al*, RHS = my choice,

Application: angular coefficients of Z-boson production

In CMS and ATLAS experiments $s = 8$ TeV, $Q = 80 - 100$ GeV and $Q_\perp$ varies from 0 to 120 GeV.

Our analysis is valid at $Q_\perp = 10 - 30$ GeV and $Y \simeq 0$ ($x_A \sim x_B \sim 0.1$) so that power corrections are small but sizable.

Angular distribution of DY leptons in the Collins-Soper frame ($c_\phi \equiv \cos\phi$, $s_\phi \equiv \sin\phi$ etc.)

$$\frac{d\sigma}{dQ^2 dy d\Omega_l} = \frac{3}{16\pi}\frac{d\sigma}{dQ^2 dy}\Big[(1+c_\theta^2) + \frac{A_0}{2}(1-3c_\theta^2) + A_1 s_{2\theta} c_\phi + \frac{A_2}{2} s_\theta^2 c_{2\phi} + A_3 s_\theta c_\phi + A_4 c_\theta + A_5 s_\theta^2 s_{2\phi} + A_6 s_{2\theta} s_\phi + A_7 s_\theta s_\phi\Big]$$

Back-of-the envelope estimation: take only f_1 contribution at large N_c, use "factorization hypothesis" for TMD $f_1(x,k_\perp) \simeq f(x)g(k_\perp)$ and calculate integrals over $k_\perp$ in the leading log approximation using $f_1(x,k_\perp^2) \simeq \frac{f(x)}{k_\perp^2}$

Comparison of A_0 with LHC results

Logarithmic estimate of A_0 (m_z - Z-boson mass, m - proton mass)

$$A_0 = \frac{Q_\perp^2}{m_z^2}\,\frac{1+2\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}}{1+\frac{Q_\perp^2}{m_z^2}\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}} \qquad (*)$$

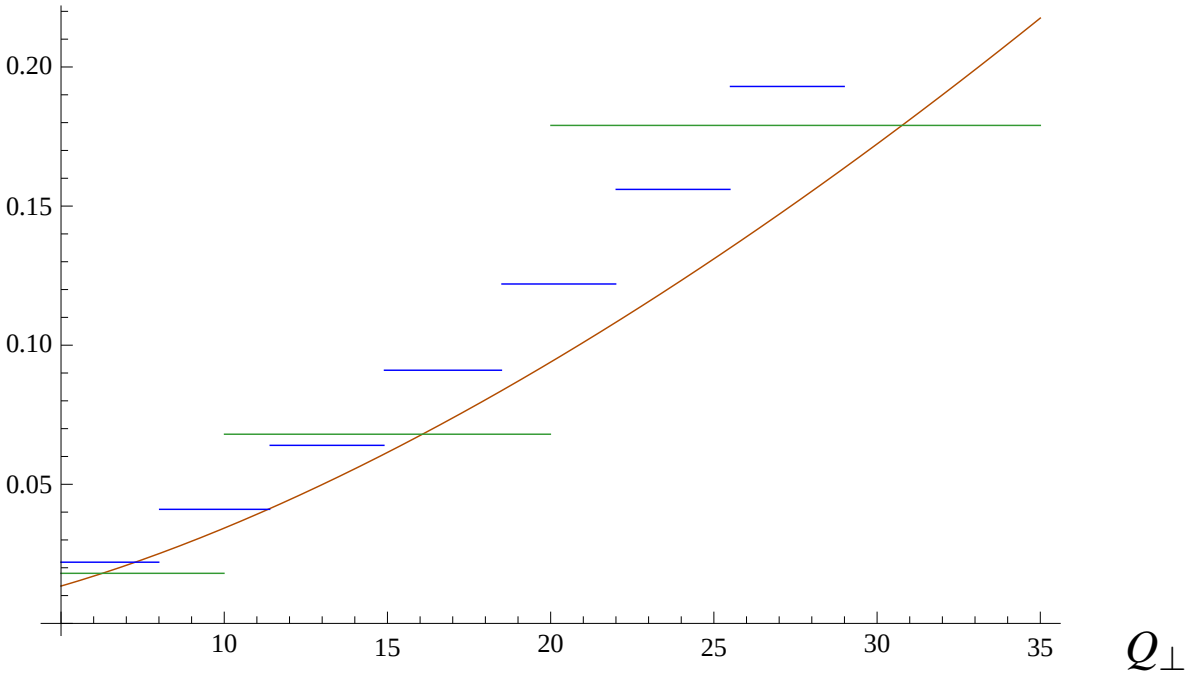


Figure: Comparison of prediction $(*)$ with lines depicting angular coefficient A_0 in bins of $Q_\perp$ and $Y < 1$ from CMS (arXiv:1504.03512) and ATLAS (arXiv1606.00689)

Comparison of A_2 with LHC results

Logarithmic estimate of A_2

$$A_2 = \frac{Q_\perp^2}{m_z^2} \frac{1}{1 + \frac{Q_\perp^2}{m_z^2} \frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}} \qquad (**)$$

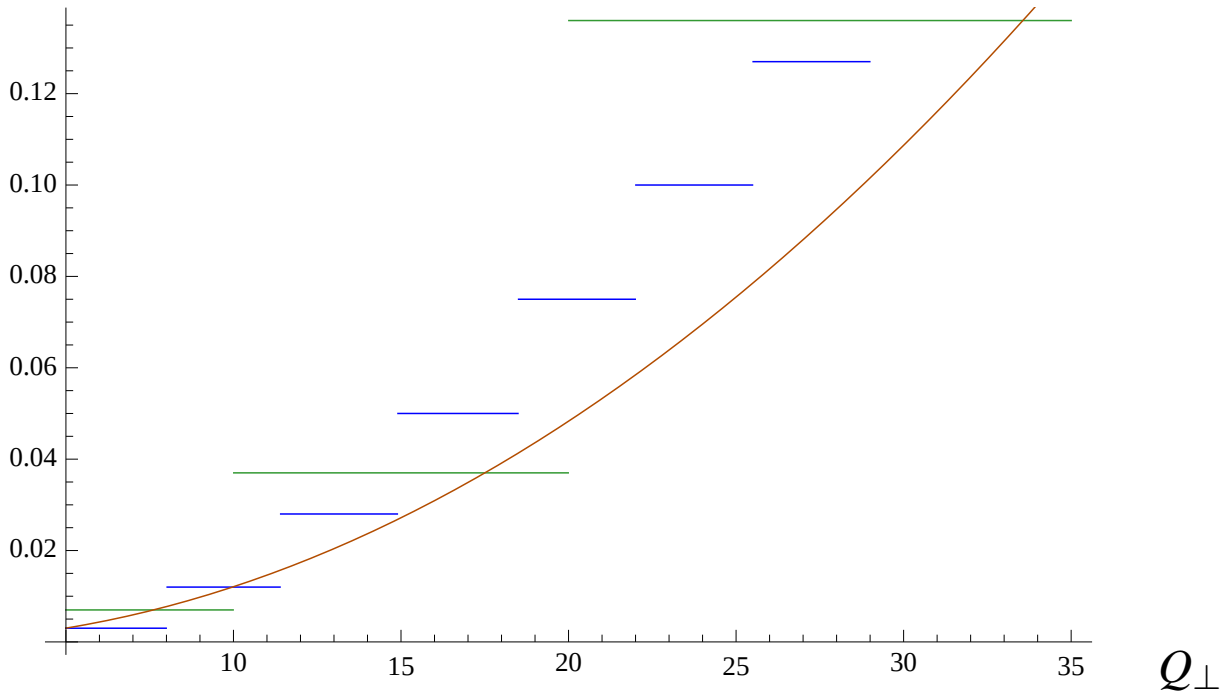


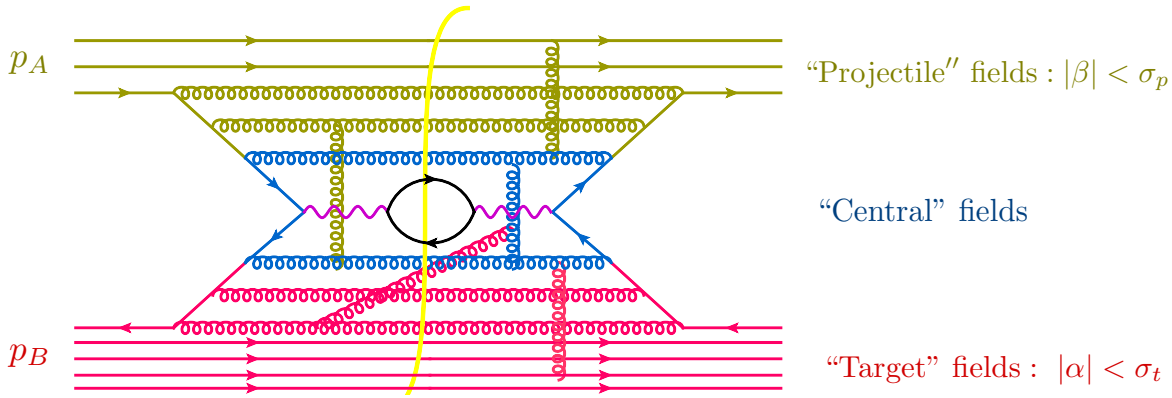
Figure: Comparison of prediction $(**)$ with lines depicting angular coefficient A_2 in bins of $Q_\perp$ and $Y < 1$ from CMS (arXiv:1504.03512) and ATLAS (arXiv1606.00689)

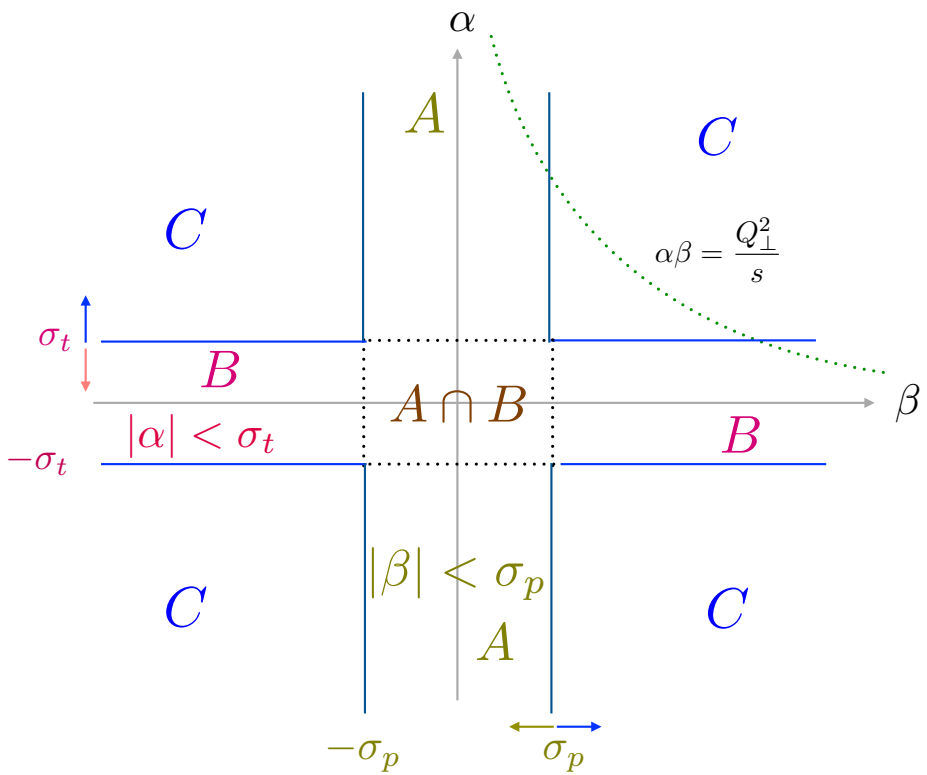
Result of estimation

Result of the estimation: A_0 and A_2 in agreement with data, rest of $A_i = 0$.
Experimentally, at $q_\perp \leq \frac{1}{4} m_z$ other A_i are an order of magnitude smaller that $A_0 \sim A_2$
$\Rightarrow$ it looks like f_1 is numerically the most important TMD for unpolarized DY cross sections.

Part 2: Rapidity factorization and rapidity evolution of TMDs

Rapidity-only cutoffs and matching of logs





Matching: $\ln \sigma_p$ in the projectile TMDs and $\ln \sigma_t$ in the target TMDs should cancel with $\ln \sigma_p$ and $\ln \sigma_t$ in the coefficient functions.

$A \cap B$, $k_\perp \sim m_\perp$: Glauber gluons

$A \cap B$, $k_\perp \ll m_\perp$: soft gluons

$A \cap B$ gluons $\equiv$ soft/Glauber (sG) gluons cancel out

Rapidity-only cutoff

Typical diagram in the background field $\Psi(\beta_B, p_{B_\perp}) = \varrho \int dz^+ dz_\perp \, \Psi(z^+, z_\perp) e^{i\varrho\beta_B z^+ - i(p_B, z)_\perp}$

$$\langle [x^+, -\infty]_x \Gamma \psi(y^+, y_\perp) \rangle_\Psi = g^2 c_F \int đ\beta_B đp_{B_\perp} e^{-ip_B y} \Gamma \Psi(\beta_B, p_{B_\perp})$$

$$\times \int_0^\infty đ\alpha \int \frac{đp_\perp}{p_\perp^2} \frac{\beta_B s e^{-i\frac{p_\perp^2}{\alpha s}\varrho\Delta^+ + i(p,\Delta)}}{\alpha\beta_B s + (p - p_B)_\perp^2 + i\epsilon} \qquad \leftarrow \text{ divergent as } \alpha \to \infty$$

$$\langle [x^+, -\infty]_x \Gamma \psi(y^+, y_\perp, -\delta^-) \rangle_\Psi = g^2 c_F \int đ\beta_B đp_{B_\perp} e^{-ip_B y} \Gamma \Psi(\beta_B, p_{B_\perp}) \qquad \sigma \equiv \frac{1}{\varrho\delta^-}$$

$$\times \int_0^\infty đ\alpha \int \frac{đp_\perp}{p_\perp^2} \frac{\beta_B s e^{-i\frac{p_\perp^2}{\alpha s}\varrho\Delta^+ + i(p,x-y)_\perp}}{\alpha\beta_B s + (p - p_B)_\perp^2 + i\epsilon} e^{-i\frac{\alpha}{\sigma}} \qquad \leftarrow \text{ convergent as } \alpha \to \infty$$

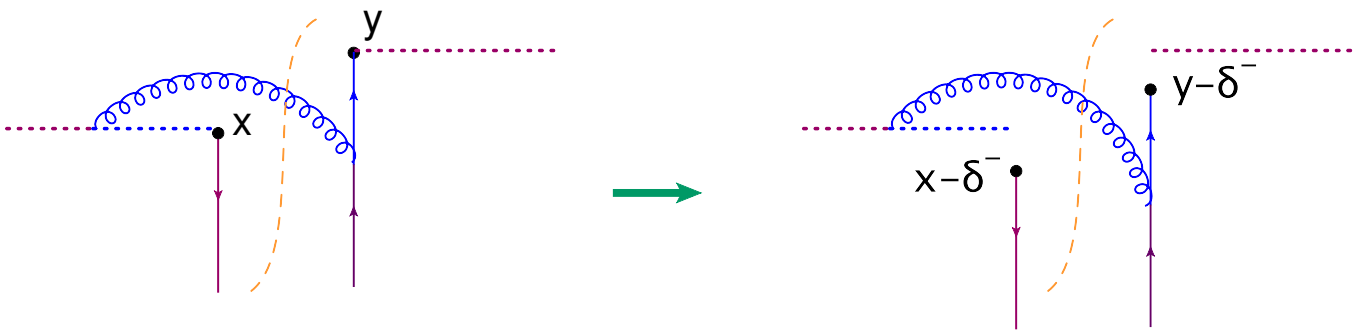


Figure: Point-splitting visualization of "smooth" rapidity-only cutoff.

Rapidity-only cutoff *vs* UV+rapidity regularization

Typical divergent integral ($\varepsilon = \frac{d}{2} - 2$, $đ^n p \equiv \frac{d^n p}{(2\pi)^n}$)

$$-i\mu^{-2\varepsilon}\int đ\alpha đ\beta đ p_\perp \frac{1}{\beta - i\epsilon}\frac{1}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\left(1 - e^{i(p,x)_\perp}\right)$$

$$= \mu^{-2\varepsilon}\int \frac{đ p_\perp}{p_\perp^2}\left(1 - e^{i(p,x)_\perp}\right)\int_0^{\beta_B} đ\beta \frac{\beta_B - \beta}{\beta_B}\frac{1}{\beta - i\epsilon} = -\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B} d\beta \frac{\beta_B-\beta}{\beta_B}\frac{1}{\beta - i\epsilon}$$

Regularization with $A^-(z^+) \to A^-(z^+)e^{\pm\delta z^+}$

$$-\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B} d\beta \frac{\beta_B-\beta}{\beta_B}\frac{1}{\beta - i\delta} \simeq \frac{1}{8\pi^2}\left(-\frac{1}{\varepsilon} + \ln\mu^2\frac{x_\perp^2}{4} + \gamma_E\right)\left(\ln\frac{\beta_B}{-i\delta} - 1\right)$$

Rapidity-only cutoff

$$-i\int đ\alpha đ\beta đ p_\perp \frac{1}{\beta - i\epsilon}\frac{e^{-i\frac{\alpha}{\sigma}}}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\left(1 - e^{i(p,x)_\perp}\right)$$

$$= \int \frac{đ p_\perp}{p_\perp^2}\left(1 - e^{i(p,x)_\perp}\right)\int_0^\infty đ\alpha \frac{\beta_B s}{\alpha\beta_B s + p_\perp^2}e^{-i\frac{\alpha}{\sigma}} = \frac{1}{16\pi^2}\ln^2\left(-i\beta_B\sigma s\frac{x_\perp^2}{4}e^{\gamma_E}\right)$$

Rapidity evolution of TMDs

Quark TMD operator

$$\mathcal{O}(z_{1+}, z_{1\perp}, z_{2+}, z_{2\perp}) \equiv \bar{\psi}(z_{1+}, z_{1\perp})[z_{1+}, -\infty]_{z_1} \Gamma [-\infty, z_{2+}]_{z_2} \psi(z_{2+}, z_{2\perp})$$

Sudakov regime: $Q^2 \gg Q_\perp^2 \Leftrightarrow z_{12+} z_{12-} \ll z_{12\perp}^2$

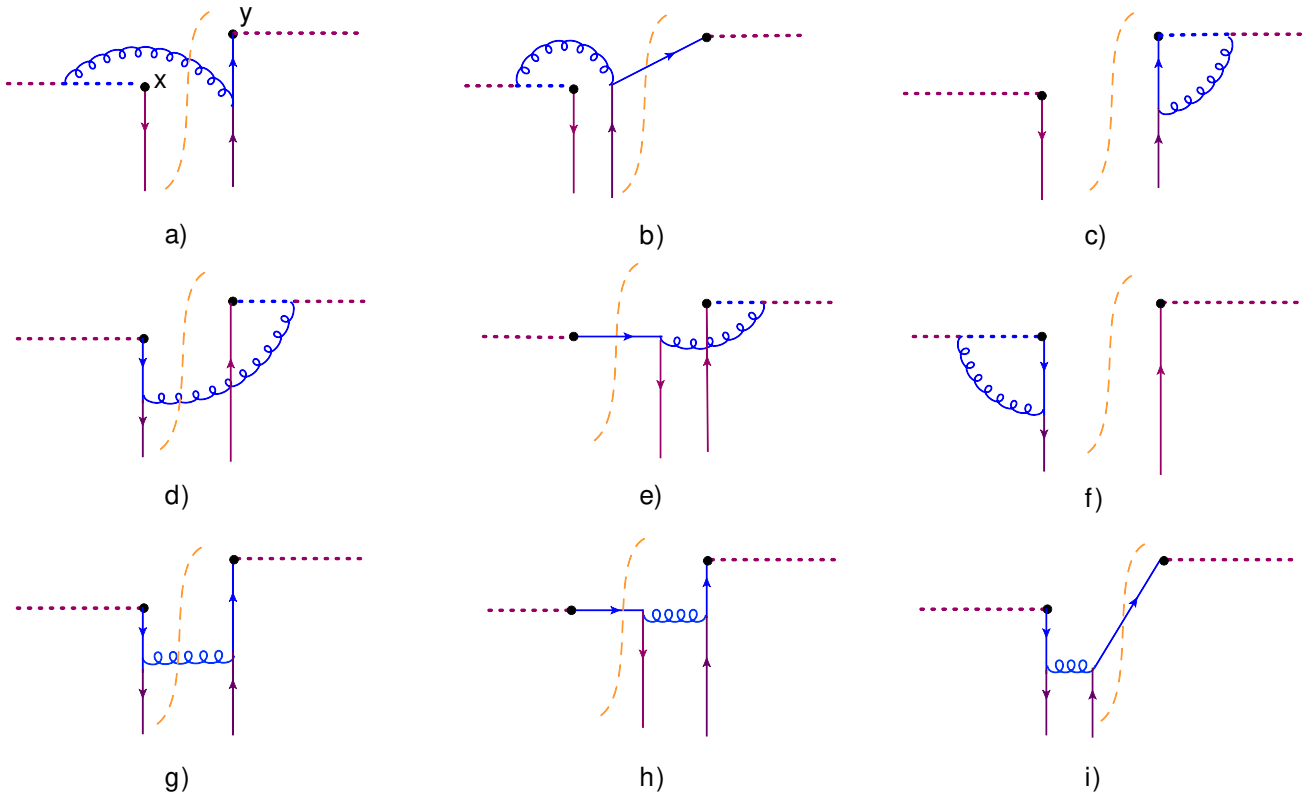


Figure: Diagrams for leading-order rapidity evolution of quark TMD in the Sudakov regime.

Conformally invariant evolution at the LO (G.A. Chirilli & I.B.,2019)

Evolution equation ($\lambda \equiv \sigma (x-y)^2_\perp \frac{s}{4}$)

$$\lambda \frac{d}{d\lambda}\mathcal{O}(x_+,y_+;\lambda)$$
$$= \left[\int_{x_+}^{\infty} dx'_+ \frac{1}{x'_+ - y_+} e^{i\frac{\lambda\sqrt{2/s}}{x'_+ - y_+}} \mathcal{O}(x_+,y_+;\lambda) - \int_{y_+}^{\infty} dy'_+ \frac{\mathcal{O}(x_+,y_+;\lambda) - \mathcal{O}(x_+ t, y'_+;\lambda)}{y'_+ - y_+}\right.$$
$$\left. + \int_{y_+}^{\infty} dy'_+ \frac{1}{y'_+ - x_+} e^{i\frac{\lambda\sqrt{2/s}}{y'_+ - x_+}} \mathcal{O}(x_+,y_+;\lambda) - \int_{x_+}^{\infty} dx'_+ \frac{\mathcal{O}(x_+,y_+;\lambda) - \mathcal{O}(x'_+,y_+;\lambda)}{x'_+ - x_+}\right]$$

If we use rapidity cutoff at $\sigma = \frac{8\varsigma}{|x-y|_\perp\sqrt{s}} \Leftrightarrow \lambda = \varsigma |x-y|\sqrt{s}$,
the solution

$$\mathcal{O}(x_+,y_+;\sigma) = e^{-\frac{\tilde{\alpha}_s}{2}\left(\ln^2\frac{2(x-y)^2_\perp \varsigma^2}{x_+ y_+} - \ln^2 \frac{2(x-y)^2_\perp \varsigma_0^2}{x_+ y_+}\right)} e^{4\tilde{\alpha}_s \psi(1)\ln\frac{\varsigma}{\varsigma_0}} \int dx'_+ dy'_+ \; \mathcal{O}(x'_+,y'_+;\sigma_0)$$
$$\times (x_+ y_+)^{-\tilde{\alpha}_s \ln\frac{\varsigma}{\varsigma_0}} \left[\frac{i\Gamma\left(1-\tilde{\alpha}_s\ln\frac{\varsigma}{\varsigma_0}\right)}{(x_+ - x'_+ + i\epsilon)^{1-\tilde{\alpha}_s\ln\frac{\varsigma}{\varsigma_0}}} - \frac{i\Gamma\left(1-\tilde{\alpha}_s\ln\frac{\varsigma}{\varsigma_0}\right)}{(x_+ - x'_+ - i\epsilon)^{1-\tilde{\alpha}_s\ln\frac{\varsigma}{\varsigma_0}}}\right]$$
$$\times \left[\frac{i\Gamma\left(1-\tilde{\alpha}_s\ln\frac{\varsigma}{\varsigma_0}\right)}{(y_+ - y'_+ + i\epsilon)^{1-\tilde{\alpha}_s\ln\frac{\varsigma}{\varsigma_0}}} - \frac{i\Gamma\left(1-\tilde{\alpha}_s\ln\frac{\varsigma}{\varsigma_0}\right)}{(y_+ - y'_+ - i\epsilon)^{1-\tilde{\alpha}_s\ln\frac{\varsigma}{\varsigma_0}}}\right]$$

is obviously invariant under the inversion $x_+ \to \frac{x_+}{x_+^2}$, $y_+ \to \frac{y_+}{y_+^2}$.

Argument of coupling constant by BLM/renormalon method

A problem with leading-order rapidity evolution: what is the argument of coupling constant?

In CSS approach - no problem, argument is defined by renormgroup

With rapidity-only evolution (BFKL, BK and the like) - argument of α_s may be obtained from the NLO calculations. BLM approach: calculate the small part of the NLO result, namely quark loop contribution to gluon propagator, and promote $-\frac{2}{3}n_f$ to the full $b = \frac{11}{3}N_c - \frac{2}{3}n_f$.

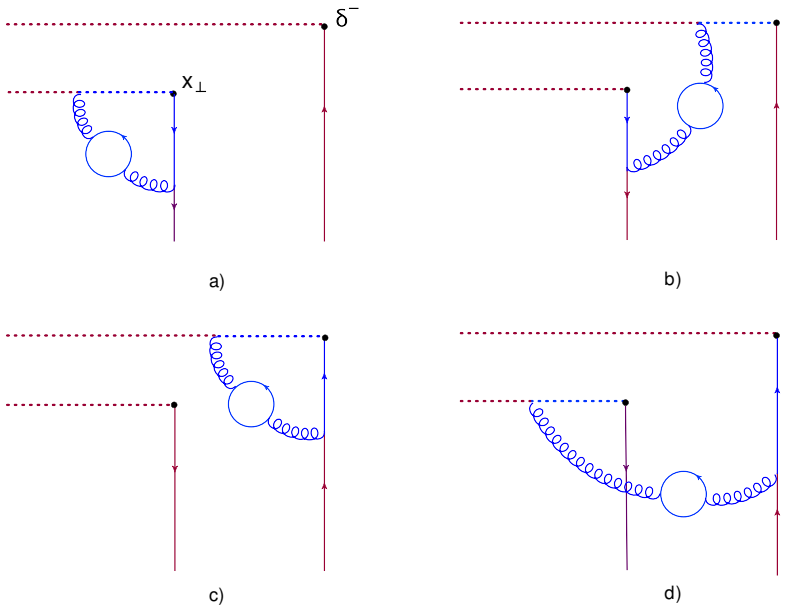
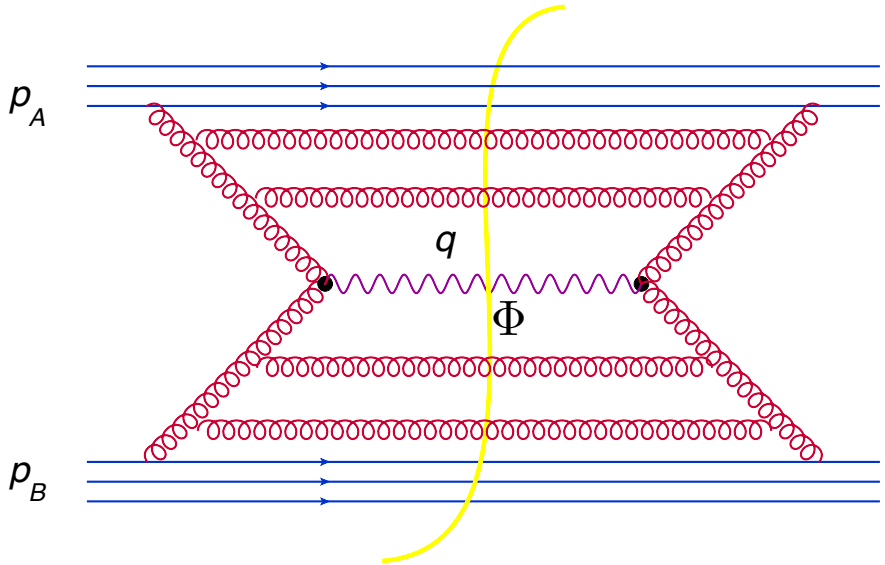


Figure: Quark loop correction to quark TMD evolution

Result: BLM optimal scale is logarithmically halfway between transverse momentum ($b_\perp^{-1/2}$) and energy ($\sigma\beta_B s$) of TMD both for quarks and gluons (G.A.Chirilli and I.B.)

Coefficient function for TMD factorization at one loop

Particle production by gluon fusion



$$s \gtrsim Q^2 \quad \gg \quad Q_\perp^2 \gtrsim m^2$$
$$q^2 \equiv Q^2 = M_\Phi^2, \quad Q_\perp^2 \equiv q_\perp^2$$

Goal: one-loop TMD factorization formula for hadronic tensor.

Result of calculations:

$$W(p_A, p_B; q) = \int db_\perp \, e^{i(q,b)_\perp} \mathcal{D}_{g/A}(x_A, b_\perp; \sigma_a) \mathcal{D}_{g/B}(x_B, b_\perp; \sigma_b)$$
$$\times \exp\left\{ \frac{\alpha_s N_c}{2\pi} \left[\ln^2 \frac{b_\perp^2 s \sigma_p \sigma_t}{4} - 2\left(\ln \frac{x_A}{\sigma_t} + \gamma \right)\left(\ln \frac{x_B}{\sigma_p} + \gamma \right) + \frac{\pi^2}{2} \right] \right\}$$
$$+ \text{NLO terms} \sim O(\alpha_s^2) + \text{power corrections}$$

One-loop coefficient function

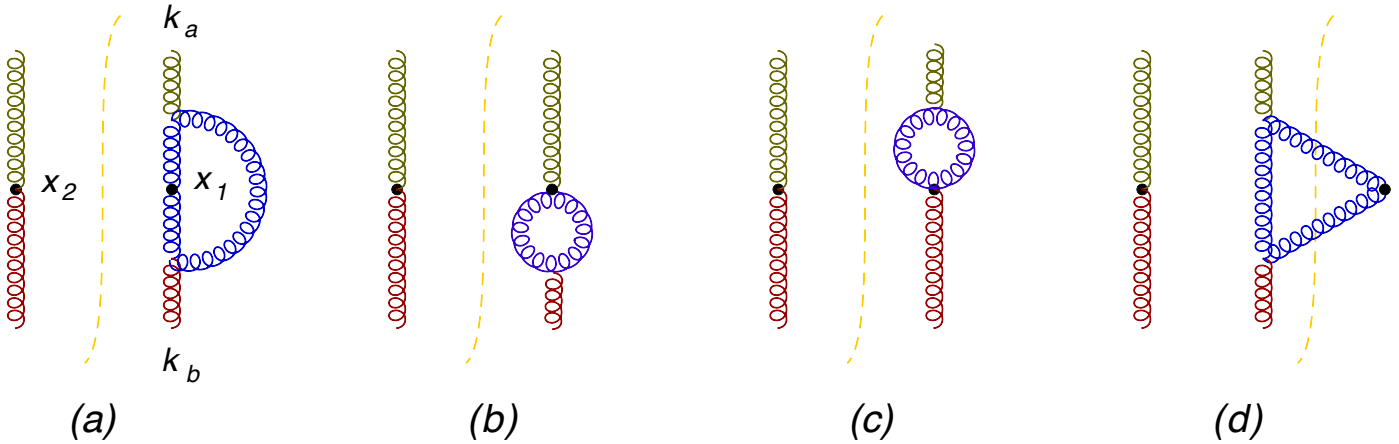
Calculation of coefficient function $\mathfrak{C}_1$ in the background field $\mathbb{A} = \bar{A} + \bar{B} + \bar{C}$

$$\int dz_2^- dz_{2\perp} dz_1^- dz_{1\perp} dw_1^+ dw_{1\perp} dw_2^+ dw_{2\perp} \frac{\alpha_s N_c}{2\pi} \mathfrak{C}_1(x_1, x_2; z_i^-, z_{i\perp}, w_i^+, w_{i\perp}; \sigma_p, \sigma_t)$$
$$\times F^{-i,a}(z_2^+, z_{2\perp}) F^{-j,a}(z_1^+, z_{1\perp}) F^{+i,a}(z_2^-, z_{2\perp}) F^{+j,a}(z_1^-, z_{1\perp})$$
$$= \frac{N_c^2 - 1}{16} g^4 \langle \tilde{F}^a_{\mu\nu} \tilde{F}^{a\mu\nu}(x_2) F^b_{\lambda\rho} F^{b\lambda\rho}(x_1) \rangle_{\mathbb{A} = \bar{A}+\bar{B}}$$
$$- \langle \hat{\mathcal{O}}^{ij,\sigma_p}(x_2^-, x_{2\perp}; x_1^-, x_{1\perp}) \hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2\perp}; x_1^+, x_{1\perp}) \rangle_{\mathbb{A} = \bar{A}+\bar{B}}$$

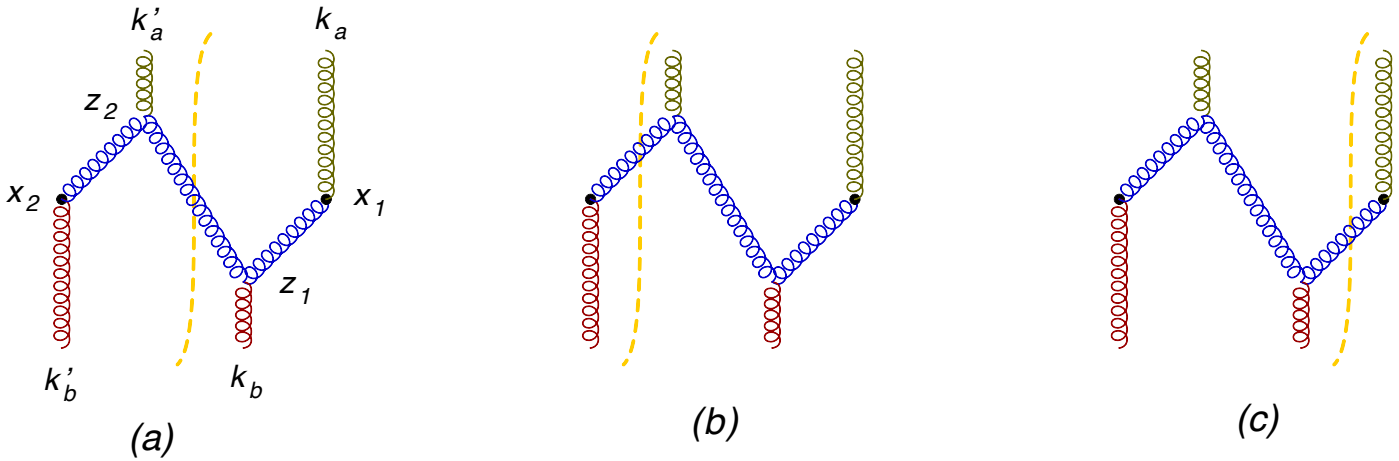
(for the purpose of calculating leading-twist coefficient function the "correction field" C can be neglected: $\mathbb{A} = \bar{A} + \bar{B}$)

Diagrams for $\langle \tilde{F}^a_{\mu\nu} \tilde{F}^{a\mu\nu}(x_2) F^b_{\lambda\rho} F^{b\lambda\rho}(x_1) \rangle_{\mathbb{A}}$ in background fields

"Virtual" diagrams



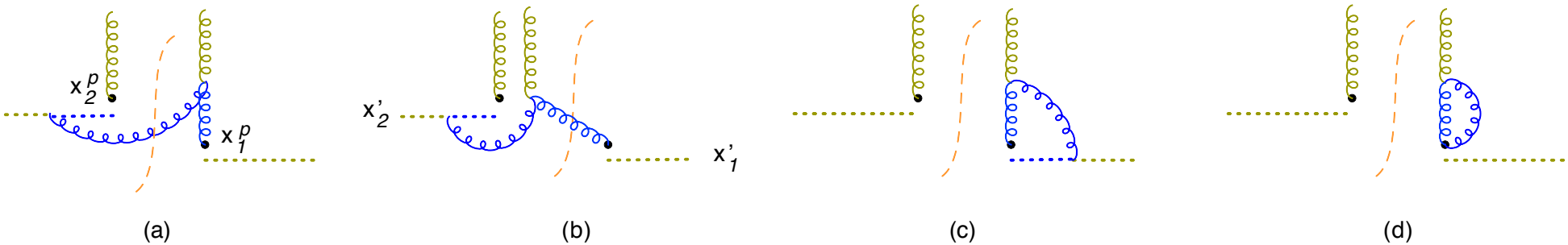
"Real" diagrams



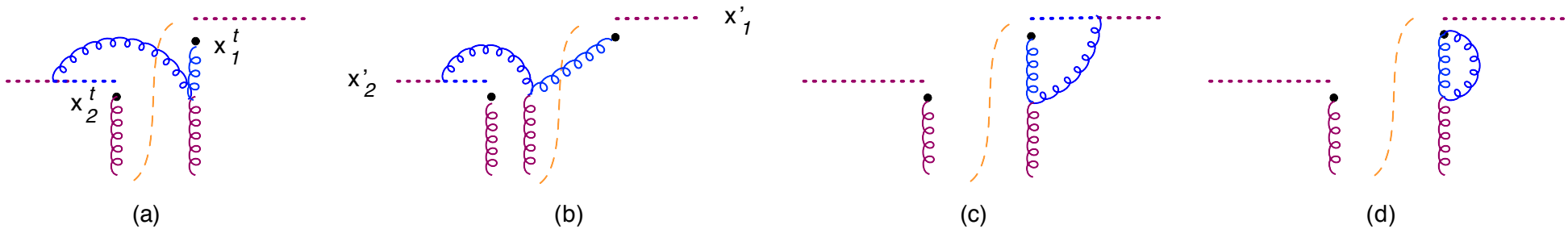
Diagrams for subtracted TMD matrix elements

"Projectile" TMD matrix elements.

The $e^{-i\frac{\beta}{\sigma_p}}$ regularization is depicted by point splitting: positions of $F's$ are separated from the beginnings of gauge links. (Violations of gauge invariance are power corrections).



"Target" TMD matrix elements. The $e^{-i\frac{\alpha}{\sigma_t}}$ regularization is depicted by point splitting.



Result for the coefficient function

Result of calculations:

$$\frac{1}{16}(N_c^2-1)\langle p_A', p_B'|g^2 F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2 F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle$$
$$= \int \mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p_A'}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p_B'}(t_i)\Psi_{p_B}(t_i)\Big[\mathcal{O}^{\sigma_p}_{ij}(x_2^-,x_{2\perp};z_1^-,x_{1\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+,x_{2\perp};x_1^+,x_{1\perp})$$
$$+\int dz_1^- dz_2^- dw_1^+ dw_2^+ \frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\ \mathcal{O}^{\sigma_p}_{ij}(z_2^-,x_{2\perp};z_1^-,x_{1\perp})\mathcal{O}^{ij;\sigma_t}(z_2^+,x_{2\perp};z_1^+,x_{1\perp})\ +\ O(\alpha_s^2)\Big]$$

This functional integral is not yet TMD formula since soft/Glauber gluons connect "projectile" and "target" gluons.

It is well known that Glauber gluons cancel and soft gluons form soft factors.

With rapidity-only cutoffs, soft factors are power corrections $\Rightarrow$ TMD formula

$$\frac{1}{16}(N_c^2-1)\langle p_A', p_B'|g^2 F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2 F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle$$
$$= \langle p_A'|\hat{\mathcal{O}}^{\sigma_p}_{ij}(x_2^-,x_{2\perp};x_1^-,x_{1\perp})|p_A\rangle\langle p_B'|\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+,x_{2\perp};x_1^+,x_{1\perp})|p_B\rangle$$
$$+\int dz_1^- dz_2^- dw_1^+ dw_2^+ \frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\ \langle p_A'|\hat{\mathcal{O}}^{\sigma_p}_{ij}(z_2^-,x_{2\perp};z_1^-,x_{1\perp})|p_A\rangle\langle p_B'|\hat{\mathcal{O}}^{ij;\sigma_t}(z_2^+,x_{2\perp};z_1^+,x_{1\perp})|p_B\rangle$$

Matching of coefficient function and TMDs

TMD evolution equations

$$\sigma_p \frac{d}{d\sigma_p} \hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})$$

$$= -\frac{\alpha_s N_c}{2\pi}\left[2\ln\frac{s x_{12_\perp}^2}{4} + \ln(-i\alpha'_a\sigma_p + \epsilon) + \ln(-i\alpha_a\sigma_p + \epsilon) + 2\gamma\right]\hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})$$

$$\sigma_t \frac{d}{d\sigma_t} \hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$

$$= -\frac{\alpha_s N_c}{2\pi}\left[2\ln\frac{s x_{12_\perp}^2}{4} + \ln(-i\beta'_b\sigma_t + \epsilon) + \ln(-i\beta_b\sigma_t + \epsilon) + 2\gamma\right]\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$

Matching of σ_p and σ_t evolutions $\Rightarrow$

$$\sigma_t \frac{d}{d\sigma_t}\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) = \frac{\alpha_s N_c}{2\pi}\left[2\ln\frac{s x_{12_\perp}^2}{4}\right.$$
$$\left. + \ln(-i\beta'_b\sigma_t + \epsilon) + \ln(-i\beta_b\sigma_t + \epsilon) + 2\gamma\right]\mathfrak{C}(x_1, x_2; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)$$

$$\sigma_p \frac{d}{d\sigma_p}\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) = \frac{\alpha_s N_c}{2\pi}\left[2\ln\frac{s x_{12_\perp}^2}{4}\right.$$
$$\left. + \ln(-i\alpha'_a\sigma_p + \epsilon) + \ln(-i\alpha_a\sigma_p + \epsilon) + 2\gamma\right]\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)$$

Matching of coefficient function and TMDs

The solution of this equations compatible with our first-order result is

$$\mathfrak{C}(x_{1\perp}, x_{2\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) \;=\; e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$

$\Rightarrow$ hadronic tensor is

$$W(\alpha'_a, \alpha_a, \beta'_b, \beta_b, x_{1\perp}, x_{2\perp}) \;=\; \int đ\alpha'_a đ\alpha_a đ\beta'_b đ\beta_b \; e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$
$$\times \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha'_a, \alpha_a, x_{2\perp}, x_{1\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2\perp}, x_{1\perp})|p_B\rangle \;+\ldots$$

Reminder

$$\mathfrak{C}_1(\alpha'_a, \alpha_a, \beta'_b, \beta_b; x_1, x_2; \sigma_p, \sigma_t)$$
$$= \ln^2\frac{x^2_{12\perp} s\sigma_p\sigma_t}{4} - \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2$$

Forward case ($\equiv$ particle production by gluon fusion)

Recall $\alpha_q \equiv x_A$, $\beta_q \equiv x_B$.

$$W(p_A,p_B;q) \;=\; \int db_\perp \; e^{i(q,b)_\perp} W(p_A,p_B;\alpha_q,\beta_q,b_\perp),$$

$$\begin{aligned} W(p_A,p_B;\alpha_q,\beta_q,b_\perp) \;=\; & \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\sigma_p}(\alpha_q,b_\perp;p_A)\mathcal{G}^{ij;\sigma_t}(\beta_q,b_\perp;p_B) \\ & \times \exp\left\{\frac{\alpha_s N_c}{2\pi}\left[\ln^2\frac{b_\perp^2 s\sigma_p\sigma_t}{4} - 2\Big(\ln\frac{\alpha_q}{\sigma_t}+\gamma\Big)\Big(\ln\frac{\beta_q}{\sigma_p}+\gamma\Big) + \frac{\pi^2}{2}\right]\right\} \\ & + \text{NLO terms} \sim O\big(\alpha_s^2\big) \;+\; \text{power corrections} \qquad (*) \end{aligned}$$

where $\mathcal{G}_{ij}^{\sigma_p}$, $\mathcal{G}_{ij}^{\sigma_t}$ are gluon TMDs:

$$\langle p_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(z^-,0^-,b_\perp)|p_A\rangle \;=\; -g^2\varrho^2\int_0^1 du\; u\mathcal{G}_{ij}^{\sigma_p}(u,b_\perp)\cos u\varrho z^-,$$

$$\langle p_B|\hat{\mathcal{O}}_{ij}^{\sigma_t}(z^-,0^-,b_\perp)|p_B\rangle \;=\; -g^2\varrho^2\int_0^1 du\; u\mathcal{G}_{ij}^{\sigma_t}(u,b_\perp)\cos u\varrho z^-.$$

Matching of coefficient function and TMDs

The r.h.s. of this evolution formula $(*)$ does not depend on cutoffs σ_p and σ_t as long as $\sigma_p \geq \tilde{\sigma}_p = \frac{4b_\perp^{-2}}{x_A s}$ and $\sigma_t \geq \tilde{\sigma}_t \equiv \frac{4b_\perp^{-2}}{x_B s}$. Thus, the result of double-log Sudakov evolution reads

$$W(p_A,p_B;x_A,x_B,b_\perp) \;=\; \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\tilde{\sigma}_p}(x_A,b_\perp;p_A)\mathcal{G}^{ij;\tilde{\sigma}_t}(x_B,b_\perp;p_B)$$

$$\times\; \exp\left\{ -\frac{\alpha_s N_c}{2\pi}\left[\left(\ln\frac{Q^2 b_\perp^2}{4} + 2\gamma \right)^2 - 2\gamma^2 - \frac{\pi^2}{2} \right] \right\} + \; O\left(\alpha_s^2\right) \text{ terms } + \text{ power corrections}$$

This result is universal for moderate x and small-x hadronic tensor. The difference lies in the continuation of the evolution beyond Sudakov region.

Double-log Sudakov evolution should stop at $x_B \tilde{\sigma}_p s \simeq b_\perp^{-2}$. After that:

- If $x_B \sim 1$ - DGLAP-type evolution from $\tilde{\sigma}_t = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{m_N^2}{s}$: summation of $\left(\alpha_s \ln \frac{b_\perp^{-2}}{m_N^2}\right)^n$
- If $x_B \ll 1$ - BFKL-type evolution from $\tilde{\sigma}_t = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{b_\perp^{-2}}{s}$: summation of $\left(\alpha_s \ln x_B\right)^n$

Interpolation between Sudakov and BFKL evolutions

The evolution equation for $\mathcal{G}(\beta_B, b_\perp) \equiv \mathcal{G}_i^i(\beta_B, b_\perp)$ valid at any $b_\perp$ and $\beta \equiv x_B$ is very complicated. It has three correct limits: DGLAP, Sudakov, and BK. (A. Tarasov and I.B.)

$$
\begin{aligned}
&\frac{d}{d\eta}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}_j^a(\beta_B, y_\perp)|p\rangle^\eta \qquad (5.5)\\
&= -\alpha_s\langle p|\mathrm{Tr}\Big\{\int \bar{d}^2k_\perp\theta\Big(1-\beta_B-\frac{k_\perp^2}{\sigma s}\Big)\Big[(x_\perp|\Big(U\frac{1}{\sigma\beta_B s+p_\perp^2}(U^\dagger k_k + p_k U^\dagger)\\
&\times\frac{\sigma\beta_B s g_{\mu i}-2k_\mu^\perp k_i}{\sigma\beta_B s+k_\perp^2} - 2k_\mu^\perp g_{ik}U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger - 2g_{\mu k}U\frac{p_i}{\sigma\beta_B s+p_\perp^2}U^\dagger\Big)\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)\\
&\times(k_\perp|\mathcal{F}^l\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)\Big(\frac{\sigma\beta_B s\delta_j^\mu - 2k_\perp^\mu k_j}{\sigma\beta_B s+k_\perp^2}(k_l U + U p_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger\\
&-2k_\perp^\mu g_{jl}U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger - 2\delta_l^\mu U\frac{p_j}{\sigma\beta_B s+p_\perp^2}U^\dagger\Big)|y_\perp)\\
&+2(x_\perp|\tilde{\mathcal{F}}_i\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)(k_\perp|\mathcal{F}^l\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)\Big(\frac{k_j}{k_\perp^2}\frac{\sigma\beta_B s+2k_\perp^2}{\sigma\beta_B s+k_\perp^2}(k_l U+U p_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger\\
&+2U\frac{g_{jl}}{\sigma\beta_B s+p_\perp^2}U^\dagger - 2\frac{k_l}{k_\perp^2}U\frac{p_j}{\sigma\beta_B s+p_\perp^2}U^\dagger\Big)|y_\perp)\\
&+2(x_\perp|\Big(U\frac{1}{\sigma\beta_B s+p_\perp^2}(U^\dagger k_k + p_k U^\dagger)\frac{k_i}{k_\perp^2}\frac{\sigma\beta_B s+2k_\perp^2}{\sigma\beta_B s+k_\perp^2} + 2U\frac{g_{ik}}{\sigma\beta_B s+p_\perp^2}U^\dagger\\
&-2U\frac{p_i}{\sigma\beta_B s+p_\perp^2}U^\dagger\frac{k_k}{k_\perp^2}\Big)\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)(k_\perp|\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|y_\perp)\Big]\\
&+2\tilde{\mathcal{F}}_i(\beta_B, x_\perp)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l + U_l)(2\delta_m^k\delta_j^l - g_{jm}g^{kl})U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger|y_\perp)\\
&+2(x_\perp|U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger(2\delta_i^k\delta_m^l - g_{im}g^{kl})(i\partial_k - U_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2}|x_\perp)\mathcal{F}_j(\beta_B, y_\perp)\\
&-4\int\frac{\bar{d}^2k_\perp}{k_\perp^2}\Big[\theta\Big(1-\beta_B-\frac{k_\perp^2}{\sigma s}\Big)\tilde{\mathcal{F}}_i\Big(\beta_B+\frac{k_\perp^2}{\sigma s}, x_\perp\Big)\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\sigma s}, y_\perp\Big)e^{i(k,x-y)_\perp}\\
&-\frac{\sigma\beta_B s}{\sigma\beta_B s+k_\perp^2}\tilde{\mathcal{F}}_i(\beta_B, x_\perp)\mathcal{F}_j(\beta_B, y_\perp)\Big]\Big\}|p\rangle^\eta + O(\alpha_s^2)
\end{aligned}
$$

Interpolation between Sudakov and BFKL evolutions

However, if we want to describe transition between Sudakov and BFKL evolutions, the equation simplifies

$$\sigma\frac{d}{d\sigma}\mathcal{G}_\sigma(\beta_q,b_\perp) \;=\; \frac{\alpha_s N_c}{\pi^2}\int dz_\perp\left[\frac{e^{i\beta_q s\sigma\frac{(b-z)_\perp^2}{4}}}{(b-z)_\perp^2}\mathcal{G}_\sigma(\beta_q,z_\perp)-\frac{(b,z)_\perp}{(b-z)_\perp^2 z_\perp^2}\mathcal{G}_\sigma(\beta_q,b_\perp)\right]$$

It has two correct limits:

- Sudakov logs at $\sigma\gg\frac{b_\perp^{-2}}{\beta_q s}=\frac{q_\perp^2}{x_B s}\sim\frac{q_\perp^2}{Q^2}$

$$\sigma\frac{d}{d\sigma}\mathcal{G}_\sigma(\beta_q,b_\perp) \;=\; \frac{\alpha_s N_c}{\pi^2}\mathcal{G}_\sigma(\beta_q,b_\perp)\int dz\left[\frac{e^{i\beta_q s\sigma\frac{(b-z)_\perp^2}{4}}}{(b-z)_\perp^2}-\frac{(b,z)_\perp}{(b-z)_\perp^2 z_\perp^2}\right]$$

$$=\;-\frac{\alpha_s N_c}{\pi^2}\left[\ln\frac{\sigma b_\perp^2}{4}+\ln(-i\beta_q+\epsilon)+\gamma_E\right]\mathcal{G}_\sigma(\beta_q,x)$$

- BFKL evolution at $\sigma\ll\frac{b_\perp^{-2}}{\beta_q s}=\frac{q_\perp^2}{x_B s}$

$$\sigma\frac{d}{d\sigma}\mathcal{G}_\sigma(\beta_q,b_\perp) \;=\; \frac{\alpha_s N_c}{\pi^2}\int dz_\perp\left[\frac{1}{(b-z)_\perp^2}\mathcal{G}_\sigma(\beta_q,z_\perp)-\frac{(b,z)_\perp}{(b-z)_\perp^2 z_\perp^2}\mathcal{G}_\sigma(\beta_q,b_\perp)\right]$$

- Hope: use this interpolation equation to obtain complete result for $W(p_A,p_B;x_A,x_B,b_\perp)$

Conclusions

1. Rapidity-only TMD factorization works for both small and moderate x:
   - Full list of $\frac{1}{Q^2}$ power corrections for DY and SIDIS.
   - Back-of-the-envelope estimates of DY corrections seems to agree with exp. data.
   - Rapidity-only evolution with BLM prescription for running coupling gives the same universal formula for Sudakov double logs at both small and moderate x for both quark and gluon TMDs.
   - Rapidity factorization at the one-loop level gives Sudakov-type double logs for both small and intermediate x_B
2. Outlook: connect Sudakov double logs with BFKL and/or DGLAP evolutions

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Thank you for attention!