

Consistent treatment of bound and scattering states for Borromean nuclei

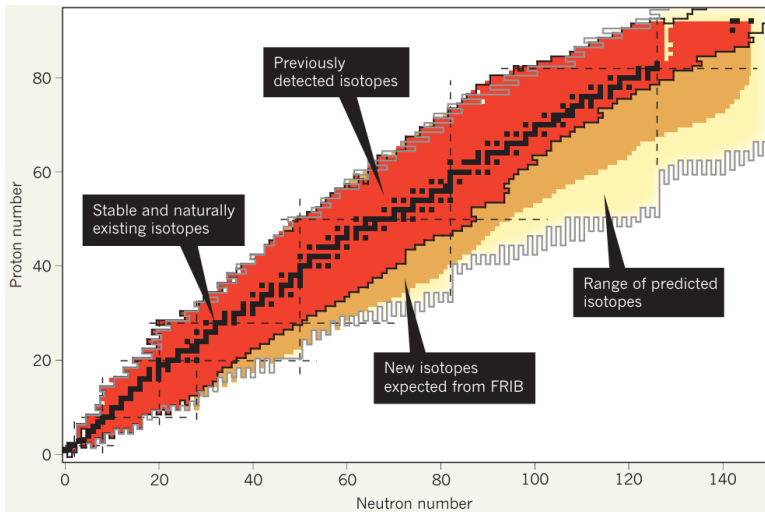
Using ^{22}C as an example

Patrick McGlynn

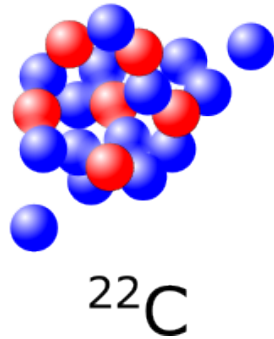
Facility for Rare Isotope Beams
Michigan State University

October 15, 2025

The future of isotope discovery is at the limits of the nuclear chart



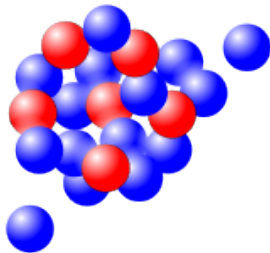
What do these have in common?



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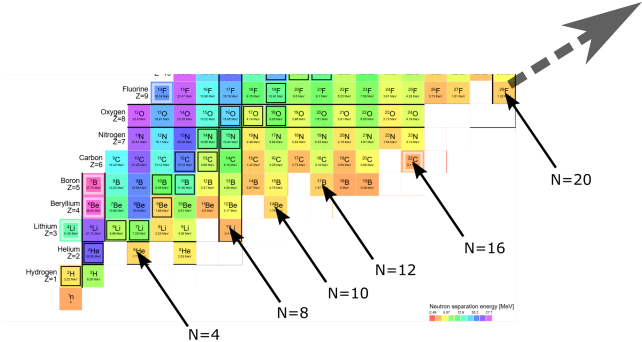
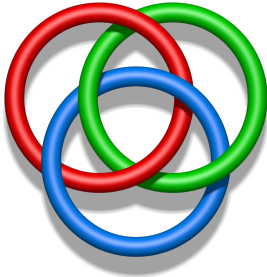


Palazzo
Borromeo



^{22}C

Near-dripline isotopes are strongly coupled to the continuum



Why do we want to know about Borromean systems?

Reasons for interest in Borromean systems:

- Presence across nuclear chart
- Universal properties of loosely-bound systems
- Importance of the continuum

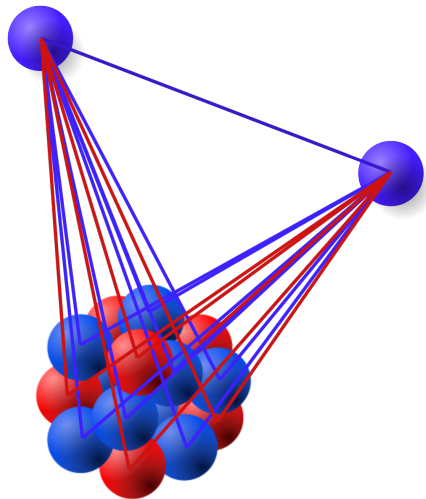
Note

Borromean nuclei are not always halo nuclei, e.g. ^{12}C as three α particles

Two neutron halos:

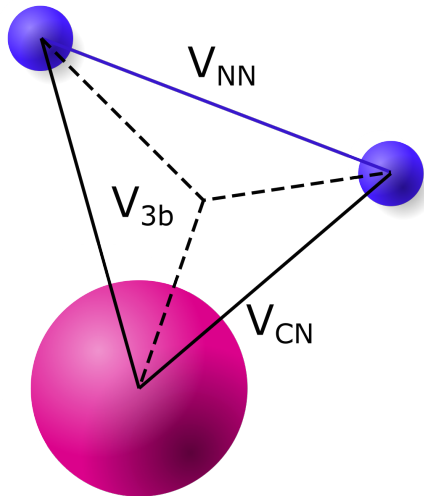
- Insight into dipole strengths of neutron-rich systems
- Support for upcoming experimental campaigns
- Possibility of extension to increasing complexity

Ab initio methods struggle to predict clusterised structures accurately



Ab-initio calculations treat all nucleons equally: challenging to compute.

Few-body methods are suited to nuclei near the dripline



Few-body representation: effective interactions

V_{CN} is the mean-field of the core

We need a consistent model of four-body reactions to unveil clusterised systems

Understanding halo and cluster structures are crucial to the progress of nuclear physics.



Experimental probes of exotic systems are model-dependent studies using reactions.



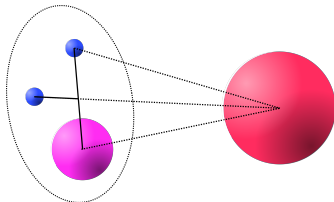
The models absolutely must treat bound and scattering states consistently.



I have built a framework that does this.



^{22}C is an excellent case study.



What can we ask for from a three-body model?

Wishlist:

- Bound **and** scattering states
- Non-resonant continuum
- Ability to extract detailed information about partial waves
- Scalable to larger, more complicated systems
- Fast, parallel code

Three-body models have been developed before:

- Faddeev calculations for structure and quasi-bound states (*Thompson, Nunes, Danilin and Lazauskas, Carbonell*)
- 3-body CDCC using Gaussian basis and transformed harmonic oscillator wavefunctions (*Rodriguez-Gallardo, Casal, Arias,...*)
- Gaussian expansion method (applied to > 3 body systems) (*Hiyama, Kamimura,...*)
- Gamow-coupled channel approach using complex momenta (*Wang, Nazarewicz, Michel, ...*)

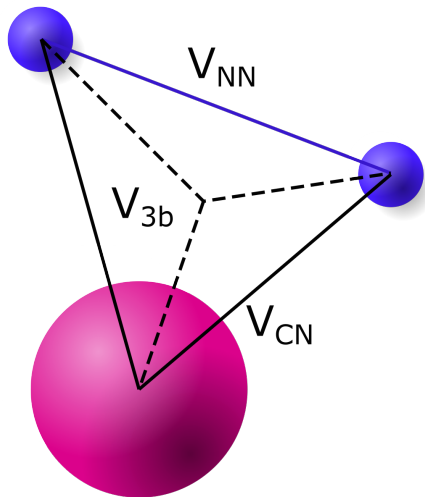
Reaction observables can be calculated once we know about states

General expression for an observable describing anything* we care about

$$A = \langle \Psi_{scat} | \hat{A} | \Psi_{bound} \rangle$$

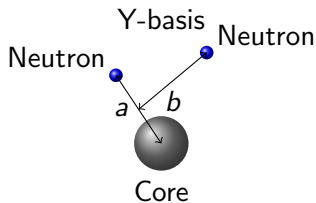
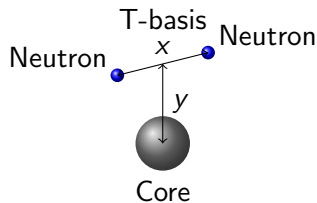
*anything includes interaction with a target for breakup, knockout,...

Computing states of the three-body system: what are the ingredients?



- Neutron-neutron potential
- Core-neutron potential
- Three-body potential
- Inert core (plan to extend to core deformation)

Hyperspherical harmonics are a useful tool to describe three-body dynamics



Hyperradius:

$$\rho = \sqrt{x^2 + y^2} = \sqrt{a^2 + b^2}$$

Hyperangle:

$$\alpha = \arctan(y/x)$$

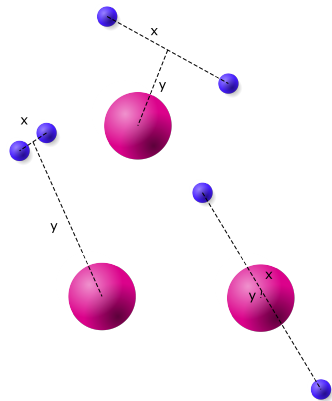
$$\Psi^{JM\pi}(\rho, \Omega_{5\rho}) = \rho^{-5/2} \sum_{K=0}^{\infty} \sum_{\gamma} \chi_{\gamma K}^{J\pi}(\rho) \mathcal{Y}_{\gamma K}^{JM}(\Omega_{5\rho})$$

$$\Omega_5 = (\alpha, \Omega_x, \Omega_y)$$

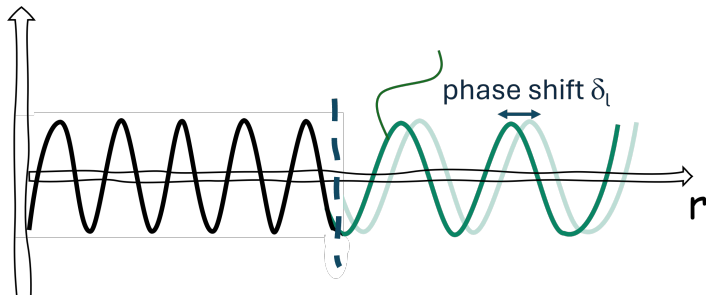
$$-\frac{\hbar^2}{2m_N} \left(\frac{\partial^2}{\partial \rho^2} - \frac{(K + 3/2)(K + 5/2)}{\rho^2} \right) \chi_{\gamma K}^{J\pi}(\rho) + \sum_{K'\gamma'} V_{K\gamma; K'\gamma'} \chi_{\gamma' K'}^{J\pi}(\rho) = E \chi_{\gamma K}^{J\pi}(\rho)$$

K_{max} is finite so this is an approximation

The three-body bound and scattering solutions are computed using the R-matrix



Channel radius in terms of
 $\rho = \sqrt{x^2 + y^2}$



Consistent treatment of the bound and scattering states $\rightarrow$ crucial for Borromean systems

R-matrix for bound states

Numerically solve for $0 < \rho \leq a$

$$(\mathcal{H} + \mathcal{L}(B) - E)\chi^{int}(\rho) = \mathcal{L}(B)\chi^{ext}(\rho)$$

$$\chi^{ext}(E < 0, \rho) = (\kappa\rho)^{1/2} \sum_K c_K K_{K+2}(\kappa\rho)$$

where $\mathcal{L}(B) = \frac{\hbar^2}{2\mu}\delta(\rho - a)\left(\frac{\partial}{\partial\rho} - \frac{B}{\rho}\right)$ is a matching operator at the boundary a , K_n is a modified Bessel function and $\kappa = \sqrt{\frac{2m|E|}{\hbar^2}}$. If we choose $B(E) = a\frac{\chi^{ext'}(a)}{\chi^{ext}(a)}$ then this becomes a homogeneous equation which we can solve:

$$(\mathcal{H} + \mathcal{L}(B(E)) - E)\chi^{int}(\rho) = 0$$

In practice we find eigenvectors of $\mathcal{H} + \mathcal{L}(B(E))$ with negative eigenvalues and iterate until converged.

For more details see Descouvemont & Baye, Rep. Prog. Phys. 2010

R-matrix for scattering states

$$\chi^{\text{ext}}(E > 0, \rho) = \sum_{K\gamma} i^{K+1} \left(\frac{2\pi}{k}\right)^{5/2} [H_K^-(k\rho) \delta_{KK'} \delta_{\gamma\gamma'} - U_{K\gamma; K'\gamma'} H_K^+(k\rho)]$$

If we choose $B = 0$ the Bloch-Schrödinger equation becomes:

$$(\mathcal{H} + \frac{\hbar^2}{2\mu} \delta(\rho - a) \frac{\partial}{\partial \rho} - E) \chi^{\text{int}}(\rho) = \frac{\hbar^2}{2\mu} \chi^{\text{ext}'}(a)$$

Using $G(\rho, \rho')$ as the Green's function which is defined as the solution to

$$(\mathcal{H} + \frac{\hbar^2}{2\mu} \delta(\rho - a) \frac{\partial}{\partial \rho} - E) G(\rho, \rho') = \delta(\rho - \rho')$$

Then

$$\chi^{\text{int}}(\rho) = \int_0^a d\rho' G(\rho, \rho') \frac{\hbar^2}{2\mu} \delta(\rho' - a) \chi^{\text{ext}'}(\rho') d\rho'$$

For more details see Descouvemont et al., Nuc. Phys. A 2006

R-matrix for scattering states (continued)

The R-matrix is the discrete form of $G(a, a)$ when we choose a finite basis

$$\chi^{int}(\rho) = \sum_i c_i \phi_i(\rho),$$

$$R_{ij} = \frac{\hbar^2}{2ma} \sum_{ij} \phi_i(a) (\mathcal{H}_{ij} + \mathcal{L}_{ij}(0) - E)^{-1} \phi_j(a)$$

We now have an equation for $\chi_{int}(a)$

$$c_i \phi_i(a) = R_{ij} ka \sum_{K\gamma} i^{K+1} \left(\frac{2\pi}{k}\right)^{5/2} \left[H_K^{-'}(ka) \delta_{KK'} \delta_{\gamma\gamma'} - U_{K\gamma; K'\gamma'} H_K^{+'}(ka) \right]$$

which depends only on the scattering matrix U , and we also have the boundary condition

$$\chi^{int}(a) = \chi^{ext}(\rho) \rightarrow c_i \phi_i(a) = \sum_{K\gamma} i^{K+1} \left(\frac{2\pi}{k}\right)^{5/2} \left[H_K^{-}(ka) \delta_{KK'} \delta_{\gamma\gamma'} - U_{K\gamma; K'\gamma'} H_K^{+}(ka) \right]$$

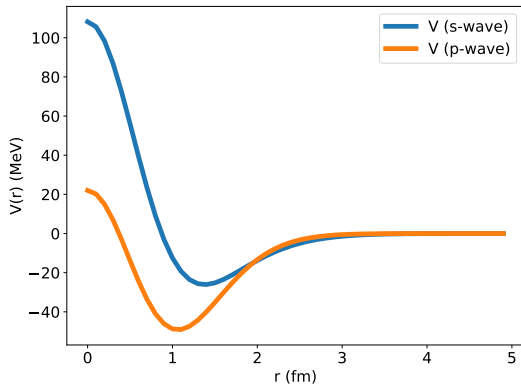
so we can compute U , and hence χ^{int} .

Operator definitions

$$B(E1) \sim \langle \Psi_{\text{scat}} | \hat{E}1 | \Psi_{\text{bound}} \rangle$$

$$\begin{aligned} \frac{dB(E\lambda)}{dE} = & \frac{(2\lambda+1)}{2(2\pi)^8} \frac{(2m_N)^3 E^2}{\hbar^6} \sum_J (2J+1) \sum_{K'\gamma'} \left| \sum_{K\gamma; K_0\gamma_0} \delta_{SS_0} (-1)^{L+S+K} \begin{Bmatrix} L_0 & S & J_0 \\ J & \lambda & L \end{Bmatrix} \right. \\ & \sqrt{2L_0+1} \sqrt{2L+1} \sqrt{(2l_{x0}+1)(2l_{y0}+1)} \int d\rho \rho^\lambda \chi_{K\gamma(\kappa',\gamma')}^{J\pi}(\rho) \chi_{K_0\gamma_0}^{J_0\pi_0}(\rho) \\ & \sum_{k=0}^{\lambda} \alpha_{\lambda k} \tilde{Z}^{\lambda k} \int \sin^{(2+k)} \alpha \cos^{(2+\lambda-k)} \alpha \phi_{l_x l_y}^K(\alpha) \phi_{l_{x0} l_{y0}}^{K_0}(\alpha) d\alpha \\ & \left. \begin{Bmatrix} \lambda-k & k & \lambda \\ l_x & l_y & L \\ l_{x0} & l_{y0} & L_0 \end{Bmatrix} \sqrt{(2(\lambda-k)+1)(2k+1)} C_{l_{x0}0(\lambda k)0}^{l_x 0} C_{l_{y0}0 k 0}^{l_y 0} \right|^2 \end{aligned}$$

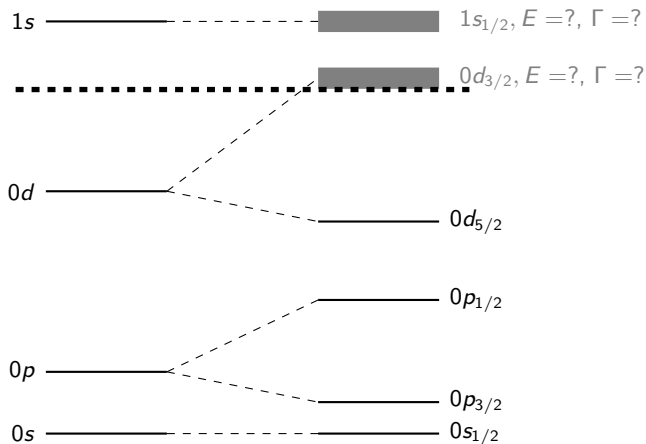
How we choose the neutron-neutron interaction



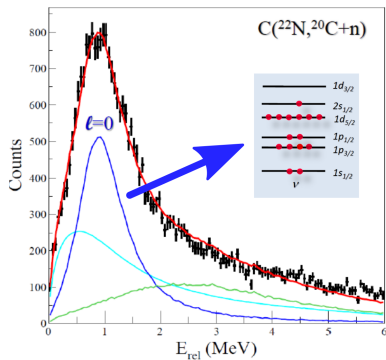
Minnesota potential reproduces scattering length of neutrons, has different interactions for different partial waves.

The core-neutron interaction needs some more constraints

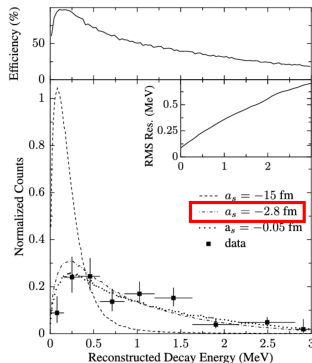
- s-wave virtual state?
- d-wave resonance?
- spin-orbit splitting
- ^{22}C bound state properties



Knockout and decay reactions tell us the structure of ^{21}C



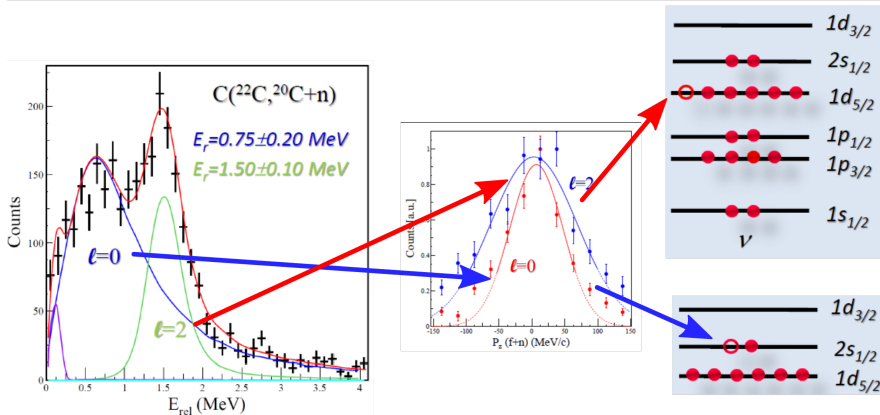
S. Leblond et al. NP1106-SAMURAI04



Mosby et al. 2013

Suggests ^{21}C has a $1/2^+$ virtual state.

Knockout reactions from ^{22}C tell us a bit more

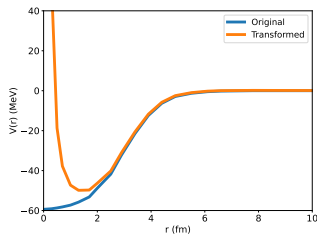


S. Leblond et al. NP1106-SAMURAI04

Tension!

There are also conflicting measurements → opportunity to find resolving experiment

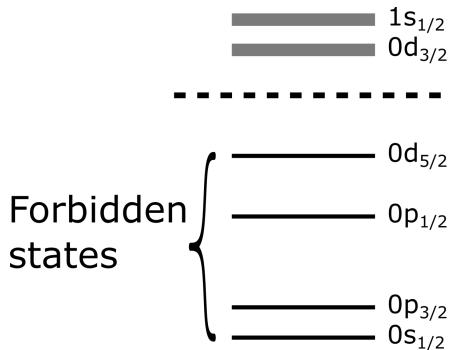
Why and how do we remove “forbidden” states?



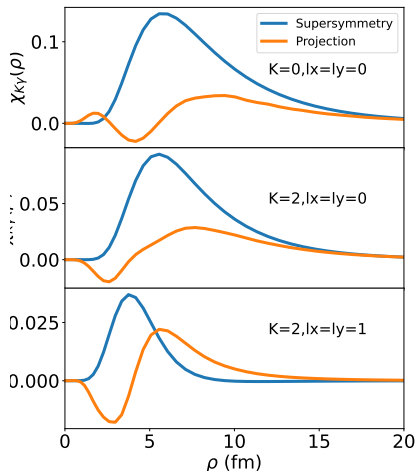
Supersymmetry: modifies the potential

$$\hat{P} = E_{\infty} \underbrace{|u_{\text{forbidden}}\rangle \langle u_{\text{forbidden}}|}_{\text{two-body state (neutron-core)}} \otimes \underbrace{\hat{1}}_{\text{the other neutron}}$$

Projection: Adds a three-body non-local operator.



Projecting out forbidden states is the right approach but it's a lot harder



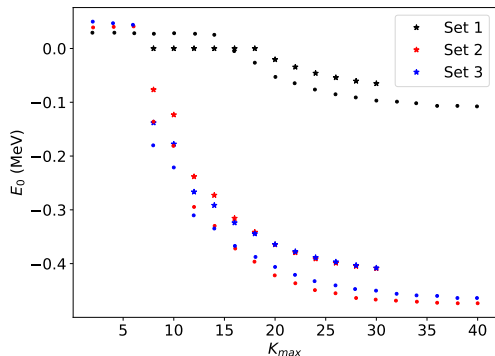
What is the supersymmetric approach?

Transformation of the potential that preserves **two-body** phase shifts while removing unwanted bound states.

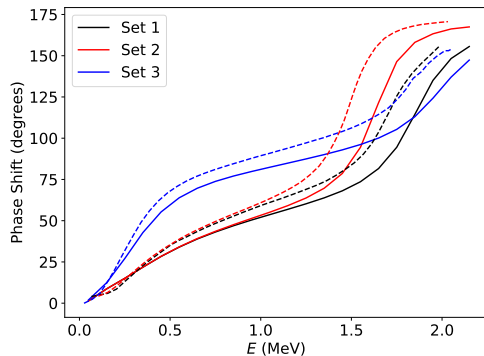
The good: Projection gives nodes corresponding to the forbidden states.

The bad: Projection operator is very difficult to compute exactly and leads to convergence difficulties.

How do we know the code is working?



Stars: this work, circles: benchmark



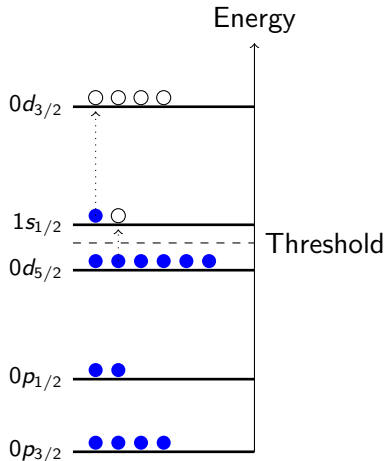
Solid: this work, dashed: benchmark

Comparison is using supersymmetry to compare with E. Pinilla, P. Descouvemont PRC 2016

We compare two different models of the structure of ^{21}C

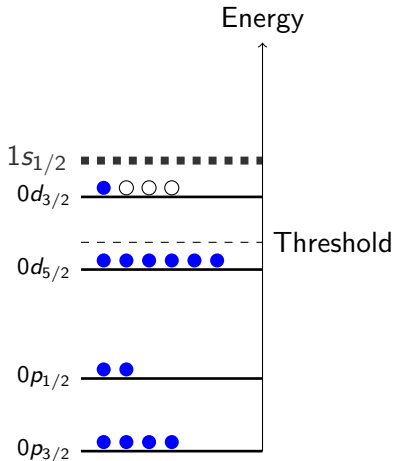
Model A:

^{21}C ground state is $1/2+$ resonance, excited states are $5/2+$ then $3/2+$ resonance

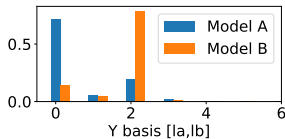
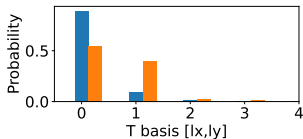
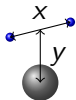
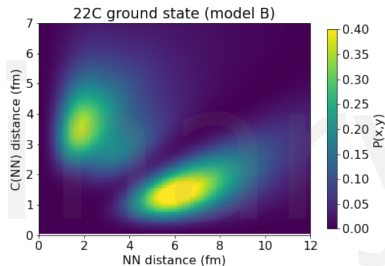
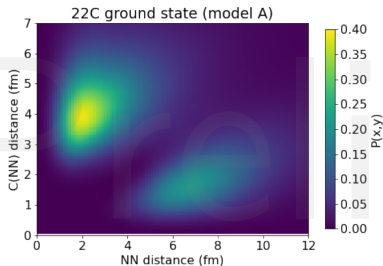


Model B:

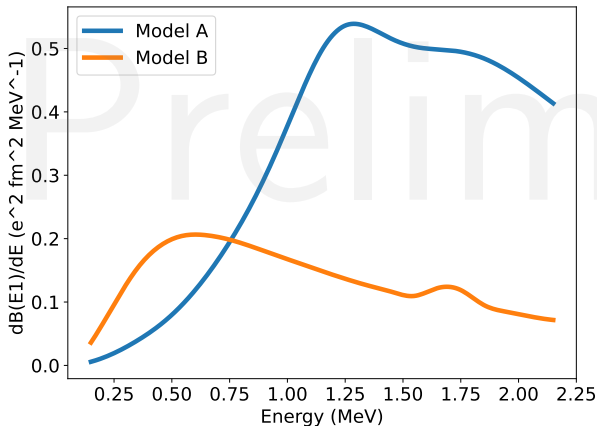
^{21}C ground state is $3/2+$ resonance, $1s_{1/2}$ is a virtual state



Carbon-22 can look very different depending on what we think Carbon-21 looks like



When we know about the structure and the scattering states, we can predict observables



Three-body force:

The binding energy in model A is fixed to that of model B (-0.43 MeV) using a three-body force.

Model uncertainty

The models we chose are not unique, we need to quantify the uncertainty from our choice of parameters.

Conclusions

Fundamentally, structure and reactions cannot be separated

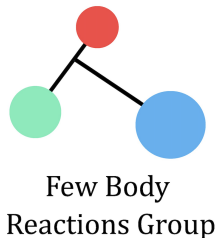
- Loosely bound (clusterised) systems couple continuum and bound states.
- Making use of the interplay between structure and reactions can help us understand the extremes of the nuclear chart.

Future work:

- Related: describe mid-mass nuclei and $3+1$ body reactions using three-body model, and quantify uncertainties.
- Unrelated: investigate multinucleon transfer using beyond mean-field methods.
- Long term: unify few- and many-body methods for a more comprehensive picture of nuclear physics.

Acknowledgments

Thanks to **Chloë Hebborn**, the FRIB few-body group and the FRIB Theory Alliance.



Extra slides

How do we actually compute the projection operator?

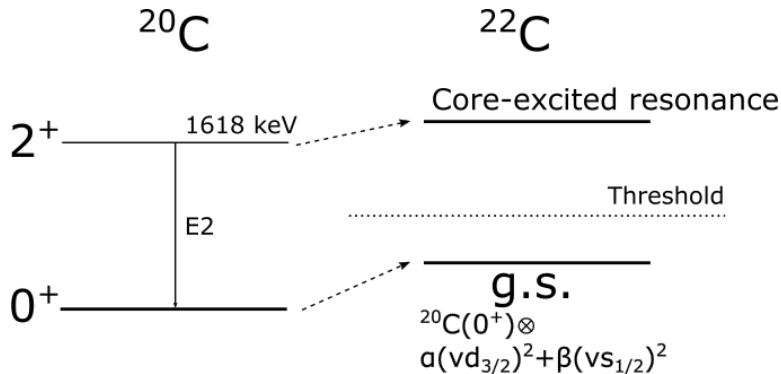
Three-body inner product

$$\langle \Psi_1(\rho, \alpha) | \Psi_2(\rho, \alpha) \rangle = \int_0^\infty \rho^5 d\rho \int_0^{\pi/2} d\alpha \sin^2 \alpha \cos^2 \alpha \rho^{-5} \Psi_1(\rho, \alpha) \Psi_2(\rho, \alpha)$$

with states $\Psi(\rho, \alpha) = \sum_K \chi_K(\rho) \phi_K(\alpha)$ (ϕ_K are basis functions for hyperangle). We want to construct three-body states which contain a given two-body (core-neutron) state $u(x)$. We construct a (two-body) basis $f_n(y)$ where $\sum_n |f_n(y)\rangle \langle f_n(y)| = 1_y$ and then the three-body forbidden states are $U_n(\rho, \alpha) = u(\rho \cos \alpha) f_n(\rho \sin \alpha)$. Using a Gauss-Lagrange expansion in ρ such that $\psi(\rho) = \sum_i c_i g_i(\rho)$ we obtain projector elements

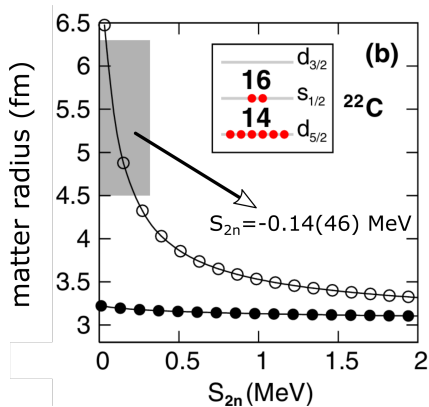
$$P_{Ki, K'i'} = (\lambda_i \lambda_{i'})^{1/2} \sum_n \left(\int_0^{\pi/2} d\alpha \sin^2 \alpha \cos^2 \alpha u(\rho_i \cos \alpha) f_n(\rho_i \sin \alpha) \phi_K(\alpha) \right) \\ \left(\int_0^{\pi/2} d\alpha \sin^2 \alpha \cos^2 \alpha u(\rho_{i'} \cos \alpha) f_n(\rho_{i'} \sin \alpha) \phi_{K'}(\alpha) \right)$$

The three-body system is built on top of the 0^+ ground state of the core.



Core excitation to be investigated in future

Binding energy and rms



L. Gaudefroy et al. 2012