Mirroring nuclei at the unitary limit

A. Kievsky

Istituto Nazionale di Fisica Nucleare, Sezione di Pisa

Pan-American Few-Body Physics Boot Camp: Fostering Collaboration

ECT* Workshop, Trento, October 13 - 24 2025

Career recap

- PhD at University of Buenos Aires (1987)
- post-doc at INFN, Pisa (1987-1989)
- researcher at INFN, Pisa (from 1989)
  Initially I worked on the solution of the three- and four-nucleon problem

Career recap

- PhD at University of Buenos Aires (1987)
- post-doc at INFN, Pisa (1987-1989)
- researcher at INFN, Pisa (from 1989)
  Initially I worked on the solution of the three- and four-nucleon problem
- At that time this argument was an intense subject of research:

  In the 90's the first high quality NN potentials appeared (Nijmegen, AV18, CDBonn)

  In parallel with those efforts, the first steps toward a description within the framework of EFT's appear

  In this context, managing the solution of the three- and four-nucleon problem it was possible to extend those studies in the description of the few-nucleon dynamics

Career recap

- Very little experience were in Pisa to attack this problem. Together with my colleague M. Viviani we started from the beginning deciding which method we prefer to use
- We decided to use the variational principle together with Hyperspherical Harmonic basis

$$\Psi_N = \sum_{[m,K]} A_{[m,K]} f_m(\rho) Y_{[K]}(\Omega_N)$$

Bound states: $H - EN = 0$

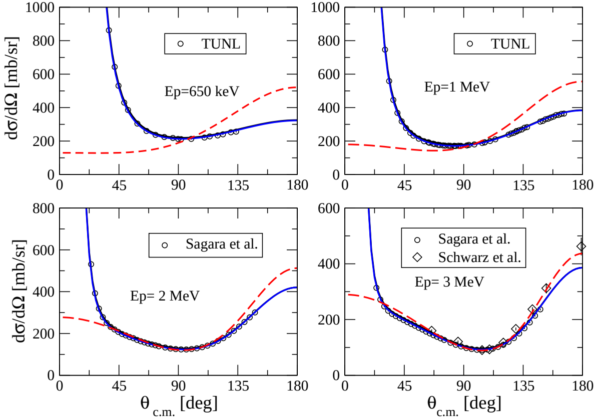
Scattering states: $[S_{ij}] = S_{ij} - \mathrm{i} < \Psi^i_N | H - E | \Psi^j_N >$

^{3}H and ^{4}He Bound States and $n-d$ scattering length

Potential(NN)	Method	^{3}H[MeV]	^{4}He[MeV]	$^2a_{nd}$[fm]
AV18	HH	7.624	24.22	1.258
	FE/FY Bochum	7.621	24.23	1.248
	FE/FY Lisbon	7.621	24.24	
CDBonn	HH	7.998	26.13	
	FE/FY Bochum	8.005	26.16	0.925
	FE/FY Lisbon	7.998	26.11	
	NCSM	7.99(1)		
N3LO-Idaho	HH	7.854	25.38	1.100
	FE/FY Bochum	7.854	25.37	
	FE/FY Lisbon	7.854	25.38	
	NCSM	7.852(5)	25.39(1)	
Potential(NN+NNN)				
AV18/UIX	HH	8.479	28.47	0.590
	FE/FY Bochum	8.476	28.53	0.578
CDBonn/TM	HH	8.474	29.00	
	FE/FY Bochum	8.482	29.09	0.570
N3LO-Idaho/N2LO	HH	8.474	28.37	0.675
	NCSM	8.473(5)	28.34(2)	
Exp.		8.48	28.30	0.645±0.010

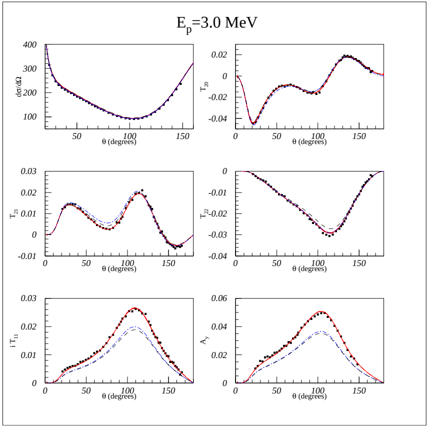
N-d scattering

- - - - nd —— pd



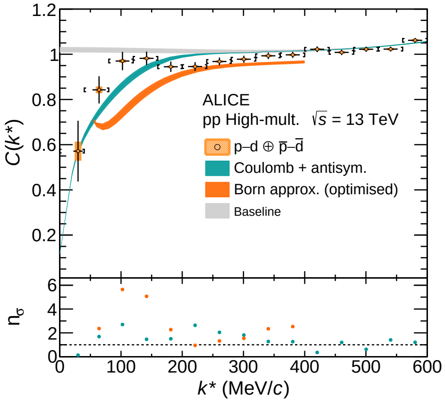
Some recent developments

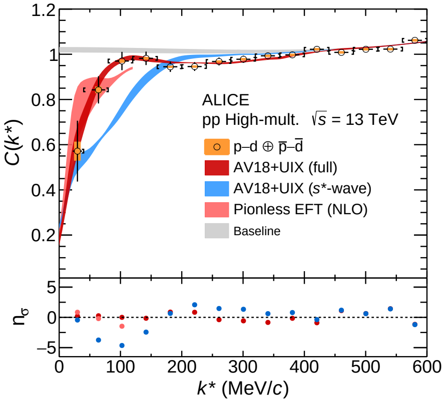
p-d scattering at 3 MeV fitting the subleading TNI terms obtaining a χ^2 per datum ≈ 1.7



L. Girlanda, A. Kievsky, L. E. Marcucci, and M. Viviani Phys. Rev. C 102, 064003 (2020)

The pd Correlation Function: comparison to experiment





M. Viviani, S. König, A. Kievsky, L.E. Marcucci, B. Singh, O. Vázquez Doce, Phys. Rev. C 108, 064002 (2023)
ALICE collaboration, Physical Review X 14, 031051 (2024)

Some new aspects

- Doing these detailed calculations I missed some simple questions

Some new aspects

- Doing these detailed calculations I missed some simple questions
- For example I did not know that the deuteron binding energy, $E_d = 2.224575(9)\,\text{MeV}$ is strictly related to the triplet scattering length, $a = 5.419(7)\,\text{fm}$, and effective range, $r_e = 1.753(8)\,\text{fm}$:

$$E_d \approx -\frac{\hbar^2}{ma^2} = 1.412\,\mathrm{MeV}$$

or, much better

$$E_d \approx -\frac{\hbar^2}{mr_e^2}(1 - \sqrt{1 - 2r_e/a})^2 = 2.223\,\mathrm{MeV}$$

Some new aspects

- Doing these detailed calculations I missed some simple questions
- For example I did not know that the deuteron binding energy, $E_d = 2.224575(9)\,\mathrm{MeV}$ is strictly related to the triplet scattering length, $a = 5.419(7)\,\mathrm{fm}$, and effective range, $r_e = 1.753(8)\,\mathrm{fm}$:

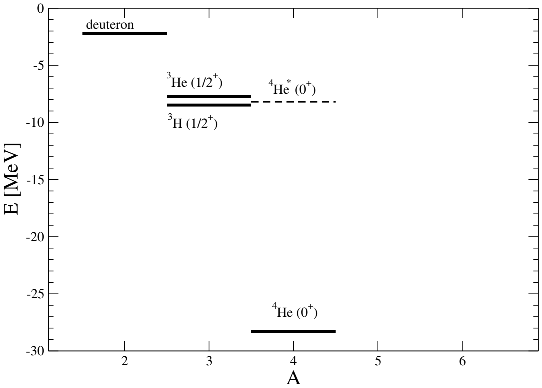
$$E_d \approx -\frac{\hbar^2}{ma^2} = 1.412\,\mathrm{MeV}$$

or, much better

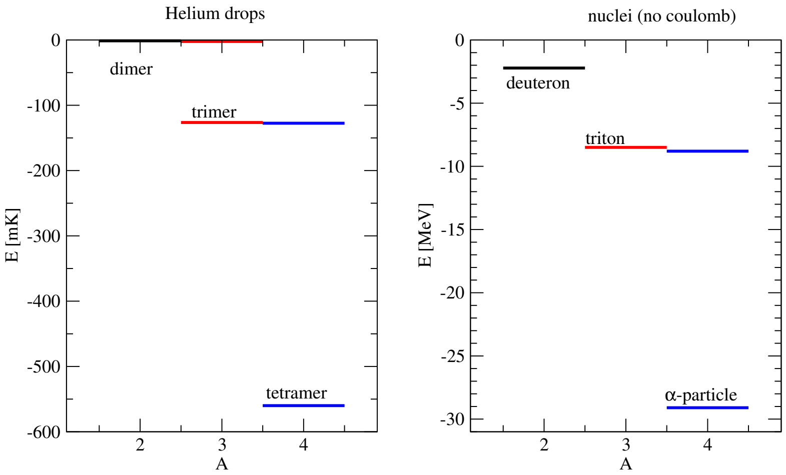
$$E_d \approx -\frac{\hbar^2}{mr_e^2}(1 - \sqrt{1 - 2r_e/a})^2 = 2.223\,\mathrm{MeV}$$

- The two-nucleon systems is inside the universal window
- A continuous scale invariance dominate in this region

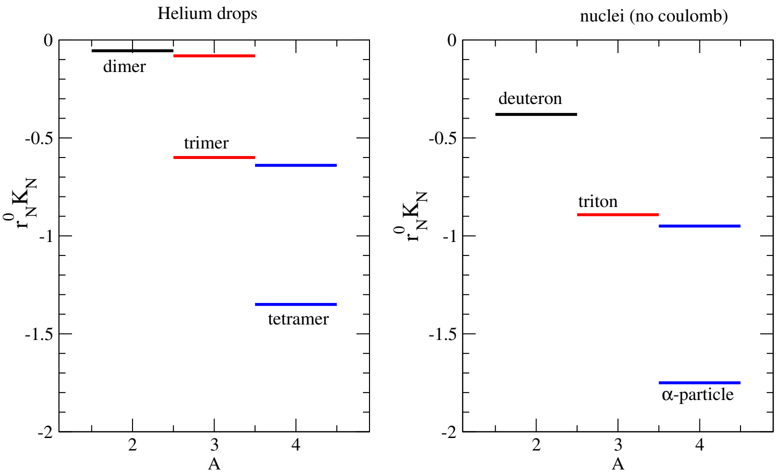
Nuclear spectrum $A \leq 4$



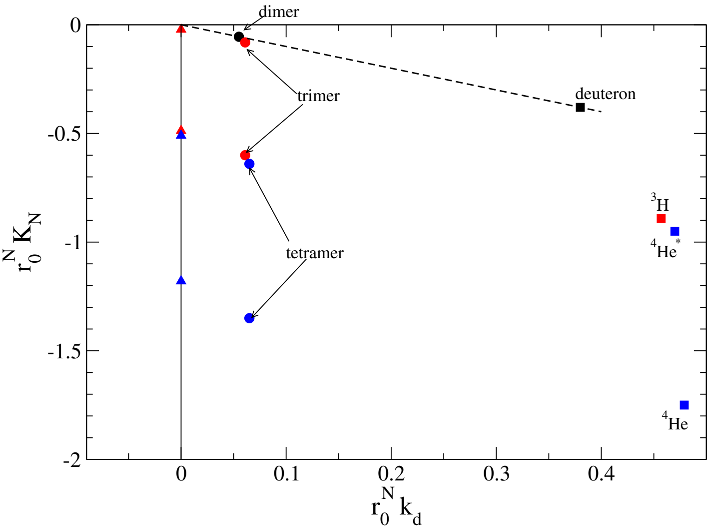
Helium drops and Nuclear spectrum $A \leq 4$



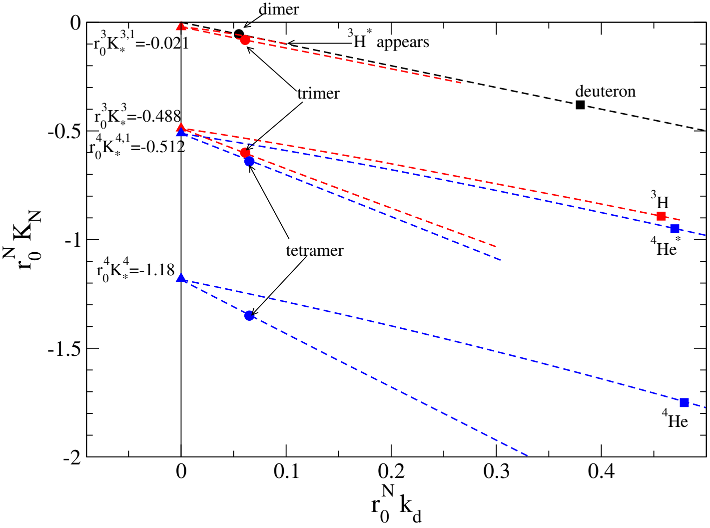
Using a scale to make the spectrum dimensionless



Helium and Nuclear spectrum inside the universal window



Helium and Nuclear spectrum inside the universal window



What we have observed?

- Appearance of universal behavior
  $\rightarrow$ independence of the interaction details
- There is a window in which universal behavior can be observed
  $\rightarrow$ It is formed by the appearence of a shallow two-body bound state
  $\rightarrow$ Correlation between bound and scattering states
- Dynamics governed by a few parameters (control parameters)
  $\rightarrow$ Continuous (or discrete) scale invariance
  $\rightarrow$ The systems can move along the window

What we have observed?

- Appearance of universal behavior
  $\rightarrow$ independence of the interaction details
- There is a window in which universal behavior can be observed
  $\rightarrow$ It is formed by the appearence of a shallow two-body bound state
  $\rightarrow$ Correlation between bound and scattering states
- Dynamics governed by a few parameters (control parameters)
  $\rightarrow$ Continuous (or discrete) scale invariance
  $\rightarrow$ The systems can move along the window

Interplay of two aspects

- Weakly bound systems are strongly correlated
- In the universal regime details of the interaction are not important
  $\rightarrow$ Effective interactions
  $\rightarrow$ Gaussian (or other) characterizations

The two-body characteristic length $r_0^{(2)}$

- The path from the physical point to the unitary point is characterized by the two-pole S-matrix representing one shallow state, virtual or bound

$$S(k) = \frac{k + i/a_B}{k - i/a_B} \frac{k + i/r_B}{k - i/r_B}$$

- The energy pole is described by the energy length a_B

$$1/k_d = a_B \longrightarrow E_2 = -\hbar^2 k_d^2/m = -\hbar^2/m a_B^2$$

- E_2 is a bound or virtual state when $a_B > 0$ or $a_B < 0$
- the second pole is described by the length $r_B = a - a_B$
- For example, for the deuteron at the physical point
  $a = 5.4\,\text{fm}$, $a_B = 4.3\,\text{fm}$
  $r_B = a - a_B = 1.1\,\text{fm}$

The two-body characteristic length $r_0^{(2)}$

- The two-pole S-matrix

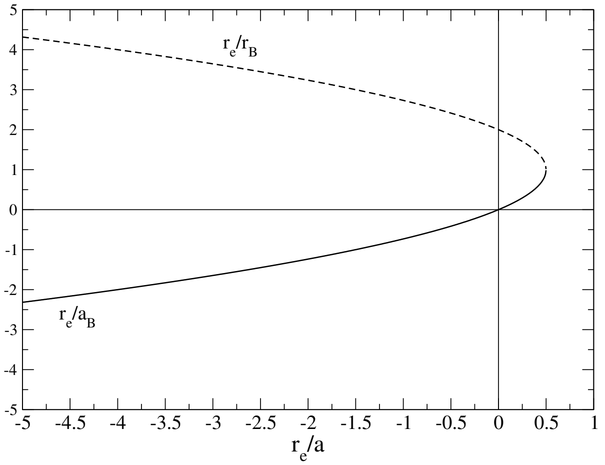
$$S(k) = e^{2i\delta} = \frac{e^{i\delta}}{e^{-i\delta}} = \frac{\cos\delta + i\sin\delta}{\cos\delta - i\sin\delta} = \frac{k\cot\delta + ik}{k\cot\delta - ik} = \frac{k + i/a_B}{k - i/a_B}\frac{k + i/r_B}{k - i/r_B}$$

  is equivalent to the second-order effective range expansion
  $k\cot\delta_0 = -1/a + r_e k^2/2$
- The two-poles are in the immaginary axes $k = ik_d$, verifying the pole equation
  $k_d = 1/a + r_e k_d^2/2$
- they are (remember $k_d = 1/a_B$ and $r_B = a - a_B$):

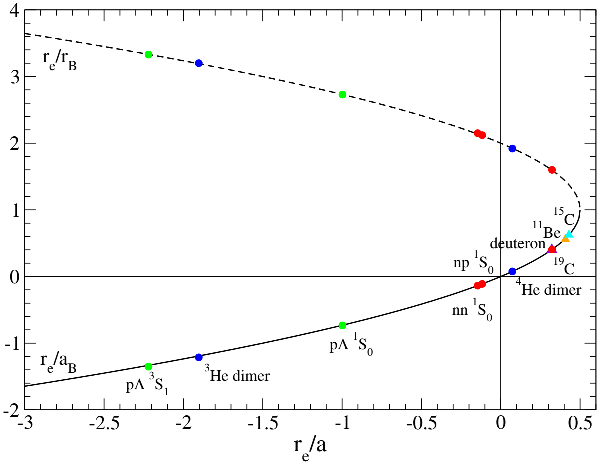
  $\frac{r_e}{a_B} = 1 - \sqrt{1 - 2r_e/a} \qquad \to r_e/a < 0.5$

  $\frac{r_e}{r_B} = 1 + \sqrt{1 - 2r_e/a}$

The two poles form the universal window

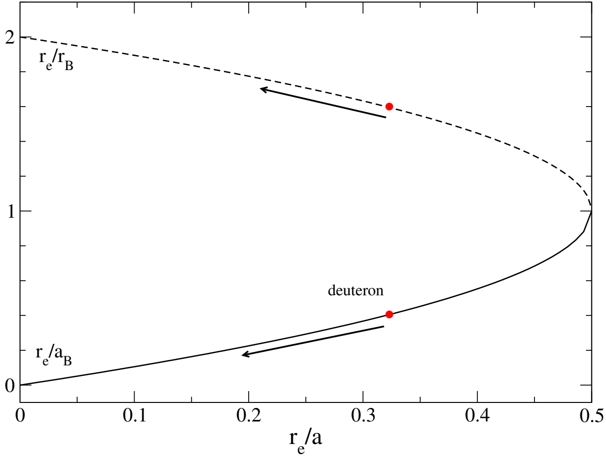


Physical systems inside the universal window



Systems are put on the figure by their experimental data

Moving along the window: $r_0^{(2)} \equiv$constant $\Longrightarrow$ $r_B \equiv$constant



Effective description (scale invariance)

- The S-matrix

$$S(k) = \frac{k + i/a_B}{k - i/a_B}\frac{k + i/r_B}{k - i/r_B}$$

is exactly represented by the Eckart potential:

$$V(r) = -2\frac{\hbar^2}{mr_0^2}\frac{\beta e^{-r/r_0}}{(1+\beta e^{-r/r_0})^2}$$

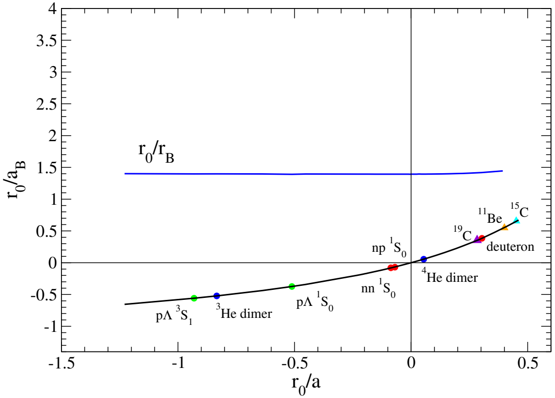
$$\begin{cases} a = 4r_0\frac{\beta}{\beta-1} \\ \\ a_B = 2r_0\frac{\beta+1}{\beta-1} \end{cases} \qquad \begin{cases} r_e = 2r_0\frac{\beta+1}{\beta} \\ \\ r_B = 2r_0 \to \text{the second pole!} \end{cases}$$

The universal window

- The figure shows the universal character of the window delimited by $-\infty < r_e/a < 0.5$ and $-\infty < r_e/a_B < 1$
- The systems can be related along the curve: Systems with similar values of r_e/a, or equivalently similar values of β, are related by scale transformation: $r_0 \to \lambda r_0$
- Many observables depend by the position on the curve:

	helium dimer		deuteron	
	exp.	calc.	exp.	calc.
$-\frac{\hbar^2}{ma_B^2} \approx -\frac{\hbar^2}{mr_e^2}(1-\sqrt{1-2r_e/a})^2$	1.3 mK	1.3mK	2.224MeV	2.223 MeV
$< r^2 > \approx \frac{a^2}{8}\left[1+(\frac{r_B}{a})^2\right]$	$67.015a_0$	$67.017a_0$	1.967fm	1.955fm
$C_a^2 \approx \frac{2}{a_B}\frac{1}{1-r_e/a_B}$	$0.10898a_0^{-1/2}$	$0.10899a_0^{-1/2}$	$0.885\text{fm}^{-1/2}$	$0.883\text{fm}^{-1/2}$

The universal window in terms of the Gaussian parameters



$$V(r) = -\frac{\hbar^2}{mr_0^2}\beta e^{-(r/r_0)^2}$$

Motivation of the movements along the window

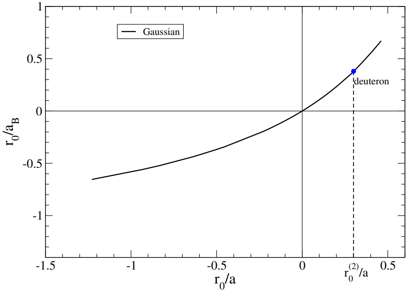
- The nuclear system, as well many other systems, are inside the universal window
- The universal window is characterized by scale invariance
- Scale invariance is not a symmetry of the underlying theory but appears for particular values of the interaction parameters
- Movements along the window help to see how scale invariance manifests and, hopefully, how to incorporate it in the effective description of nuclei.

Motivation of the movements along the window

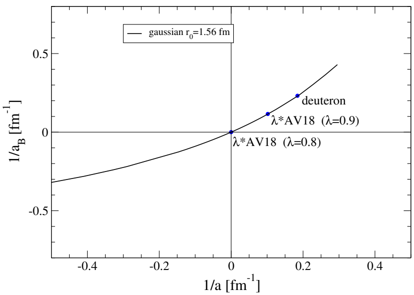
- The nuclear system, as well many other systems, are inside the universal window
- The universal window is characterized by scale invariance
- Scale invariance is not a symmetry of the underlying theory but appears for particular values of the interaction parameters
- Movements along the window help to see how scale invariance manifests and, hopefully, how to incorporate it in the effective description of nuclei.

- For example, looking at nuclei, we have these two ingredients
  - The microscopic theory for the nuclear interaction, chiral EFT
  - The scale invariance

The two-body scale $r_0^{(2)}$ → assigning dimensions → the deuteron trip



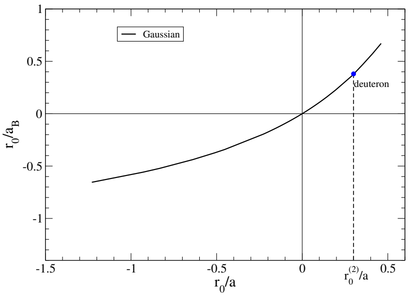
$\rightarrow$



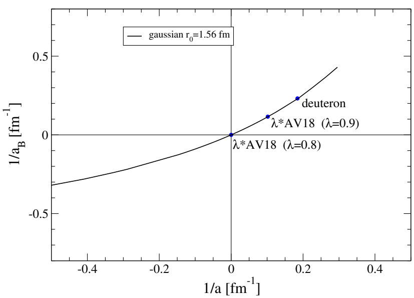
$r_0^{(2)}/a = 0.288 \rightarrow r_0 = 1.56\,\mathrm{fm}$ → The $T = 0$ np plot ($r_0 = 1.56\,\mathrm{fm}$)

$$V_{r_0}(r) = -\frac{\hbar^2}{mr_0^2}\beta e^{-r^2/r_0^2}$$

The two-body scale $r_0^{(2)} \rightarrow$ assigning dimensions $\rightarrow$ the deuteron trip



$\rightarrow$

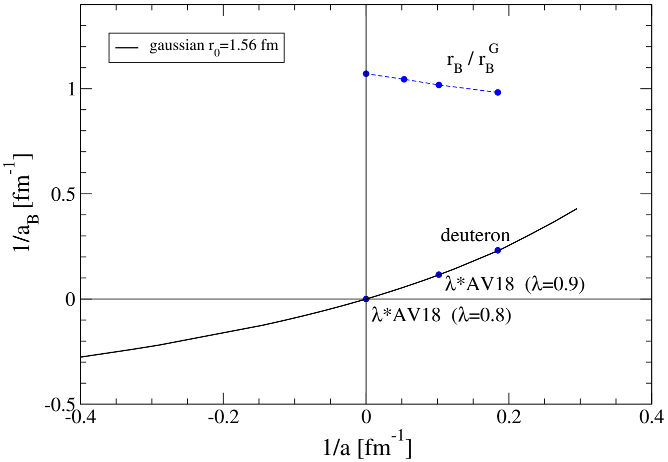


$$r_0^{(2)}/a = 0.288 \rightarrow r_0 = 1.56\,\mathrm{fm} \quad \rightarrow \text{The } T = 0 \; np \text{ plot } (r_0 = 1.56\,\mathrm{fm})$$

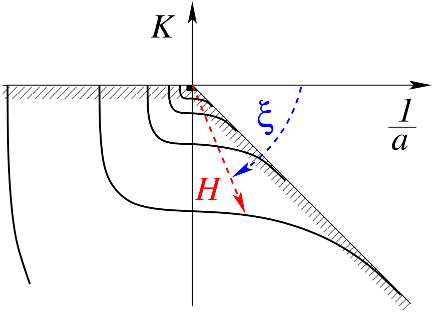
$$V_{r_0}(r) = -\frac{\hbar^2}{m r_0^2}\beta e^{-r^2/r_0^2}$$

The *np* data, represented by AV18 have been moved from the physical point to the unitary point

Moving along the window with constant resolution



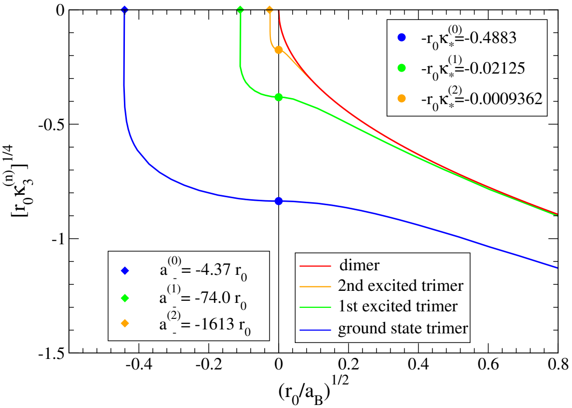
The three-body scale $r_0^{(3)}$ and K_*, the three-body parameter



The Efimov plot: The three-body sector is scale invariant, K_*, is the binding momentum at the unitary limit. It fixes the branch in which the system is located

The three-body scale $r_0^{(3)}$ using the gaussian characterization

The case of three bosons: $V = \sum_{ij} V_0 e^{-(r_{ij}/r_0)^2}$

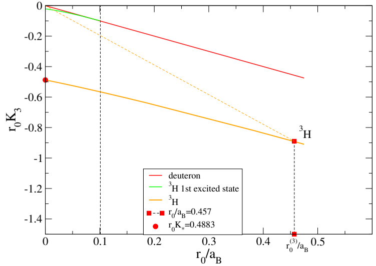


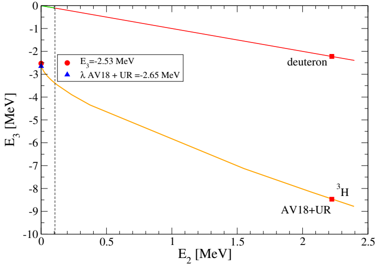
$$a_-^{(0)}\kappa_*^{(0)} = -2.14$$
$$a_-^{(1)}\kappa_*^{(1)} = -1.57$$
$$a_-^{(2)}\kappa_*^{(2)} = -1.51$$

The three-body scale $r_0^{(3)}$ using the gaussian characterization

$$V(1,2,3) = \sum_{i<j} V(i,j) = \sum_{i<j} \left(V_0 e^{-(r/r_0)^2} \mathcal{P}_{01} + V_1 e^{-(r/r_0)^2} \mathcal{P}_{10} \right)$$

To construct the plot we follow the nuclear path defined as ${}^0a_{np}/{}^1a_{np} = -4.38$



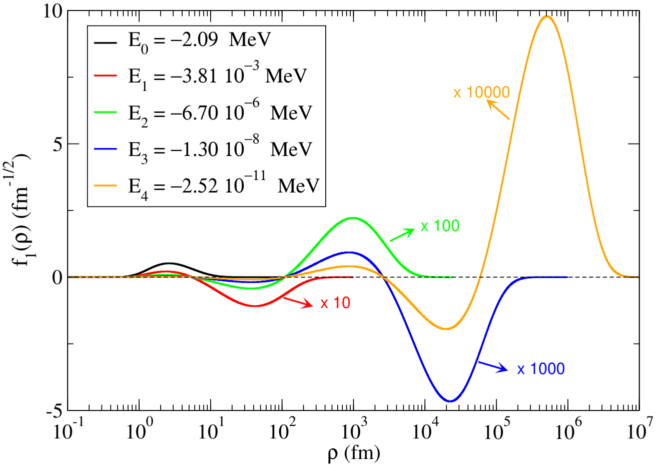


$\tan\xi = K a_B$

$r_0^{(3)} = 1.98$ fm

The ^{3}H plot ($\lambda_1 = 0.8$, $\lambda_0 = 1.06$)

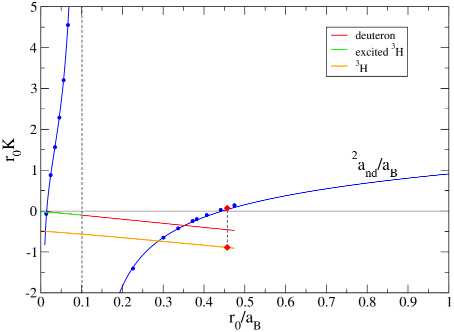
The three-nucleon Efimov effect



$E_0/E_1 = 549, \ E_1/E_2 = 568, \ E_2/E_3 = 515, \ E_3/E_4 = 515$

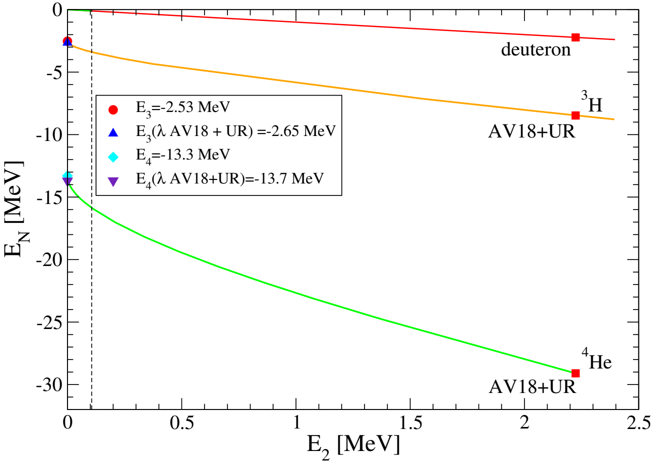
The three-nucleon system: correlations

$$V(1,2,3)=\sum_{i<j}V(i,j)=\sum_{i<j}\left(V_0 e^{-(r/r_0)^2}\mathcal{P}_{01}+V_1 e^{-(r/r_0)^2}\mathcal{P}_{10}\right)$$

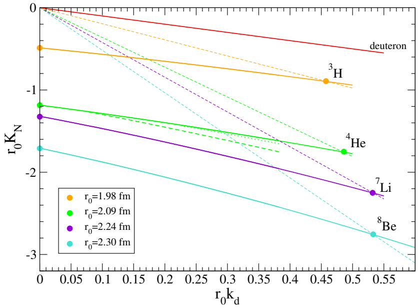


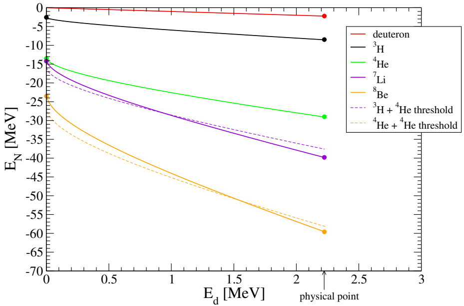
$$a_{nd}/a_B = d_1 + d_2 \tan[s_0 \ln(K_* r_0(a_B/r_0) + \Gamma_3) + d_3]$$

The three- and four-body scales, $r_0^{(3)}$ and $r_0^{(4)}$

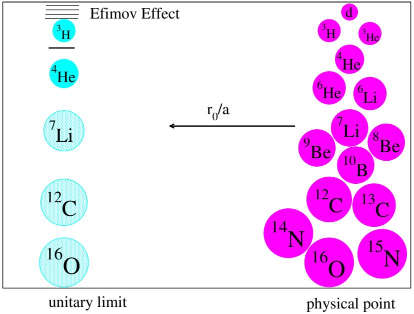


The N-body scales, $r_0^{(N)}$, for $A \leq 8$

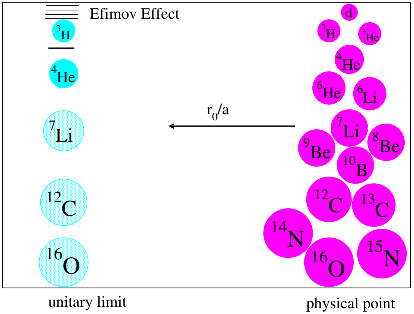




Unifying the scales



Unifying the scales



$$V_N = \sum_{i<j} V(i,j,r_0^{(N)},\beta_0)\mathcal{P}_{01} + \sum_{i<j} V(i,j,r_1^{(N)},\beta_1)\mathcal{P}_{10}$$

Unifying the scales: the LO potential

- The scale invariance is encoded in the two-pole S-matrix
- The trip to the unitary point has shown that two nuclear structures. They form the thresholds from which the other nuclei emerge

Unifying the scales: the LO potential

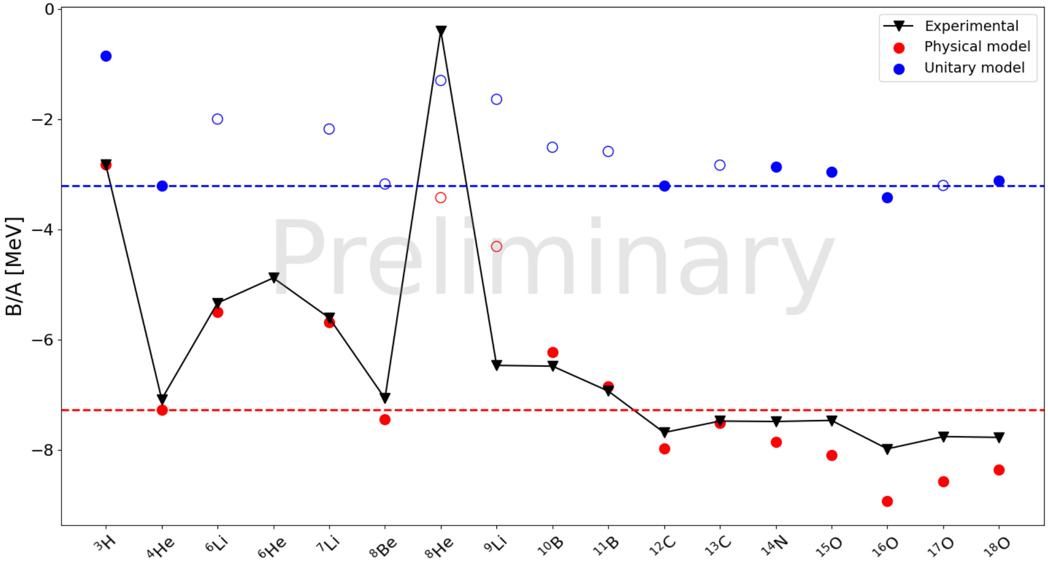
- The scale invariance is encoded in the two-pole S-matrix
- The trip to the unitary point has shown that two nuclear structures. They form the thresholds from which the other nuclei emerge
- Accordingly we propose the following potential to be considered at the lowest order

$$V_N = \sum_{i<j} V(i,j,r_0^{(N)},\beta_0)\mathcal{P}_{01} + \sum_{i<j} V(i,j,r_1^{(N)},\beta_1)\mathcal{P}_{10} \rightarrow$$

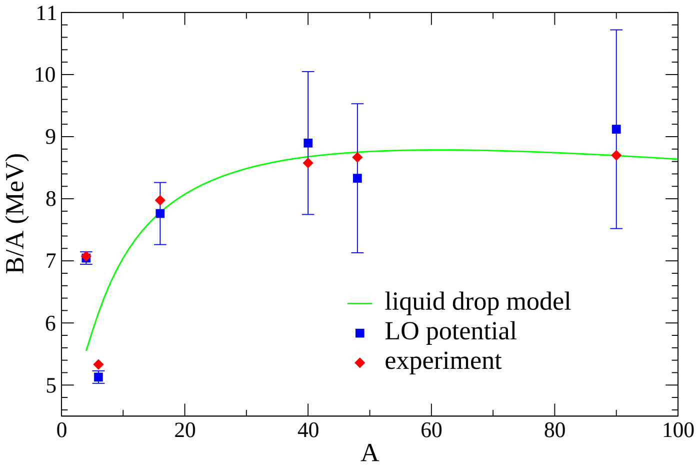
$$V_N = \sum_{i<j} V(i,j,r_0^{(2)},\beta_0)\mathcal{P}_{01} + \sum_{i<j} V(i,j,r_1^{(2)},\beta_1)\mathcal{P}_{10} + \sum_{i<j<k} W(i,j,k,r_3,\beta_3)$$

with β_3, r_3 fixed to reproduce $E(^3\mathrm{H})$ and $E(^4\mathrm{He})$

Mirroring the nuclear chart at the unitary limit



The LO potential at the physical point



Conclusions

- The nuclear system is well inside the universal window, accordingly it shows scale invariance
- Scale invariance manifests in particular correlations not well explained otherwise
- Moreover, this symmetry is independent of the microscopic theory as many different systems are located inside this window
- It will be important to incorporate this symmetry in the Ab Initio description of the nuclear structure

Conclusions

- The nuclear system is well inside the universal window, accordingly it shows scale invariance
- Scale invariance manifests in particular correlations not well explained otherwise
- Moreover, this symmetry is independent of the microscopic theory as many different systems are located inside this window
- It will be important to incorporate this symmetry in the Ab Initio description of the nuclear structure
- From our trip we have seen important structures suggesting a modification in the power counting that organizes the perturbative series
- We refer here either to chiral or to pionless EFT
- From our point of view the nuclear potential at lowest order should decribe the two-pole S-matrix plus the triton and alpha-particle binding energies