The Three-Body Limit Cycle: Universal Form for General Regulators

Feng Wu

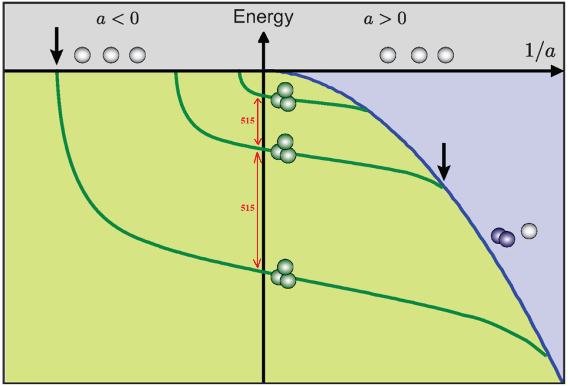
ECT* Workshop "Pan-American Few-Body Physics Boot Camp: Fostering Collaboration"

October 20, 2025

L. Chen, F. Wu, X. Lin, S. König, U. van Kolck, P. Zhang, arXiv:2509.04746.

cnrs
université PARIS-SACLAY
IJCLab Irène Joliot-Curie
Laboratoire de Physique des 2 Infinis

Efimov effect



https://physics.aps.org/articles/v3/9

- Efimov trimers,
  $B_{3,i}/B_{3,i+1} = e^{2\pi/s_0} \approx 515,\ s_0 = 1.00624$
  V. Efimov, PLB 33, 563 (1970)
- At the unitarity limit, discrete scale invariance (DSI) is exact.
- DSI of observables are expected in systems that exhibit limit cycles in their renormalization group

Renormalization group limit cycle

PHYSICAL REVIEW D VOLUME 3, NUMBER 8 15 APRIL 1971

Renormalization Group and Strong Interactions*

KENNETH G. WILSON
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
and
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850†

(Received 30 November 1970)

The renormalization-group method of Gell-Mann and Low is applied to field theories of strong interactions. It is assumed that renormalization-group equations exist for strong interactions which involve one or several momentum-dependent coupling constants. The further assumption that these coupling constants approach fixed values as the momentum goes to infinity is discussed in detail. However, an alternative is suggested, namely, that these coupling constants approach a limit cycle in the limit of large momenta. Some results of this paper are: (1) The e^+-e^- annihilation experiments above 1-GeV energy may distinguish a fixed point from a limit cycle or other asymptotic behavior. (2) If electrodynamics or weak interactions become strong above some large momentum Λ, then the renormalization group can be used (in principle) to determine the renormalized coupling constants of strong interactions, except for $U(3)\times U(3)$ symmetry-breaking parameters. (3) Mass terms in the Lagrangian of strong, weak, and electromagnetic interactions must break a symmetry of the combined interactions with zero mass. (4) The $\Delta I=\frac{1}{2}$ rule in nonleptonic weak interactions can be understood assuming only that a renormalization group exists for strong interactions.

"... A limit cycle is a periodic solution of a nonlinear set of equations of motion. ..."
"... These are not the only possibilities but other possibilities are more difficult to analyze. Studying the consequences of a fixed point or a limit cycle makes clear the importance of the renormalization group for field theory"

Short-range EFT

The most general short-range dynamics allowed by assumed spacetime symmetries

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m} \right) \psi - \frac{C_0}{2} \left(\psi^\dagger \psi\right)^2 - \frac{D_0}{6} \left(\psi^\dagger \psi\right)^3$$

- Unitarity limit: $a_0 = \pm\infty$, $r_0 = 0$, ... Corresponding to the **non-trivial fixed point** of C_0
  To reproduce the scattering length a_0, $C_0 = -\frac{4\pi}{M(\theta_1 \Lambda - a_0^{-1})}$
  RGE for C_0: $\Lambda \frac{d\hat{C}_0}{d\Lambda} = \hat{C}_0(1 - \hat{C}_0)$, $\hat{C}_0 \equiv -\frac{\theta_1 M \Lambda}{4\pi} C_0$
- The three-body LEC D_0 displays a limit cycle behavior Bedaque et al. PRL 82 (1999)

$$H_0(\Lambda/\Lambda_*) \equiv \frac{\Lambda^2 D_0(\Lambda_*, \Lambda)}{6mC_0^2(\Lambda)} \propto \frac{\sin(s_0 \ln(\Lambda/\Lambda_*) - \delta_0)}{\sin(s_0 \ln(\Lambda/\Lambda_*) + \delta_0)}\ ,\ \Lambda_* \text{ a three-body scale}$$

The three-body limit cycle

Sharp-cutoff regulator: $H_0(\Lambda/\Lambda_*) = h_0 \frac{\sin(s_0\ln(\Lambda/\Lambda_*) - \arctan(1/s_0))}{\sin(s_0\ln(\Lambda/\Lambda_*) + \arctan(1/s_0))}$

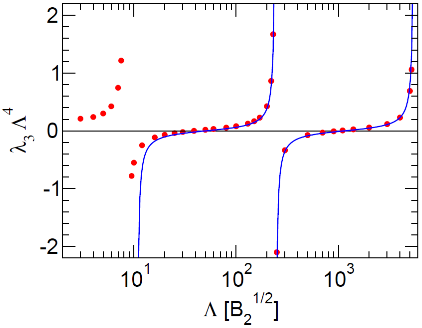
Bedaque et al. PRL 82 (1999): $h_0 = 1$

Braaten & Hammer, Phys. Rept. 428 (2006): $b_0 \equiv \Lambda_*/\kappa_* = 2.61$, $\kappa_* = \sqrt{mB_3}$

Braaten et al. PRL 106 (2011): $h_0 = 0.879$

Chen & Zhang arXiv.2506.12531: analytical expressions for h_0 and b_0

This two-parameter equation was also used for other regulators



L. Platter, thesis, 2005.

The three-body limit cycle

Sharp-cutoff regulator: $H_0(\Lambda/\Lambda_*) = h_0 \frac{\sin(s_0 \ln(\Lambda/\Lambda_*) - \arctan(1/s_0))}{\sin(s_0 \ln(\Lambda/\Lambda_*) + \arctan(1/s_0))}$

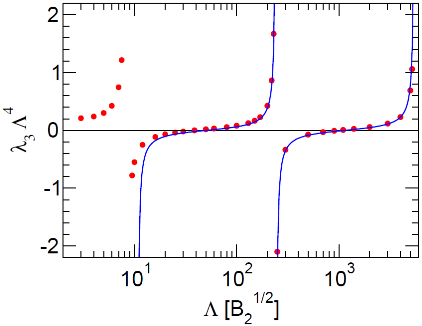
Bedaque et al. PRL 82 (1999): $h_0 = 1$

Braaten & Hammer, Phys. Rept. 428 (2006): $b_0 \equiv \Lambda_*/\kappa_* = 2.61$, $\kappa_* = \sqrt{mB_3}$

Braaten et al. PRL 106 (2011): $h_0 = 0.879$

Chen & Zhang arXiv.2506.12531: analytical expressions for h_0 and b_0

This two-parameter equation was also used for other regulators



L. Platter, thesis, 2005.

What is the general form for other regulators?

At the unitarity, for general separable regulators,

$$H_0(\Lambda/\Lambda_*) = h_0 \frac{\sin(s_0 \ln(\Lambda/\Lambda_*) - \delta_0)}{\sin(s_0 \ln(\Lambda/\Lambda_*) + \delta_0)}$$

or more generally, $\tan(s_0 \ln(\Lambda/\Lambda_*))$ is related to H_0 through a **real Möbius transformation**

3 parameters: h_0, b_0, δ_0

SREFT with dimer field

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m} \right) \psi - \frac{C_0}{2} \left(\psi^\dagger \psi\right)^2 - \frac{D_0}{6} \left(\psi^\dagger \psi\right)^3$$

$$= \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m} \right) \psi + \frac{d^\dagger d}{2mC_0} - \frac{1}{2\sqrt{m}} \left(d^\dagger \psi\psi + d\psi^\dagger\psi^\dagger \right) + h d^\dagger d \psi^\dagger \psi + \dots \ ,$$

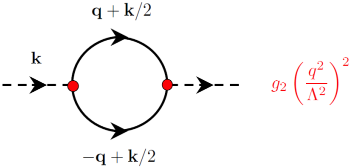
Three-body LEC: $H_0(\Lambda/\Lambda_*) \equiv \frac{\Lambda^2 D_0(\Lambda_*,\Lambda)}{6mC_0^2(\Lambda)} \equiv -\Lambda^2 h(\Lambda_*, \Lambda)$

SREFT with dimer field

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m} \right) \psi - \frac{C_0}{2} \left(\psi^\dagger \psi \right)^2 - \frac{D_0}{6} \left(\psi^\dagger \psi \right)^3$$

$$= \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2m} \right) \psi + \frac{d^\dagger d}{2mC_0} - \frac{1}{2\sqrt{m}} \left(d^\dagger \psi\psi + d\psi^\dagger\psi^\dagger \right) + h d^\dagger d \psi^\dagger \psi + \dots ,$$

Three-body LEC: $H_0(\Lambda/\Lambda_*) \equiv \frac{\Lambda^2 D_0(\Lambda_*,\Lambda)}{6mC_0^2(\Lambda)} \equiv -\Lambda^2 h(\Lambda_*, \Lambda)$



self-energy: $\Sigma(E,\mathbf{k}) = \frac{1}{2m} \int \frac{d^3q}{(2\pi)^3} \frac{g_2^2(q^2/\Lambda^2)}{E_+ - \epsilon_{\mathbf{q}-\mathbf{k}/2} - \epsilon_{\mathbf{q}+\mathbf{k}/2}}$

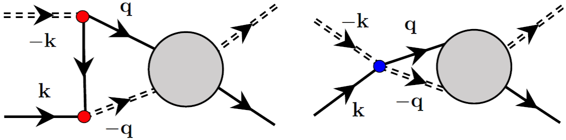
$E_+ = E + i0^+$ and $\epsilon_{\mathbf{k}} = k^2/2m$

dimer propagator:
$D_r(E,\mathbf{k}) = [(2mC_0)^{-1} - \Sigma(E,\mathbf{k})]^{-1}$

At unitarity,
$D_r(E,\mathbf{k})^{-1} = -\frac{1}{8\pi}\sqrt{-mE_r}\,\chi(-mE_r/\Lambda^2)$,
$E_r = E_+ - k^2/4m$
$\chi(x)$: high-order correction

STM with three-body force

Focus on the bound-state sector with total energy $E < 0$ and zero total momentum



$g_2\left(\frac{|\mathbf{k}+\mathbf{q}/2|^2}{\Lambda^2}\right) g_2\left(\frac{|\mathbf{q}+\mathbf{k}/2|^2}{\Lambda^2}\right)$ $\quad$ $g_3\left(\frac{k^2}{\Lambda^2}\right) g_3\left(\frac{q^2}{\Lambda^2}\right)$

Skorniakov-Ter-Martirosian equation with three-body force (STM3):

$$k\phi(k) = -\int \frac{dq}{2\pi^2}\left(G_r(k,q;E) - \frac{kq}{\Lambda^2} H_0(\Lambda) g_3\left(\frac{k^2}{\Lambda^2}\right) g_3\left(\frac{q^2}{\Lambda^2}\right)\right) D_r(E-\epsilon_{\mathbf{q}}, -\mathbf{q})\, q\phi(q)\ ,$$

where

$$G_r(k,q;E) \equiv kq \int \frac{d\Omega_{\mathbf{q}}}{4\pi} \frac{g_2\left(\frac{|\mathbf{k}+\mathbf{q}/2|^2}{\Lambda^2}\right) g_2\left(\frac{|\mathbf{q}+\mathbf{k}/2|^2}{\Lambda^2}\right)}{m(\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{q}}+\epsilon_{\mathbf{k}+\mathbf{q}}-E)}$$

Focusing on the limit $E \to 0$ and introducing $k \equiv \Lambda \exp(-t)$, $q \equiv \Lambda \exp(-s)$, $\xi(t) \equiv k\phi(k)$, $\tilde{g}_3(t) \equiv \exp(-t)\, g_3(\exp(-2t))$, $\tilde{\chi}_r(t) \equiv \chi(3\exp(-2t)/4)$, and $\lambda \equiv \sqrt{3}\pi/8$, the STM3 equation can be recast as

$$\int_{-\infty}^{\infty} ds \Big(G_r(t,s) - H_0\, \tilde{g}_3(t)\, \tilde{g}_3(s) \Big) \tilde{\chi}_r^{-1}(s) \xi(s) = \lambda \xi(t) \ .$$

The function $G_r(t,s) = G_r(k,q;0)$ approaches

$$G(t-s) \equiv \frac{1}{2} \ln \left(\frac{\cosh(t-s) + 1/2}{\cosh(t-s) - 1/2} \right),$$

if both s and t are large and positive. If either $t \ll 0$ or $s \ll 0$, $G_r(t,s)$ goes to 0.

To eliminate $\tilde{\chi}_r(t)$, we make the redefinitions

$$B(t,s) \equiv G_r(t,s)/\sqrt{\tilde{\chi}_r(t)\tilde{\chi}_r(s)}\,, \psi(t) \equiv \xi(t)/\sqrt{\tilde{\chi}_r(t)},\; v(t) \equiv \tilde{g}_3(t)/\sqrt{\tilde{\chi}_r(t)}\,.$$

$$\Rightarrow \int_{-\infty}^{\infty} ds \left[B(t,s) - H_0 v(t) v(s)\right] \psi(s) = \lambda \psi(t)\ .$$

The asymptotic behavior of $\psi(t)$ is the same as that for a sharp cutoff regulator,

$$\psi(t) \sim \cos(s_0 t + \tilde{\varphi}),\; \text{with } \tilde{\varphi} = s_0 \ln[\tilde{\Lambda}_*/\Lambda], \tilde{\Lambda}_* \equiv \exp(\varphi_0/s_0)\Lambda_*$$

Determining the running of H_0 is equivalent to analyzing how the phase of the solution $\psi(k)$ depends on H_0

Choosing the normalization of ψ as $\int_{-\infty}^{\infty} ds\, v(s)\psi(s) = 1$, we get

$$\int_{-\infty}^{\infty} ds \left[B(t,s) - \lambda \delta(s-t)\right] \psi(s) = H_0 v(t)\ .$$

Only possible when the three-body regulator is separable

$$\int_{-\infty}^{\infty} ds\,[B(t,s)-\lambda\delta(s-t)]\,\psi(s) = H_0 v(t)$$

$\psi_0(s)$, $\psi_1(s)$: the solutions for $H_0 = 0, 1$
General solution:

$$\psi(t) = (1-H_0)\psi_0(t) + H_0\psi_1(t)$$

This solution depends linearly on H_0

In the low-energy regime $t \gg 1$: $\psi_0(t) \sim \text{Re}\big[A_0 e^{is_0 t}\big]$, $\psi_1(t) \sim \text{Re}\big[(A_0+A_1)e^{is_0 t}\big]$
Matching at the low-energy regime $\psi(t) \sim \cos(s_0 t + \tilde{\varphi})$

$$\Rightarrow \tan\tilde{\varphi} = \tan\left(\arg(A_0 + H_0 A_1)\right) = \frac{\text{Im}(A_0) + \text{Im}(A_1)H_0}{\text{Re}(A_0) + \text{Re}(A_1)H_0}$$

A **real Möbius transformation**, involving **three independent parameters**
Conversely, H_0 can be expressed as a real Möbius transformation of $\tan\tilde{\varphi}$

Real Möbius Transformation

Definition

A *real Möbius transformation* is a map on the extended real line $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of the form

$$f(x) = \frac{ax+b}{cx+d}, \qquad a, b, c, d \in \mathbb{R},\ ad - bc \neq 0.$$

It is represented (up to scalar) by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$.

The set of all real Möbius transformations forms a group under composition.
It is naturally identified with the projective linear group

$$\mathrm{PGL}(2, \mathbb{R}) = \mathrm{GL}(2, \mathbb{R})/\{\alpha I : \alpha \in \mathbb{R}^\times\}.$$

$\dim \mathrm{PGL}(2, \mathbb{R}) = 3$

A special parameterization

$$\tan\tilde{\varphi} = \frac{\mathrm{Im}(A_0) + \mathrm{Im}(A_1)H_0}{\mathrm{Re}(A_0) + \mathrm{Re}(A_1)H_0}, \quad \tilde{\varphi} = s_0 \ln\left[\Lambda_*/\Lambda\right] + \varphi_0$$

The parametrization of this relation is arbitrary

Choosing $\arg(A_0 A_1) = 2\varphi_0$ and setting $\mathrm{Re}(\exp(-i\varphi_0)A_1) = 1$, we can write

$$\begin{aligned} A_0 &= -h_0\,(1 - i\tan\delta_0)\,e^{i\varphi_0}\ , \\ A_1 &= (1 + i\tan\delta_0)\,e^{i\varphi_0}\ , \end{aligned}$$

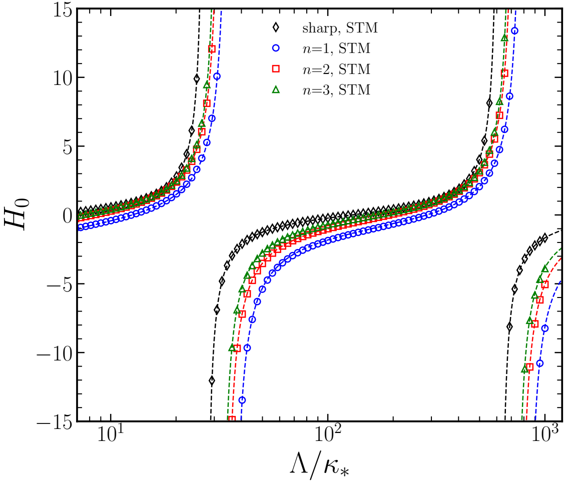
δ_0 and h_0 are real and can be determined numerically

$$\begin{aligned} \Rightarrow H_0 &= h_0 \frac{\tan(\varphi_0 - \tilde{\varphi}) - \tan\delta_0}{\tan(\varphi_0 - \tilde{\varphi}) + \tan\delta_0} \\ &= h_0 \frac{\sin(s_0\ln(\Lambda/\Lambda_*) - \delta_0)}{\sin(s_0\ln(\Lambda/\Lambda_*) + \delta_0)} \end{aligned}$$

3 parameters: h_0, $b_0 \equiv \Lambda_*/\kappa_*$, δ_0

Numerical demonstration

Consider (super-)Gaussian regulators: $g_2(x^2) = g_3(x^2) = \exp(-x^{2n})$



$$H_0 = h_0 \frac{\sin(s_0 \ln(\Lambda/\Lambda_*) - \delta_0)}{\sin(s_0 \ln(\Lambda/\Lambda_*) + \delta_0)}$$

regulator	δ_0	h_0	b_0
sharp	0.7823	0.879	2.61
$n = 1$	1.0463	1.8024	4.4436
$n = 2$	0.8869	1.4744	3.4930
$n = 3$	0.8361	1.2804	3.2042

The fit is excellent

An approximation for (super-)Gaussian regulators

$$\text{STM3} \Rightarrow h(\Lambda) \simeq \frac{\int d(\ln\Lambda)\left(\frac{\partial}{\partial \ln\Lambda}\int\frac{d^3q}{(2\pi)^3} g_2(|\mathbf{k}+\mathbf{q}/2|^2/\Lambda^2) g_2(|\mathbf{q}+\mathbf{k}/2|^2/\Lambda^2) G_0 D_r \phi(q)\right)_{k\ll\Lambda}}{\int d(\ln\Lambda)\left(\frac{\partial}{\partial \ln\Lambda}\int\frac{d^3q}{(2\pi)^3} g_3(k^2/\Lambda^2) g_3(q^2/\Lambda^2) D_r \phi(q)\right)_{k\ll\Lambda}}$$

To evaluate it analytically, we perform the approximations

$$g_2\left(\frac{|\mathbf{k}+\mathbf{q}/2|^2}{\Lambda^2}\right) g_2\left(\frac{|\mathbf{q}+\mathbf{k}/2|^2}{\Lambda^2}\right) \sim \exp\left[-\left(1+2^{-2n}\right)\left(\frac{q}{\Lambda}\right)^{2n}\right]$$

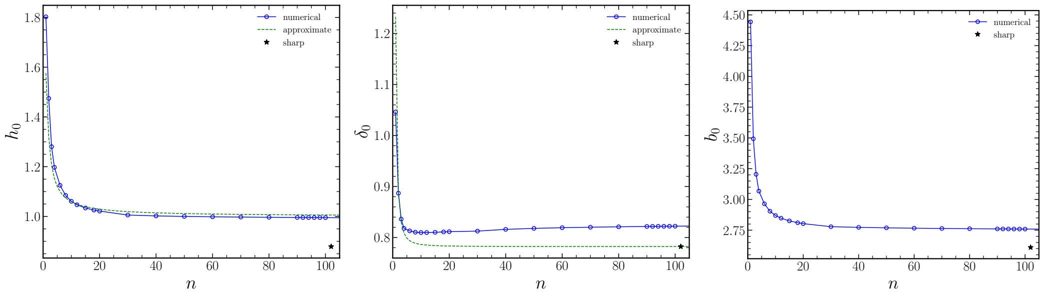
$$g_3\left(\frac{k^2}{\Lambda^2}\right) g_3\left(\frac{q^2}{\Lambda^2}\right) \sim \exp\left[-\left(\frac{q}{\Lambda}\right)^{2n}\right]$$

$$G_0 \sim -\frac{1}{q^2},\quad D_r \sim \frac{1}{q},\quad \phi(q) \sim \mathrm{Re}\left(\left(\frac{q}{\tilde{\Lambda}_*}\right)^{is_0-1}\right)$$

This gives: $\delta_0 \simeq \dfrac{1}{2}\arg\left(\left(1+2^{-2n}\right)^{is_0/2n}\dfrac{\Gamma\left(\frac{is_0+1}{2n}\right)}{\Gamma\left(\frac{is_0-1}{2n}\right)}\right)$

$$h_0 \simeq \left(1+2^{-2n}\right)^{1/2n}\frac{\left|\Gamma\left(\frac{is_0-1}{2n}\right)\right|}{\left|\Gamma\left(\frac{is_0+1}{2n}\right)\right|}$$

An approximation for (super-)Gaussian regulators



- As n increases, the approximation provides a good description of h_0
- The approximate δ_0 converges to the sharp cutoff value, which is different from the limit of the super-Gaussian regulators
- The super-Gaussian regulators do not converge to the sharp cutoff $\Theta(1 - q/\Lambda)$

Main idea of the derivation

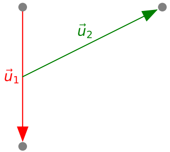
STM3 equation:

$$k\phi(k) = -\int \frac{dq}{2\pi^2}\left(G_r(k,q;E) - \frac{kq}{\Lambda^2} H_0(\Lambda) g_3\left(\frac{k^2}{\Lambda^2}\right) g_3\left(\frac{q^2}{\Lambda^2}\right)\right) D_r(E-\epsilon_{\mathbf{q}}, -\mathbf{q})\, q\phi(q)\ ,$$

$$\Rightarrow H_0(\Lambda/\Lambda_*) = h_0 \frac{\sin(s_0 \ln(\Lambda/\Lambda_*) - \delta_0)}{\sin(s_0 \ln(\Lambda/\Lambda_*) + \delta_0)}$$

Main idea: Faddeev equation $\rightarrow$ STM3 $\rightarrow$ $H_0(\Lambda/\Lambda_*)$

Faddeev equation



Jacobi momenta: $\mathbf{u}_1 = (\mathbf{p}_1 - \mathbf{p}_2)/2$

$$\mathbf{u}_2 = 2\left[\mathbf{p}_3 - (\mathbf{p}_1 + \mathbf{p}_2)/2\right]/3$$

Potentials are obtained from SREFT

Two-body potential: $V_2 = C_0 \left|g_2\right\rangle \left\langle g_2\right\rangle$, $\left\langle \mathbf{u}_1 | g_2 \right\rangle = g_2(u_1^2/\Lambda^2)$
Three-body potential: $V_3 = D_0 \left|\zeta\right\rangle \left\langle\zeta\right|$, $\left\langle \mathbf{u}_1\mathbf{u}_2 | \zeta \right\rangle \equiv \zeta(u_1, u_2) = \zeta\left((u_1^2 + \frac{3}{4}u_2^2)/\Lambda^2\right)$.

Faddeev equation: $\left|\psi\right\rangle = G_0 t P \left|\psi\right\rangle + 3 G_0 t G_0 t_3 \left|\psi\right\rangle$
G_0: free three-body Green's function
$P = P_{12}P_{23} + P_{13}P_{23}$. Total wave function $\left|\Psi\right\rangle = (1 + P)\left|\psi\right\rangle$
$t(z) = \left|g_2\right\rangle \tau(z) \left\langle g_2\right|$, $\tau(z) = D_r(z, \mathbf{0})/2m$. D_r is the dimer propagator.
$t_3(E) = \left|\zeta\right\rangle \tau_3(E) \left\langle\zeta\right|$, $\tau_3(E) = -H_0'(E,\Lambda)/I_2^\zeta(E)$ with $I_2^\zeta(E) = \left\langle \zeta | G_0(E) | \zeta \right\rangle$ and

$$H_0'(E,\Lambda) = \frac{H_0(\Lambda)}{H_0(\Lambda) - \Lambda^2/6mC_0^2 I_2^\zeta(E)} .$$ **(another Möbius transformation)**

Define a reduced Faddeev component $F(u_2)$ via

$$\langle u_1 u_2|\psi\rangle = g_2(u_1^2/\Lambda^2)\, G_0(E; u_1, u_2)\, \tau\big(E - \tfrac{3}{4}u_2^2\big)\, F(u_2) \ ,$$

$$\Rightarrow u_2 F(u_2) = -\int \frac{du_2'}{2\pi^2} \left(G_r(u_2, u_2', E) - \frac{u_2 u_2'}{\Lambda^2}\, H_0'(E,\Lambda)\, g_3'(E, u_2)\, g_3'(E, u_2') \right) \times D_r(E - \epsilon_{\mathbf{u}_2'}, -\mathbf{u}_2')\, u_2' F(u_2') \ ,$$

where

$$g_3'(E, u_2) = -\sqrt{3}\Lambda I_0^\zeta(E, u_2)/\sqrt{-2m I_2^\zeta(E)}$$

$$I_0^\zeta(E, u_2) = \int \frac{d^3 u_1}{(2\pi)^3}\, g_2\!\left(\frac{u_1^2}{\Lambda^2}\right) G_0(E; u_1, u_2)\, \zeta(u_1, u_2) \ .$$

STM3 equation:

$$k\phi(k) = -\int \frac{dq}{2\pi^2} \left(G_r(k,q;E) - \frac{kq}{\Lambda^2} H_0(\Lambda) g_3\left(\frac{k^2}{\Lambda^2}\right) g_3\left(\frac{q^2}{\Lambda^2}\right)\right) D_r(E-\epsilon_{\mathbf{q}}, -\mathbf{q})\, q\phi(q)$$

The reduced Faddeev component $F(u_2)$ satisfies

$$u_2 F(u_2) = -\int \frac{du_2'}{2\pi^2} \left(G_r(u_2,u_2',E) - \frac{u_2 u_2'}{\Lambda^2} H_0'(E,\Lambda)\, g_3'(E,u_2)\, g_3'(E,u_2')\right) \times D_r(E-\epsilon_{\mathbf{u}_2'}, -\mathbf{u}_2')\, u_2' F(u_2') \ ,$$

It has the same structure of the STM3 equation

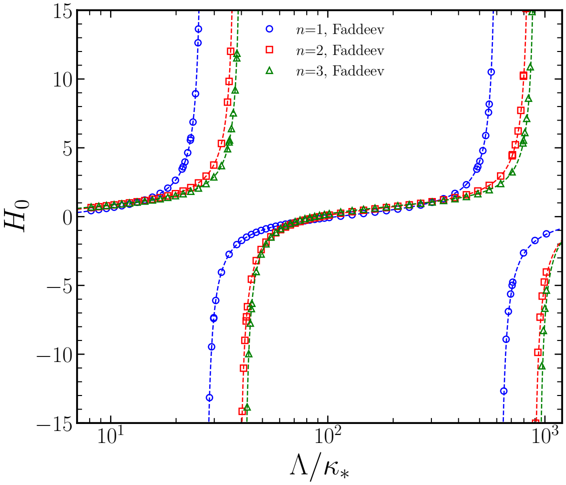
$\Rightarrow H_0'(0,\Lambda)$ is related to $\tan(s_0 \ln(\Lambda/\Lambda_*))$ through a real Möbius transformation

$\Rightarrow H_0$ is related to $\tan(s_0 \ln(\Lambda/\Lambda_*))$ through a real Möbius transformation

$$\Rightarrow H_0 = h_0 \frac{\sin(s_0 \ln(\Lambda/\Lambda_*) - \delta_0)}{\sin(s_0 \ln(\Lambda/\Lambda_*) + \delta_0)}$$

Numerical demonstration

Consider (super-)Gaussian regulators: $g_2(x^2) = \zeta(x^2) = \exp(-x^{2n})$



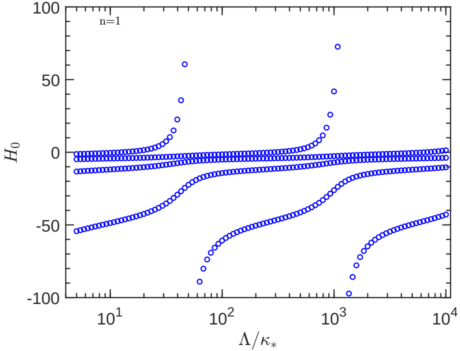
$$H_0 = h_0 \frac{\sin(s_0 \ln(\Lambda/\Lambda_*) - \delta_0)}{\sin(s_0 \ln(\Lambda/\Lambda_*) + \delta_0)}$$

regulator	δ_0	h_0	b_0
$n = 1$	0.7094	0.7976	2.3965
$n = 2$	0.4455	1.0189	2.6236
$n = 3$	0.3766	1.0037	2.5985

The fit is also excellent

More discussions

- When DSI is weakly broken by a finite scattering length a_0, numerical results show that the ratios between adjacent poles and zeros of H_0 gradually approach the universal value $\exp(\pi/s_0) \approx 22.69$ as either the cutoff or the scattering length increases, implying that the universal form we derived holds up to corrections that are suppressed by inverse powers of $a_0\Lambda$.
- For (non-separable) local regulators, preliminary results show that there could be multi-branches.



Summary

- We prove that the running of the three-body interaction strength universally follows a real Möbius transformation, characterized by just three parameters.
- Our findings broaden the class of three-body limit cycles and provide a solid theoretical foundation for SREFT with general regulators.

Outlook

- Corrections to the universal functional form due to large scattering length and effective range
- For non-separable (local) regulators, preliminary results indicate a more intricate limit-cycle structure, warranting further investigation

Thank you!

- Langxuan Chen (Fudan University, Stanford University)
- Pengfei Zhang (Fudan University, Hefei National Laboratory)
- Xincheng Lin (NCSU), Sebastian König (NCSU)
- U. van Kolck (ECT* Trento, IJCLab Orsay, U. Arizona)