Hybrid high-energy factorization and evolution at NLO from the high-energy limit of collinear factorization

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Collinear factorisation at high energy

Let's consider an inclusive hard process:

$$p(P^{(+)}) + p(\bar{P}^{(-)}) \to \mathcal{H}(p) + X,$$

where $\mathcal{H}$ could be a jet, Higgs, heavy hadron, ..., with a hard scale $Q \gg \Lambda_{\rm QCD}$. Up to corrections power-suppressed by Q^2 the collinear factorisation (CF) (hopefully...) holds:

$$d\sigma = \sum_{i,\overline{i}=q,\bar{q},g} \int_0^1 dX d\bar{x}\, f_i(X,\mu_F) f_{\overline{i}}(\bar{x},\mu_F) d\hat{\sigma}_{i\overline{i}}(X,\bar{x},\mu_F),$$

The CF coefficient function $d\hat{\sigma}_{i\overline{i}}$ is computable order-by-order in α_s. What happens in the limit when $x_{\mathcal{H}} = p^+/P^+ \ll 1$? ($p(P)$ – "target", flies *forward*, $p(\bar{P})$ – "projectile", flies *backward*).

Very deeply in the low-$x_{\mathcal{H}}$ regime, the CF and twist expansion are expected to break-down, due to saturation/multiple scattering effects. But today we are concerned about the regime where CF still holds, but higher-order corrections to $d\hat{\sigma}$ become large.

Collinear factorisation at high energy

$$\frac{d\sigma}{dx_{\mathcal{H}}} = \sum_{i,\overline{\imath}=q,\bar{q},g} \int_{x_{\mathcal{H}}}^{1} dX\, f_i(X,\mu_F) \int_0^1 d\bar{x}\, f_{\overline{\imath}}(\bar{x},\mu_F) \frac{d\hat{\sigma}_{i\overline{\imath}}}{dx_{\mathcal{H}}}(x_{\mathcal{H}}/X,\bar{x},\mu_F),$$

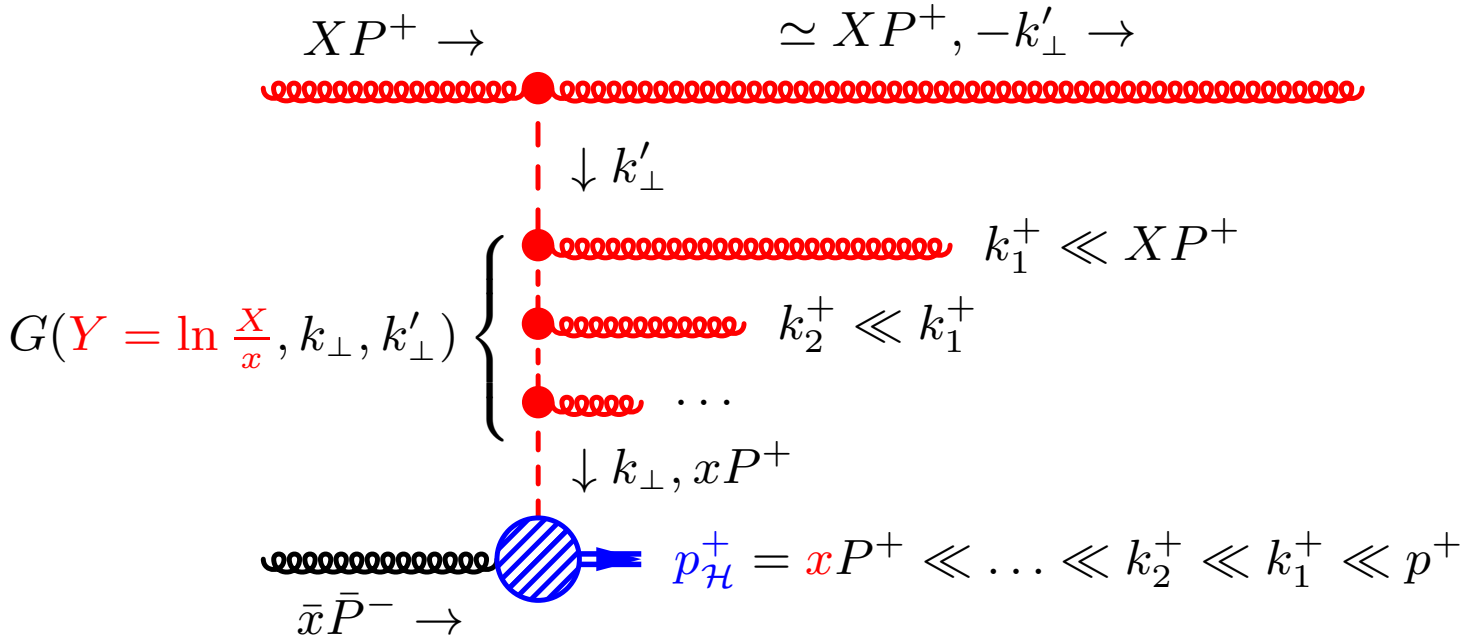
For $\xi = x_{\mathcal{H}}/X \ll 1$ the $d\hat{\sigma}_{i\overline{\imath}}$ develops large logarithmic corrections $\propto \alpha_s^{n+k} \ln^{n_0+n}(x_{\mathcal{H}}/X)$ (usually $n_0 = -1$, $k = 0$ – LLA, $k = 1$ – NLLA, etc.) **at leading power in** $\xi \ll 1$. The LP LLA series can be resummed by the ("hybrid") High-Energy Factorisation (HEF) formula [Collins, Ellis, '91; Catani, Ciafaloni, Hautmann, '91,'94] :

$$\frac{d\hat{\sigma}^{(\mathrm{HEF})}_{i\overline{\imath}}}{dx_{\mathcal{H}}}(\xi,\bar{x},\mu_F) = \int d^2k_\perp \int_\xi^1 \frac{dx}{x}\, \mathcal{C}_{i\star}(\xi/x,k_\perp,\mu_F) \frac{d\mathrm{B}_{\star\overline{\imath}}}{dx_{\mathcal{H}}}\big(x,k_\perp,\bar{x}\,;\{p\}_n\big),$$

where $\mathcal{C}_{i\star}$ is the *resummation factor* (precisely known in LLA in the $\overline{MS}$-scheme) , $\sum_n \alpha_s^n \ln^{n-1}(\xi/z)$, the $d\mathrm{B}_{\star\overline{\imath}}$ is the *off-shell cross section* (or *HEF coefficient function*) and $\{p\}_n$ are momenta of final-state partons defining $\mathcal{H}$ at LO in α_s.
$d\hat{\sigma}^{(\mathrm{HEF})}_{i\overline{\imath}} \to d\sigma \Rightarrow$"hybrid $k_\perp$-factorisation" formula:

$$\frac{d\sigma}{dx_{\mathcal{H}}} = \sum_{\overline{\imath}=q,\bar{q},g} \int_0^1 d\bar{x}\, f_{\overline{\imath}}(\bar{x},\mu_F) \int d^2k_\perp \int_{x_{\mathcal{H}}}^1 dx\, F(x,k_\perp) \frac{d\mathrm{B}_{\star\overline{\imath}}}{dx_{\mathcal{H}}}\big(x,k_\perp,\bar{x}\,;\{p\}_n\big),$$

where the *unintegrated PDF* $F = f_i \otimes \mathcal{C}_{i\star}$. **How to compute** $d\mathrm{B}_{\star\overline{\imath}}$ **?**


$XP^+ \to$
$\simeq XP^+, -k'_\perp \to$
$\downarrow k'_\perp$
$k_1^+ \ll XP^+$
$G(Y = \ln \frac{X}{x}, k_\perp, k'_\perp)$
$k_2^+ \ll k_1^+$
$\cdots$
$\downarrow k_\perp, xP^+$
$p_{\mathcal{H}}^+ = xP^+ \ll \ldots \ll k_2^+ \ll k_1^+ \ll p^+$
$\bar{x}\bar{P}^- \to$

Two ways to define LLA

The **"CCH k_T-factorisation":**

$$d\sigma \propto \int d^2k_\perp \int_{x_{\mathcal{H}}}^{1} dx\, F(x,k_\perp) d\mathrm{B}_{\star\bar{\imath}}(x,k_\perp)$$

1. Includes some NLLA (and NLP) corrections in $x_{\text{sch.}}$ already at LO
2. **Virtual** NLO corrections to dB are the same (**have RDs**)
3. **Real** NLO corrections **have no RD** (cut by $F(x \to 1, k_\perp)$)
4. Looks similar to TMD
5. For experts: is related to the "γ-shift" approaches in BFKL [Salam; Sabio Vera; Colferai, Stasto; ...] ($\chi(\gamma) \to \chi(\gamma \pm \omega/2)$, $I(\gamma) \to I(\gamma \pm \omega/2)$)

The **strict LP+LLA** (or **"BFKL(/MRK/CGC/Shockwave) way":**

$$d\sigma \propto \int d^2k_\perp F(x_{\text{sch.}}, k_\perp) \int_0^{\infty} dx\, d\mathrm{B}_{\star\bar{\imath}}(x,k_\perp)$$

1. Strict (N)LLA and LP in $x_{\text{sch.}}$
2. **Virtual** NLO corrections to dB are the same (**have RDs**)
3. **Real** NLO corrections **have RD** (generated by $x \sim 1$)
4. Hard to relate with TMD

Three common schemes:

- *Projectile scheme:* $x_{\text{sch.}} = p^-_{\mathcal{H}}/P^-$ (CGC/JIMWLK/Shockwave)
- *Target scheme:* $x_{\text{sch.}} = p^+_{\mathcal{H}}/P^+$ (DIS, HEF)
- *Symmetric scheme:* $x_{\text{sch.}} \to Y$ (BFKL, Mueller-Navelet)

LLA evolution w.r.t. $\ln 1/\xi$

In the LL($\ln 1/\xi$)-approximation, the $Y = \ln 1/\xi$-evolution equation for *collinearly un-subtracted* $\tilde{\mathcal{C}}$-factor has the form:

$$\tilde{\mathcal{C}}(\xi, \mathbf{q}_T) = \delta(1-\xi)\delta(\mathbf{q}_T^2) + \hat{\alpha}_s \int\limits_{\xi}^{1} \frac{dz}{z} \int d^{2-2\epsilon}\mathbf{k}_T K(\mathbf{k}_T^2, \mathbf{q}_T^2) \tilde{\mathcal{C}}\left(\frac{\xi}{z}, \mathbf{q}_T - \mathbf{k}_T\right)$$

with $\hat{\alpha}_s = \alpha_s C_A/\pi$ and

$$K(\mathbf{k}_T^2, \mathbf{q}_T^2) = \frac{1}{\pi(2\pi)^{-2\epsilon}\mathbf{k}_T^2} + \delta^{(2-2\epsilon)}(\mathbf{k}_T)\ 2\omega_g(\mathbf{q}_T^2),$$

where $\omega_g(\mathbf{q}_T^2)$ – 1-loop Regge trajectory of a gluon. It is convenient to go from $(z, \mathbf{q}_T)$-space to $(N, \mathbf{x}_T)$-space:

$$\tilde{\mathcal{C}}(N, \mathbf{x}_T) = \int d^{2-2\epsilon}\mathbf{q}_T\ e^{i\mathbf{x}_T\mathbf{q}_T} \int\limits_0^1 dx\ x^{N-1}\ \tilde{\mathcal{C}}(x, \mathbf{q}_T),$$

because:

- Mellin convolutions over z turn into products: $\int \frac{dz}{z} \to \frac{1}{N}$
- Large logs map to poles at $N = 0$: $\boxed{\alpha_s^{k+1} \ln^k \frac{1}{\xi} \to \frac{\alpha_s^{k+1}}{N^{k+1}}}$
- All *collinear divergences* are contained inside $\mathcal{C}$ in $\mathbf{x}_T$-space.

Exact LL solution and the DLA

In $(N, \mathbf{q}_T)$-space, subtracted $\mathcal{C}$, which resums all terms $\propto (\hat{\alpha}_s/N)^n$ (complete LLA) has the form [Collins, Ellis, '91; Catani, Ciafaloni, Hautmann, '91,'94]:

$$\mathcal{C}(N, \mathbf{q}_T, \mu_F) = R(\gamma_{gg}(N, \alpha_s)) \frac{\gamma_{gg}(N, \alpha_s)}{\mathbf{q}_T^2} \left(\frac{\mathbf{q}_T^2}{\mu_F^2} \right)^{\gamma_{gg}(N, \alpha_s)},$$

where $\gamma_{gg}(N, \alpha_s)$ is the solution of [Jaroszewicz, '82]:

$$\frac{\hat{\alpha}_s}{N} \chi(\gamma_{gg}(N, \alpha_s)) = 1, \text{ with } \chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma),$$

where $\psi(\gamma) = d \ln \Gamma(\gamma)/d\gamma$ – Euler's ψ-function. The first few terms:

$$\gamma_{gg}(N, \alpha_s) = \underbrace{\underbrace{\frac{\hat{\alpha}_s}{N}}_{\text{DLA [Blümlein, '95]}} + 2\zeta(3) \frac{\hat{\alpha}_s^4}{N^4} + 2\zeta(5) \frac{\hat{\alpha}_s^6}{N^6} + \ldots}_{\text{LLA}}$$

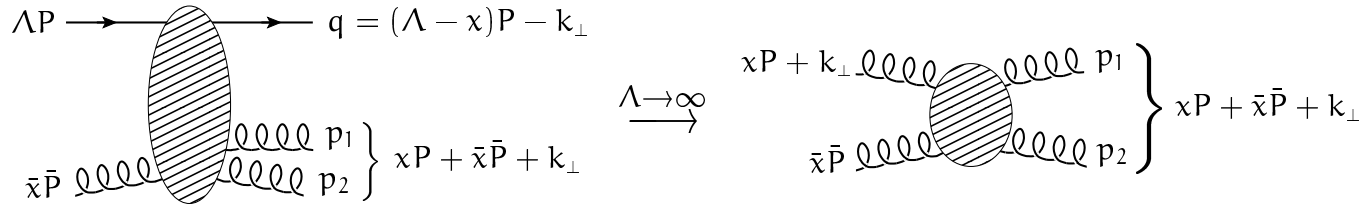
$\frac{\hat{\alpha}_s}{N} \leftrightarrow P_{gg}(z \to 0) = \frac{2C_A}{z} + \ldots$

The function $R(\gamma)$ is

$$R(\gamma_{gg}(N, \alpha_s)) = 1 + O(\alpha_s^3).$$

Auxiliary parton method

[van Hameren, Kotko, Kutak, 2012] , implemented into the **KaTie** parton-level MC generator. Off-shell tree-level MEs agree with ones obtained from Lipatov's EFT [Lipatov '95] .



The momentum of an off-shell (Reggeised) gluon:

$$k_\star^\mu = xP^\mu + k_\perp^\mu,$$

The $d\mathrm{B}_{\star\bar{\imath}}$ is computed with the tree-level off-shell matrix element $\left|\overline{M}_{\star\bar{\imath}}\right|^2$, which is related with the matrix element with an auxiliary parton $i = q, g$ as:

$$\left|\overline{M}_{\star\bar{\imath}}\right|^2\left(k_\star, k_{\bar{\imath}}\,; \{p\}_n\right) = \lim_{\Lambda\to\infty} \frac{x^2|k_\perp|^2}{g_{\mathrm{s}}^2 C_i \Lambda^2} \left|\overline{M}_{i\bar{\imath}}\right|^2\left(k_\Lambda, k_{\bar{\imath}}\,; q_\Lambda, \{p\}_n\right) \ ,$$

with $C_g = 2N_{\mathrm{c}}$, $C_q = 2C_F = \frac{N_{\mathrm{c}}^2-1}{N_{\mathrm{c}}}$.

$$k_\Lambda^2 = q_\Lambda^2 = 0 \quad , \quad k_\Lambda^\mu - q_\Lambda^\mu \overset{\Lambda\to\infty}{=} xP^\mu + k_\perp^\mu \ .$$

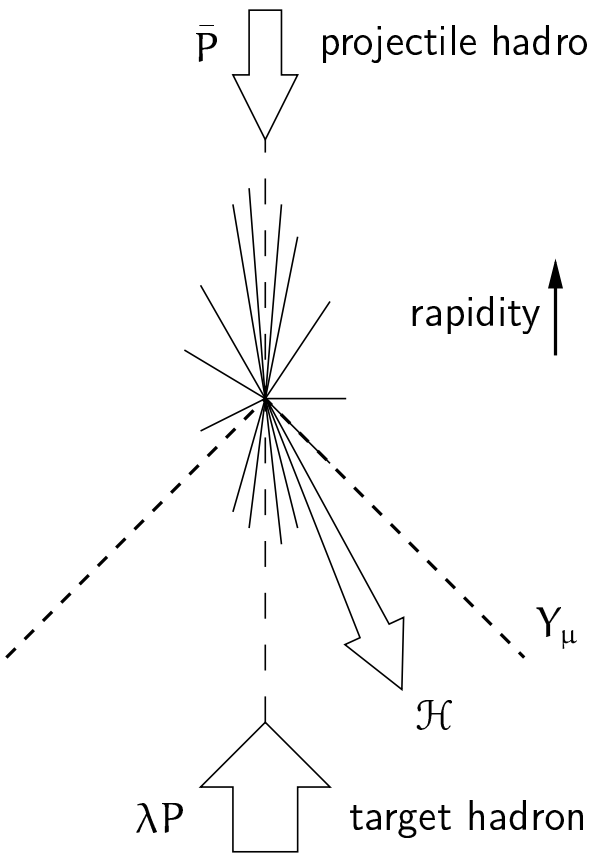
$$\begin{aligned} k_\Lambda^\mu &= \Lambda P^\mu + \alpha k_\perp^\mu + \beta \bar{P}^\mu \ , \\ q_\Lambda^\mu &= (\Lambda - x)P^\mu + (\alpha - 1)k_\perp^\mu + \beta \bar{P}^\mu \ , \end{aligned}$$

Strategy to go to NLO for $d\mathrm{B}_{\star\bar{\imath}}$

To extract NLO $d\mathrm{B}_{\star\bar{\imath}}$ we either need to compute $d\sigma/dx_{\mathcal{H}}$ in CF for $x_{\mathcal{H}} \ll 1$, at **NNLO** in CF, which is hard to do, or we need to define the **observable** (=IRC-safe) cross section

$$\frac{d\sigma}{dx d^2k_\perp}$$

and compute it in CF **NLO** in the limit $x \ll 1$, which is much easier!



IRC-safe definitions of x and $k_\perp$ in terms of observable hadron momenta p_j:

$$x = \sum_j \theta\big(y_j < Y_\mu\big)\, \frac{p_j \cdot \bar{P}}{P \cdot \bar{P}} \quad \left(= \sum_j \theta\big(y_j < Y_\mu\big)\, \frac{p_j^+}{P^+} \right) ,$$

$$\vec{k_\perp} = -\sum_j \theta\big(y_j < Y_\mu\big)\, \vec{p}_{j\perp} ,$$

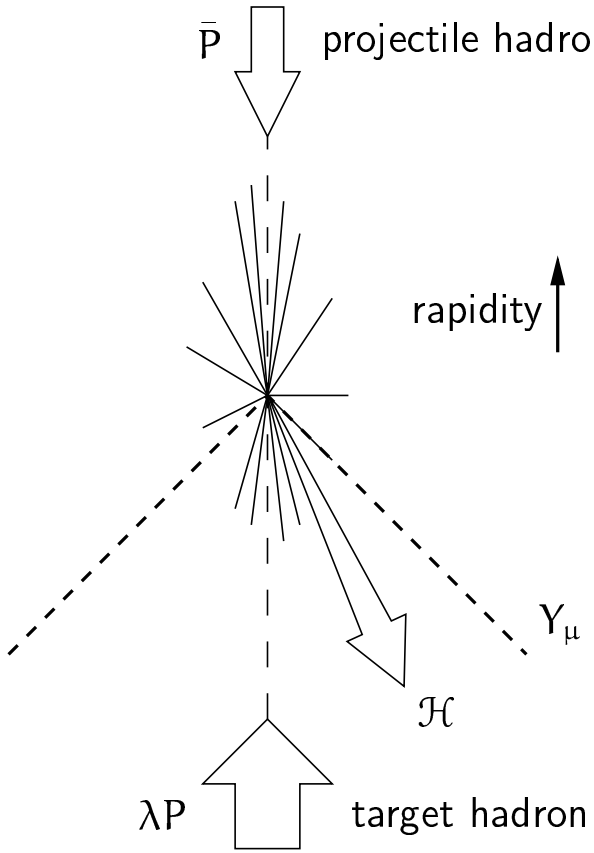
The $k_\perp$ is IRC-safe because low-$p_\perp$ hadrons give a small contribution. The x is IRC-safe, because contributions of hadrons with negative rapidity is supressed by e^{y_i}, while hadrons with positive rapidity are cut-out by Y_μ.

Strategy to go to NLO for $d\mathrm{B}_{\star\bar{\imath}}$

We define the **observable** (=IRC-safe) cross section

$$\frac{d\sigma^{(\mathrm{CF})}(Y_\mu)}{dx d^2k_\perp}[p(\lambda P)+p(\bar{P})\to\mathcal{H}+X]$$

and compute it in CF **NLO** in the limit $\lambda\gg 1$ for x-fixed $\Leftrightarrow x\ll 1$.



IRC-safe definitions of x and $k_\perp$ in terms of observable hadron momenta p_j:

$$x=\sum_j \theta\big(y_j<Y_\mu\big)\,\frac{p_j\cdot\bar{P}}{P\cdot\bar{P}}\ ,$$

$$\vec{k_\perp}=-\sum_j \theta\big(y_j<Y_\mu\big)\,\vec{p}_{j\perp}\ ,$$

One can relate the *Collins-Soper (rapidity) scale* μ_Y with x and Y_μ:

$$xP_+=\mu_Y e^{Y_\mu},$$

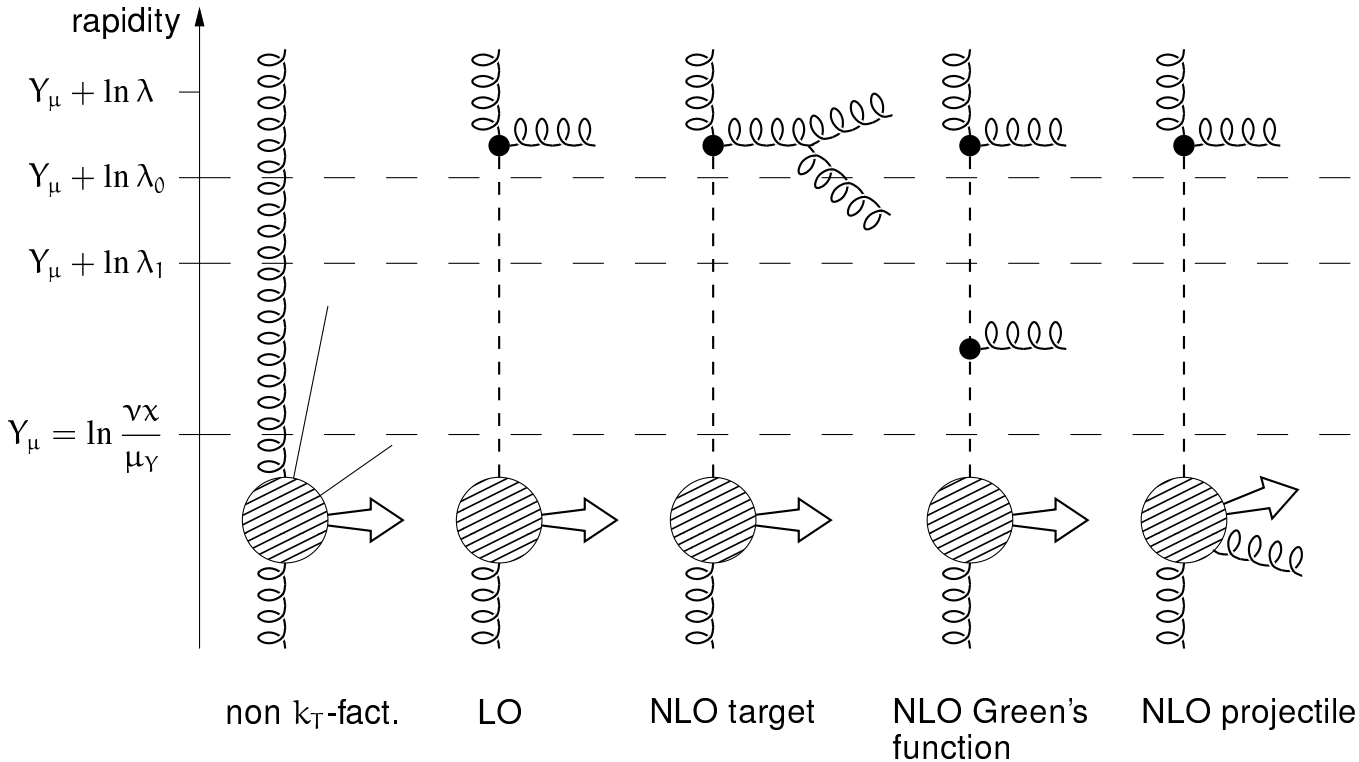
later, P_+ will be often denoted as ν.

Obviously, in reality $\int d^2k_\perp \int\limits_0^1 dx\frac{d\sigma(Y_\mu)}{dx d^2k_\perp}$ does not depend on Y_μ or μ_Y. In QCD this puts nontrivial constraints on HEF-formula, order-by-order in α_s.

Contributions to $d\sigma^{(\mathrm{CF})}$ up to NLO

For $\lambda \to \infty$, x-fixed and $\lambda \succ \lambda_0 \succ \lambda_1 \to \infty$ (meaning $\lambda_0/\lambda \to 0$ and $\lambda_1/\lambda_0 \to 0$) :

$$\frac{d\sigma^{(\mathrm{CF,\ NLO})}}{dx d^2k_\perp} = {}_{(\mathrm{non}\ k_\perp\text{-fact})} + \int_0^1 d\bar{x}\, f_{\bar{\imath}}(\bar{x},\mu_F) F(x,k_\perp,\mu_Y) dH_{\star\bar{\imath}}(x,k_\perp,\bar{x},\mu_Y;...) + O(1/\lambda),$$



NLO target and **NLO Green's function** contributions contribute to F at NLO, while NLO projectile contribution contributes to $dH_{\star\bar{\imath}}$ at NLO.

Non $k_\perp$-factorisable contribution?

At LO the X can be $\sim 1/\lambda$:

$$d\sigma^{(\mathrm{CF,\ LO})} = \int_0^1 dX\, f(X)\, d\hat{\sigma}^{\mathrm{LO}}(\lambda X) = \int_0^1 dX\, [f^{>}(X) + f^{<}(X)] d\hat{\sigma}^{\mathrm{LO}}(\lambda X),$$

where $f^{>}(X) = f(X)\,\theta(X > \delta_0)$, with $\delta_j = \frac{\lambda_j x |k_\perp|}{\lambda \mu_Y}$, such that:

$$\int_0^1 dX\, f^{>}(X)\, d\hat{\sigma}^{\mathrm{LO}}(\lambda X) \overset{\lambda\to\infty}{\longrightarrow} \left[\int_0^1 dX\, f^{>}(X)\right] d\hat{\sigma}^{\mathrm{LO}}_{k_T-\mathrm{fact}}\ ,$$

The $\lambda \gg 1$-limit for the LO $k_\perp$-factorisable contribution:

$$\frac{d\sigma^{\mathrm{CF,B}}_{\lambda\to\infty}\big(\{p\}_n\big)}{dx d^2k_\perp} = F^{\mathrm{LO}}(\delta_0, k_\perp) \sum_{\overline{i}} \int_0^1 d\bar{x}\, f_{\overline{i}}(\bar{x})\, d\mathrm{B}_{\star\overline{i}}\big(x, k_\perp, \bar{x}\,; \{p\}_n\big)\ ,$$

with the LO in α_s unintegrated PDF being defined as

$$F^{\mathrm{LO}}(\delta_0, k_\perp) = \sum_i \frac{\alpha_\mathrm{s} C_i}{2\pi^2 |k_\perp|^2}\, \mathcal{N}_i(\delta_0) \quad \text{and} \quad \mathcal{N}_i(\delta_0) = \int_0^1 dX\, f_i^{>}(X)\ ,$$

Non $k_\perp$-factorisable contribution?

At NLO, the collinear divergence cancels in:

$$d\sigma^{(\mathrm{CF,\ NLO})} = \int_0^1 dX\, f(X)\, d\hat{\sigma}^{\mathrm{NLO}}(\lambda X) + \int_0^1 dX\, \left[\mathcal{Z}^{(1)} \otimes f\right](X)\, d\hat{\sigma}^{\mathrm{LO}}(\lambda X)\ ,$$

integral over X again conflicts with $\lambda \to \infty$ limit and we separate it as:

$$\begin{aligned} d\sigma^{\mathrm{NLO}} = & \int_0^1 dX\, f^{>}(X)\, d\hat{\sigma}^{\mathrm{NLO}}(\lambda X) + \int_{\delta_1}^1 dX\, \left[\mathcal{Z}^{(1)} \otimes f^{>}\right](X)\, d\hat{\sigma}^{\mathrm{LO}}(\lambda X) \\ & + \int_0^1 dX\, f^{<}(X)\, d\hat{\sigma}^{\mathrm{NLO}}(\lambda X) + \int_0^1 dX\, \left[\mathcal{Z}^{(1)} \otimes f^{<}\right](X)\, d\hat{\sigma}^{\mathrm{LO}}(\lambda X) \\ & + \int_0^{\delta_1} dX\, \left[\mathcal{Z}^{(1)} \otimes f^{>}\right](X)\, d\hat{\sigma}^{\mathrm{LO}}(\lambda X) \end{aligned}$$

only the first line is $k_\perp$-factorisable, *independently on the behaviour of* $f(X)$, and we have:

$$\int_{\delta_1}^1 dX\, \left[\mathcal{Z}^{(1)} \otimes f^{>}\right](X)\, d\hat{\sigma}^{\mathrm{LO}}(\lambda X) \overset{\lambda\to\infty}{\longrightarrow} \left[\int_0^1 dX\, f^{>}(X)\, p(\delta_1/X)\right] d\hat{\sigma}^{\mathrm{LO}}_{k_T-\mathrm{fact}}\ ,$$

which cancels the collinear divergence in the NLO target contribution. Everything is fine, since all large logarithms $\ln\lambda \sim \ln(1/x)$ are in the $k_\perp$-factorisable contribution.

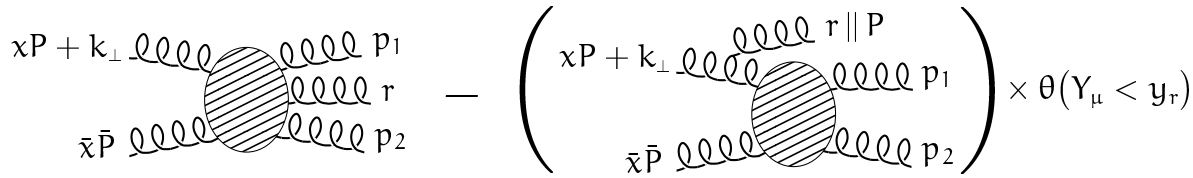
Separation of the virtual part

The most general decomposition, consistent with separate IR-safety of target, projectile and Green's function contributions ($\Lambda = X\lambda$):

$$
\begin{aligned}
&d\mathrm{V}_{i\bar{\imath}}^{\mathrm{targ}}(\epsilon, \Lambda, \lambda_1, \mu_Y) \\
&\qquad = d\mathrm{V}_{i\bar{\imath}}^{\mathrm{unf}}(\epsilon, \Lambda) - d\mathrm{B}_{\star\bar{\imath}} \times a_\epsilon \left[\left(\frac{\mu^2}{\mu_Y^2} \right)^\epsilon \frac{N_\mathrm{c}}{\epsilon^2} + \frac{\gamma_g}{\epsilon} + \left(\frac{\mu^2}{|k_\perp|^2} \right)^\epsilon \frac{2N_\mathrm{c} \ln \lambda_1}{\epsilon} \right] \\
&d\mathrm{V}_{\star\bar{\imath}}^{\mathrm{Green}}(\epsilon, \lambda_1) \\
&\qquad = \qquad\qquad d\mathrm{B}_{\star\bar{\imath}} \times a_\epsilon \left[\qquad\qquad \left(\frac{\mu^2}{|k_\perp|^2} \right)^\epsilon \frac{2N_\mathrm{c} \ln \lambda_1}{\epsilon} \right] \\
&d\mathrm{V}_{\star\bar{\imath}}^{\mathrm{proj}}(\epsilon, \mu_Y) \\
&\qquad = d\mathrm{V}_{\star\bar{\imath}}^{\mathrm{fam}}(\epsilon) \quad + d\mathrm{B}_{\star\bar{\imath}} \times a_\epsilon \left[\left(\frac{\mu^2}{\mu_Y^2} \right)^\epsilon \frac{N_\mathrm{c}}{\epsilon^2} + \frac{\gamma_g}{\epsilon} \right] .
\end{aligned}
$$

The "familiar" virtual part [van Hameren, Motyka, Ziarko, 2022] is defined as the part of the one-loop auxiliary parton amplitude, IR-divergences of which are the same as if the off-shell parton was on-shall.

Subtraction scheme for the real-emision on the projectile side



The subtraction term ($\nu = P^+$)

$$
\begin{aligned}
&d\mathrm{R}^{\mathrm{fam,coll}}_{\star\bar{\imath}}\big(\epsilon, \mu_Y\,; x, k_\perp, \bar{x}\,; \{p\}_n\big) \\
&\qquad = \int \frac{\mu^{2\epsilon} d^{4-2\epsilon} r}{(2\pi)^{3-2\epsilon}}\, \delta_+(r^2)\, \theta(x_r < x)\, \theta\bigg(y_r > \ln \frac{\nu(x - x_r)}{\mu_Y}\bigg) \\
&\qquad\qquad \times g_{\mathrm{s}}^2\, \frac{4N_{\mathrm{c}}}{|r_\perp|^2(1 - x_r/x)}\, d\Sigma_{\star\bar{\imath}}\big(k_\star - x_r P - r_\perp, \bar{x}\bar{P}\,; \{p\}_n\big)\, J_{\mathrm{B}}\big(\{p\}_n\big)\ .
\end{aligned}
$$

The μ_Y-evolution equation

Now the projectile contribution is completely defined. From the μ_Y-independence of the x and $k_\perp$-integrated cross section we derive the evolution equation (generalised CSS equation):

$$\frac{d\hat{F}(x,k_\perp;\mu_Y)}{d\ln\mu_Y^2} = \frac{\alpha_{\rm s}N_{\rm c}}{2\pi}\int\frac{d^2r_\perp}{\pi|r_\perp|^2}\left\{\hat{F}\left(x\left[1+\frac{|r_\perp|}{\mu_Y}\right],k_\perp+r_\perp\,;\mu_Y\right)\right.$$
$$\left. -\,\theta\big(\mu_0-|r_\perp|\big)\,\hat{F}(x,k_\perp;\mu_Y)\right\} + \frac{\alpha_{\rm s}N_{\rm c}}{2\pi}\,\hat{F}(x,k_\perp;\mu_Y)\ln\frac{\mu_0^2}{\mu_Y^2}\ .$$

In $x_\perp$-space we have:

$$\frac{d\tilde{F}(x,x_\perp;\mu_Y)}{d\ln\mu_Y^2} = \frac{\alpha_{\rm s}N_{\rm c}}{\pi}\int_x^1\frac{dz}{z(1-z)_+}J_0\left(\mu_Y|x_\perp|\frac{1-z}{z}\right)\tilde{F}\left(\frac{x}{z},x_\perp;\mu_Y\right)\ ,$$

In the TMD limit $k_\perp \ll \mu_Y$ and in $x_\perp$-space we have:

$$\frac{d}{d\ln\mu_Y^2}\tilde{F}(x,x_\perp;\mu_Y) = \frac{\alpha_{\rm s}}{2\pi}\Big[-N_{\rm c}\ln\left(\mu_Y^2\overline{x}_\perp^2\right)\Big]\tilde{F}(x,x_\perp;\mu_Y)\ ,$$

which is just the LO CSS equation.

UPDF at the starting scale $\mu_Y = |k_\perp|$

The sum of NLO target and Green's function contributions at NLO in α_s is finite:

$$F^{\mathrm{LO+NLO}}(x,k_\perp;\mu_Y)$$
$$= \sum_i \frac{\alpha_s C_i}{2\pi^2} \int_x^1 dX\, f_i(X,\mu_F) \Bigg\{ \frac{1}{|k_\perp|^2} \Big[1 + a_\epsilon\, \Delta I_i\big(1,k_\perp,\mu_F\big) + a_\epsilon\, \Delta U\big(k_\perp,\mu_Y\big) \Big]$$
$$+ a_\epsilon \ln\left(\frac{X\mu_Y}{x|k_\perp|}\right) \int d^{2-2\epsilon} r_\perp\, \frac{1}{|k_\perp + r_\perp|^2}\, K(k_\perp + r_\perp, k_\perp, \mu_F) \Bigg\},$$

$$\Delta I_i(z,k_\perp,\mu_F) = \left[\mathcal{J}_i^{(0)} - 2N_c \ln\frac{1}{z}\right] \ln\frac{\mu_F^2}{|k_\perp|^2} + 2\gamma_g \ln\frac{\mu^2}{|k_\perp|^2} + 2\mathcal{K} + \mathcal{J}_i^{(1)} \ ,$$

$$\Delta U(k_\perp,\mu_Y) = -\frac{N_c}{2} \ln^2\frac{\mu_Y^2}{|k_\perp|^2} \ ,$$

$$K(k'_\perp,k_\perp,\mu_F) = 4N_c \frac{\mu^{2\epsilon}}{2\pi_\epsilon} \left[\frac{1}{|k'_\perp - k_\perp|^2} - \frac{\theta(\mu_F^2 - |k'_\perp|^2)}{|k_\perp|^2} + \delta^{(2-2\epsilon)}(k'_\perp - k_\perp) \frac{\pi_\epsilon}{\epsilon} |k_\perp^2|^{-\epsilon} \right] .$$

We have checked that it satisfies the μ_Y-evolution equation up to NLO.

UPDF at the starting scale $\mu_Y = |k_\perp|$

From this result we conjecture the following all-order initial condition for the μ_Y-evolution at $\mu_Y = |k_\perp|$:

$$F\big(x, k_\perp, \mu_Y = |k_\perp|\big) = \sum_i \int_x^1 dX\, f_i(X, \mu_F) \int d^2k'_\perp I_i\big(k'_\perp, \mu_F\big)\, G\left(k'_\perp, k_\perp, \frac{X}{x}, \mu_F\right),$$

where $I_i(k_\perp, \mu_F) = \frac{\alpha_s C_i}{2\pi^2 |k_\perp|^2}\Big[1 + a_\epsilon \Delta I_i(1, k_\perp, \mu_F) + \mathcal{O}\big(\alpha_s^2\big)\Big]$ and the Green's function satisfies the BFKL-Collins-Ellis evolution equation:

$$G\big(k'_\perp, k_\perp, x, \mu_F\big) = \delta^{(2-2\epsilon)}(k'_\perp - k_\perp)$$
$$+ a_\epsilon \int_x^1 \frac{dz}{z} \int d^{2-2\epsilon} q_\perp \Big[K_{\rm BFKL}\big(k'_\perp, q_\perp\big) - \theta\big(\mu_F^2 - |k'_\perp|^2\big)\, K_{\rm BFKL}\big(0, q_\perp\big)\Big] G\left(q_\perp, k_\perp, \frac{x}{z}, \mu_F\right)$$

where $K_{\rm BFKL}\big(k'_\perp, k_\perp\big) = 4N_c \frac{\mu^{2\epsilon}}{2\pi_\epsilon}\left[\frac{1}{|k'_\perp - k_\perp|^2} + \delta^{(2-2\epsilon)}\big(k'_\perp - k_\perp\big) \frac{\pi_\epsilon}{\epsilon} \big|k_\perp^2\big|^{-\epsilon}\right]$

This initial condition resums $\ln(X/x)$, while the μ_Y evolution resums $\ln(\mu_Y/|k_\perp|)$.

Conclusions and outlook

- The scheme of NLO computations of HEF coefficient functions (impact-factors) for arbitrary QCD hard processes is derived
- From the derivation, the necessity to introduce the rapidity scale μ_Y emerges and the picture:

$$\mathrm{TMD}(x, k_\perp, \mu_Y) = [\text{low-x UPDF}](x, k_\perp) \otimes \mathrm{Sudakov}(k_\perp, \mu_Y),$$

in the TMD limit is supported. In certain DLA, $[\text{low-x UPDF}](x, k_\perp) = \frac{\partial}{\partial \ln k_\perp^2} f_g(x, k_\perp^2)$, so something similar to the KMRW formula is true.

Thank you for your attention!