Discrete and continuous scale invariance in quantum few-body systems

Rafael Mendes Francisco

Technological Institute of Aeronautics

Pan-American Few-Body Physics Boot Camp: Fostering Collaboration

Italy - Trento

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The Efimov effect

- Was proposed in 1970 by Vitaly Efimov

- At the unitarity limit ($a \to \infty$), an infinite tower of three-body bound states appears

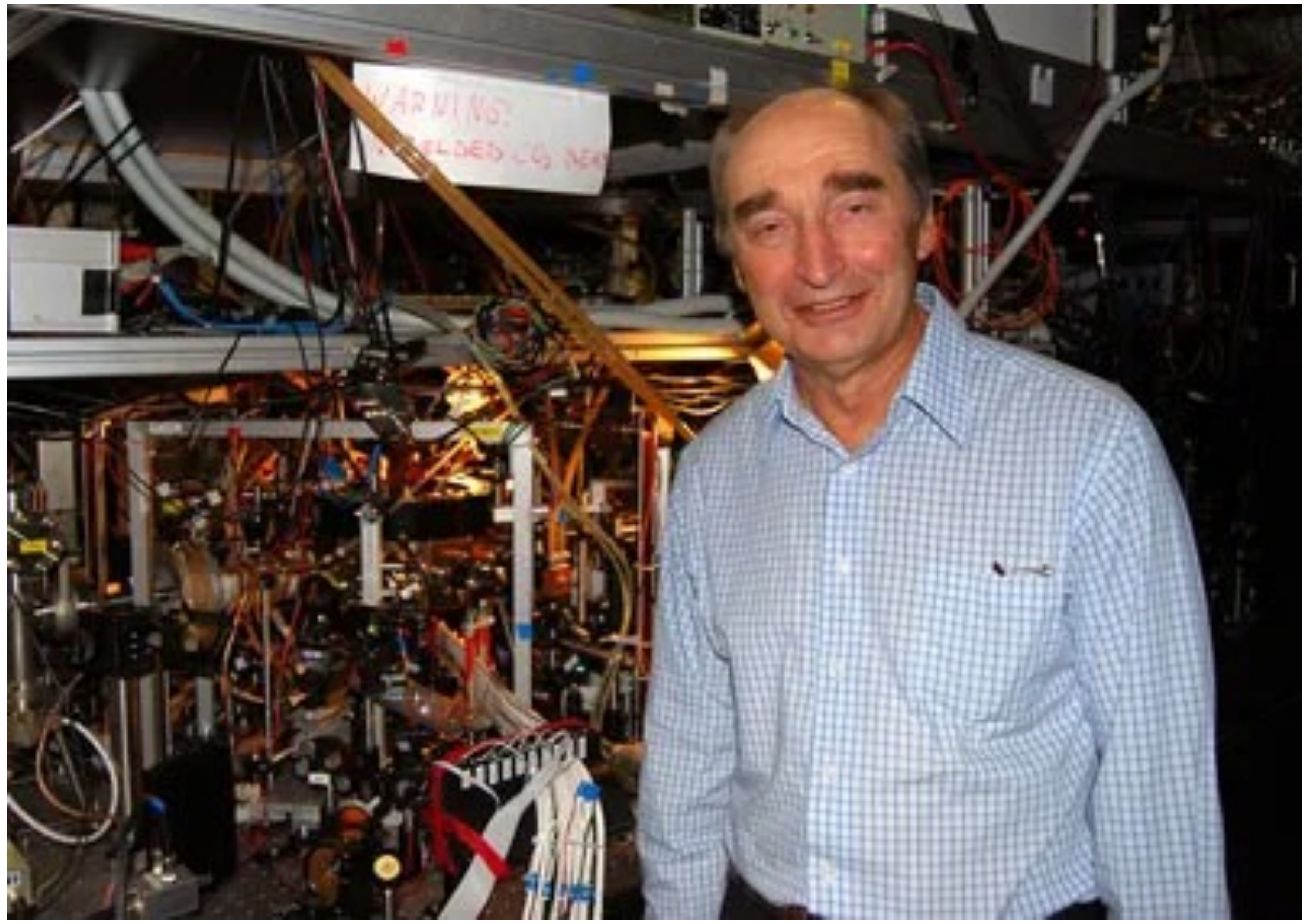
- The ratio between the energies of successive states is a universal constant (discrete scale symmetry)

$$\frac{E_{N+1}}{E_N} = e^{-2\pi/s_0}$$

- For three identical bosons in three-dimensions, we have

$$\frac{E_{N+1}}{E_N} \approx \frac{1}{(22.7)^2} \approx \frac{1}{515}$$

- For finite scattering length, a finite number of Efimov states appear



V. Efimov, Phys. Lett. B, 33, 563 (1970)

Two heavy impurities immersed in light few-boson systems with noninteger dimensions

- The Born-Oppenheimer approach

Light particles Schrodinger-like equation

$$\left[\sum_{i=1}^{N-2}\left(-\frac{\hbar^2}{2\mu_{B,AA}}\nabla^2_{r_i}+\sum_{j=1}^{2}V_{BA}\left(\left|\mathbf{r}_i+(-1)^j\frac{\mathbf{R}}{2}\right|\right)\right)\right]\psi_R(\mathbf{r}_i)=\epsilon(R)\psi_R(\mathbf{r}_i)$$

Heavy particles Schrodinger-like equation

$$\left[-\frac{\hbar^2}{2\mu_{AA}}\nabla^2_R+V_{AA}(|\mathbf{R}|)+(N-2)\epsilon(R)\right]\phi(\mathbf{R})=E_N^{(D)}\phi(\mathbf{R})$$

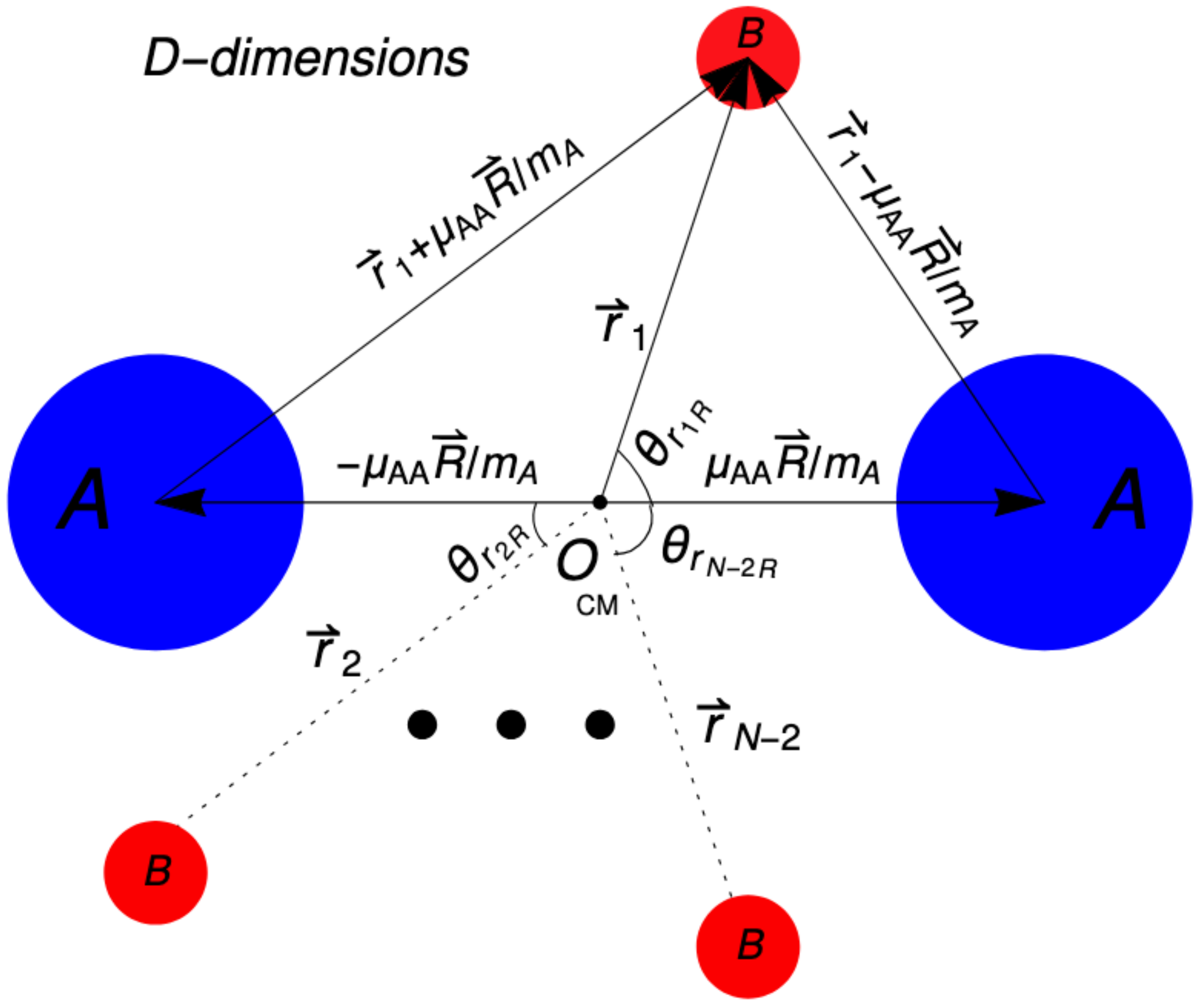


FIG. 1. Representation of the AABB...B structure forming the N-body system composed by two identical heavy-atoms with masses m_A, and $(N-2)$ light-atoms with masses m_B.

R. M. Francisco, D. S Rosa, T. Frederico. Phys. Rev. A, 106, 063305 (2022)

The effective potential in the BO picture

- The Born-Oppenheimer approach (we are considering finite scattering length)

Effective potential for small distances ($\bar{R} \to 0$)

$$|\bar{\epsilon}(\bar{R})|_{R\to 0} = \frac{g(D)}{\bar{R}^2} \quad \text{with} \quad g(D) = \left[-\frac{\pi \csc(D\pi/2)}{2^{D/2}\Gamma(D/2)K(D/2-1,\sqrt{g(D)})} \right]^{\frac{4}{2-D}}$$

, being K the modified Bessel function of the second kind

We also have that $\bar{R} = R/L,\ L = \sqrt{\hbar^2/2\mu_{B,AA}|E_2^{(D)}|}$ and $|E_2^{(D)}| = \frac{\hbar^2}{2\mu_{AB}\ a_{(D)}^2} \left(\frac{2^{D-2}}{\pi} \frac{\Gamma\left(\frac{D-2}{2}\right)\Gamma\left(\frac{D}{2}\right)}{\cot\left(\frac{D\pi}{2}\right) - i} \right)^{\frac{2}{D-2}}$

Effective potential for large distances ($\bar{R} \to \infty$)

$$|\epsilon(\bar{R})|_{\bar{R}\to\infty} = |E_2^D| + \frac{K\left(D/2-1,\bar{R}\right)}{d(D,\bar{R})}|E_2^D|$$

where $d(D,\bar{R}) = \frac{\bar{R}}{2}K(D/2,\bar{R}) + (1-D/2)\left[\frac{\pi \csc(D\pi/2)}{2^{D/2}\bar{R}^{(1-D/2)}\Gamma(D/2)} + K(D/2-1,\bar{R}) \right]$

R. M. Francisco, D. S Rosa, T. Frederico. Phys. Rev. A, 106, 063305 (2022)

The effective potential in the BO picture

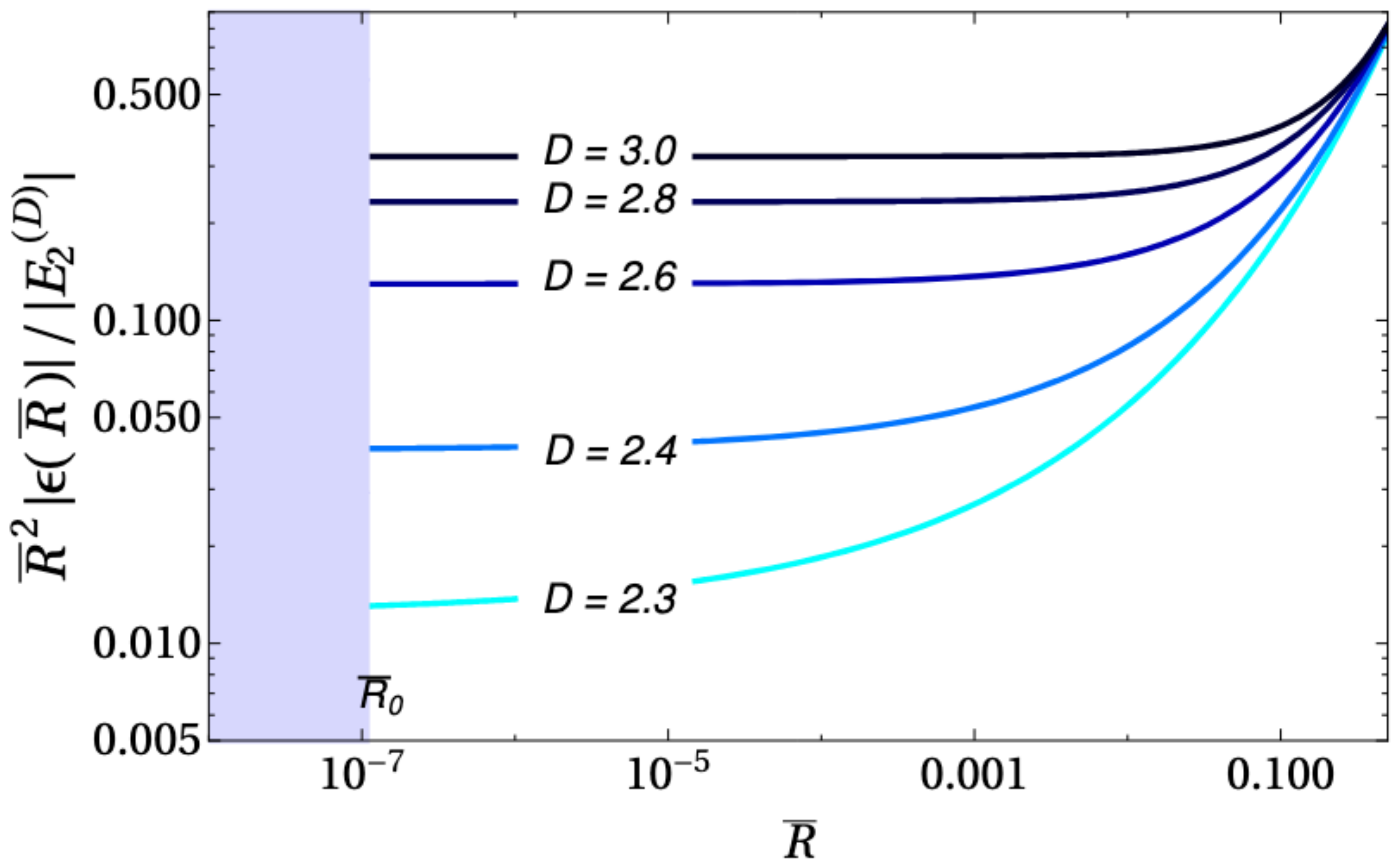


FIG. 6. Effective potential for different dimensions D. The band corresponds to the region where the short-range potential is repulsive to avoid the Thomas collapse.

R. M. Francisco, D. S Rosa, T. Frederico. Phys. Rev. A, 106, 063305 (2022)

The scale parameter in the BO

- At the unitarity limit, after some manipulation, we can write the heavy particle equation as

$$\left[-\frac{d^2}{d\bar{R}^2} - \frac{s_0^2 - 1/4}{\bar{R}^2}\right]\chi(\bar{R}) = -\kappa^2\,\chi(\bar{R})$$

where $\frac{m_A E_N}{\hbar^2} = -\kappa^2$

- This method gives as the dependence of the scale parameter on the variables of the system

$$s_0 = \sqrt{\frac{m_A}{2\mu_{B,AA}}(N-2)g(D) - \frac{(D-2+2l)^2}{4}}$$

- We can obtain the condition for the effective potential to be attractive

$$\frac{m_A}{2\mu_{B,AA}}(N-2)g(D) > \frac{(D-2+2l)^2}{4}$$

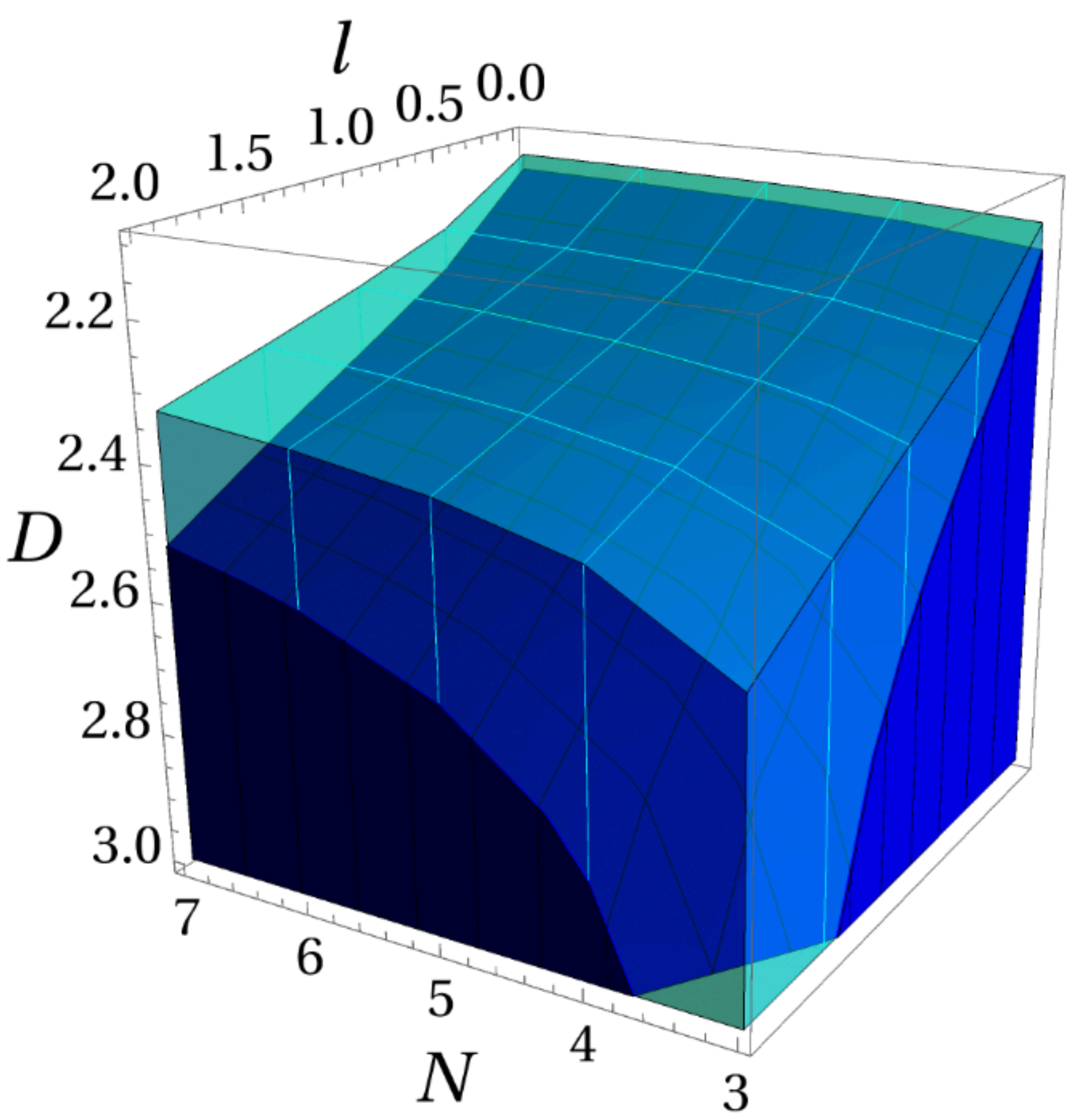
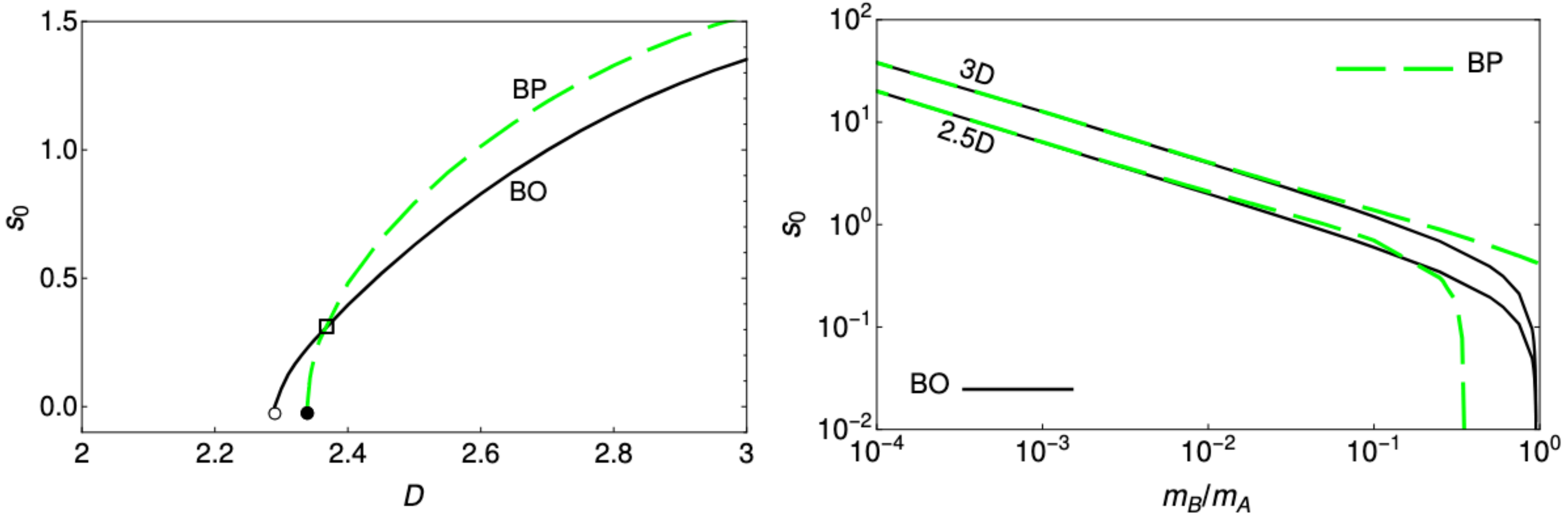


FIG. 4. Regions for the existence of the Efimov effect in the unitary limit for different dimensions, angular momentum and number of light-atoms. Lower blue and upper cyan regions presents results for $m_B/m_A = 6/133$ and $m_B/m_A = 0.1$, respectively.

R. M. Francisco, D. S Rosa, T. Frederico. Phys. Rev. A, 106, 063305 (2022)

The scale parameter in the BO



^{7}Li-^{87}Rb$_2$

D. S. Rosa, T. Frederico, R. M. Francisco, G. Krein, M. T. Yamashita, *Few-Body Syst* 65, 77 (2024)

Why noninteger dimensions?

- We can think in cold atomic gases being *squeezed* from 3d to 2d by means of **external confining potentials**

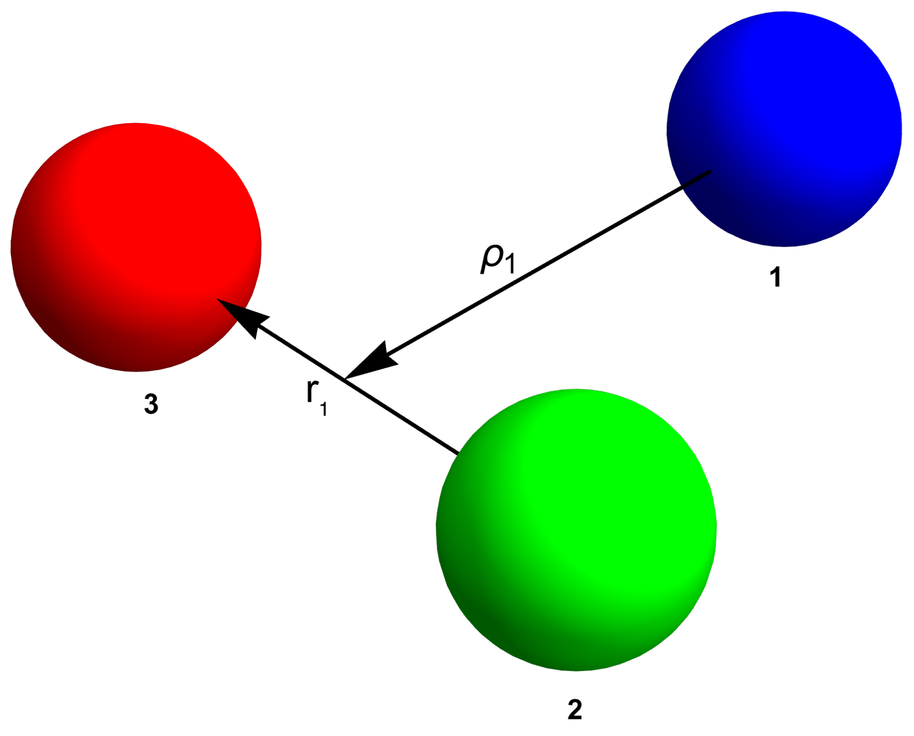
- The crucial development for practical application is the **translation function** established between the external potential and the noninteger dimensional method

$$\frac{b_{ho}}{r_{2D}} = \sqrt{\frac{3(D-2)}{(D-1)(3-D)}} \text{ where } \omega = \frac{\hbar}{m_\omega b_{ho}^2}$$

Let's look at the equations for more general systems

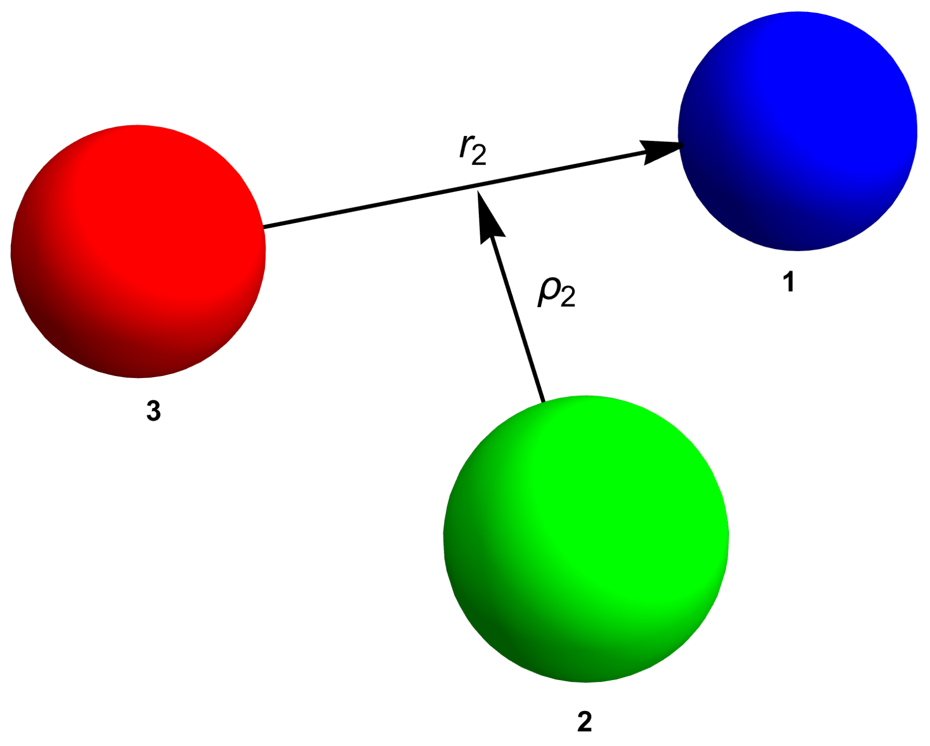
Garrido, E., Jensen, A. S. Physical Review Research, 2(3), 033261 (2020)

Faddeev decomposition and Jacobi Coordinates



$$\Psi = \psi^{(1)}(r_1, \rho_1) + \psi^{(2)}(r_2, \rho_2) + \psi^{(3)}(r_3, \rho_3)$$

- We obtain a system of coupled equations

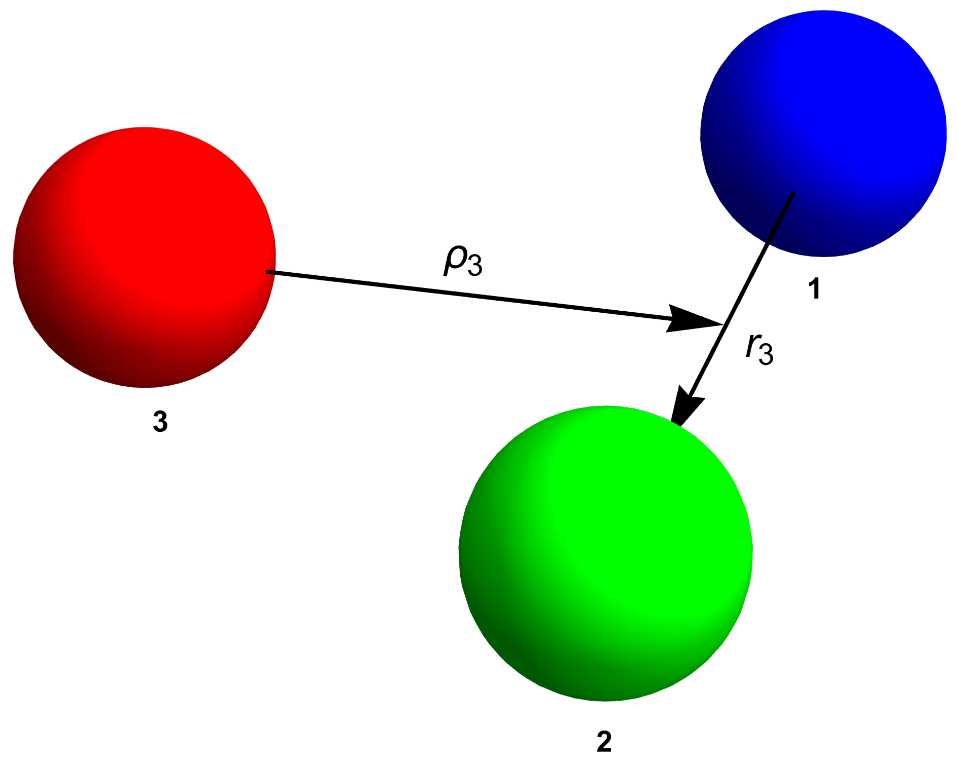


$$\begin{cases} (E - H_0)(\psi^{(1)}(r_1, \rho_1) = V_1[(\psi^{(1)}(r_1, \rho_1) + \psi^{(2)}(r_2, \rho_2) + \psi^{(3)}(r_3, \rho_3)] \\ (E - H_0)\psi^{(2)}(r_2, \rho_2) = V_2[(\psi^{(1)}(r_1, \rho_1) + \psi^{(2)}(r_2, \rho_2) + \psi^{(3)}(r_3, \rho_3)] \\ (E - H_0)\psi^{(3)}(r_3, \rho_3) = V_3[(\psi^{(1)}(r_1, \rho_1) + \psi^{(2)}(r_2, \rho_2) + \psi^{(3)}(r_3, \rho_3)] \end{cases}$$

- In D dimensions we can write a reduced wave function as

$$u^{(i)}(r_i, \rho_i) = (r_i, \rho_i)^{\frac{(D-1)}{2}} \psi^{(i)}(r_i, \rho_i)$$

- This gives for the radial equation (with null angular momentum)



$$\left[-\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial \rho^2} + \frac{(D-1)(D-3)}{4} \left(\frac{1}{r} + \frac{1}{\rho} \right) - k^2 \right] u(r, \rho) = 0$$

We consider contact interactions via the Bethe-Peierls boundary condition

- In hiperradial coordinates ($r = R\sin\alpha$ and $\rho = R\cos\alpha$), we can write

$$\left[-\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} - \frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2} + \frac{(D-1)(D-3)}{4R^2}\left(\frac{1}{\sin^2\alpha\cos^2\alpha}\right) - k^2\right] u(R,\alpha) = 0$$

- Writing the solution as $u(R,\alpha) = F(R)G(\alpha)$, we can separate the Schrodinger equation as

$$\left[-\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} - k^2\right] F(R) = -\frac{s_0^2 - \frac{1}{4}}{R^2}F(R) \quad \text{and} \quad \left(-\frac{\partial^2}{\partial \alpha^2} - s_0^2\right) G(\alpha) = -\frac{(D-1)(D-3)}{4\sin^2\alpha\cos^2\alpha}G(\alpha)$$

- The solution of the radial part is the modified Bessel function of second kind and order s_n

$$F(R) = \frac{K_{s_n}\left(\sqrt{2}\,\kappa_N R\right)}{R}$$

- The angular equation can be written in the form of the associated Legendre differential equation, which has the solution

$$G^{(i)}(\alpha_{(i)}) = C^{(i)} \sin 2\alpha_{(i)}{}^{1/2}\left(P^{D/2-1}_{s/2-1/2}(\cos 2\alpha_{(i)}) - \frac{2}{\pi}\tan\left[\pi\left(\frac{s-1}{2}\right)\right] Q^{D/2-1}_{s/2-1/2}(\cos 2\alpha_{(i)})\right)$$

where $P^{D/2-1}_{s_n/2-1/2}$ e $Q^{D/2-1}_{s_n/2-1/2}$ are the associated Legendre functions

D-dimensional Bethe-Peierls boundary condition

$$\frac{C^{(i)}}{2}\left[\left(\cot\alpha_i\right)^{\frac{D-1}{2}}\left(\sin 2\alpha_i\frac{\partial}{\partial\alpha_i}+D-3\right)G^{(i)}(\alpha_i)\right]_{\alpha_i\to 0}+(D-2)\left[\frac{C^{(j)}\,G^{(j)}(\theta_k)}{\left(\sin\theta_k\cos\theta_k\right)^{\frac{D-1}{2}}}+\frac{C^{(k)}\,G^{(k)}(\theta_j)}{\left(\sin\theta_j\cos\theta_j\right)^{\frac{D-1}{2}}}\right]=0$$

D-dimensional Faddeev component

$$\psi(\mathbf{r})\xrightarrow[r\to 0]{}\left(\frac{1}{r}-\frac{1}{a}\right)$$

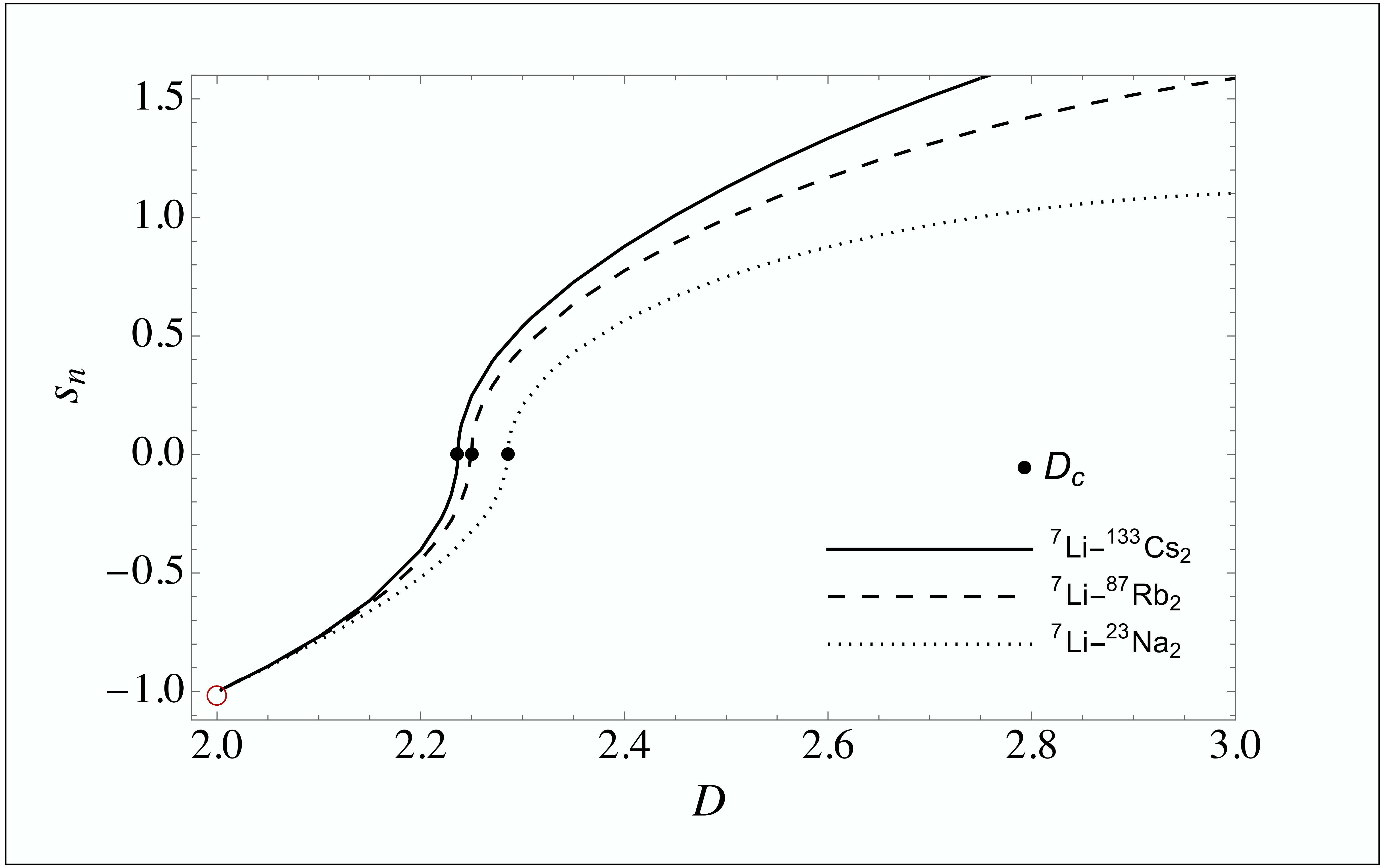
$$\psi^{(i)}(r_i',\rho_i')=C^{(i)}\frac{K_{s_n}\left(\sqrt{2}\kappa_N\sqrt{r_i'^2+\rho_i'^2}\right)}{\left(r_i'^2+\rho_i'^2\right)^{D/2-1/2}}\frac{\sqrt{\sin\left[2\arctan\left(r_i'/\rho_i'\right)\right]}}{\left\{\cos\left[\arctan\left(r_i'/\rho_i'\right)\right]\ \sin\left[\arctan\left(r_i'/\rho_i'\right)\right]\right\}^{D/2-1/2}}$$

$$\times\left[P^{D/2-1}_{s_n/2-1/2}\left\{\cos\left[2\arctan(r_i'/\rho_i')\right]\right\}-\frac{2}{\pi}\tan\left[\pi(s_n-1)/2\right]Q^{D/2-1}_{s_n/2-1/2}\left\{\cos\left[2\arctan(r_i'/\rho_i')\right]\right\}\right]$$

E. Garrido, A. S. Jensen. Physical Review Research 2, 033261 (2020)

D. S. Rosa, T. Frederico, G. Krein, M. T. Yamashita. Physical Review A 108, 032214 (2023)

Scale parameter as a function of the noninteger dimension



R. M. Francisco, D. S. Rosa, T. Frederico, M. T. Yamashita, G. Krein. Phys. Rev. A 112, 033315 (2025)

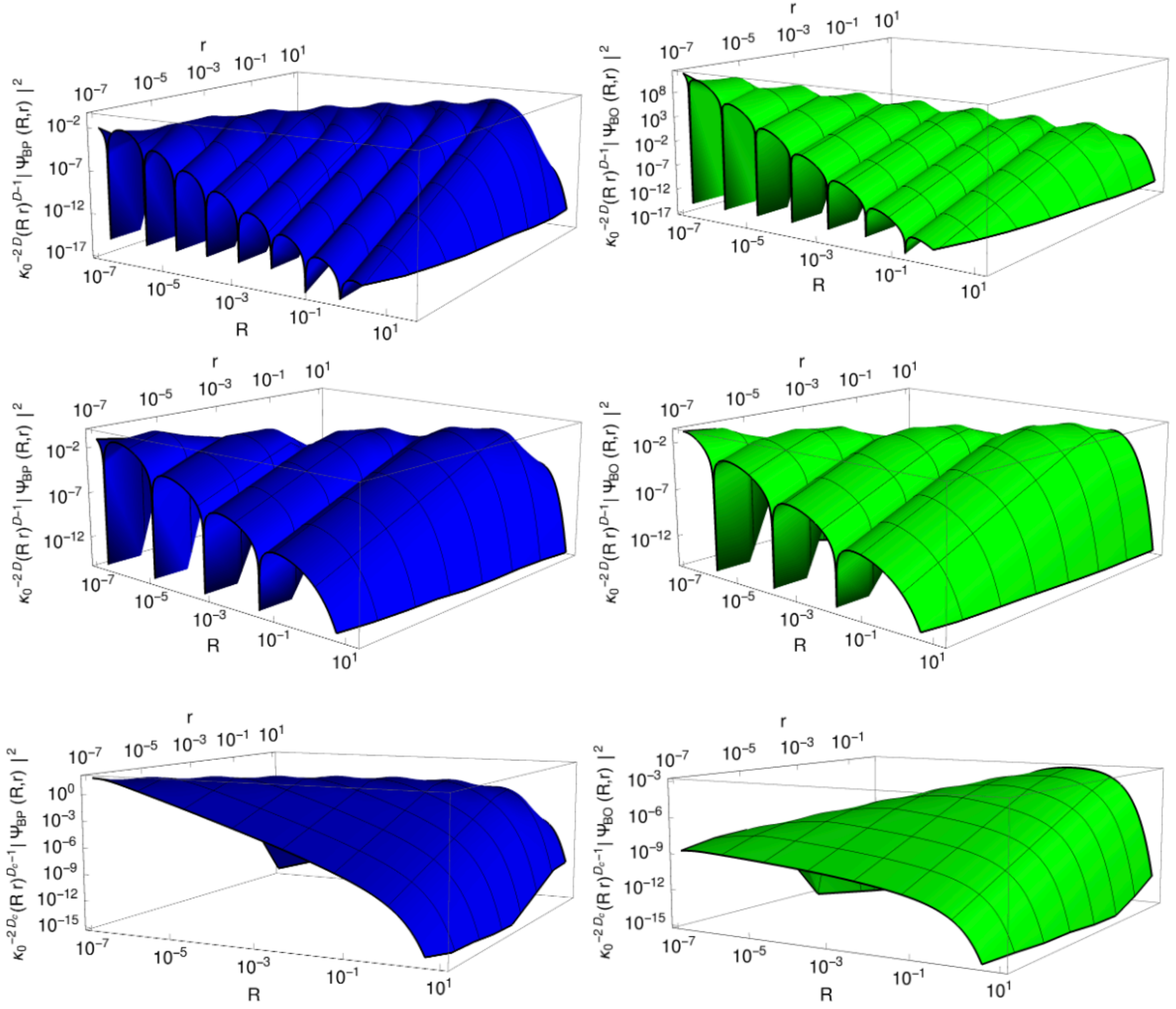


Fig. 3 Dimensionless radial distributions as a function of dimensionless quantities $R = \kappa_0 R_3$ (^{7}Li-^{87}Rb relative distance) and $r = \kappa_0 r_3$ (^{7}Li relative distance to the ^{7}Li-^{87}Rb system). We consider the three-body system ^{87}Rb$_2$-^{7}Li embedded in three different dimensions $3D$ (upper panels), $2.5D$ (middle panels) and at the critical dimension (lower panels). Left panels show the results from BP approach and right panels present results obtained with the BO approximation. The angle between $\vec{r}$ and $\vec{R}$ is fixed to $\pi/3$.

Unatomic states

The three-body wave function with the condition $\kappa_0 R \to 0$ is given by

$$\psi(R,\alpha)\big|_{\kappa_0 R\to 0} \propto \frac{\mathcal{T}(\alpha, s_n)}{R^{D-1}}\Lambda(R, s_n) \quad \text{em que} \quad \Lambda(R, s_n) = \kappa_0^{s_n}\frac{\Gamma(-s_n)}{2^{s_n/2}}R^{s_n} + \frac{\kappa_0^{-s_n}\Gamma(s_n)}{2^{-s_n/2}}R^{-s_n}$$

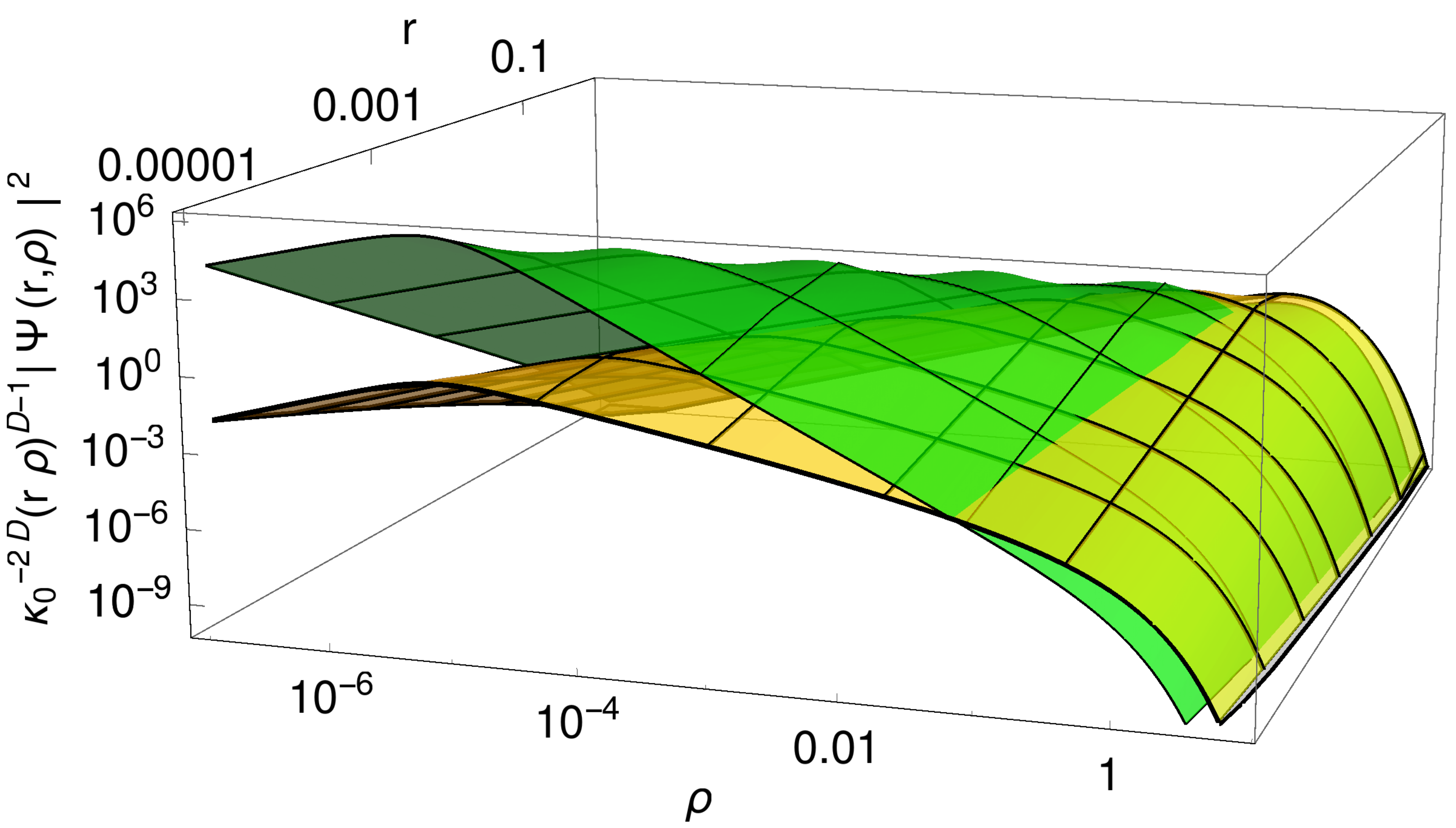
Efimov regime: s_n is imaginary $\rightarrow$ well known oscillatory behavior

Unatomic Regime: s_n is real $\rightarrow$ power-law behavior

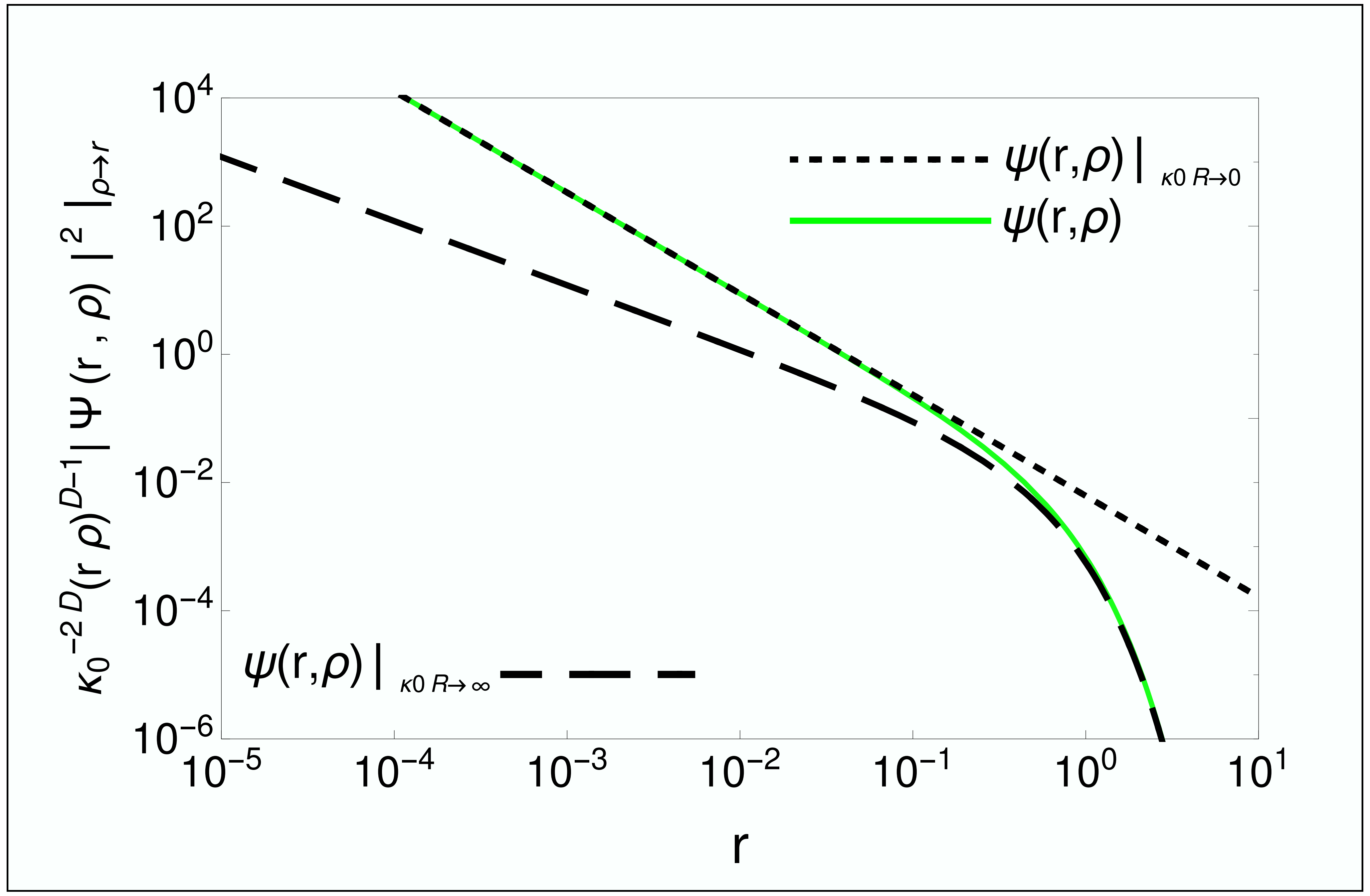
$$\psi(R,\alpha)\big|_{\kappa_0 R\to 0} \propto \frac{\mathcal{T}(\alpha, s_1)}{R^{D-s_1-1}} \quad \Longrightarrow \quad \psi(\lambda\bar{r}, \lambda\bar{\rho}) \propto \psi(\bar{r}, \bar{\rho}) \Longrightarrow$$ Continuous scale invariance

D. S. Rosa, R. M. Francisco, T. Frederico, G. Krein, M. T. Yamashita. Phys. Rev. A, 110, 042803 (2024)

Unatomic states



Probability density at the critical dimension (yellow curve) and in the unatomic regime (green curve) for a system with three identical bosons



Probability density in the unatomic regime for a system with three identical bosons. Black curves for asymptotic functions and green curve for regular function

D. S. Rosa, R. M. Francisco, T. Frederico, G. Krein, M. T. Yamashita. Phys. Rev. A, 110, 042803 (2024)

Momentum space

We can write the D-dimensional Faddeev component in momentum space for shallow bound states as

$$\chi^{(i)}(q_i') \underset{q_i \gg \kappa_0}{=} C^{(i)} \kappa_0^{1-D} \mathfrak{F}_{(D,s_n)} \left(\frac{q_i'}{\sqrt{2}\kappa_0} \right)^{1-D} \left[\mathscr{G}_{(+s_n)} \left(\frac{q_i'}{\sqrt{2}\kappa_0} \right)^{s_n} + \mathscr{G}_{(-s_n)} \left(\frac{q_i'}{\sqrt{2}\kappa_0} \right)^{-s_n} \right]$$

Where $\mathscr{G}_{(\pm s_n)} = \dfrac{\Gamma(\pm s_n)}{\Gamma\left[(D-1 \pm s_n)/2\right] \Gamma\left[(1 \pm s_n)/2\right]}$

and $\mathfrak{F}_{(D,s_n)} \equiv \dfrac{2^{1+D/2} \pi^{D-1} \, \Gamma\left[\mathscr{F}^{+}_{(D,s_n)}\right] \Gamma\left[\mathscr{F}^{-}_{(D,s_n)}\right]}{(2-D)} \cos\left[\frac{\pi}{2}(D - s_n)\right] \csc\left(\frac{\pi}{2} s_n\right)$

D. S. Rosa, R. M. Francisco, T. Frederico, G. Krein, M. T. Yamashita. Phys. Rev. A, 110, 042803 (2024)
R. M. Francisco, D. S. Rosa, T. Frederico, M. T. Yamashita, G. Krein. Phys. Rev. A 112, 033315 (2025)

Momentum space

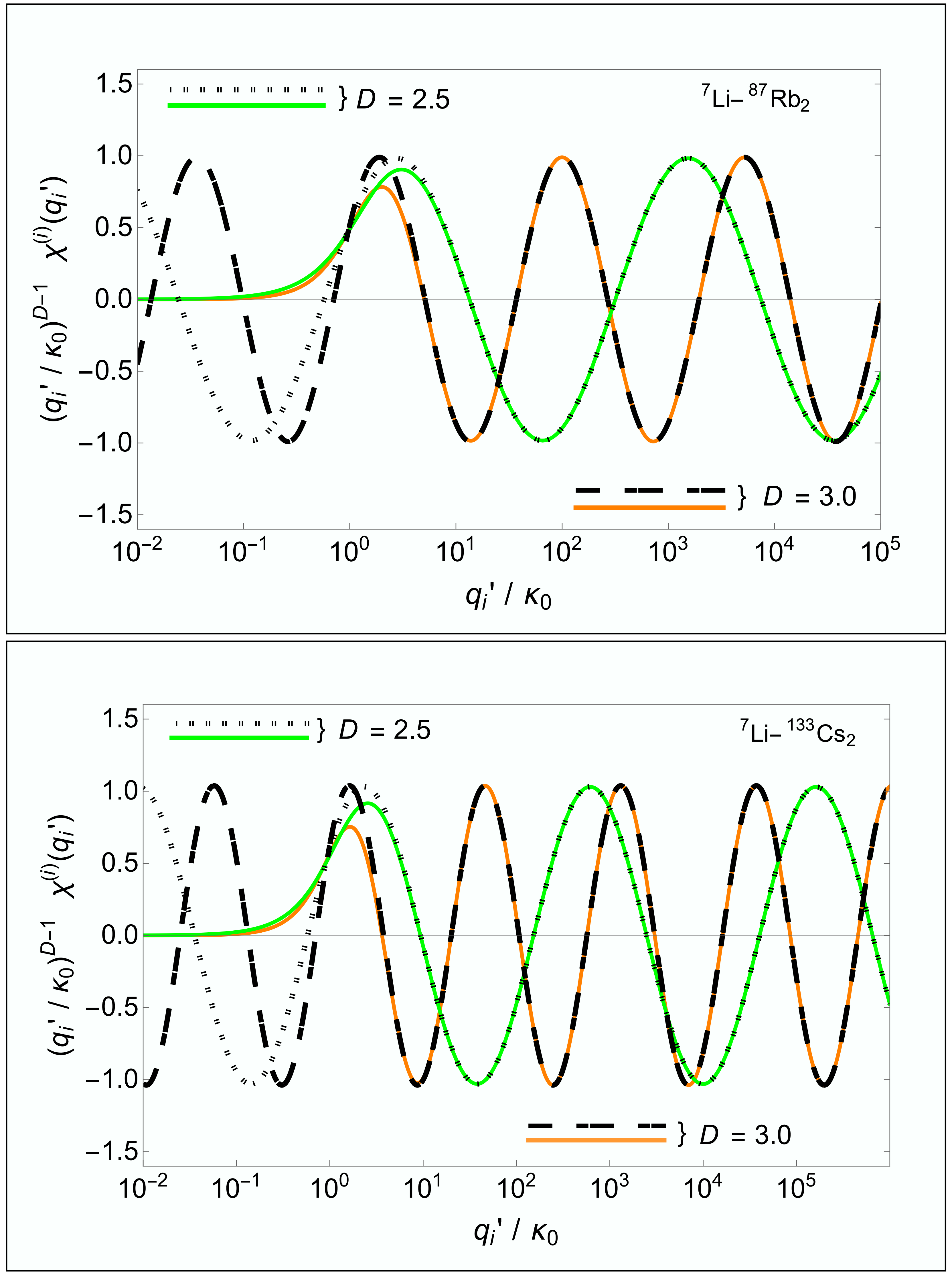
In the unatomic regime ($2 < D \leq D_c$), where the values of s_n are real, the spectator function reads

$$\chi^{(i)}(q_i') \underset{q_i \gg \kappa_0}{=} C^{(i)} \kappa_0^{1-D} \mathfrak{F}_{(D,s_{0_u})} \mathcal{G}_{(-s_{0_u})} \left(\frac{q_i'}{\sqrt{2}\kappa_0} \right)^{1-D-s_{0_u}}$$

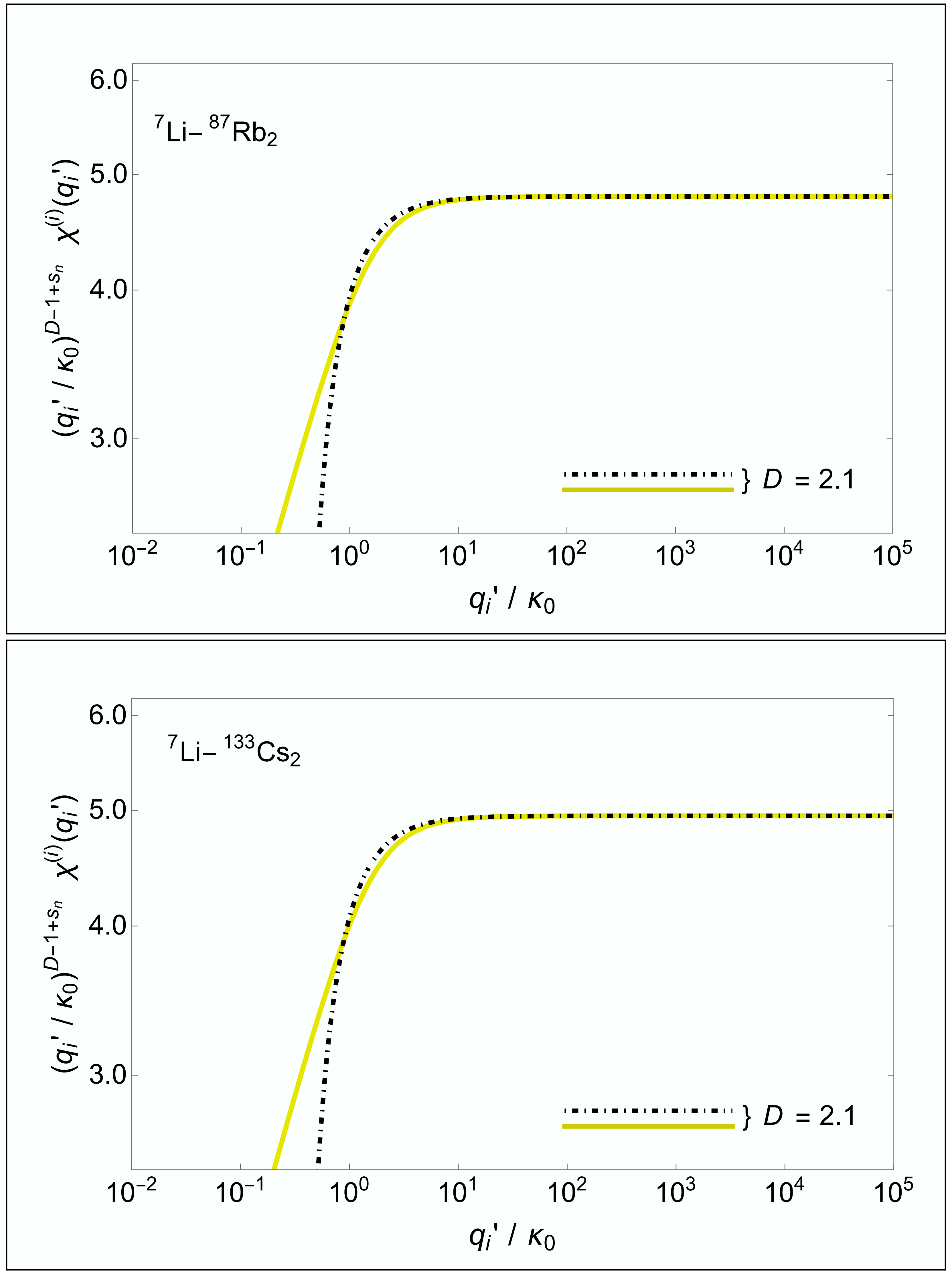
The presence of the power-law symmetry, which is the signature of unatomic states, can be appreciated as

$$\chi^{(i)}(\lambda\, q_i') \rightarrow \lambda^{1-D-s_{0_u}} \chi^{(i)}(q_i')$$

D. S. Rosa, R. M. Francisco, T. Frederico, G. Krein, M. T. Yamashita. Phys. Rev. A, 110, 042803 (2024)
R. M. Francisco, D. S. Rosa, T. Frederico, M. T. Yamashita, G. Krein. Phys. Rev. A 112, 033315 (2025)



Faddeev components in momentum space for the Efimov regime: black curves for asymptotic functions and colored for the regular ones



Faddeev components in momentum space for the Unatomic regime: black curves for asymptotic functions and colored for the regular ones

R. M. Francisco, D. S. Rosa, T. Frederico, M. T. Yamashita, G. Krein. Phys. Rev. A 112, 033315 (2025)

Probability density in momentum space

Efimov regime $$n_B(q_B) \underset{q_B \gg \kappa_0}{=} \frac{C_2}{q_B^4} + \frac{C_3'}{q_B^{D+2}} + \frac{C_3}{q_B^{D+2}} \cos\left[2s_{0_E} \ln\left(\frac{q_B/\kappa_0^*}{(4\mu_A\mu_B)^{1/4}}\right) + \Phi\right] + \cdots$$

Unatomic regime $$n_B(q_B) \underset{q_B \gg \kappa_0}{=} \frac{C_2}{q_B^4} + C_3\left[\frac{q_B/\kappa_0}{(4\mu_A\mu_B)^{1/4}}\right]^{-(D+2+2s_{0_u})} + \cdots$$

$$\kappa_0^* \equiv \kappa_0/\exp\left\{\frac{1}{s_{0_E}} \arctan\left[\frac{\mathrm{Im}\ \mathcal{G}_{(+is_{0_E})}}{\mathrm{Re}\ \mathcal{G}_{(+is_{0_E})}}\right]\right\}$$

For $2 < D \leq \overline{D}$, the tail of the momentum distribution is entirely defined by the three-body parameter

$$n_B(q_B) \underset{q_B \gg \kappa_0}{=} \frac{C_3}{q_B^{D+2+2s_{0_u}}} + \cdots$$

where $\Delta_n \equiv -(D + 2 + 2s_{0_u})$ is a scaling dimension for $n(\lambda\ q) \to \lambda^{\Delta_n} n(q)$

H. Hammer, D.T. Son, *Proc. Natl. Acad. Sci. U.S.A.* 118 35 (2021)

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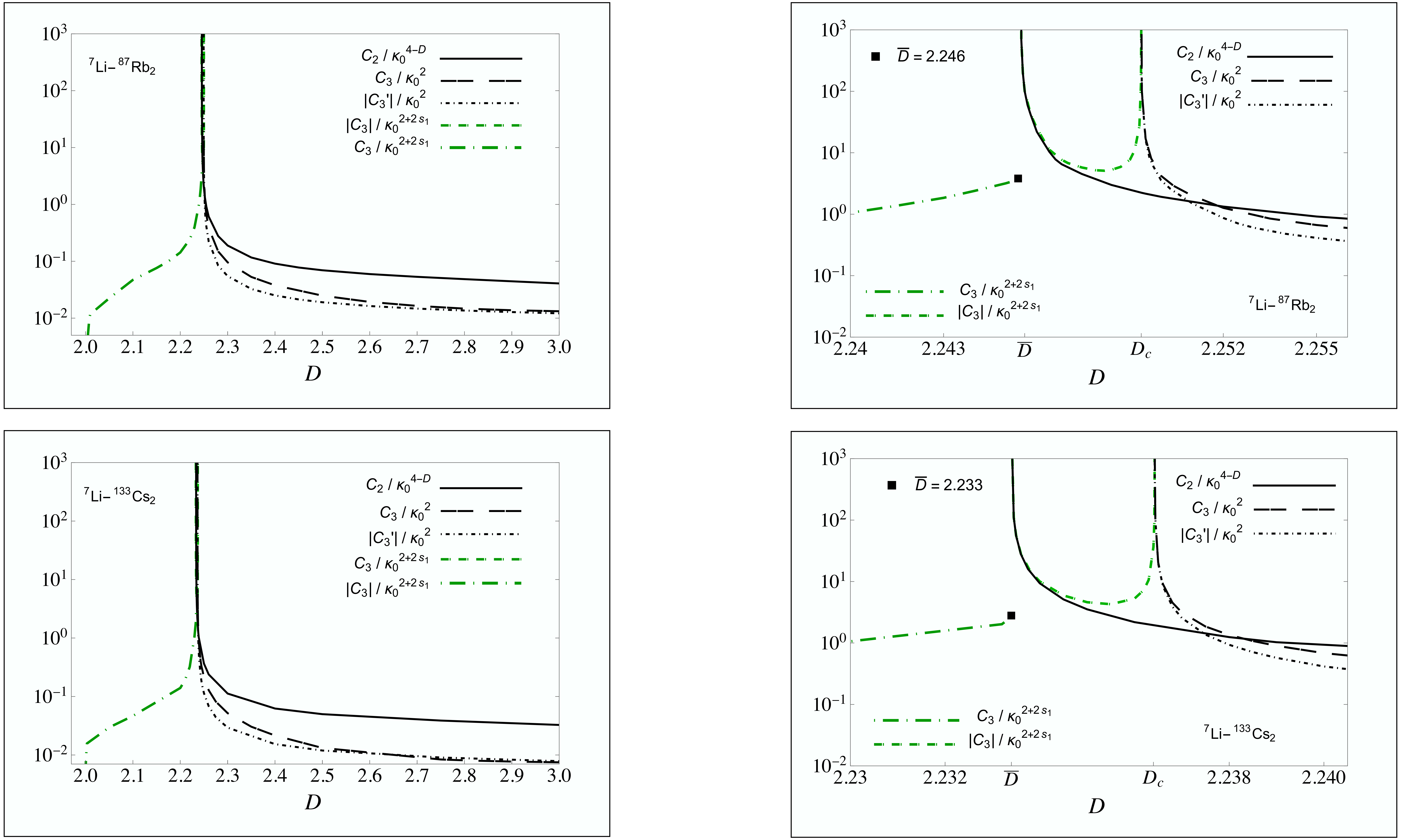

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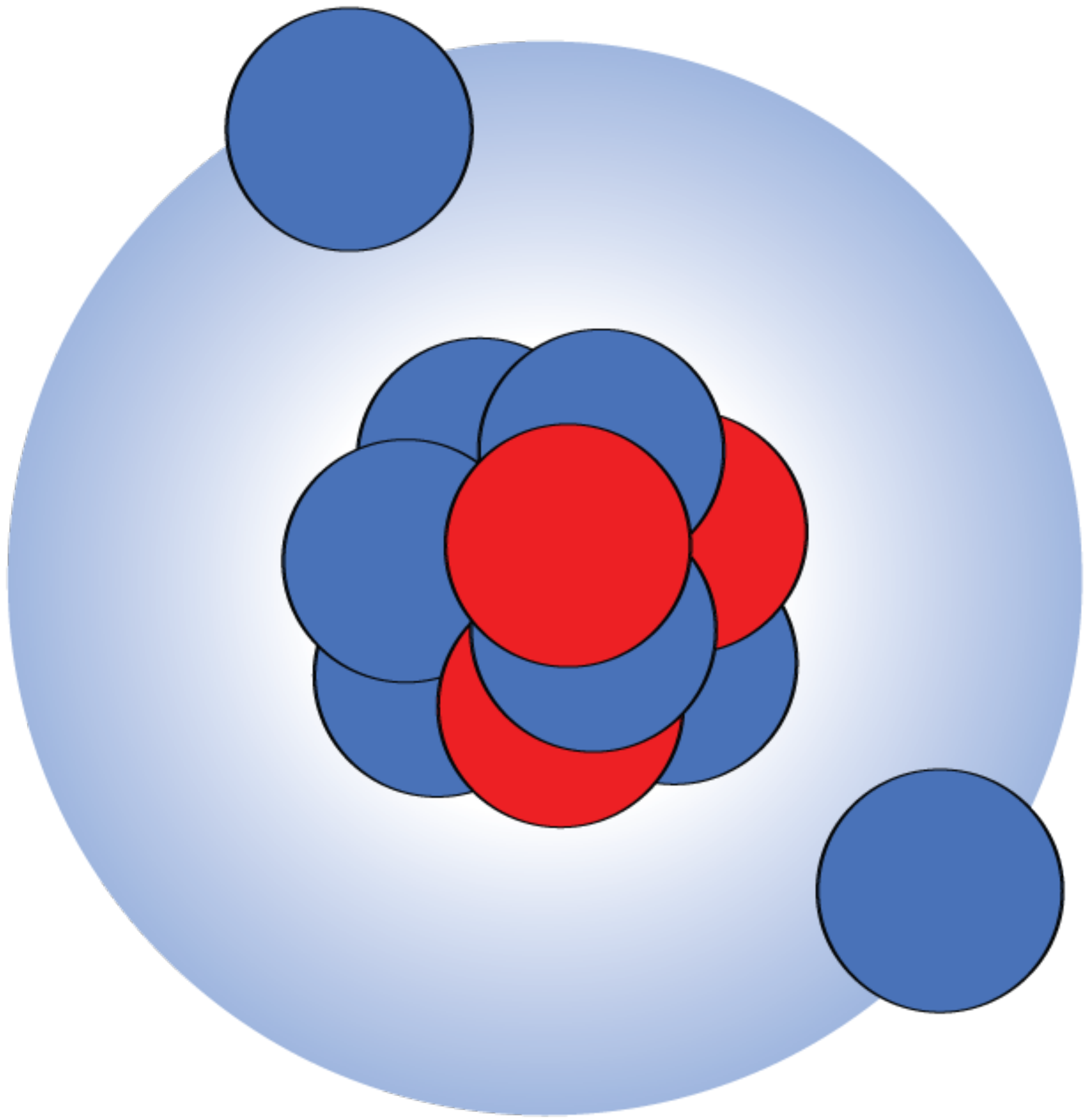
Two-neutron halo nuclei

- A dense, tightly-bound core with two valence neutrons orbiting at a great distance, forming a diffuse “halo."

- The halo neutrons are very weakly bound, characterized by an extremely low two-neutron separation energy ($s2n$)



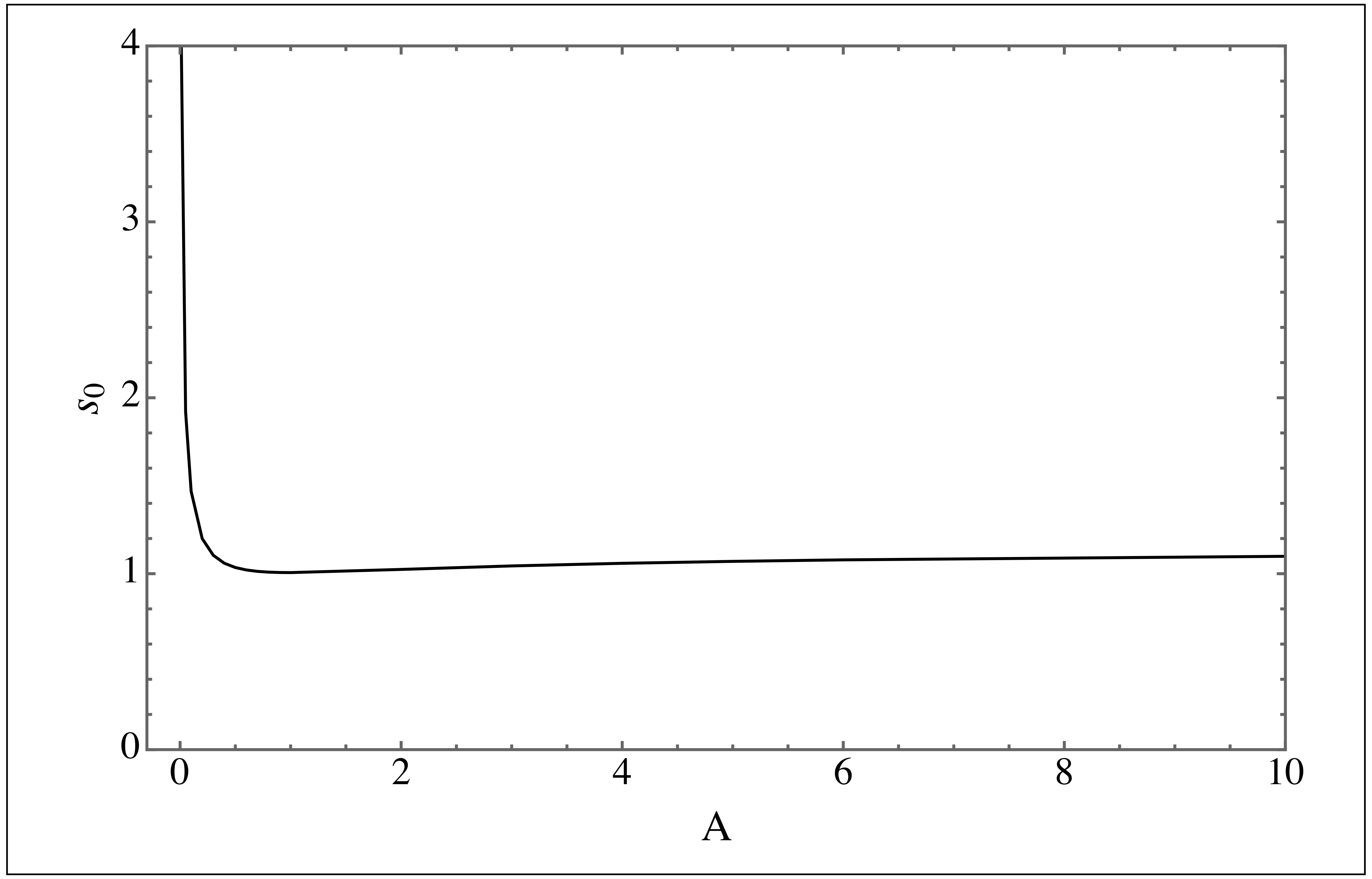
Credits: https://www.sciencenews.org/article/rare-isotope-elements-new-particle-accelerator-atom-nucleus (T. Tibbits)

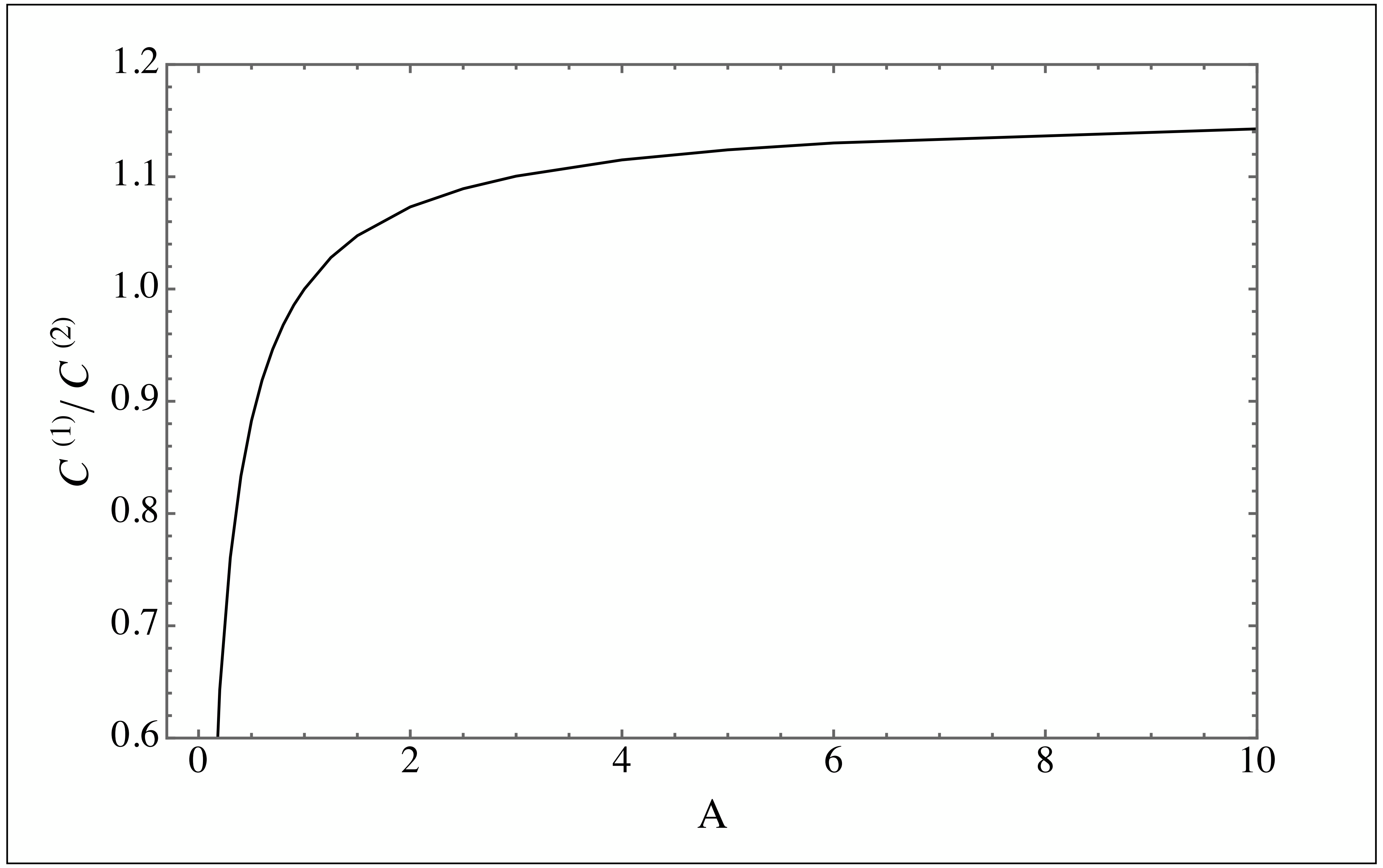
T. Kobayashi et al., *Breakup of a new type of nuclear halo in 11Li.*, Phys. Rev. Lett. **60**, 2599 (1988)
Tanihata et al., *Measurements of interaction cross sections and nuclear radii in the light p-shell region*., Phys. Rev. Lett. **55**, 2676 (1985)
M. Fukuda et al., *Neutron halo in 14Be*., Phys. Lett. B **268**, 339 (1991)

Wave function for D = 3

$$\Psi(r_i,\rho_i,\theta_{r\rho}) = \frac{K_{s_n}\left(\sqrt{2}\,\kappa_N\sqrt{r_i^2+\rho_i^2}\right)}{\left(r_i^2+\rho_i^2\right)}\left(\sum_{j=1}^{3} C^{(j)}\frac{\sin\left(s_n\left(\frac{\pi}{2}-\arctan\left(\frac{r_j}{\rho_j}\right)\right)\right)}{\sin\left(2\arctan\left(\frac{r_j}{\rho_j}\right)\right)}\right)_i$$

where K_{s_n} is the Bessel function of the second kind and order s_n and κ_N is the three-body binding momentum

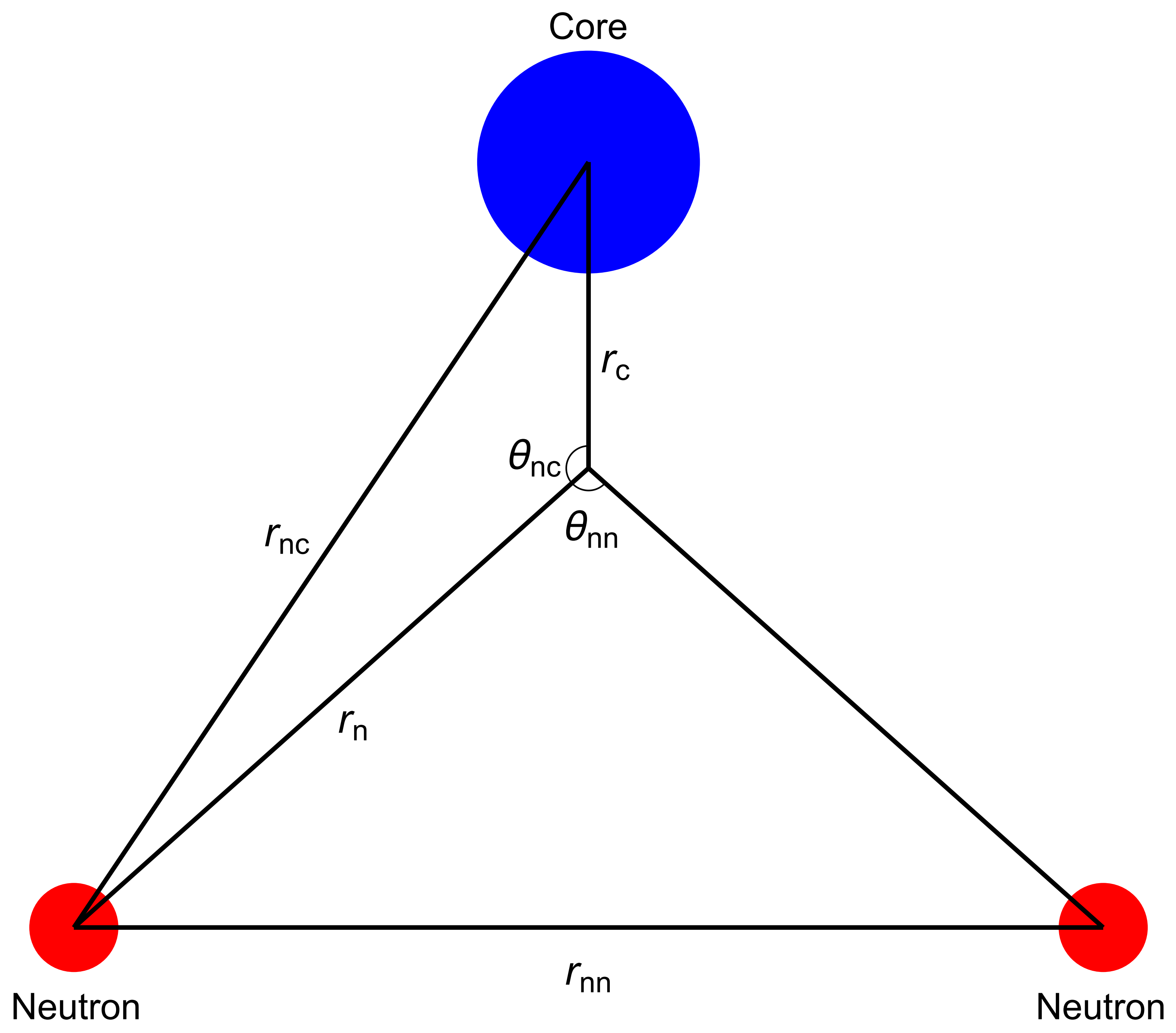


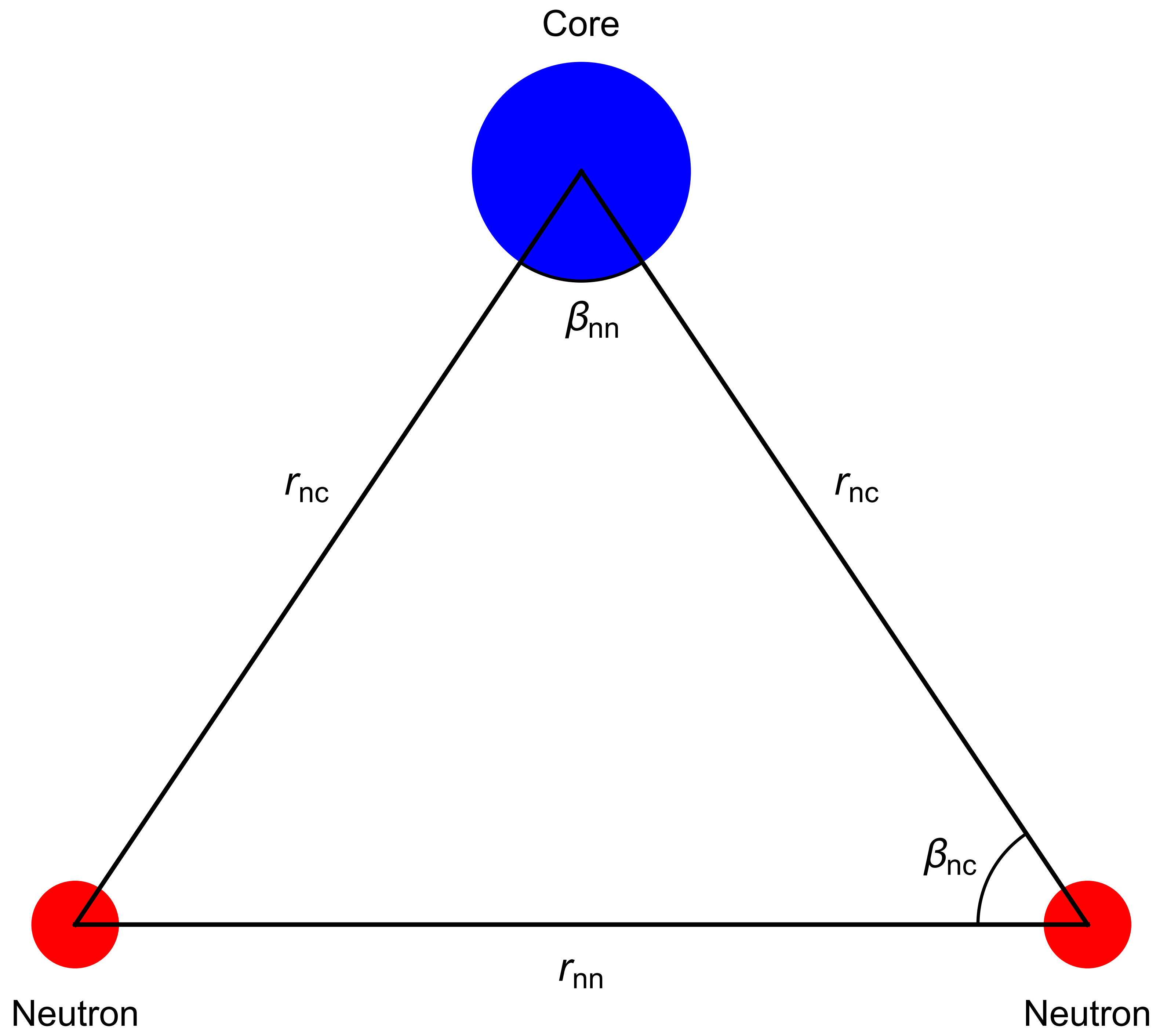


R. Y. Castin, F. Werner, Phys. Rev. A 83 (2011) 063614)

V. Efimov, Soviet Journal of Nuclear Physics 12 (5) (1971) 589–595, original: Yadernaya Fizika 12 (1970) 1080–1091.

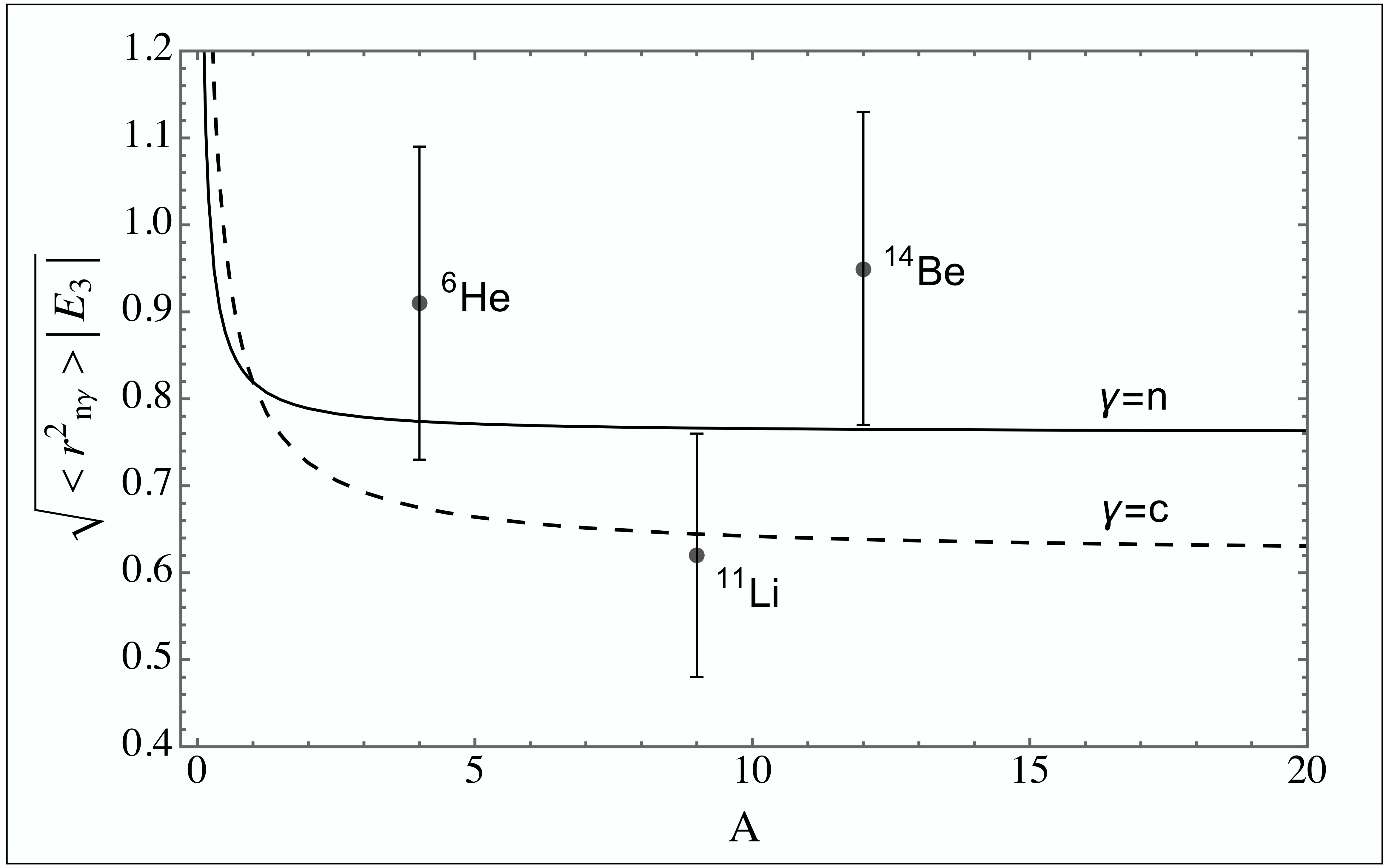
Separation distances

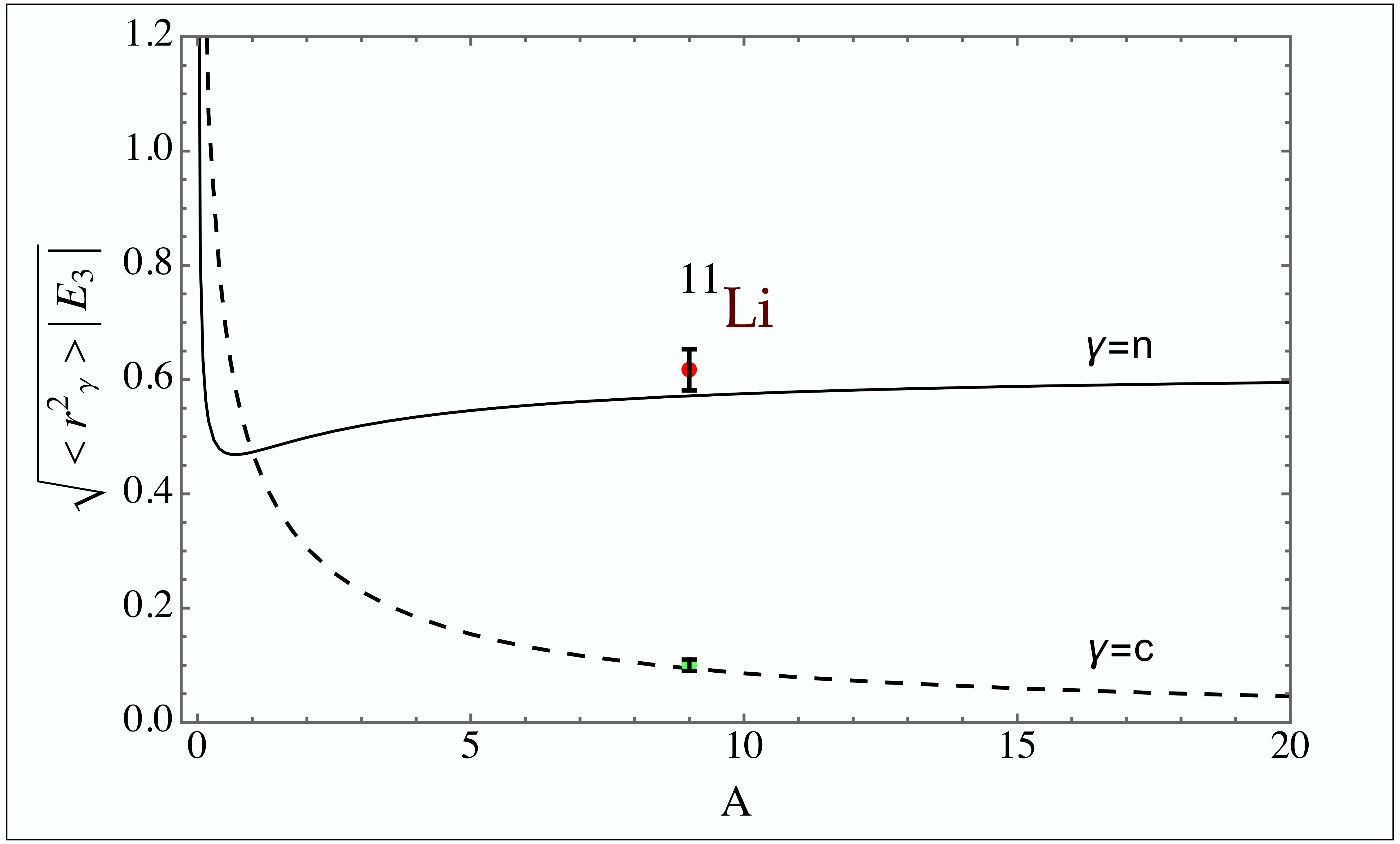




Dimensionless root mean square distances

This is a universal quantity: $\sqrt{\langle O^2 \rangle |E_3|}$





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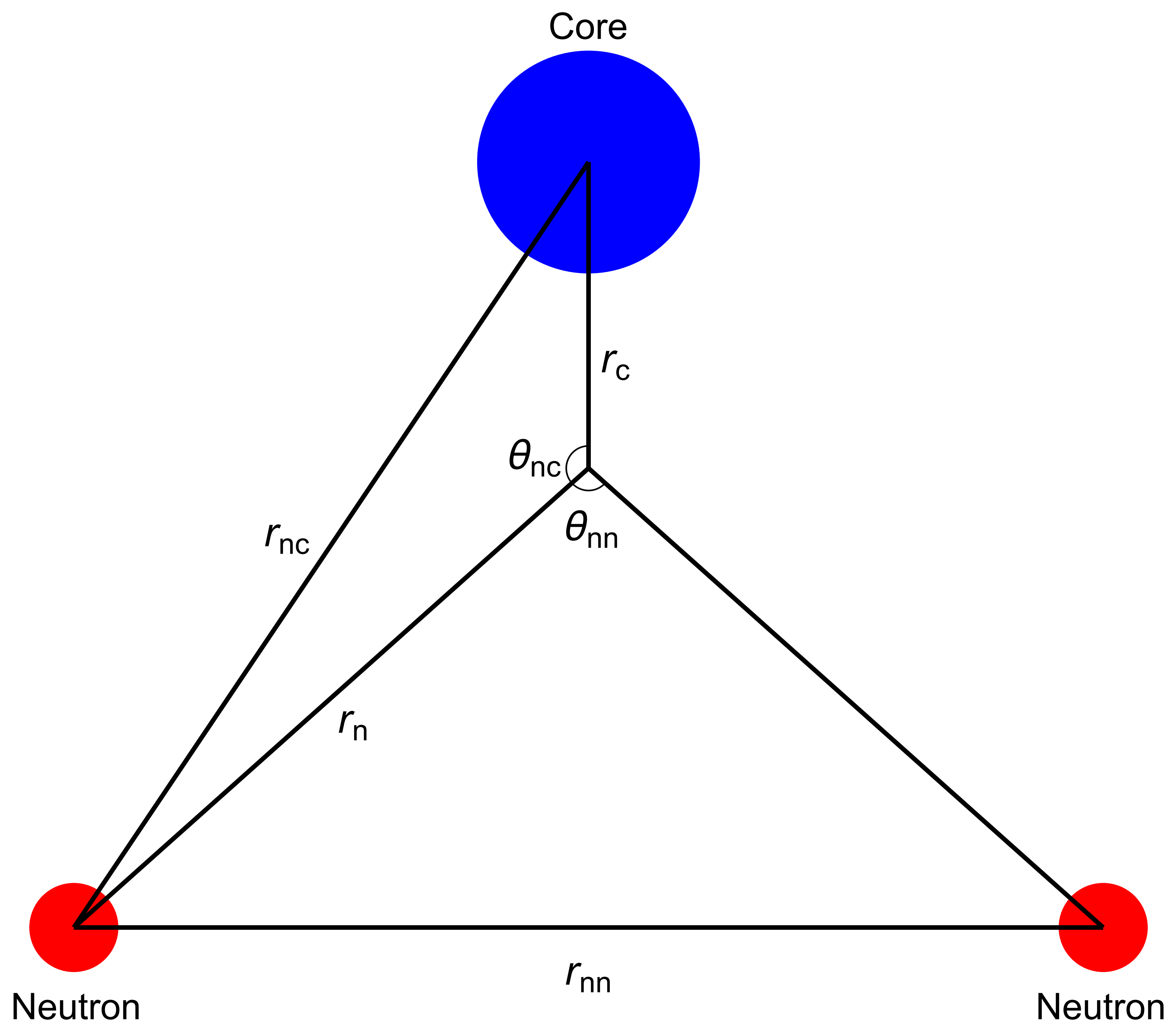
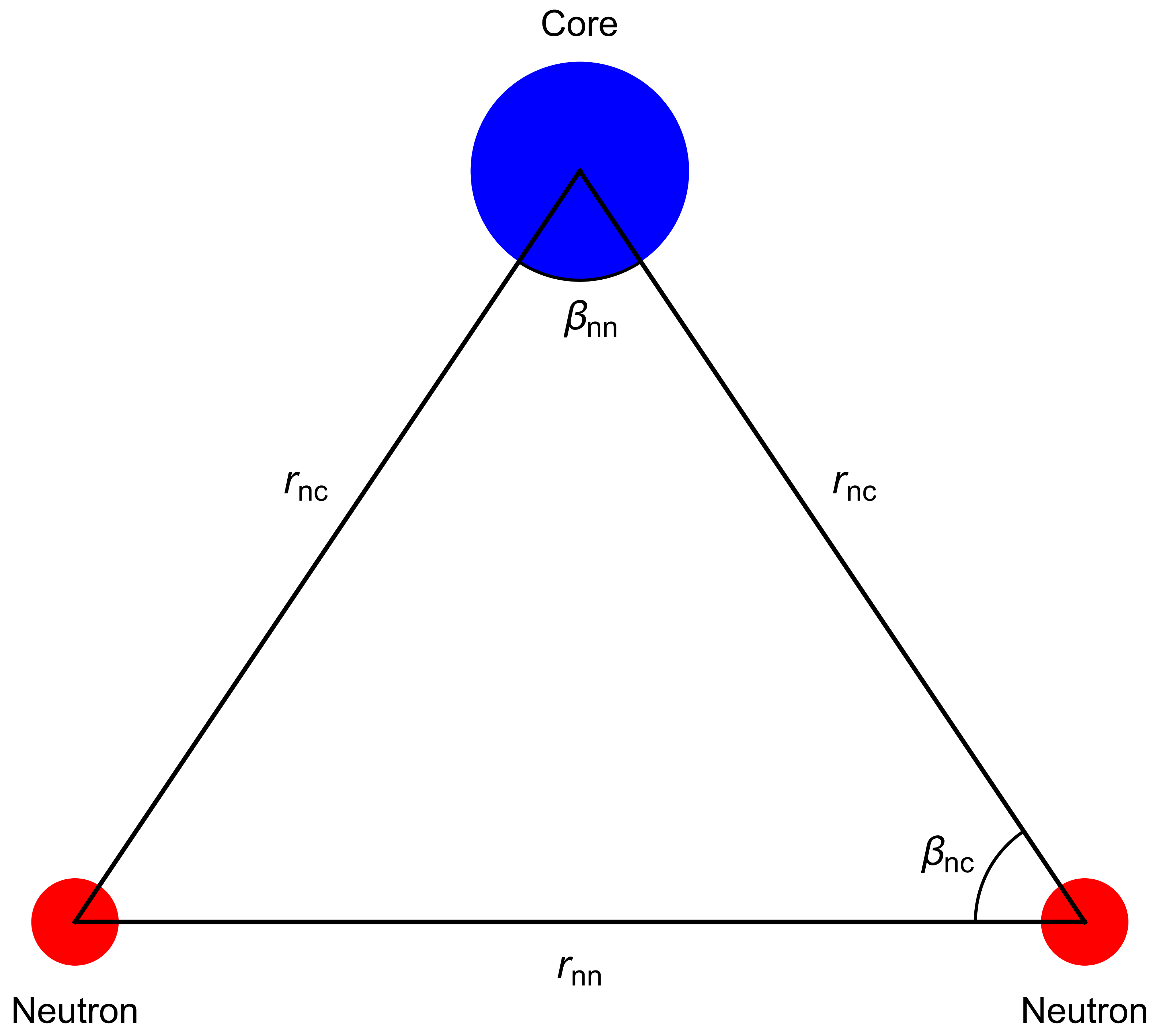
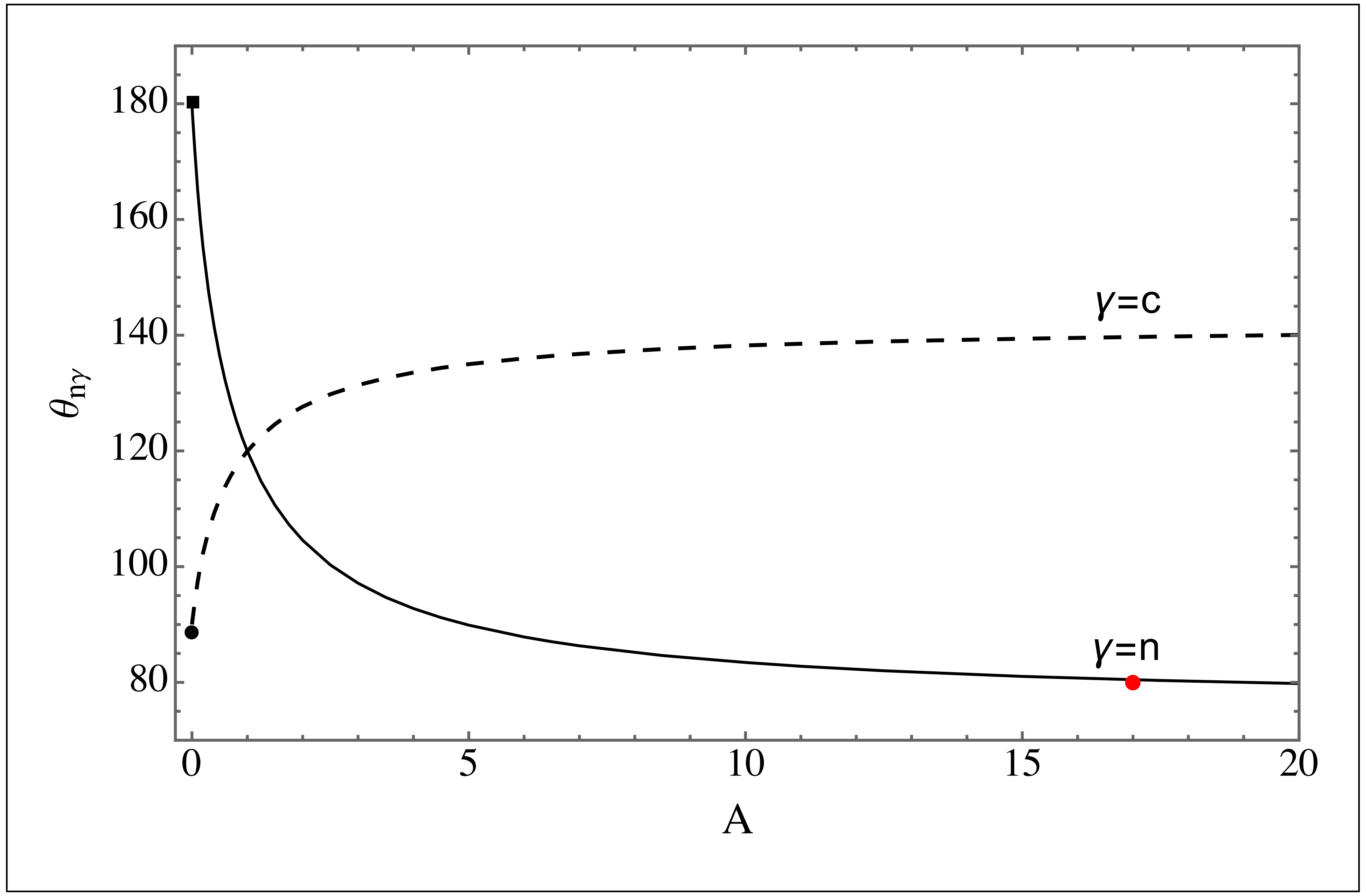
Opening angles





Opening angles



$$\theta_{nc} = \cos^{-1}\frac{\langle \vec{r}_c \cdot \vec{r}_n \rangle}{\sqrt{\langle r_c^2\rangle\langle r_n^2\rangle}} = \cos^{-1}\left(\frac{\langle r_c^2\rangle + \langle r_n^2\rangle - \langle r_{nc}^2\rangle}{2\sqrt{\langle r_c^2\rangle\langle r_n^2\rangle}}\right)$$

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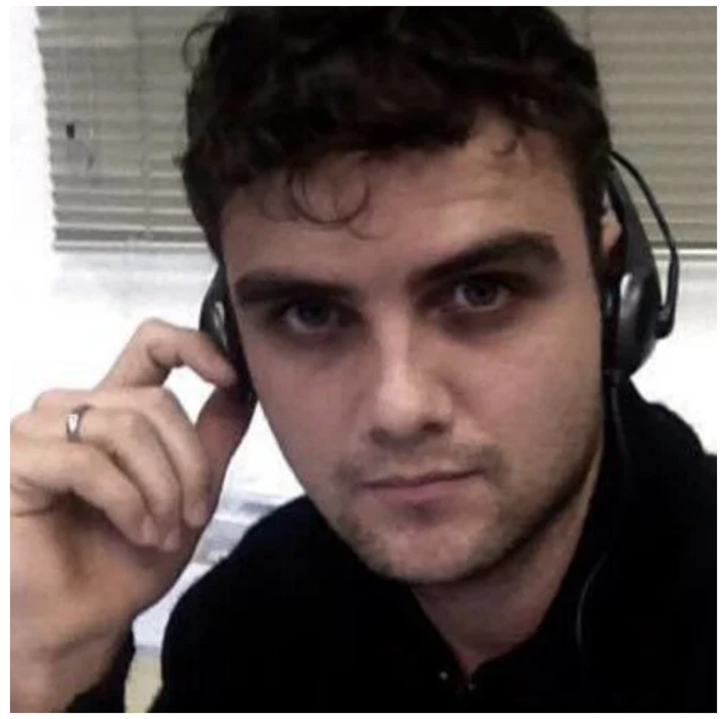
Dr. Tobias Frederico - ITA

Dr. Marcelo Takeshi Yamashita - IFT/Unesp



Dr. Derick dos Santos Rosa

Dr. Gastão Krein - IFT/Unesp

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Aknowledgments

Thank you for your attention!