Gluon splitting at small-x: unified derivation for the JIMWLK, DGLAP and CSS equations

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Bridging TMD frameworks - Trento ECT* - March 10, 2026

with E. Iancu, F. Salazar and F. Yuan / based on 2510.08454 (JHEP)

Motivations

- TMD factorisation at small x:

  1. non-linear QCD evolution goes beyond the usual moderate-x framework
     $\Rightarrow$ conceptual interest for the TMD communities.

  2. pQCD motivated ansatz for the initial condition to CSS **including nucleus targets**.
     E.g: MV model, with few parameters (Q_s, etc) be eventually fitted to data.
     $\Rightarrow$ phenomenological interest at the future EIC.

- For this workshop: we should be able to speak the same language as TMD factorization theorems have a well-defined $x \ll 1$ limit.

This talk: a NLO study of the Weisacker-Williams gluon TMD

- Small-x limit dominated by gluon contribution.
  (and sea quarks, see also talks by Jamal and Swaleha tomorrow)

- To build bridges with TMD factorization at moderate $x \Rightarrow$ NLO.

- Our strategy:
  1. We consider a process which is known to factorize at LO at small x.
  2. We compute the NLO corrections to the **observable** (cross-section)
  3. $\Rightarrow$ would-be divergences regulated by physical scales.

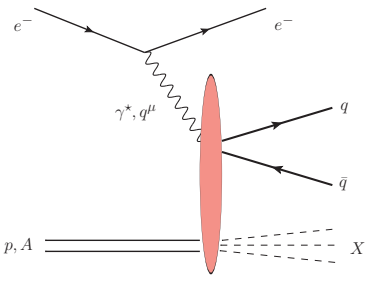
- NOT a NLO calculation of the Weizsäcker-Williams operator only.
  Balitsky, Tarasov, 1505.02151, Xiao, Yuan, Zhou, 1703.06163, Mukherjee, Skokov, Tarasov, Tiwari, 2311.16402
  see also talk by Shaswat on Thursday.

Inclusive dijet production in DIS at small-x

$\Rightarrow$ a "standard candle" of the saturated regime of QCD.

$\Rightarrow$ sensitive to the Weizsäcker-Williams gluon TMD in the back-to-back limit.

$\Rightarrow$ azimuthal correlations sensitive to linearly polarised gluon.
Boer, Brodksy, Mulders, Pisano, 1011.4225



Back-to-back limit

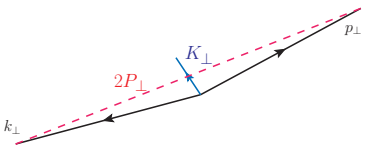
- $\boldsymbol{k}_\perp$ quark transverse momentum, $\boldsymbol{p}_\perp$ antiquark transverse momentum
- Hard scale: dijet relative transverse momentum

$$\boldsymbol{P}_\perp = z_2 \boldsymbol{k}_\perp - z_1 \boldsymbol{p}_\perp \,, \quad z_{1,2} = k^+_{q,\bar{q}}/q^+$$

- Soft scale: dijet transverse momentum imbalance.

$$\boldsymbol{K}_\perp = \boldsymbol{k}_\perp + \boldsymbol{p}_\perp$$

- Back-to-back limit $||\boldsymbol{P}_\perp||| \gg ||\boldsymbol{K}_\perp||$



- No hierarchy between $Q^2 = -q^2$ and $\boldsymbol{P}^2_\perp$: applies in the photo-production limit (eg UPCs) so long as a hard jet pair is produced.

Structure of the factorization theorem at moderate x

See del Castillo, Echevarria, Makris, Scimemi, 2008.07531

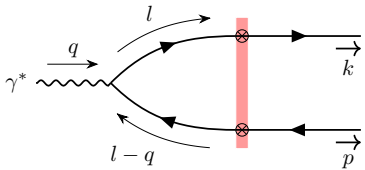
- E.g. contributions from unpolarised gluons

$$\frac{\mathrm{d}\sigma^{\gamma^* g\to \mathrm{dijet}+X}}{\mathrm{d}\boldsymbol{P}_\perp \mathrm{d}\boldsymbol{K}_\perp \mathrm{d}\eta_1 \mathrm{d}\eta_2} = H(\boldsymbol{P}_\perp,\mu)\int \mathrm{d}^2\boldsymbol{b}\; e^{i\boldsymbol{K}_\perp\cdot\boldsymbol{b}}\; f_{WW}(x,\boldsymbol{b},\mu,\zeta)$$
$$\times\, C_q(\boldsymbol{b},R,\mu) J_q(p_\perp R,\mu) C_{\bar q}(\boldsymbol{b},R,\mu) J_{\bar q}(p_\perp R,\mu)$$

- Similar expression for linearly polarised gluon contribution.
- f_{WW}: unpolarised WW gluon TMD, includes CSS evolution resumming $\ln^2(P_\perp/K_\perp)$ and $\ln(P_\perp/K_\perp)$.
- $C_{q,\bar q}$: collinear-soft function. Resums $\ln(1/R)\ln(P_\perp/K_\perp)$.
- $J_{q,\bar q}$: jet function. Resums $\ln(1/R)$.
- **Can we start from the $x\to 0$ limit and obtain a similar expression?**

Color Glass Condensate approach to the $x \to 0$ limit

- We work in the dipole picture of DIS, large q^+. Nucleus moves along the minus direction.



- CGC effective vertex, projectile light-cone gauge $A^+ = 0$
  $= (2\pi)\delta(k^- - l^-)\gamma^- \int \mathrm{d}^2\boldsymbol{x}_\perp e^{-i(\boldsymbol{k}_\perp - \boldsymbol{l}_\perp)\boldsymbol{x}_\perp} V_{ij}(\boldsymbol{x}_\perp)$

- Multiple gluon interactions with the target resummed by Wilson lines:

$$V(\boldsymbol{x}_\perp) \equiv T \exp\left(ig \int_{-\infty}^{\infty} \mathrm{d}x^+ A_a^-(x^+, \boldsymbol{x}_\perp) t^a \right)$$

See also talks by Yuri and Dionisi this morning.

LO cross-section in the CGC

- Differential cross-section at leading order:

$$\left.\frac{\mathrm{d}\sigma^{\gamma^*_\lambda + A \to q\bar{q}+X}}{\mathrm{d}^2\boldsymbol{k}_\perp \mathrm{d}^2\boldsymbol{p}_\perp \mathrm{d}\eta_1 \mathrm{d}\eta_2}\right|_{\mathrm{LO}} = \frac{\alpha_{\mathrm{em}} e_f^2 N_c}{(2\pi)^6} \int \mathrm{d}^8 \boldsymbol{X}_\perp e^{-i\boldsymbol{k}_\perp \boldsymbol{r}_{xx'}} e^{-i\boldsymbol{p}_\perp \boldsymbol{r}_{yy'}} \Xi_{\mathrm{LO}}(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp; \boldsymbol{y}'_\perp, \boldsymbol{x}'_\perp) \mathcal{R}^{\lambda}_{\mathrm{LO}}(\boldsymbol{r}_{xy}, \boldsymbol{r}'_{xy})$$

- Convolution between perturbative factor describing the $\gamma^\star \to q\bar{q}$ splitting...

$$\mathcal{R}^{\mathrm{L}}_{\mathrm{LO}}(\boldsymbol{r}_{xy}, \boldsymbol{r}'_{xy}) = 8 z_1^3 z_2^3 Q^2 K_0(\bar{Q} r_{xy}) K_0(\bar{Q} r_{xy'})$$

- ... and a color structure describing the interaction of $q\bar{q}$ with the dense target

$$\Xi_{\mathrm{LO}}(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp; \boldsymbol{x}'_\perp, \boldsymbol{y}'_\perp) = \left\langle \underbrace{Q(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp; \boldsymbol{y}'_\perp, \boldsymbol{x}'_\perp)}_{\textit{quadrupole}} - D(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp) - \underbrace{D(\boldsymbol{y}'_\perp, \boldsymbol{x}'_\perp)}_{\textit{dipole}} + 1 \right\rangle_Y$$

$$\text{Dipole:} \qquad D(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp) = \frac{1}{N_c} \mathrm{Tr}(V(\boldsymbol{x}_\perp) V^\dagger(\boldsymbol{y}_\perp))$$

- Leading power in x, **but all orders in** $K_\perp/P_\perp$.

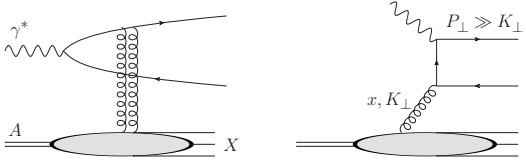
The back-to-back limit at LO

- Power expansion of LO cross-section: keep the LP term in $K_\perp/P_\perp$ and $Q_s/P_\perp$.
- $\Rightarrow$ TMD factorization Dominguez, Marquet, Xiao, Yuan, 1101.0715

$$\left.\frac{\mathrm{d}\sigma^{\gamma^*\to q\bar{q}+X}}{\mathrm{d}^2\boldsymbol{P}_\perp \mathrm{d}^2\boldsymbol{K}_\perp \mathrm{d}\eta_1 \mathrm{d}\eta_2}\right|_{\mathrm{LO}} = \alpha_s \mathcal{H}^{ij}_{\mathrm{LO}}(\boldsymbol{P}_\perp) \int \frac{\mathrm{d}^2\boldsymbol{b}_\perp \mathrm{d}^2\boldsymbol{b}'_\perp}{(2\pi)^4} e^{-i\boldsymbol{K}_\perp\cdot(\boldsymbol{b}_\perp-\boldsymbol{b}'_\perp)} \mathcal{W}^{ij}(\boldsymbol{b}_\perp,\boldsymbol{b}'_\perp) + \mathcal{O}\left(\frac{q_\perp}{P_\perp}\right) + \mathcal{O}\left(\frac{Q_s}{P_\perp}\right)$$

- LO hard factor **identical** to moderate x case. $\boldsymbol{K}_\perp$-dependence in the WW correlator.

$$\mathcal{W}^{ij}(\boldsymbol{b}_\perp,\boldsymbol{b}'_\perp) \equiv -\frac{2}{\alpha_s}\left\langle \frac{1}{N_c}\mathrm{Tr}\left[\partial^i V^\dagger(\boldsymbol{b}_\perp) V(\boldsymbol{b}'_\perp) \partial^j V^\dagger(\boldsymbol{b}'_\perp) V(\boldsymbol{b}_\perp)\right]\right\rangle_x$$



Gluon TMDs at small x

WW gluon TMD

$$\mathcal{F}_{WW}(x,\boldsymbol{K}_\perp) = \int \frac{\mathrm{d}\xi^+\mathrm{d}^2\boldsymbol{\xi}}{4\pi^3 P^-} e^{-ixP^-\xi^+ + i\boldsymbol{K}_\perp\cdot\boldsymbol{\xi}} \langle P|\mathrm{Tr} F^{-i}(\xi^+,\boldsymbol{\xi}) W_{\sqsupset}(\xi^\mu,0) F^{-i}(0,0) W^\dagger_{\sqsupset}(\xi^\mu,0)|P\rangle$$

$\Rightarrow$ Limit $x \to 0$, projectile light-cone gauge

$$\mathcal{F}_{WW}(x,\boldsymbol{K}_\perp) = -\frac{2}{\alpha_s}\int \frac{\mathrm{d}^2\boldsymbol{b}\mathrm{d}^2\boldsymbol{b}'}{(2\pi)^4} e^{-i\boldsymbol{K}_\perp\cdot(\boldsymbol{b}-\boldsymbol{b}')} \frac{1}{N_c} \left\langle \mathrm{Tr}\left[\partial^i V^\dagger(\boldsymbol{b}) V(\boldsymbol{b}') \partial^j V^\dagger(\boldsymbol{b}') V(\boldsymbol{b})\right]\right\rangle_x$$

Dipole gluon TMD

$$\mathcal{F}_D(x,\boldsymbol{K}_\perp) = \int \frac{\mathrm{d}\xi^+\mathrm{d}^2\boldsymbol{\xi}}{4\pi^3 P^-} e^{-ixP^-\xi^+ + i\boldsymbol{K}_\perp\cdot\boldsymbol{\xi}} \langle P|\mathrm{Tr} F^{-i}(\xi^+,\boldsymbol{\xi}) W_{\sqsubset}(\xi^\mu,0) F^{-i}(0,0) W^\dagger_{\sqsupset}(\xi^\mu,0)|P\rangle$$

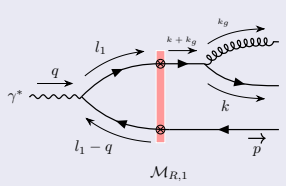
$\Rightarrow$ Limit $x \to 0$, projectile light-cone gauge

$$\mathcal{F}_D(x,\boldsymbol{K}_\perp) = \frac{2\boldsymbol{K}_\perp^2}{\alpha_s}\int \frac{\mathrm{d}^2\boldsymbol{b}\mathrm{d}^2\boldsymbol{b}'}{(2\pi)^4} e^{-i\boldsymbol{K}_\perp\cdot(\boldsymbol{b}-\boldsymbol{b}')} \frac{1}{N_c} \left\langle \mathrm{Tr}\left[V(\boldsymbol{b}) V^\dagger(\boldsymbol{b}')\right]\right\rangle_x$$

NLO computation: real amplitudes

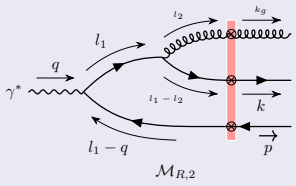
- Consider the real NLO corrections first.
- Gluon emission either before or after the shockwave.

Real diagrams



$\mathcal{M}_{R,1}$

$+(q \longleftrightarrow \bar{q})$



$\mathcal{M}_{R,2}$

$+(q \longleftrightarrow \bar{q})$

- First computed by Ayala, Hentschinski, Jalilian-Marian, Tejeda-Yeomans, 1701.07143.
- Now, need to isolate the LP contribution at the cross-section level.

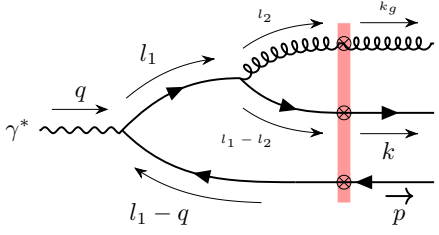
Gluon emission before the shock-wave

- 4 transverse momentum scales: $\boldsymbol{P}_\perp$, $\boldsymbol{K}_\perp$, $\boldsymbol{\ell}$, and Q_s.
- $\boldsymbol{\ell}$ momentum transferred from the target.

$$\boldsymbol{\ell} \equiv \underbrace{\boldsymbol{k}_\perp + \boldsymbol{p}_\perp}_{\equiv \boldsymbol{K}_\perp} + \boldsymbol{k}_{g\perp}$$

.

- $\boldsymbol{\ell}$ (or $\boldsymbol{k}_{g\perp}$) and z_g integrated over.
- LP term is controlled by
  1. $||\boldsymbol{\ell}|| \lesssim K_\perp \ll P_\perp$
  2. gluon with $z_g \sim K_\perp^2 / P_\perp^2$
- Technical. **My goal: explain the emergence of the dipole gluon TMD.**



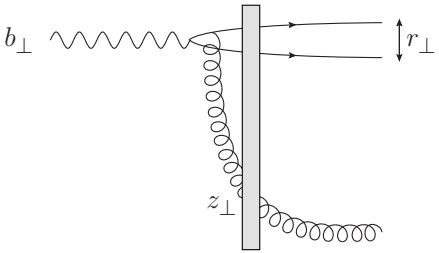
$\mathcal{M}_{R,2}$

Gluon emission before the shockwave

- $q\bar{q}$ transverse size conjugate to $\boldsymbol{P}_\perp$.
- $q\bar{q}$ dipole-gluon transverse size conjugate to $\boldsymbol{\ell}$.
- Colour structure reduces to a gluon-gluon dipole.

$$U^{ac}_{\boldsymbol{z}_\perp} V_{\boldsymbol{x}_\perp} t^c V^\dagger_{\boldsymbol{y}_\perp} - t^a \simeq U^{ac}_{\boldsymbol{z}_\perp} V_{\boldsymbol{b}} t^c V^\dagger_{\boldsymbol{b}} - t^a = (U_{\boldsymbol{z}_\perp} U^\dagger_{\boldsymbol{b}})^{ac} t^c - t^a$$

- Up to $K^2_\perp/P^2_\perp$ power corrections:
$\mathcal{M}_B = g\epsilon^{\bar{\lambda}*,n}\epsilon^{\lambda,j} H^{mj}_{\sigma\bar{\sigma}}(\boldsymbol{P}_\perp, z_1, z_2)\mathcal{G}^{mn}_B(\boldsymbol{K}_\perp, \boldsymbol{\ell}, \mathcal{M}_g)$



$$\mathcal{G}^{mn}_B(\boldsymbol{K}_\perp, \boldsymbol{\ell}, \mathcal{M}_g) = \int \frac{\mathrm{d}^2\boldsymbol{k}_\perp}{(2\pi)^2} \mathcal{C}^a_B(\boldsymbol{k}_\perp, \boldsymbol{\ell} - \boldsymbol{k}_\perp) \frac{(\boldsymbol{k}_\perp - \boldsymbol{K}_\perp)^m (\boldsymbol{k}_\perp - \boldsymbol{K}_\perp)^n - \frac{1}{2}(\boldsymbol{k}_\perp - \boldsymbol{K}_\perp)^2 \delta^{mn}}{(\boldsymbol{k}_\perp - \boldsymbol{K}_\perp)^2 + \mathcal{M}_g^2},$$

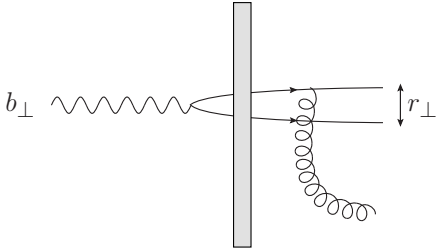
$$\mathcal{C}^a_B(\boldsymbol{\ell}, \boldsymbol{\ell}') = \int \mathrm{d}^2\boldsymbol{b}\, \mathrm{d}^2\boldsymbol{z}_\perp\, e^{-i\boldsymbol{\ell}\cdot\boldsymbol{b} - i\boldsymbol{\ell}'\cdot\boldsymbol{z}} \left[(U_{\boldsymbol{z}_\perp} U^\dagger_{\boldsymbol{b}})^{ac} t^c - t^a \right].$$

Gluon effective virtuality: $\mathcal{M}_g^2 \equiv z_g \left(Q^2 + \frac{P^2_\perp}{z_1 z_2} \right)$

Gluon emission after the shock-wave

- Assume here $z_g \ll k_{g\perp}/P_\perp$: simplify the combination of emissions by q and $\bar{q}$.
- Colour structure after small $r_\perp$ expansion:

$$\left[V_{\boldsymbol{b}}\partial^m V_{\boldsymbol{b}}^\dagger, t^a\right] = -\left(U_{\boldsymbol{b}}\partial^m U_{\boldsymbol{b}}^\dagger\right)^{ac} t^c$$



$$\mathcal{M}_A = g\boldsymbol{\epsilon}^{\bar{\lambda}*,n}\boldsymbol{\epsilon}^{\lambda,j} H^{mj}_{\sigma\bar{\sigma}}(\boldsymbol{P}_\perp, z_1, z_2)\mathcal{G}_A^{mn}(\boldsymbol{K}_\perp, \boldsymbol{\ell}),$$

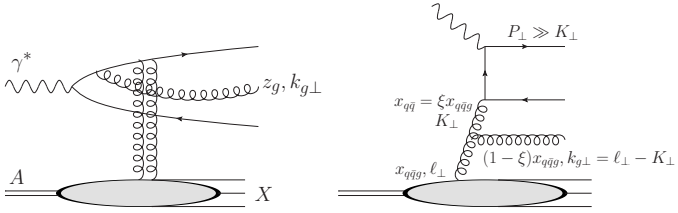
$$\mathcal{G}_A^{mn}(\boldsymbol{K}_\perp, \boldsymbol{\ell}) \equiv \frac{\boldsymbol{k}^n_{g\perp}}{k^2_{g\perp}}\mathcal{C}_A^{a,m}(\boldsymbol{\ell}), \quad \mathcal{C}_A^{a,m}(\boldsymbol{\ell}) = \int \mathrm{d}^2\boldsymbol{b}\; e^{-i\boldsymbol{\ell}\cdot\boldsymbol{b}}\left(U_{\boldsymbol{b}}\partial^m U_{\boldsymbol{b}}^\dagger\right)^{ac} t^c\,.$$

Final result for the real amplitude

$$\mathcal{M}_B = g\epsilon^{\bar{\lambda}*,n}\epsilon^{\lambda,j}H^{mj}_{\sigma\bar{\sigma}}(\boldsymbol{P}_\perp, z_1, z_2)\mathcal{G}^{mn}_B(\boldsymbol{K}_\perp, \ell, \mathcal{M}_g)\,, \quad \mathcal{M}_A = g\epsilon^{\bar{\lambda}*,n}\epsilon^{\lambda,j}H^{mj}_{\sigma\bar{\sigma}}(\boldsymbol{P}_\perp, z_1, z_2)\mathcal{G}^{mn}_A(\boldsymbol{K}_\perp, \ell)\,,$$

- TMD factorization not yet manifest, as $\mathcal{M}_g$ depends on the hard scale.
  Hauksson, Iancu, Mueller, Triantafyllopoulos, Wei, 2402.14748 See Dionisi's talk
- When integrating over z_g: $z_g \to \xi =$ − longitudinal momentum fraction wrt parent.

$$z_g = \frac{\xi}{1-\xi}\frac{k^2_{g\perp}}{Q^2 + P^2_\perp/(z_1 z_2)}$$



Contribution to the real cross-section

- Compute the amplitude squared $\mathrm{d}\sigma \propto |\mathcal{M}_B + \mathcal{M}_A|^2$.
- Same hard factor $\alpha_s \mathcal{H}^{mn}_{\mathrm{LO}}$ as the LO cross-section.
- TMD factorization at NLO:

$$\left.\frac{\mathrm{d}\sigma^{\gamma^*_T A \to q\bar{q}(g)X}_{\mathcal{R}}}{\mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}^2\boldsymbol{P}_\perp \mathrm{d}^2\boldsymbol{K}_\perp}\right|_{\mathcal{R}} = \alpha_s \mathcal{H}^{mn}_{\mathrm{LO}} \times \alpha_s \Delta\mathcal{W}^{mn}_{\mathcal{R}}(x, \boldsymbol{K}_\perp)\,.$$

- $\Delta\mathcal{W}^{mn}_{\mathcal{R}}(x, \boldsymbol{K}_\perp)$ involves two new operators

$$\mathcal{F}^{(0)}_D(x,\ell) \equiv \frac{C_F \ell^2}{2\pi^2\alpha_s} \int \frac{\mathrm{d}^2\boldsymbol{r}_\perp}{(2\pi)^2} \frac{e^{i\boldsymbol{q}_\perp\cdot\boldsymbol{r}_\perp}}{N_c^2-1} \left\langle \mathrm{Tr}\left[U_{\boldsymbol{B}_\perp+\frac{\boldsymbol{r}_\perp}{2}} U^\dagger_{\boldsymbol{B}_\perp-\frac{\boldsymbol{r}_\perp}{2}}\right]\right\rangle_X$$

$$\mathcal{Z}^{n,(0)}(x,\boldsymbol{q}_1,\boldsymbol{q}_2) \equiv \frac{iC_F \boldsymbol{q}_2^2}{2\pi^2\alpha_s} \int \frac{\mathrm{d}^2\boldsymbol{r}_1 \mathrm{d}^2\boldsymbol{r}_2}{(2\pi)^4} \frac{e^{i\boldsymbol{q}_1\cdot\boldsymbol{r}_1 + i\boldsymbol{q}_2\cdot\boldsymbol{r}_2}}{N_c^2-1} \left\langle \mathrm{Tr}\left[U_{\boldsymbol{B}_\perp+\boldsymbol{r}_2/2} U^\dagger_{\boldsymbol{B}_\perp-\boldsymbol{r}_2/2} \partial^n U_{\boldsymbol{B}_\perp+\boldsymbol{r}_2/2+\boldsymbol{r}_1} U^\dagger_{\boldsymbol{B}_\perp+\boldsymbol{r}_2/2+\boldsymbol{r}_1}\right]\right\rangle_x$$

Contribution to the real cross-section

$$\alpha_s \Delta \mathcal{W}_{\mathcal{R}}^{mn}(x, \boldsymbol{K}_\perp) = \frac{\alpha_s N_c}{\pi^2} \int \mathrm{d}^2 \boldsymbol{k}_{g\perp} \int_x^{1-\xi_0(k_{g\perp})} \mathrm{d}\xi \Bigg\{ \frac{1}{\xi(1-\xi)} \frac{1}{\boldsymbol{k}_{g\perp}^2} \mathcal{W}^{mn,(0)}\left(\frac{x}{\xi}, \boldsymbol{K}_\perp + \boldsymbol{k}_{g\perp}\right)$$

$$+ \frac{1-\xi}{4\xi} \frac{1}{(\boldsymbol{K}_\perp + \boldsymbol{k}_{g\perp})^2} \Bigg[\frac{\xi^2 (K_\perp^2 - k_{g\perp}^2)^2 \delta^{mn}}{\Big((1-\xi)K_\perp^2 + \xi k_{g\perp}^2\Big)^2}$$

$$+ \frac{4K_\perp^2}{(1-\xi)K_\perp^2 + \xi k_{g\perp}^2} \left(\frac{\boldsymbol{k}_{g\perp}^m \boldsymbol{k}_{g\perp}^n}{k_{g\perp}^2} + \frac{\boldsymbol{K}_\perp^m \boldsymbol{K}_\perp^n}{K_\perp^2} - \frac{(\boldsymbol{k}_{g\perp}^m \boldsymbol{K}_\perp^n + \boldsymbol{k}_{g\perp}^n \boldsymbol{K}_\perp^m)(\boldsymbol{k}_{g\perp} \cdot \boldsymbol{K}_\perp)}{k_{g\perp}^2 K_\perp^2} \right) \Bigg] \mathcal{F}_D^{(0)}\left(\frac{x}{\xi}, \boldsymbol{K}_\perp + \boldsymbol{k}_{g\perp}\right)$$

$$+ \frac{1}{2} \Bigg[\int \frac{\mathrm{d}^2 \boldsymbol{k}_\perp}{\boldsymbol{k}_\perp^2} \frac{1}{\xi} \mathcal{Z}^{n,(0)}\left(\frac{x}{\xi}, \boldsymbol{K}_\perp + \boldsymbol{k}_{g\perp}, \boldsymbol{k}_\perp\right)$$

$$\times \frac{\boldsymbol{k}_{g\perp}^i (\boldsymbol{K}_\perp - \boldsymbol{k}_\perp)^2}{k_{g\perp}^2 \Big((1-\xi)(\boldsymbol{K}_\perp - \boldsymbol{k}_\perp)^2 + \xi k_{g\perp}^2\Big)} \left(\delta^{mi} - \frac{2(\boldsymbol{K}_\perp - \boldsymbol{k}_\perp)^m (\boldsymbol{K}_\perp - \boldsymbol{k}_\perp)^i}{(\boldsymbol{K}_\perp - \boldsymbol{k}_\perp)^2} \right) + \overline{(m \leftrightarrow n)} \Bigg] \Bigg\}.$$

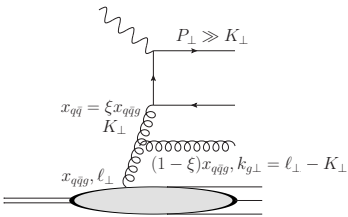
The various limits and possible scale hierarchies

Cross-check this result and physically interpret the various terms

- The limit $\xi \to 0$:
  1. without any assumption on $K_\perp$ wrt ℓ wrt Q_s: JIMWLK
  2. assuming $\ell \gg Q_s$: BFKL
- The limit $\xi \to 1$: CSS evolution
- Regime $K_\perp \gg \ell_\perp$:
  1. without any assumption on ξ: one DGLAP step of the gluon PDF
  2. $\xi \to 0$: DLA approximation (common limit to BFKL and DGLAP)
- Regime $K_\perp$, $\ell_\perp \gg Q_s$: k_t- factorisation (dilute limit)

Limit $\xi \to 0$: (real) JIMWLK evolution

- LLx dominated by strong ordering $1 \gg \xi \gg x$.
- Set $\xi = 0$ everywhere but in the $1/\xi$ divergence.



$$\alpha_s \Delta \mathcal{W}^{mn}_{\mathcal{R}}(x, \boldsymbol{K}_\perp, P_\perp^2) \underset{x \ll 1}{\simeq} \frac{\alpha_s N_c}{\pi^2} \ln\left(\frac{1}{x}\right) \int \mathrm{d}^2\boldsymbol{\ell} \left\{ \frac{\mathcal{W}^{mn,(0)}(x,\boldsymbol{\ell})}{(\boldsymbol{\ell}-\boldsymbol{K}_\perp)^2} + \frac{1}{\ell_\perp^2} \mathcal{F}_D^{(0)}(x,\boldsymbol{\ell}) \right.$$
$$\times \left[\frac{(\boldsymbol{\ell}-\boldsymbol{K}_\perp)^m (\boldsymbol{\ell}-\boldsymbol{K}_\perp)^n}{(\boldsymbol{\ell}-\boldsymbol{K}_\perp)^2} + \frac{\boldsymbol{K}_\perp^m \boldsymbol{K}_\perp^n}{K_\perp^2} - \frac{((\boldsymbol{\ell}-\boldsymbol{K}_\perp)^m \boldsymbol{K}_\perp^n + (\boldsymbol{\ell}-\boldsymbol{K}_\perp)^n \boldsymbol{K}_\perp^m)((\boldsymbol{\ell}-\boldsymbol{K}_\perp)\cdot\boldsymbol{K}_\perp)}{(\boldsymbol{\ell}-\boldsymbol{K}_\perp)^2 K_\perp^2} \right]$$
$$\left. - \left[\int \frac{\mathrm{d}^2\boldsymbol{k}_\perp}{k_\perp^2} \mathcal{Z}^{n,(0)}(x,\boldsymbol{\ell},\boldsymbol{k}_\perp) \frac{(\boldsymbol{\ell}-\boldsymbol{K}_\perp)^i}{(\boldsymbol{\ell}-\boldsymbol{K}_\perp)^2} \frac{(\boldsymbol{K}_\perp-\boldsymbol{k}_\perp)^m (\boldsymbol{K}_\perp-\boldsymbol{k}_\perp)^i}{(\boldsymbol{K}_\perp-\boldsymbol{k}_\perp)^2} + \overline{(m \leftrightarrow n)} \right] \right\}$$

$\Rightarrow$ High-energy evolution of the WW TMD in momentum space.
Agrees with Dominguez, Mueller, Munier, Xiao, 1108.1752 (real part).

Connection with BFKL

- If one further assumes that Q_s is small, then the ℓ integrant can be approximated by the universal tail of all gluon TMDs ("UGD")

$$\mathcal{F}_{WW}(x,\ell) \sim \mathcal{F}_D(x,\ell) \sim \mathcal{F}_g(x,\ell), \quad \ell_\perp \gg Q_s$$

- The non-linear evolution becomes linear.

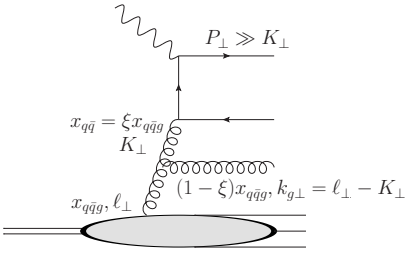
- Real part of the BFKL equation

$$\alpha_s \Delta\mathcal{F}_{WW,\mathcal{R}}(x,K_\perp,P_\perp^2) = \frac{\alpha_s N_c}{\pi^2} \ln\left(\frac{1}{x}\right) \int \frac{\mathrm{d}^2\ell}{(\ell-\boldsymbol{K}_\perp)^2} \mathcal{F}_g^{(0)}(x,\ell) \ .$$

Divergence at $\xi \to 1$: Sudakov logarithms

- $\xi \to 1$ endpoint singularity regulated by physical scales in dijet production.
- gluon emission well outside the jets $\theta_g \gg \theta_q \Rightarrow z_g \ll k_{g\perp}/P_\perp \Rightarrow \xi \leq 1 - k_{g\perp}/P_\perp$

$$\alpha_s \Delta \mathcal{W}_{\mathcal{R}}^{mn}(x, \boldsymbol{K}_\perp, P_\perp^2) \underset{\xi_0 \ll 1}{=} \frac{\alpha_s N_c}{\pi^2} \int \frac{\mathrm{d}^2 \boldsymbol{k}_{g\perp}}{\boldsymbol{k}_{g\perp}^2} \ln\left(\frac{1}{\xi_0}\right) \mathcal{W}^{mn,(0)}(x, \boldsymbol{K}_\perp + \boldsymbol{k}_{g\perp}) \,, \quad \xi_0 \simeq \frac{k_{g\perp}}{P_\perp}$$



- Real part of one step in the CSS evolution.

A bridge: CSS equation

- Add virtual corrections.
- Take a derivative w.r.t $P_\perp$.
- Momentum space version of the CSS equation. Local in ξ, non-local in $\boldsymbol{K}_\perp$.

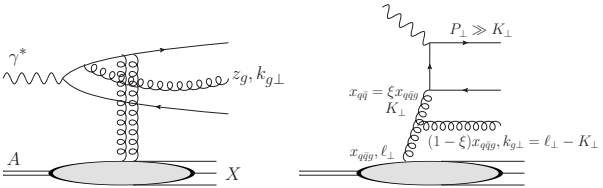
$$\frac{\partial \mathcal{F}_{WW}(x, \boldsymbol{K}_\perp, P_\perp^2)}{\partial \ln P_\perp^2} = \frac{\alpha_s N_c}{2\pi^2} \int \frac{\mathrm{d}^2 \boldsymbol{k}_{g\perp}}{\boldsymbol{k}_{g\perp}^2} \Big[\mathcal{F}_{WW}(x, \boldsymbol{K}_\perp + \boldsymbol{k}_{g\perp}, P_\perp^2) - \Theta(P_\perp^2 - k_{g\perp}^2) \mathcal{F}_{WW}(x, \boldsymbol{K}_\perp, P_\perp^2) \Big] + \frac{\alpha_s \beta_0 N_c}{\pi} \mathcal{F}_{WW}(x, \boldsymbol{K}_\perp, P_\perp^2).$$

See also talk by Maxim before. van Hameren, Nefedov, 2501.02619

- Exactly equivalent to the "diagonal" CSS equation for $f_{WW}(x, \boldsymbol{K}_\perp, \mu, \zeta)$ with $\mu^2 = \zeta = P_\perp^2$ and LO anomalous dimension γ_g and CSS kernel.
  PC, Iancu, 2406.04238

DGLAP regime

- Strong transverse ordering $P_\perp \gg K_\perp \gg \ell_\perp$, $\xi = O(1)$: onset of DGLAP evolution.



- Correction to the large $K_\perp$ tail of the WW gluon TMD. PC, Iancu, 2406.04238

$$\begin{aligned}\alpha_s \Delta\mathcal{F}_{WW,\mathcal{R}}(x, K_\perp, P_\perp^2) \underset{K_\perp \gg \ell_\perp}{\simeq} & \frac{\alpha_s N_c}{\pi^2}\frac{1}{K_\perp^2}\int \mathrm{d}^2\boldsymbol{\ell}\int_x^{1-\xi_0}\mathrm{d}\xi\left\{\frac{1}{\xi(1-\xi)}\mathcal{F}_{WW}^{(0)}(x,\boldsymbol{\ell})\left(\frac{x}{\xi},\boldsymbol{\ell}\right)\right. \\ & \left. + \frac{(1-\xi)(1+\xi^2)}{\xi}\mathcal{F}_D^{(0)}\left(\frac{x}{\xi},\boldsymbol{\ell}\right) - \frac{1+\xi}{\xi}\mathcal{F}_D^{(0)}\left(\frac{x}{\xi},\boldsymbol{\ell}\right)\right\}, \\ = & \frac{\alpha_s}{2\pi^2}\frac{1}{K_\perp^2}\int_x^{1-\xi_0}\mathrm{d}\xi\; P_{gg}(\xi)\,\frac{x}{\xi}G^{(0)}\left(\frac{x}{\xi}, K_\perp^2\right),\end{aligned}$$

Connection with k_t-factorisation

- Finally, regime $P_\perp \gg K_\perp, \ell_\perp \gg Q_s$
- All gluon TMDs in $\Delta\mathcal{F}_{WW,\mathcal{R}}$ reduce to the same universal UGD.

$$\alpha_s \Delta\mathcal{F}_{WW,\mathcal{R}}(x, K_\perp, P_\perp^2) \underset{\ell_\perp, K_\perp \gg Q_s}{\simeq} \frac{\alpha_s}{2\pi^2} \int \frac{\mathrm{d}^2\boldsymbol{\ell}}{\boldsymbol{k}_{g\perp}^2} \int_x^{1-\xi_0} \mathrm{d}\xi P_{g^*\to g^* g}(\xi, \boldsymbol{K}_\perp, \boldsymbol{k}_{g\perp}) \mathcal{F}_g^{(0)}\left(\frac{x}{\xi}, \boldsymbol{\ell}\right),$$

TMD splitting function of a gluon with $\boldsymbol{\ell}$ into two gluons with $\boldsymbol{K}_\perp$ and $\boldsymbol{k}_{g\perp} = \boldsymbol{\ell} - \boldsymbol{K}_\perp$

$$P_{g^*\to g^* g}(\xi, \boldsymbol{K}_\perp, \boldsymbol{k}_{g\perp}) \equiv 2N_c \left\{ \frac{1}{\xi(1-\xi)} + \frac{\left[(\boldsymbol{K}_\perp \cdot \boldsymbol{k}_{g\perp}) - k_{g\perp}^2\right]}{\left[(1-\xi)K_\perp^2 + \xi k_{g\perp}^2\right]} + \frac{\xi(1-\xi)k_{g\perp}^2 \left[K_\perp^2 - k_{g\perp}^2\right]^2}{2\ell_\perp^2 \left[(1-\xi)K_\perp^2 + \xi k_{g\perp}^2\right]^2} \right\}.$$

- Perfect agreement with Hentschinski, Kusina, Kutak, Serino, 1711.04587.
- New result for linearly polarised gluons.

Other real corrections and virtual corrections

- In addition to the contribution discussed in this talk
  1. Real corrections inside or near the jet boundaries.
     $\Rightarrow$ relax the approximation $z_g \ll k_{g\perp}/P_\perp$ for final state gluon emissions.
  2. Virtual corrections

- Computed in details in PC, Salazar, Schenke, Stebel, Venugopalan 2208.13872, 2304.03304, 2308.00022.

- Contribute to NLO corrections to the jet function, collinear-soft function and hard factor.

Structure of TMD factorization/resummation at small x

- We combine real and virtual corrections.
- Resum the large logs into the corresponding evolution.
- Conjecture for the general structure of the factorization "theorem" at small x:

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\mathrm{LO}}^{\gamma_T^* A\to q\bar{q}X}}{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}^2\boldsymbol{P}_\perp\mathrm{d}^2\boldsymbol{K}_\perp} &= \alpha_s\left[\mathcal{H}^{mn}_{\mathrm{LO}}(P_\perp)+\alpha_s\mathcal{H}^{mn}_{\mathrm{NLO}}(P_\perp,x_f,\mu)+...\right]\\ &\times\int\mathrm{d}^2\boldsymbol{b}\;e^{-i\boldsymbol{K}_\perp\cdot\boldsymbol{r}_{bb'}}\left\{\mathcal{W}^{mn}(x_f,\boldsymbol{r}_{bb'},\mu,\zeta)+\alpha_s\Delta\mathcal{W}^{mn}_{\mathcal{R},+}(x_f,\boldsymbol{r}_{bb'},...)+...\right\}\\ &\times C_q(\boldsymbol{b},R,\mu)J_q(p_\perp R,\mu)C_{\bar{q}}(\boldsymbol{b},R,\mu)J_{\bar{q}}(p_\perp R,\mu)\end{aligned}$$

- $\alpha_s\Delta\mathcal{W}^{mn}_{\mathcal{R},+}$ is the NLO discussed here with LL high-energy and CSS logs subtracted.

Take-home messages: towards bridging together TMD frameworks

- NLO correction to the WW gluon TMD in the non-linear regime encompasses JIMWLK, CSS, DGLAP and k_t factorization.

- NLO TMD factorisation theorem for $\gamma^* g \to \mathrm{dijet} + X$ at $x \to 0$ can be organised pretty much as at moderate x: **same soft, collinear-soft and jet functions**.

- Main differences:
  1. New operators **at LP** not present at moderate x: non-linear effects are encoded into more complicated operators generated by the BK/JIMWLK hierarchy.
  2. High-energy factorisation replaces collinear matching onto the gluon PDF: explicit x_f factorization scale dependence of the NLO hard factor.
  3. Gluon saturation constrains model for the initial condition to CSS.

Thank you for your attention!

Back-up slides

BACK-UP

Sum rules for the correlator $\mathcal{Z}$

Link with dipole gluon TMD:

$$\int \mathrm{d}^2\boldsymbol{q_1}\,\mathcal{Z}^{n,(0)}(x,\boldsymbol{q_1},\boldsymbol{q_2}) = q_2^n\,\mathcal{F}_D^{(0)}(x,\boldsymbol{q_2})\,,$$

Unitarity relation:

$$\int \frac{\mathrm{d}^2\boldsymbol{k}_\perp}{\boldsymbol{k}_\perp^2}\,\mathcal{Z}^{n,(0)}\,(x,\ell,\boldsymbol{k}_\perp) = 0\,.$$

Dilute limit:

$$\frac{1}{k_\perp^2}\mathcal{Z}^n(x,\ell,\boldsymbol{k}_\perp) \simeq \left(\delta^{(2)}(\ell-\boldsymbol{k}_\perp) - \delta^{(2)}(\boldsymbol{k}_\perp)\right)\frac{\ell^n}{\ell_\perp^2}\mathcal{F}_g^{(0)}(x,\ell)$$

Results for linearly polarised gluons

Full result:

$$\alpha_s \Delta\mathcal{H}_{WW,\mathcal{R}} = \frac{\alpha_s N_c}{\pi^2} \int \mathrm{d}^2\boldsymbol{\ell} \int_x^{1-\xi_0} \mathrm{d}\xi \left\{ \frac{1}{\xi(1-\xi)k_{g\perp}^2} \left[\frac{2(\boldsymbol{K}_\perp \cdot \boldsymbol{\ell})^2}{K_\perp^2 \ell_\perp^2} - 1 \right] \mathcal{H}_{WW}^{(0)} \left(\frac{x}{\xi}, \boldsymbol{\ell} \right) \right.$$
$$\left. + \int \frac{\mathrm{d}^2\boldsymbol{k}_\perp}{\xi k_\perp^2} \frac{\boldsymbol{k}_{g\perp}^i (\boldsymbol{K}_\perp - \boldsymbol{k}_\perp)^2 \left[\delta^{mi} - \frac{2(\boldsymbol{K}_\perp - \boldsymbol{k}_\perp)^m (\boldsymbol{K}_\perp - \boldsymbol{k}_\perp)^i}{(\boldsymbol{K}_\perp - \boldsymbol{k}_\perp)^2} \right]}{k_{g\perp}^2 \left[(1-\xi)(\boldsymbol{K}_\perp - \boldsymbol{k}_\perp)^2 + \xi k_{g\perp}^2 \right]} \left[\frac{2\boldsymbol{K}_\perp^m \boldsymbol{K}_\perp^n}{K_\perp^2} - \delta^{mn} \right] \mathcal{Z}^{n,(0)} \left(\frac{x}{\xi}, \boldsymbol{\ell}, \boldsymbol{k}_\perp \right) \right\},$$

TMD splitting function:

$$P_{g^* \to g_L^* g}(\xi, \boldsymbol{K}_\perp, \boldsymbol{k}_{g\perp})$$
$$\equiv 2N_c \left\{ \frac{1}{\xi} + \frac{1}{(1-\xi)} \left[\frac{2(\boldsymbol{K}_\perp \cdot \boldsymbol{\ell})^2}{K_\perp^2 \ell_\perp^2} - 1 \right] + \frac{\left[K_\perp^2 - k_{g\perp}^2 \right] \left[k_{g\perp}^2 - (\boldsymbol{K}_\perp \cdot \boldsymbol{k}_{g\perp}) \right]}{\ell_\perp^2 \left[(1-\xi) K_\perp^2 + \xi k_{g\perp}^2 \right]} \right\}.$$

DGLAP splitting function:

$$P_{g^* \to g_L^* g}(\xi, \boldsymbol{K}_\perp, \boldsymbol{k}_{g\perp}) = \frac{2N_c(1-\xi)}{\xi} + \mathcal{O}\left(\frac{\ell_\perp^2}{K_\perp^2} \right)$$