

Fusion reactions in collisions of weakly bound nuclei with heavy targets

Jeannie Rangel Borges

15/10/2025

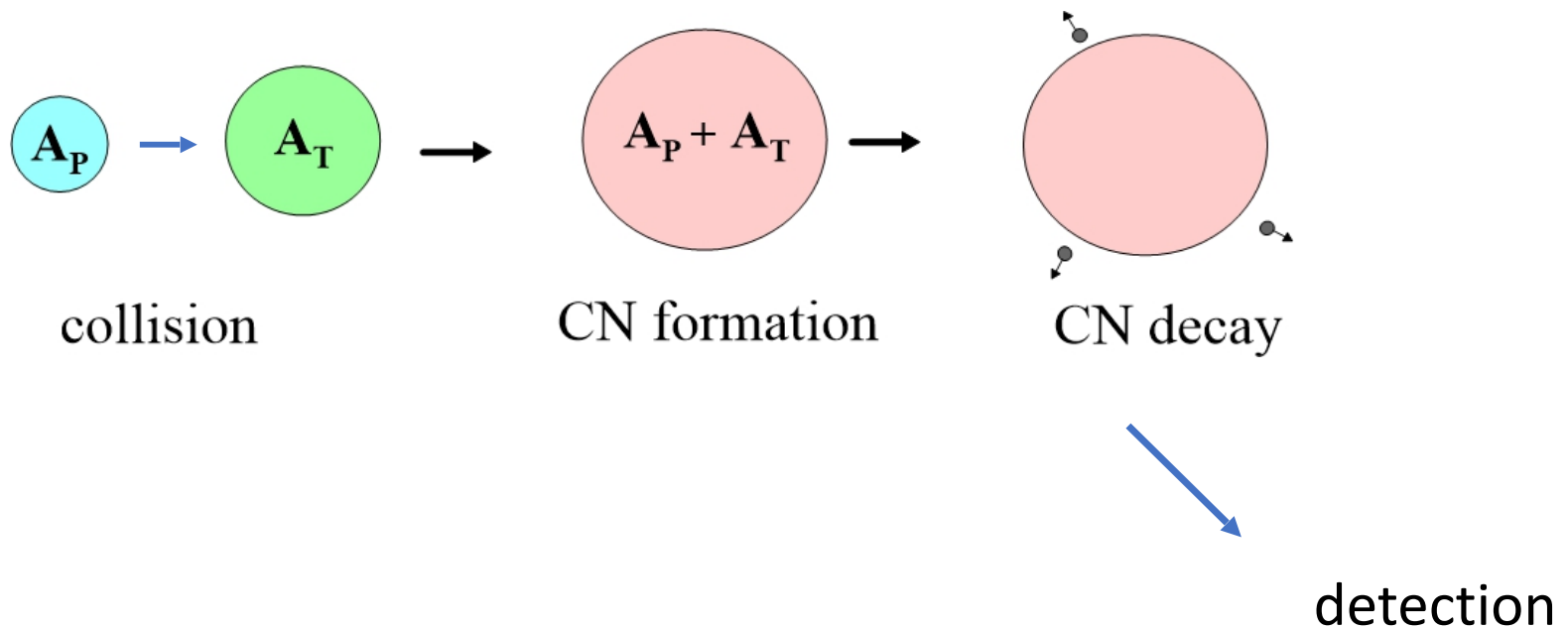


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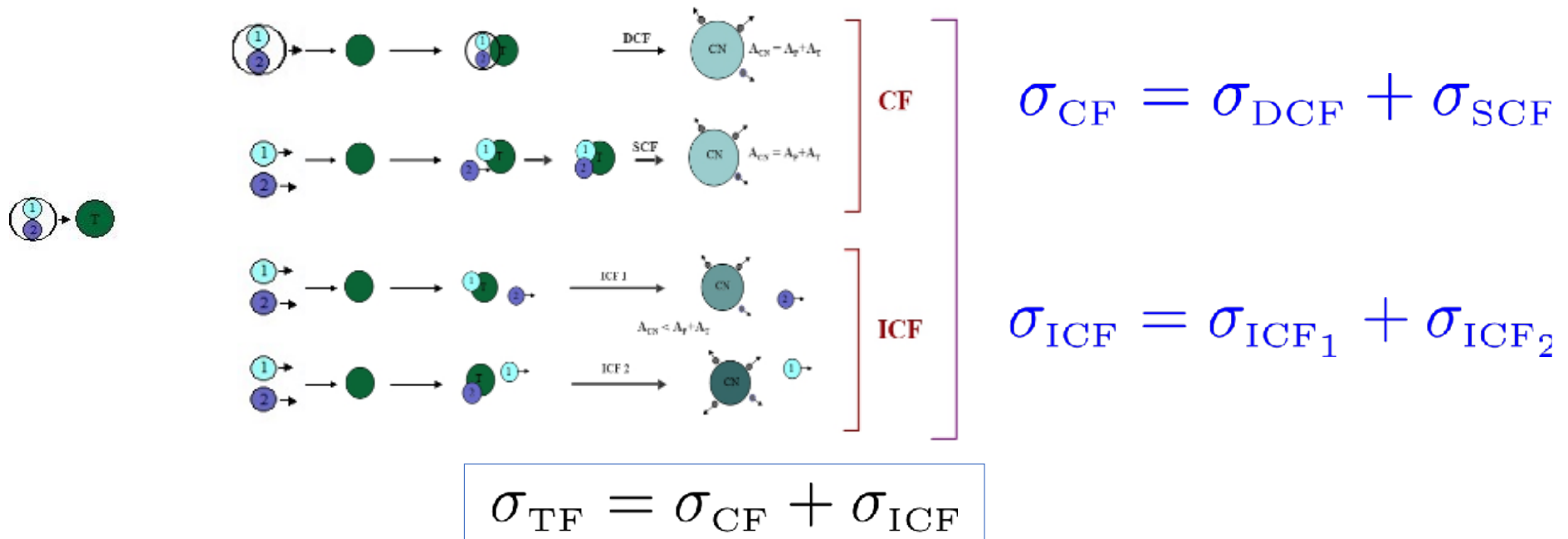
Outline

- Motivation
- Brief review of the theory
 - ✓ Classical and semiclassical model
 - ✓ Quantum mechanics models
- Our approach
- Some illustrations
- conclusions

Fusion of tightly bound nuclei



Collisions of weakly bound nuclei (different fusion processes)



Finding CF and ICF cross section is a great challenge

(both for experimentalists and theorists)

Experiment:

- CF absorption of all projectile charge ($^{11}\text{Be} = ^{10}\text{Be} + n$)
- Most experiments determine only TF
- Individual CF and/or ICF have been measured for some particular weakly bound nuclei:

^6Li : BE= 1.47 MeV ($\alpha + d$)

^7Li : BE= 2.47 MeV ($\alpha + t$)

^{11}Be : BE = 0.500 MeV ($^{10}\text{Be} + n$)

^6He : BE =0.600 MeV ($^4\text{He} + n+n$)

Finding CF and ICF cross section is a great challenge

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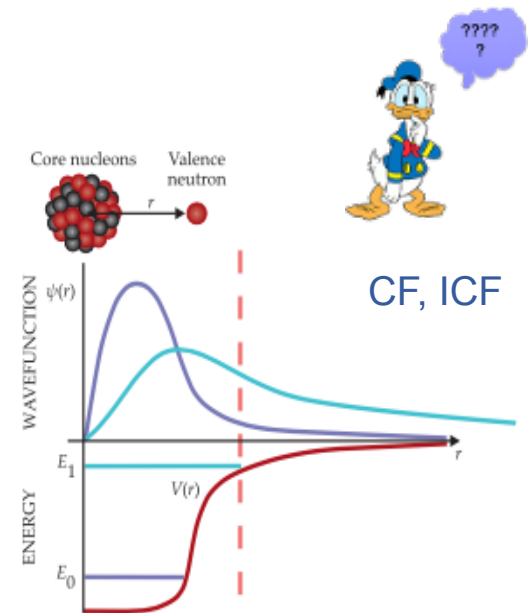
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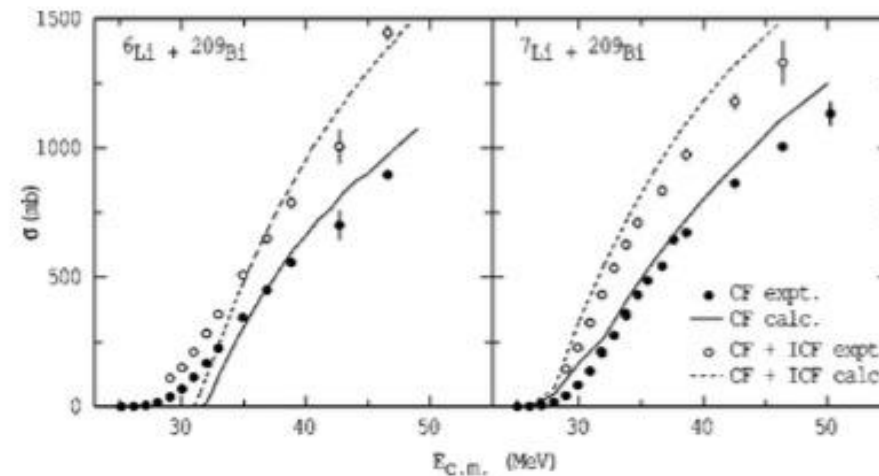
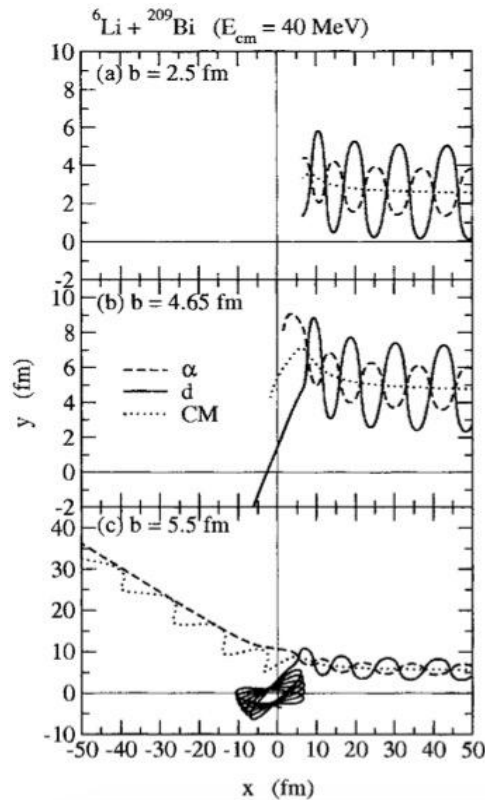
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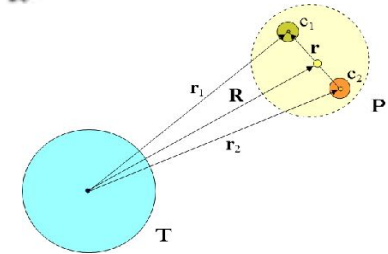
^6He : BE =0.600 MeV ($^4\text{He} + n+n$)



Fusion Estimations: classical picture

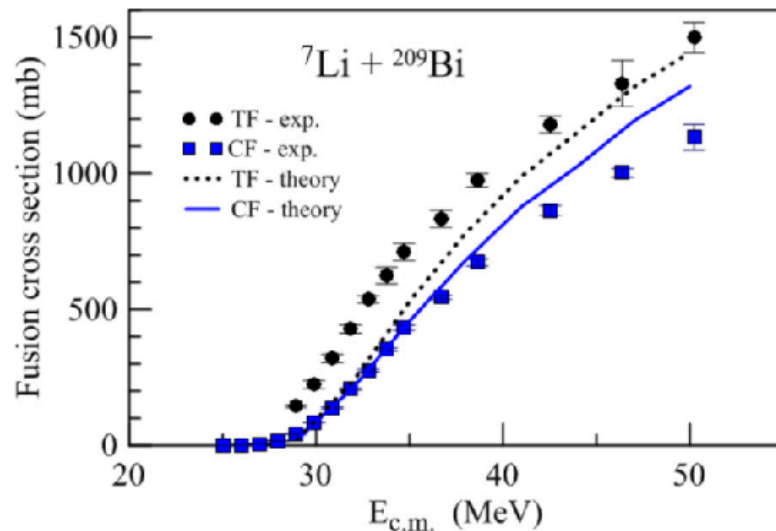


- Classical picture with stochastic parameters.



Hagino et al., NPA **238**, 475 (2004), Dasgupta et al., PRC **66**, 041602 (2002),

Fusion Estimations: semi classical models



- Classical trajectory
- Intrinsic dynamic: time dependent Schrödinger equation

$$[h_0 + V(b, \vec{r}, t)]\Psi(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, t) \quad \therefore$$

$$\Psi(b, t) = \sum c_i(b, t) \psi_i e^{-i\epsilon_i t/\hbar}$$

- Fusion: tunnelling through the barrier

* Marta et al., PRC **89**, 034625 (2014), Kolinger et al., PRC **98**, 044604 (2018)

The method of Hagino, Vitturi, Dasso and Lenzi (HVDL)

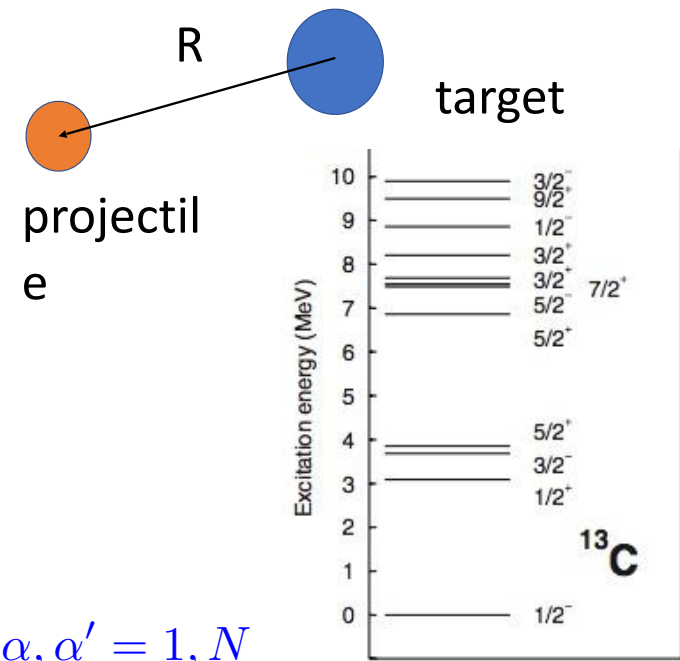
- To solve the Schrödinger

$$\mathbb{H}\Psi^+ = E\Psi^+$$

- Coupled Channels equations:

$$\Psi^+(R, \xi) = \sum \psi_\alpha^+(R) \phi(\xi)$$

$$[E - \epsilon_\alpha - T_R - U_{\alpha,\alpha}(R)]\psi_\alpha(R) = \sum_{\alpha' \neq \alpha} U_{\alpha',\alpha}(R)\psi_{\alpha'}(R), \alpha, \alpha' = 1, N$$



The method of Hagino, Vitturi, Dasso and Lenzi (HVDL)

- Set of coupled equations

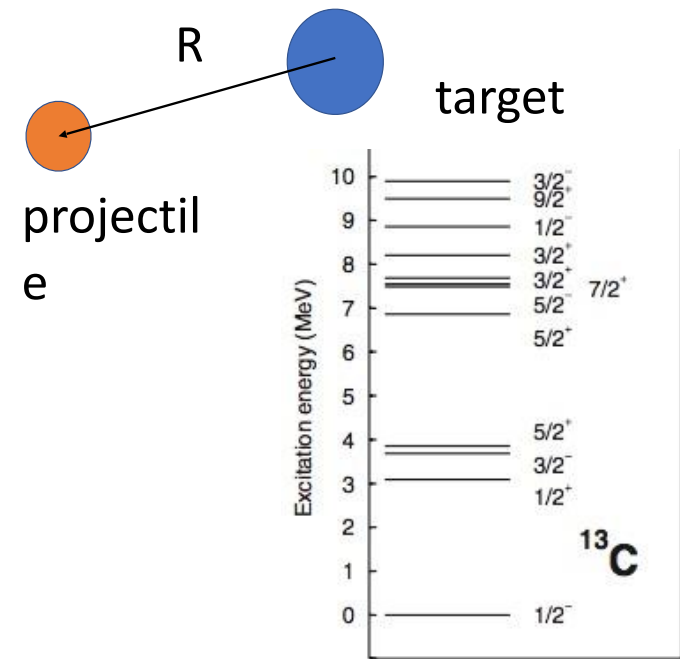
$$[E - \epsilon_0 - T_R - U_{0,0}(R)]\psi_0(R) = \sum_{\alpha' \neq 0} U_{\alpha',0}(R)\psi_{\alpha'}(R)$$

⋮

$$[E - \epsilon_N - T_R - U_{N,N}(R)]\psi_N(R) = \sum_{\alpha' \neq N} U_{\alpha',N}(R)\psi_{\alpha'}(R)$$

- Coupled potential

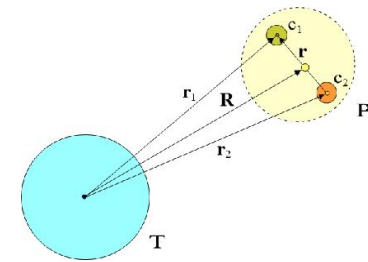
$$U_{\alpha',\alpha}(R) = \int d\xi \phi_{\alpha'}^*(\xi) \mathbb{U}_{PT}(R, \xi) \phi_{\alpha}(\xi)$$



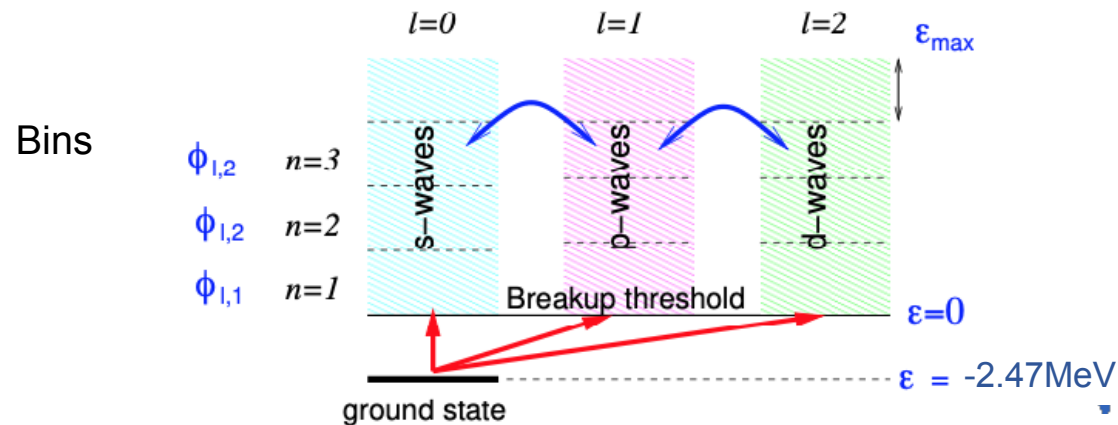
The method of Hagino, Vitturi, Dasso and Lenzi (HVDL)

CDCC :

$$U(r_1, r_2) = U_1(r_1) + U(r_2)$$



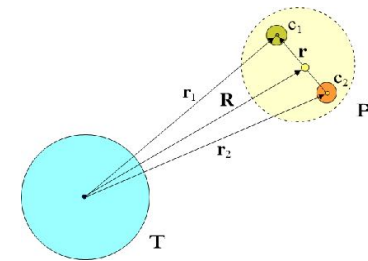
- **Continuum Discretized Coupled Channels equations:**



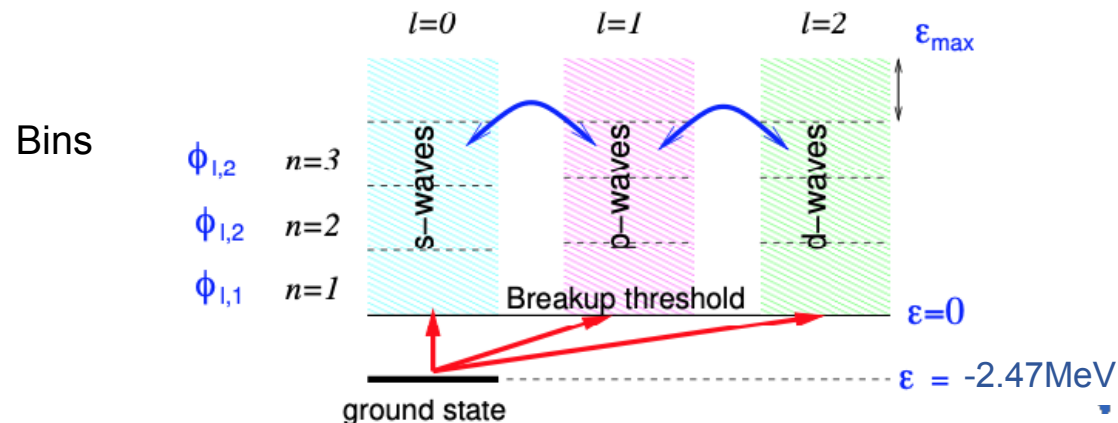
The method of Hagino, Vitturi, Dasso and Lenzi (HVDL)

CDCC :

$$U_{\alpha',\alpha}(R) = \int d\mathbf{r}(r) \phi_{\alpha'}^*(r) [\mathbb{U}_{vT}(r_1) + \mathbb{U}_{cT}(r_1)] \phi_{\alpha}(r)$$



•Continuum Discretized Coupled Channels equations:



The method of Hagino, Vitturi, Dasso and Lenzi (HVDL)

- **P-T imaginary potential (instead of $W(1) + W(2)$)**

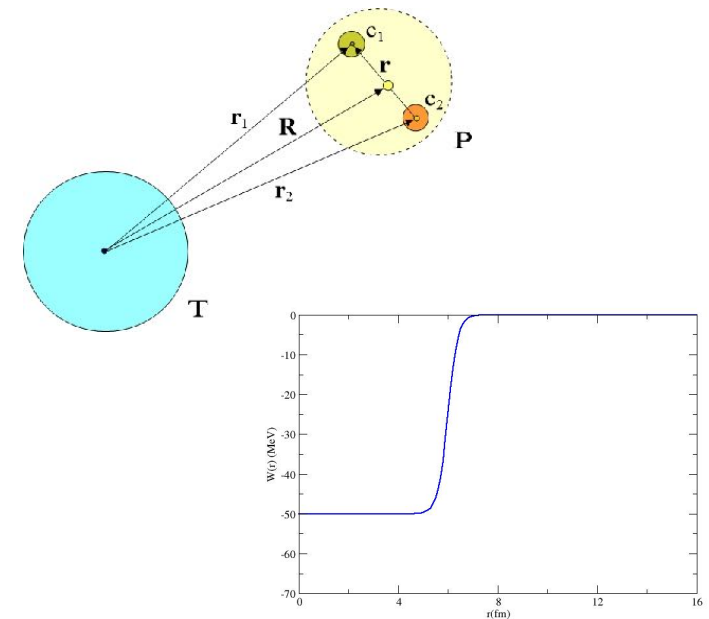
$$\sigma_F = \frac{K}{E} \sum_{\alpha=1} < \psi_{\alpha} | W(R) | \psi_{\alpha} > = \sum_{\alpha=1} \sigma_{\alpha}$$

- **Or**

$$\sigma_F = \sigma_C + \sigma_B$$

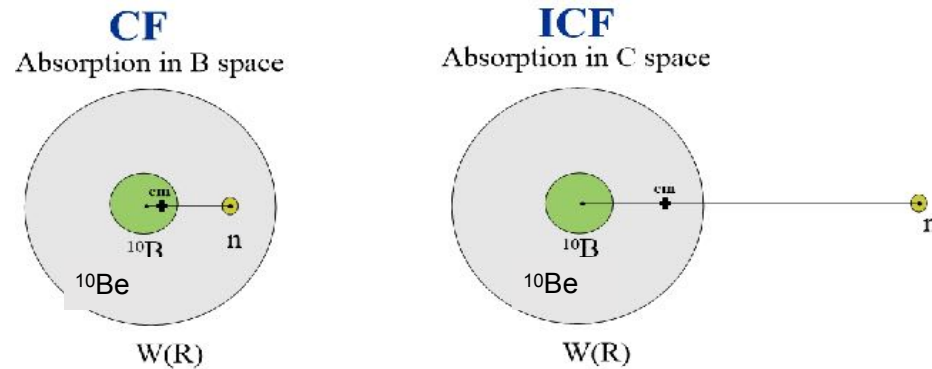
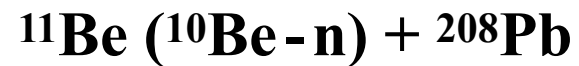
$$\sigma_B = \frac{K}{E} \sum_{\alpha \in \text{bound}} < \psi_{\alpha} | W(R) | \psi_{\alpha} >$$

$$\sigma_C = \frac{K}{E} \sum_{\alpha \in \text{continuum}} < \psi_{\alpha} | W(R) | \psi_{\alpha} >$$



Basic Assumption: $\sigma_{CF} = \sigma_B$, $\sigma_{ICF} = \sigma_C$

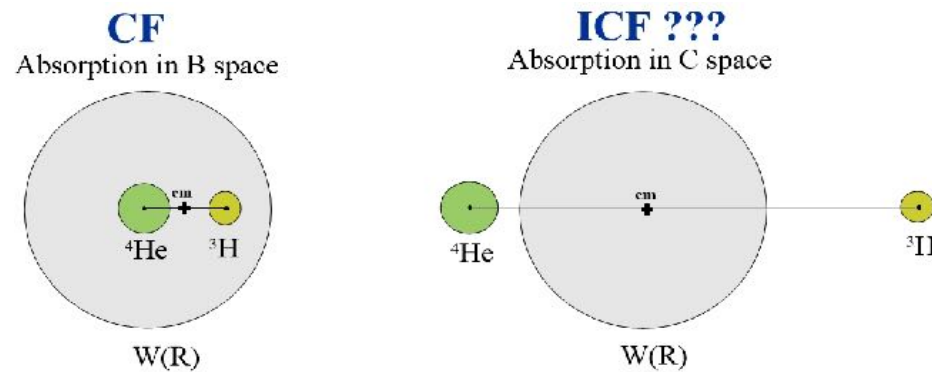
Limitation: works for a fragment much heavier than the other



Works fine !

Basic Assumption: $\sigma_{CF} = \sigma_B$, $\sigma_{ICF} = \sigma_C$

Limitation: works for a fragment much heavier than the other



Does not work !

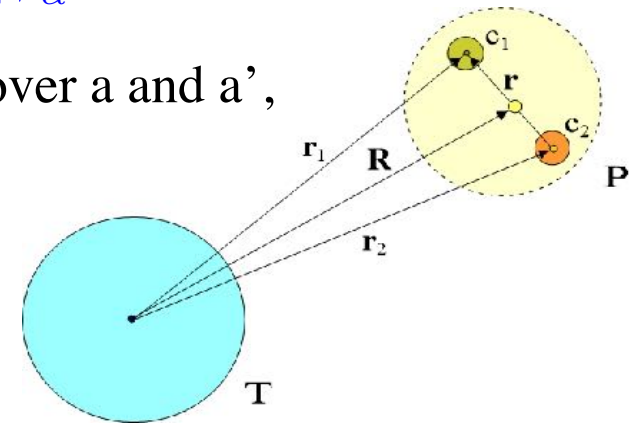
Theory: based on HVDL *

- Direct calculation using radial wave functions

$$\sigma_F = \frac{K}{E} \sum_{\alpha, \alpha'} \langle \psi_\alpha | W_{\alpha, \alpha'}^1(r_1) + W_{\alpha, \alpha'}^2(r_2) | \psi'_{\alpha'} \rangle$$

Performing ang. mom. projection and the summing over α and α' ,

$$\sigma_F = \frac{\pi}{k^2} \sum_J (2J + 1) [P^1(J) + P^2(J)]$$



* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

Theory: based on HVDL *

- Individual probability

$$\mathcal{P}^{(i)}(J) = \frac{4K}{E \hat{j}_0^2} \sum_{L_0} \sum_{\lambda} \sum_{\alpha L \alpha' L'} \mathcal{B}_{\alpha L, \alpha' L'}^{\lambda}(J) \mathcal{M}_{\alpha L, \alpha' L'}^{(i)\lambda}(L_0, J), \quad \text{where}$$

$$\mathcal{B}_{\alpha L, \alpha' L'}^{\lambda}(J) = (-)^{J-s+j_{\alpha}-j_{\alpha'}-L-L'+\lambda} \hat{\lambda}^2 \hat{L} \hat{L}' \hat{l}_{\alpha} \hat{l}_{\alpha'} \hat{j}_{\alpha} \hat{j}_{\alpha'} \\ W(LL'j_{\alpha}j_{\alpha'}; \lambda J) W(l_{\alpha}l_{\alpha'}j_{\alpha}j_{\alpha'}; \lambda s) \begin{pmatrix} \lambda & L' & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda & l_{\alpha'} & l_{\alpha} \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{M}_{\alpha L, \alpha' L'}^{(i)\lambda}(L_0, J) = i^{L'-L} \int dR u_{\alpha L, 0L_0}^{J*}(k_{\alpha}, R) \mathcal{F}_{l_{\alpha}j_{\alpha}, l_{\alpha'}j_{\alpha'}}^{(i)\lambda}(R) u_{\alpha' L', 0L_0}^J(k'_{\alpha}, R).$$

* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

A new QM method to evaluate CF and ICF*

Contribution from the B-space: $\sigma_B = \sigma_{DCF}$

$$\sigma_B = \frac{K}{E} \sum_{\alpha \in B} \langle \psi_\alpha | W(R) | \psi_\alpha \rangle$$



$$\sigma_{DCF} = \frac{\pi}{k^2} \sum_J (2J+1) P(J)$$



* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

Contribution from channels in the continuum to TF*

$$\sigma_C = \frac{K}{E} \sum_{\alpha, \alpha' \in C} \langle \psi_\alpha | W_{\alpha, \alpha'}^1(r_1) + W_{\alpha, \alpha'}^2(r_2) | \psi'_\alpha \rangle$$

$$\searrow \sigma_C = \frac{\pi}{k^2} \sum_J (2J + 1) [P^1(J) + P^2(J)]$$

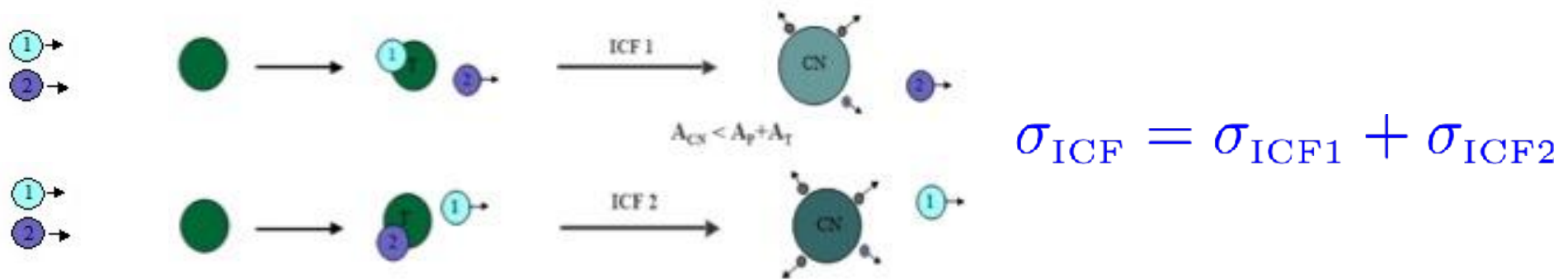
$P(i) (J)$ = abs. probability of fragment i in the C-space

* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

ICF (ICF1, ICF2), SCF cross sections*

$$\sigma_{\text{ICF1}} = \frac{\pi}{k^2} \sum_J (2J+1) P^1(J) [1 - P^2(J)]$$

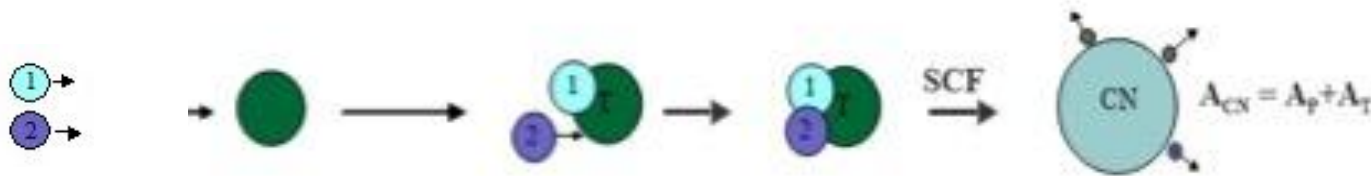
$$\sigma_{\text{ICF2}} = \frac{\pi}{k^2} \sum_J (2J+1) P^2(J) [1 - P^1(J)]$$



* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

ICF (ICF1, ICF2), SCF cross sections*

$$\sigma_{\text{SCF}} = \frac{\pi}{k^2} \sum_J (2J + 1) P^1(J) \times P^2(J)$$

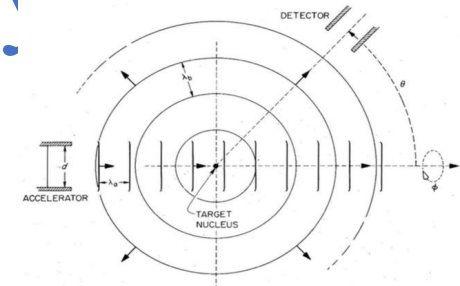


$$\sigma_{\text{CF}} = \sigma_{\text{DCF}} + \sigma_{\text{SCF}}$$

- J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)
- Further applications: PRC, **102**, 064628 (2020) and PRC **105**, 054601 (2022)

Procedure Summary

- Run FRESCO code: CDCC Method

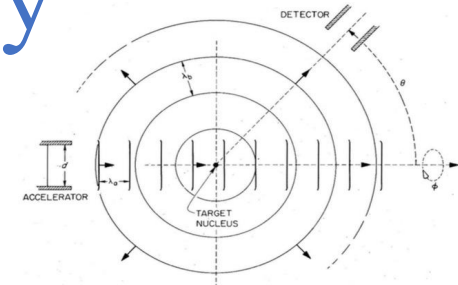


$$\sigma_{\alpha} = \frac{1}{(2I_{p_i} + 1)(2I_{t_i} + 1)} \sum |f_{\alpha, \alpha'}(\theta)|^2$$

* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

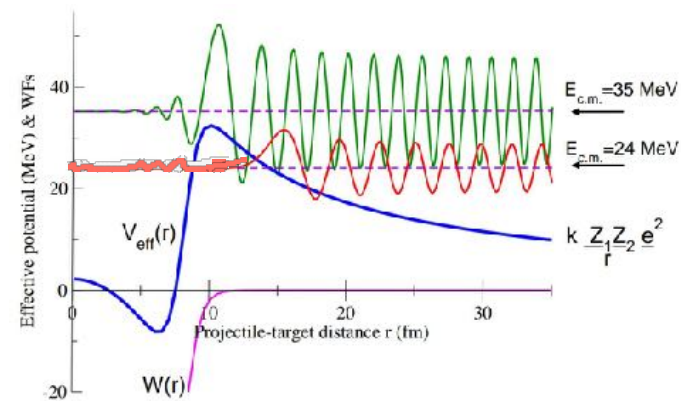
Procedure Summary

- Run FRESKO code: CDCC Method



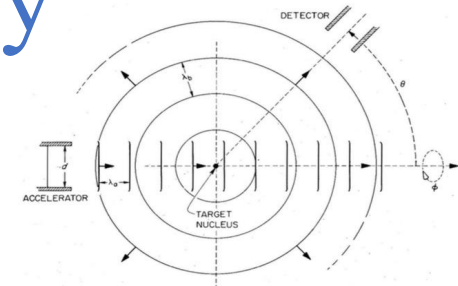
$$\sigma_{\alpha} = \frac{1}{(2I_{p_i} + 1)(2I_{t_i} + 1)} \sum |f_{\alpha, \alpha'}(\theta)|^2$$

- Extract scattering wave function



Procedure Summary

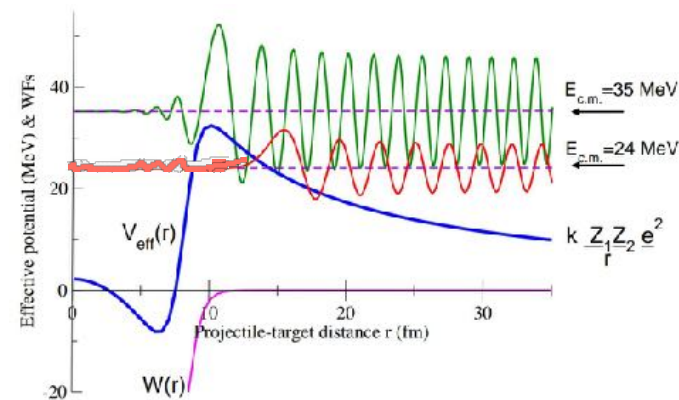
- Run FRESKO code: CDCC Method



$$\sigma_{\alpha} = \frac{1}{(2I_{p_i} + 1)(2I_{t_i} + 1)} \sum |f_{\alpha, \alpha'}(\theta)|^2$$

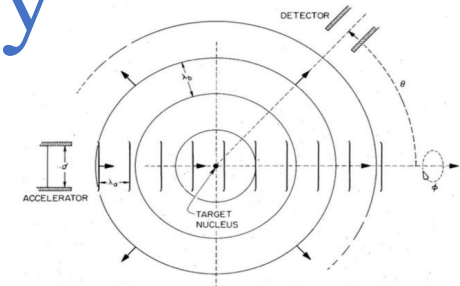
- Extract scattering wave function
- Estimate probabilities

$$\sigma_F = \frac{\pi}{k^2} \sum_J (2J + 1) [P^1(J) + P^2(J)]$$



Procedure Summary

- Run FRESKO code: CDCC Method



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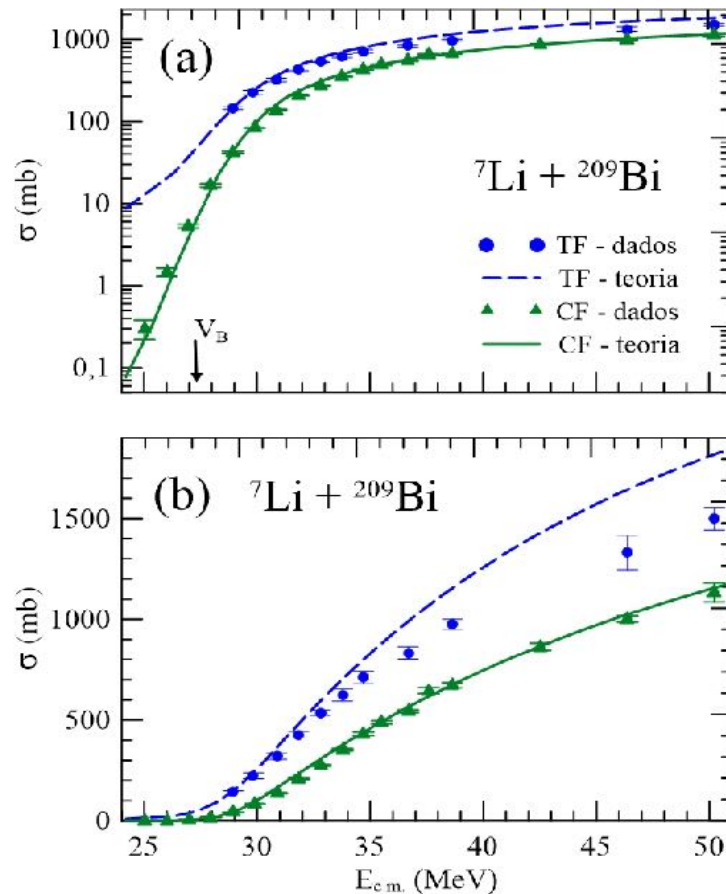
$$\sigma_F = \frac{\pi}{k^2} \sum_J (2J + 1) [P^1(J) + P^2(J)]$$

- Separate fusion processes:

$$\sigma_{\text{ICF}} = \sigma_{\text{ICF1}} + \sigma_{\text{ICF2}}$$

$$\sigma_{\text{CF}} = \sigma_{\text{DCF}} + \sigma_{\text{SCF}}$$

${}^7\text{Li} + {}^{209}\text{Bi}$ fusion – theory vs. experiment*



2.47 MeV $\alpha + t$

0.48 MeV $1/2^-$

0.0 MeV $3/2^-$

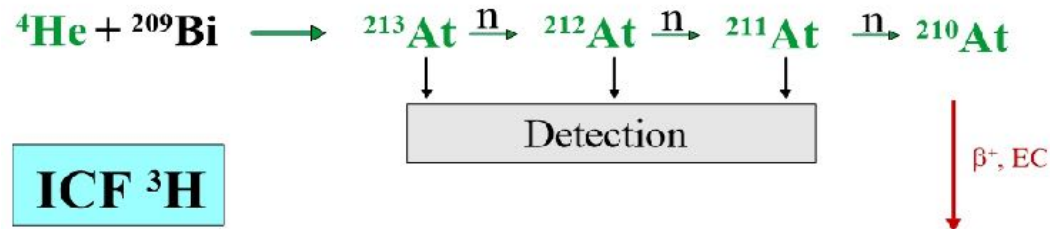
- TF and CF predictions are in excellent agreement with data below V_B , and above up to $E \sim 36$ MeV

- Predictions for TF above ~ 36 MeV overestimate the experiment. But...

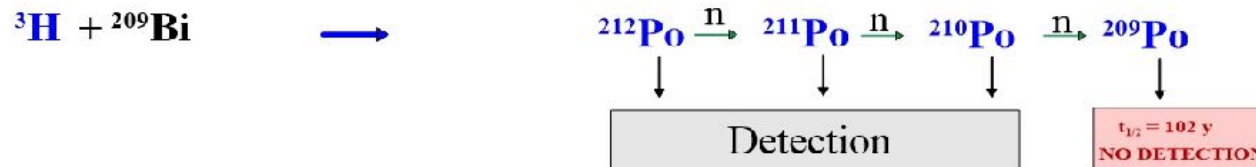
* Dasgupta *et al.*, PRC **66**, 041602(R) (2002); PRC **70**, 024606 (2004)

Decay schemes of nuclei produced by ICF

ICF ^4He



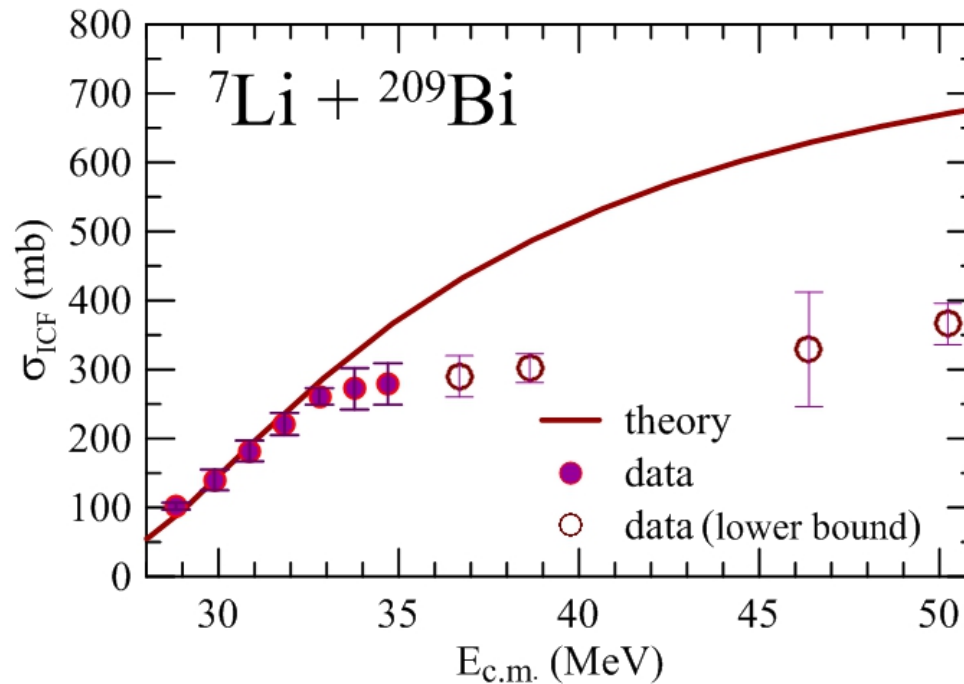
ICF ^3H



- ^{209}Po is the decay chain of both ICF processes
- Contribution from ^{209}Po is not detected
- Estimates with PACE: ^{209}Po is important above 36 MeV.
- Above $E_{\text{c.m.}} \sim 36 \text{ MeV}$, data is only a lower bound

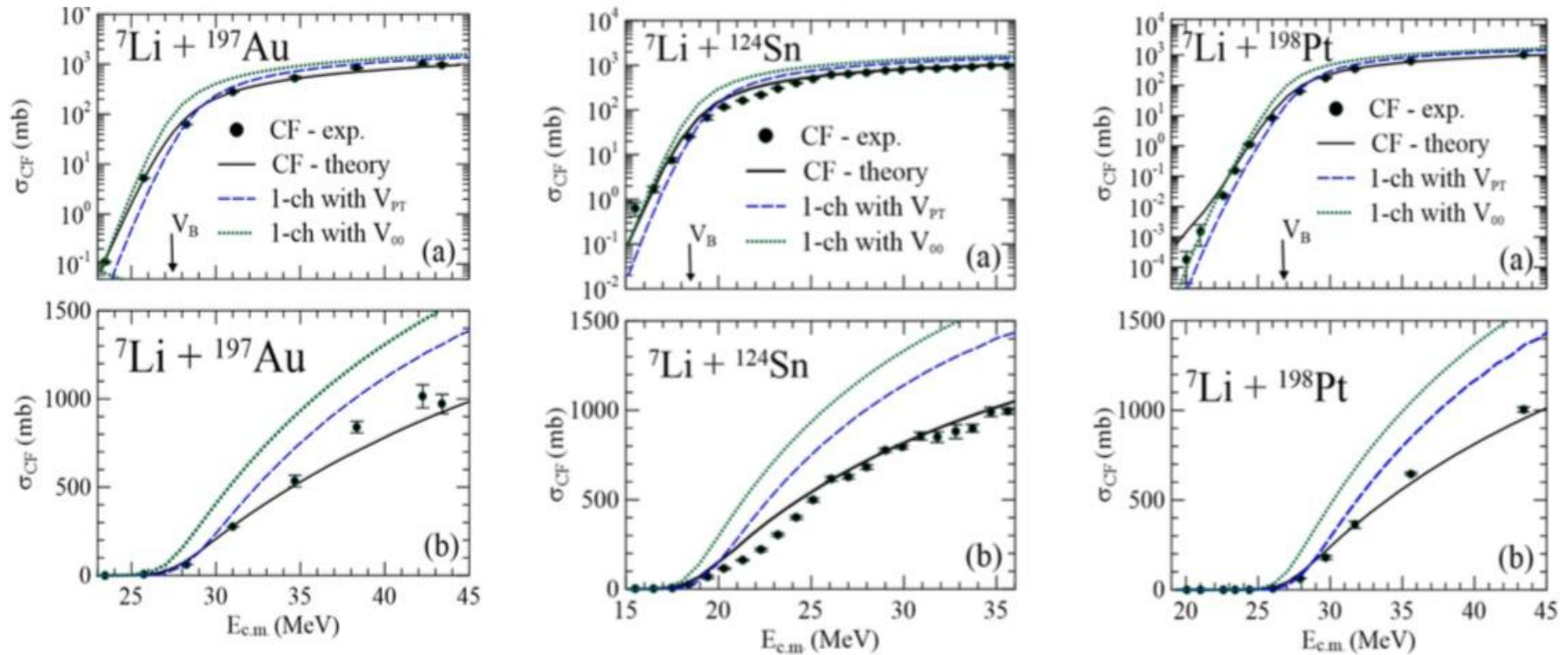
Authors' conclusions of the experiment

ICF



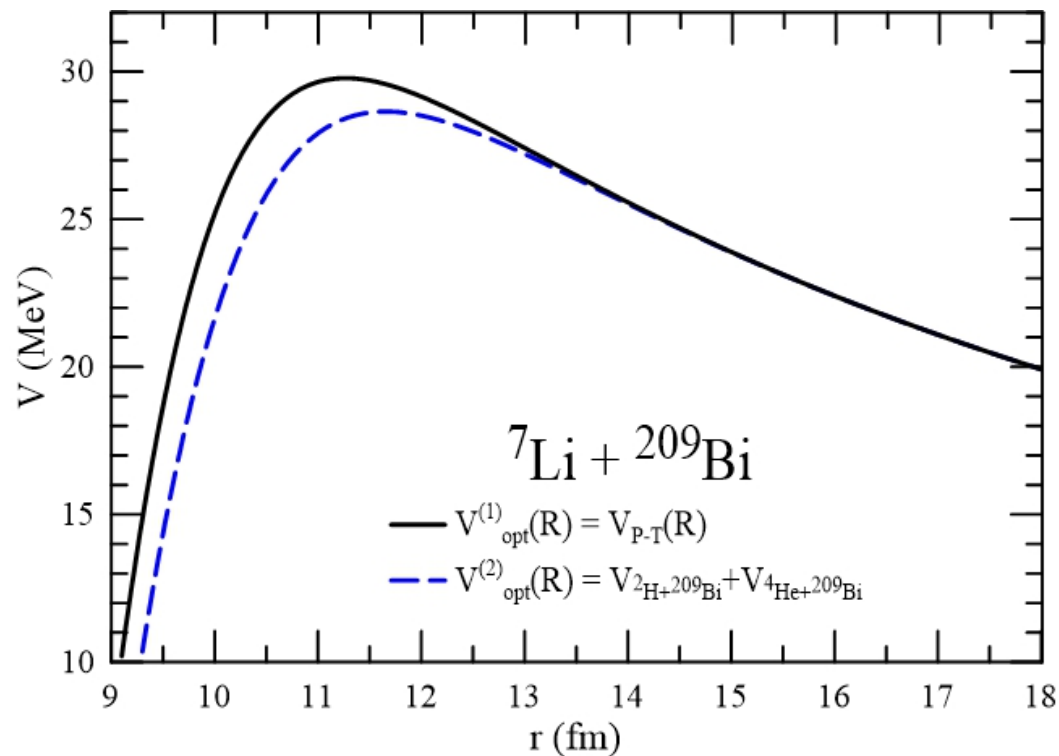
- Excellent agreement where all relevant decay channels are measured
- Consistent with data where they give a lower bound

CF – ^7Li + targets



* M. Cortes, J. Rangel, J. L. Ferreira, J. Lubian, LFC (Phys. Rev. C, 00, 004600- 2020)

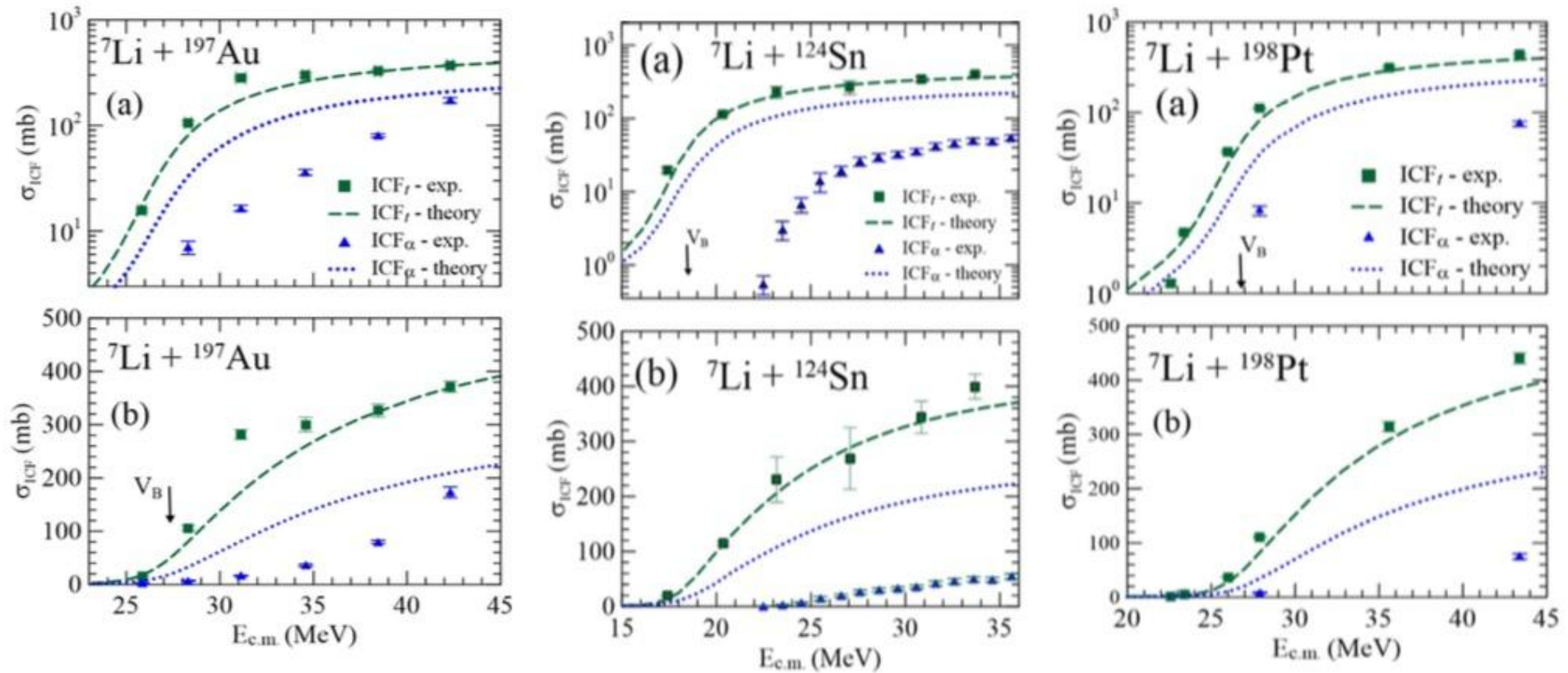
CF – ${}^7\text{Li} + \text{targets}$



projectile's effects
over Coulomb barrier

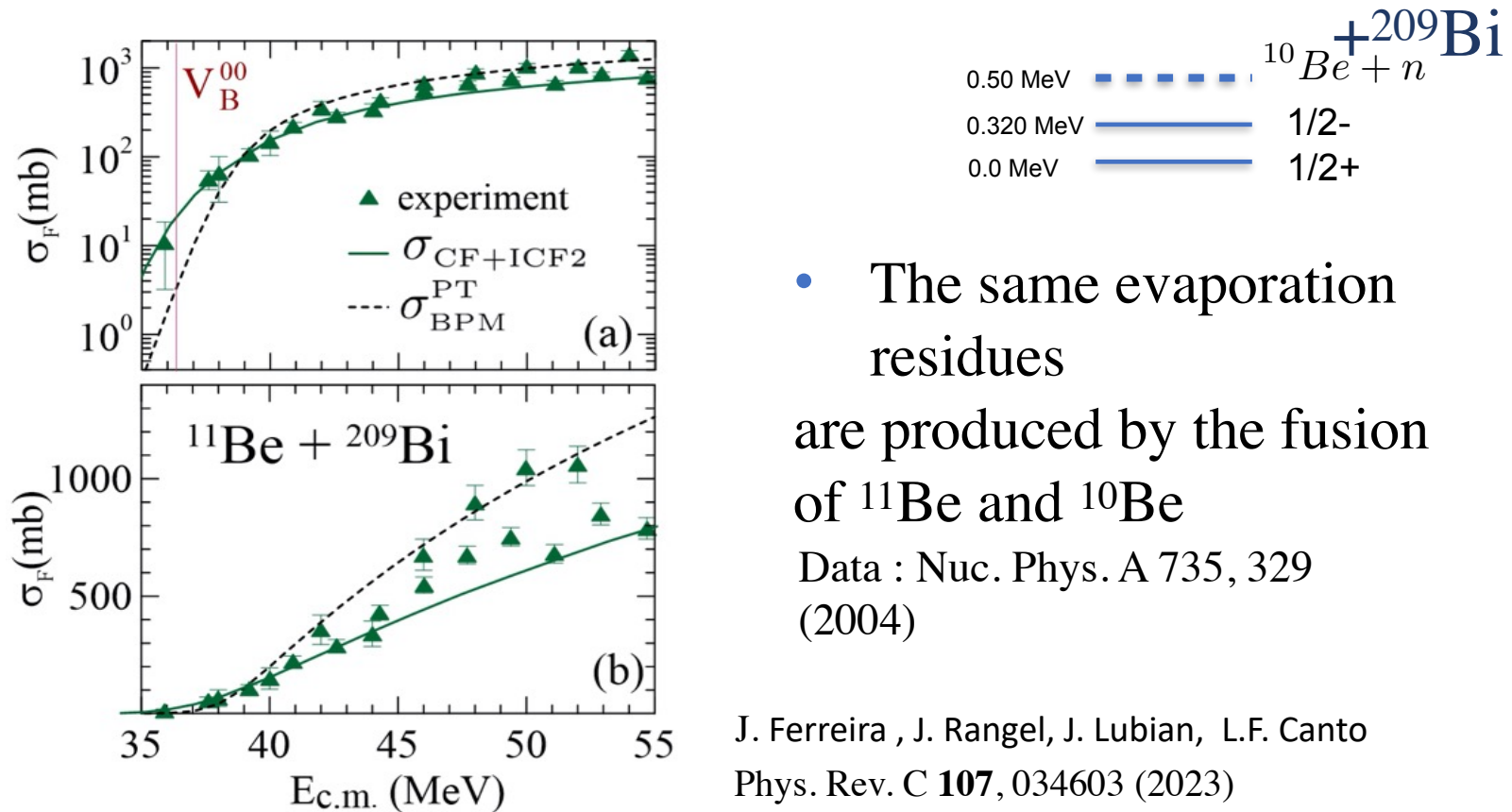
* M. Cortes, J. Rangel, J. L. Ferreira, J. Lubian, LFC (Phys. Rev. C, 00, 004600- 2020)

ICF – ${}^7\text{Li}$ + targets



* M. Cortes, J. Rangel, J. L. Ferreira, J. Lubian, LFC (Phys. Rev. C, 00, 004600- 2020)

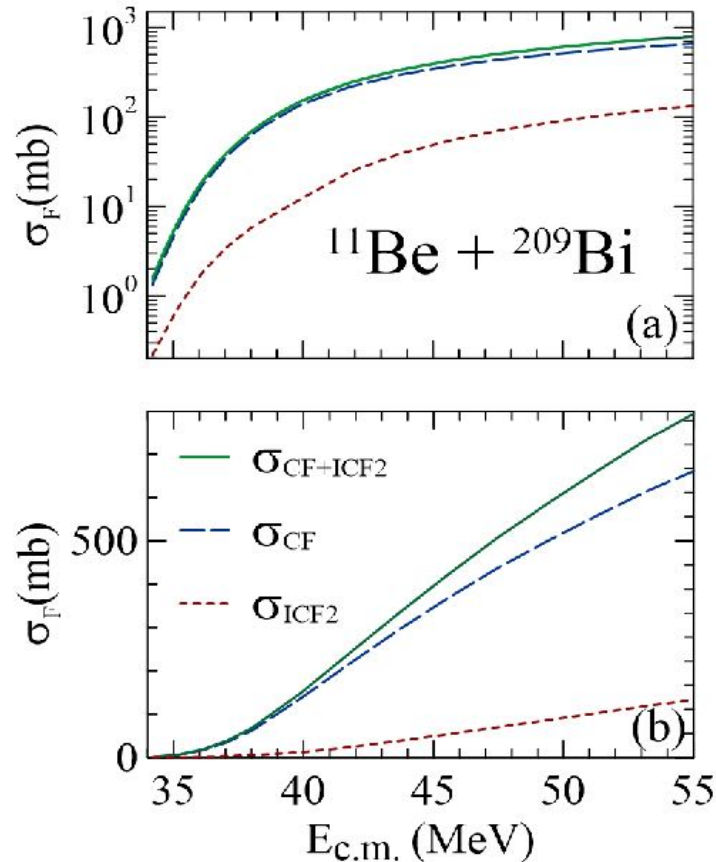
Complete fusion: neutron valence ^{11}Be



- The same evaporation residues are produced by the fusion of ^{11}Be and ^{10}Be
- Data : Nuc. Phys. A 735, 329 (2004)

J. Ferreira , J. Rangel, J. Lubian, L.F. Canto
Phys. Rev. C **107**, 034603 (2023)

Complete fusion: neutron valence ^{11}Be $+^{209}\text{Bi}$



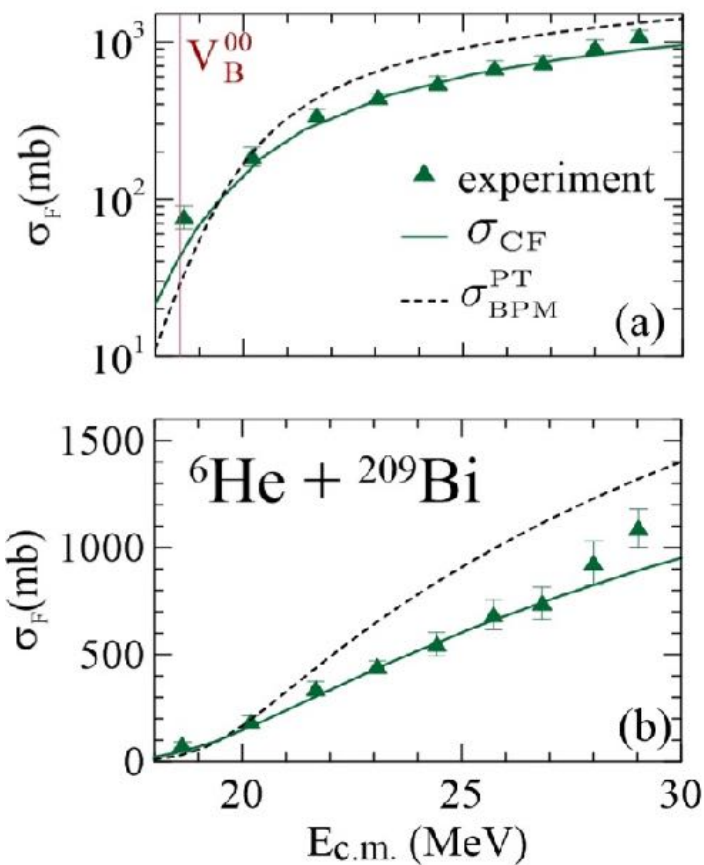
- The contribution of the absorption of ^{10}Be is too small ($\sim 20\%$)

Data : Nuc. Phys. A 735, 329
(2004)

J. Ferreira , J. Rangel, J. Lubian, L.F. Canto

Phys. Rev. C **107**, 034603 (2023)

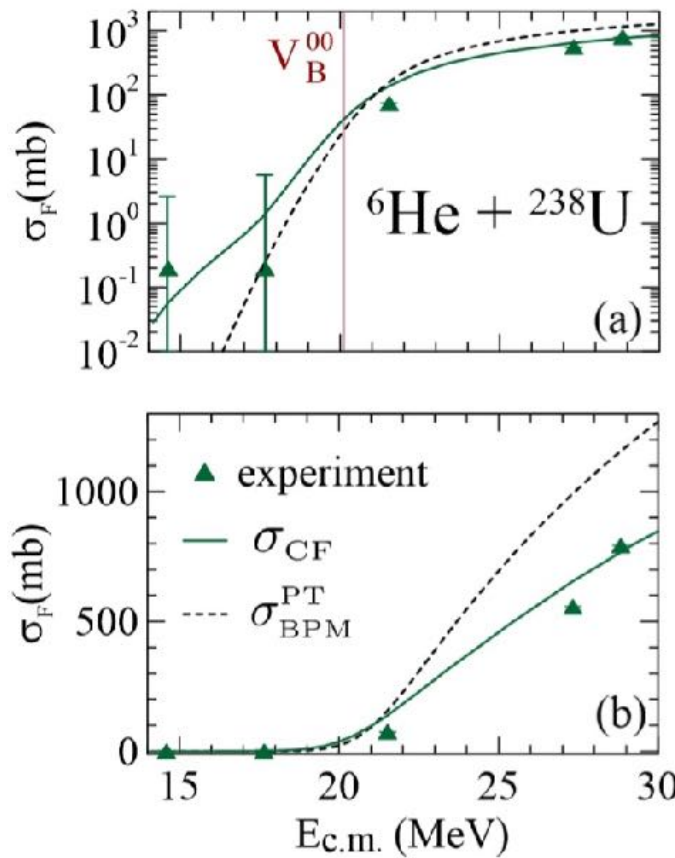
Complete fusion: neutron valence ${}^6\text{He} + {}^{209}\text{Bi}$



Data: Phys. Rev. C 58, 3442
(1998)

J. Ferreira, J. Rangel, J. Lubian, L.F. Canto
Phys. Rev. C **107**, 034603 (2023)

Complete fusion ${}^6\text{He} + {}^{238}\text{U}$



Data: Phys. Rev. Lett. 84, 2342 (2004)

J. Ferreira, J. Rangel, J. Lubian, L.F. Canto
Phys. Rev. C **107**, 034603 (2023)

Conclusions

- We have proposed a quantum mechanical method to evaluate CF and ICF in collisions of weakly bound nuclei
- The method was applied to ^7Li and neutron-valence projectiles and the results were compared with very good agreement to CF / CF + ICF experimental data
- Calculations of other light systems are under way.

People

- Universidade Federal Fluminense (UFF).

Jesus Lubian Rios

Vinicius Zagatto

Brenda Pinheiro

- Universidade Federal do Rio de Janeiro(UFRJ):

Luiz Felipe A. Canto

- Michigan state University (MSU):

Filomena Nunes



APPENDIX: CALCULATION OF THE ABSORPTION PROBABILITY

In this Appendix we evaluate the probabilities $P_B^{(i)}(J)$ and $P_C^{(i)}(J)$ of Sec. II. We consider the collision of a projectile formed by two fragments, one with spin zero and the other with s on a spinless target. In this case, the contribution from the absorption of fragment c_i to the TF cross section is given by the expression,

$$\sigma_{\text{TF}}^{(i)} = \frac{K}{E} \frac{(2\pi)^3}{(2j_0 + 1)} \sum_{v_0} \langle \Psi_{\mathbf{k}j_0v_0}^{(+)} | \mathbb{W}^{(i)} | \Psi_{\mathbf{k}j_0v_0}^{(+)} \rangle, \quad (\text{A1})$$

where $\Psi_{\mathbf{k}j_0v_0}^{(+)}$ is the scattering wave function for a collision with wave-vector $\mathbf{k}$, initiated with intrinsic angular momentum j_0 and z -component v_0 . In this equation, the normalization constant of Eq. (5) was set as $A = (2\pi)^{-3/2}$.

The angular momentum projected scattering wave function is obtained coupling the intrinsic angular momentum ($\mathbf{j}_\alpha$) with the orbital angular momentum of the projectile-target motion ($\mathbf{L}$). It is given by [40]

$$\begin{aligned} \Psi_{\mathbf{k}j_0v_0}^{(+)}(\mathbf{R}, \mathbf{r}) = & \frac{1}{(2\pi)^{3/2}} \sum_{\alpha L L_0} \frac{\mathcal{U}_{\alpha L, 0 L_0}^J(K_\alpha, R)}{K R} e^{i\sigma_{L_0}} \\ & \times \sqrt{4\pi} (2L_0 + 1) \langle J v_0 | L_0 0 j_0 v_0 \rangle \mathcal{Y}_{\alpha L}^{J v_0}(\hat{\mathbf{R}}, \mathbf{r}), \end{aligned} \quad (\text{A2})$$

where $\mathcal{U}_{\alpha L, 0 L_0}^J(k_\alpha, R)$ are the solutions of the radial equation and $\mathcal{Y}_{\alpha L}^{J v_0}(\hat{\mathbf{R}}, \mathbf{r})$ are the spin-channel wave functions (in the present case, the intrinsic coordinates $\mathbf{r}$ are simply the

components of vector $\mathbf{r}$),

$$\mathcal{Y}_{\alpha L}^{J v_0}(\hat{\mathbf{R}}, \mathbf{r}) = i^L \sum_{\nu M_L} \langle L M_L j_\alpha \nu | J v_0 \rangle Y_{L M_L}(\hat{\mathbf{R}}) \phi_{\alpha j_\alpha \nu}(\mathbf{r}). \quad (\text{A3})$$

Above, $\phi_{\alpha j_\alpha \nu}(\mathbf{r})$ is the eigenstate of the intrinsic Hamiltonian of the projectile with energy ε_α , angular momentum j_α , and projection ν (the explicit form of these states will be discussed later).

Next, we carry out the multipole expansion of the imaginary potential,

$$\mathbb{W}^{(i)}(\mathbf{R}, \mathbf{r}_i') = 4\pi \sum_{\lambda, \mu} (-)^\mu Y_{\lambda, \mu}(\hat{\mathbf{R}}) Q_{\lambda, -\mu}^{(i)}(\mathbf{r}_i'). \quad (\text{A4})$$

where $Q_{\lambda, -\mu}^{(i)}(\mathbf{r}_i')$ is the spherical tensor operator,

$$Q_{\lambda, -\mu}^{(i)}(\mathbf{r}_i') = \mathcal{W}^{(i)\lambda}(R, \mathbf{r}_i') Y_{\lambda, -\mu}(\hat{\mathbf{r}}_i'). \quad (\text{A5})$$

Using Eqs. (A2)–(A4) in Eq. (A1), $\sigma_{\text{TF}}^{(i)}$ can be put in the form

$$\sigma_{\text{TF}}^{(i)} = \frac{\pi}{K^2} \sum_J (2J + 1) \mathcal{P}^{(i)}(J), \quad (\text{A6})$$

where $\mathcal{P}^{(i)}(J)$ is the probability of absorption of fragment c_i by the target in a collision with angular momentum J , given by

$$\begin{aligned} \mathcal{P}^{(i)}(J) = & \frac{4K}{E j_0^2 j^2} \sum_{\lambda} \sum_{\alpha L L_0} \sum_{\alpha' L' L'_0} (i)^{L' - L} \hat{L}_0 \hat{L}'_0 e^{i(\sigma_{L'_0} - \sigma_{L_0})} \\ & \times \sum_{v_0} \langle L_0 0 j_0 v_0 | J v_0 \rangle \langle J v_0 | L'_0 0 j_0 v_0 \rangle \\ & \times \int dR \mathcal{U}_{\alpha L, 0 L_0}^{J*}(K_\alpha R) \mathcal{U}_{\alpha' L', 0 L'_0}^J(K_{\alpha'} R) X_{\alpha L, \alpha' L'}^{J(\lambda)}(R). \end{aligned} \quad (\text{A7})$$

Above, we denote: $j_0 = \sqrt{2j_0 + 1}$ and use an analogous notation for other angular momentum quantum numbers, and

$$X_{\alpha L, \alpha' L'}^{J(\lambda)}(R) = 4\pi (\mathcal{Y}_{\alpha L}^{J v_0} | \mathbf{Y}_\lambda \cdot \mathbf{Q}_\lambda | \mathcal{Y}_{\alpha' L'}^{J v_0}). \quad (\text{A8})$$

The above quantity seems to depend on v_0 , but it actually does not. It cannot depend on orientation because $\mathbf{Y}_\lambda \cdot \mathbf{Q}_\lambda$ is a scalar. Thus, the v_0 dependence is restricted to the Clebsh-Gordan coefficients. Then, carrying out the sum over v_0 , we get [64]

$$\begin{aligned} & \sum_{v_0} \langle L_0 0 j_0 v_0 | J v_0 \rangle \langle J v_0 | L'_0 0 j_0 v_0 \rangle \\ &= j^2 \sum_{v_0} \begin{pmatrix} L_0 & j_0 & J \\ 0 & v_0 & -v_0 \end{pmatrix} \begin{pmatrix} L'_0 & j_0 & J \\ 0 & v_0 & -v_0 \end{pmatrix} \\ &= \frac{j^2}{\hat{L}_0^2} \delta_{L_0 L'_0}. \end{aligned} \quad (\text{A9})$$

Using this result, Eq. (A7) takes the form

$$\begin{aligned} P^{(i)}(J) &= \frac{4K}{E \hat{j}_0^2} \sum_{\lambda} \sum_{\alpha L \alpha' L'} (i)^{L'-L} \int dR X_{\alpha L, \alpha' L'}^{(i)\lambda}(R) \\ &\quad \times \mathcal{U}_{\alpha L, 0 L_0}^{J*}(K_\alpha R) \mathcal{U}_{\alpha' L, 0 L'_0}^J(K_{\alpha'} R). \end{aligned} \quad (\text{A10})$$

1. Evaluation of $X_{\alpha L, \alpha' L'}^{(i)\lambda}(R)$

Using the notation of Ref. [64] for the wave functions: $\mathcal{Y}_{\alpha L}^{j_0} \rightarrow |\alpha(L j_\alpha) J\rangle$, Eq. (A8) reads

$$X_{\alpha L, \alpha' L'}^{(i)\lambda}(R) = 4\pi \langle \alpha(L j_\alpha) J || \mathbf{Y}_\lambda(\hat{\mathbf{R}}) \cdot \mathbf{Q}_\lambda(R, \mathbf{r}) || \alpha'(L' j_{\alpha'}) J \rangle, \quad (\text{A11})$$

or (Eq. (5.13) of Ref. [64])

$$\begin{aligned} X_{\alpha L, \alpha' L'}^{(i)\lambda}(R) &= 4\pi (-)^{J-L-L'} \hat{L} \hat{j}_\alpha W(LL' j_\alpha j_{\alpha'}; \lambda J) \\ &\quad \times \langle L || \mathbf{Y}_\lambda(\hat{\mathbf{R}}) || L' \rangle \langle \alpha j_\alpha || \mathbf{Q}_\lambda(R, \mathbf{r}) || \alpha' j_{\alpha'} \rangle. \end{aligned} \quad (\text{A12})$$

The first reduced matrix element is (Eq. (4.17) of Ref. [64])

$$\langle L || \mathbf{Y}_\lambda(\hat{\mathbf{R}}) || L' \rangle = (-)^{\lambda-L'} \frac{\hat{\lambda} \hat{L}'}{\sqrt{4\pi}} \begin{pmatrix} \lambda & L' & L \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A13})$$

Using this result, Eq. (A12) can be put in the form

$$X_{\alpha L, \alpha' L'}^{(i)\lambda}(R) = \mathcal{A}_{\alpha L, \alpha' L'}^{J\lambda} F_{\alpha \alpha'}^{(i)\lambda}(R), \quad (\text{A14})$$

and Eq. (A10) becomes

$$\begin{aligned} P^{(i)}(J) &= \frac{4K}{E \hat{j}_0^2} \sum_{\lambda} \sum_{\alpha L \alpha' L'} \mathcal{A}_{\alpha L, \alpha' L'}^{J\lambda} (i)^{L'-L} \\ &\quad \times \int dR \mathcal{U}_{\alpha L, 0 L_0}^{J*}(K_\alpha R) \mathcal{U}_{\alpha' L, 0 L'_0}^J(K_{\alpha'} R) F_{\alpha \alpha'}^{(i)\lambda}(R), \end{aligned} \quad (\text{A15})$$

with

$$\begin{aligned} \mathcal{A}_{\alpha L, \alpha' L'}^{J\lambda} &= \sqrt{4\pi} (-)^{J+\lambda-L-L'-j_{\alpha'}} \hat{L} \hat{L}' \hat{\lambda} \hat{j}_\alpha \\ &\quad \times W(LL' j_\alpha j_{\alpha'}; \lambda J) \begin{pmatrix} \lambda & L' & L \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (\text{A16})$$

and

$$F_{\alpha \alpha'}^{(i)\lambda}(R) = \langle \alpha j_\alpha || \mathbf{Q}_\lambda(R, \mathbf{r}) || \alpha' j_{\alpha'} \rangle. \quad (\text{A17})$$

2. Calculation of $F_{\alpha \alpha'}^{(i)\lambda}(R)$ for a two-fragment projectile

Now we consider the situation where the projectile is formed by two fragments, one with spin zero and the other with spin s . In this case the intrinsic coordinates are $\mathbf{r} \equiv \{r, \hat{\mathbf{r}}\}$. The angular momentum-projected intrinsic states are then given by

$$\phi_{\alpha j_\alpha v}(\mathbf{r}) = \frac{u_{\alpha l_\alpha j_\alpha}(r)}{r} \mathcal{J}_{l_\alpha j_\alpha v}(\hat{\mathbf{r}}), \quad (\text{A18})$$

with

$$\mathcal{J}_{l_\alpha j_\alpha v}(\hat{\mathbf{r}}) = \sum_{m_l m_s} \langle l_\alpha m_l m_s | j_\alpha v \rangle Y_{l_\alpha m_l}(\hat{\mathbf{r}}) |s m_s\rangle, \quad (\text{A19})$$

where $|s m_s\rangle$ are states in the spin space and $\langle l_\alpha m_l m_s | j_\alpha v \rangle$ are Clebsh-Gordan coefficients. In Eq. (A18), $u_{\alpha l_\alpha j_\alpha}(r)$ stands for the radial wave functions of the projectile. They are either bound states or bins generated by scattering states of the fragments $u_{\varepsilon l_\alpha j_\alpha}(r)$, where ε is the collision energy.

The tensor of Eq. (A5) then becomes

$$Q_{\lambda-\mu}^{(i)}(\mathbf{r}') \rightarrow Q_{\lambda-\mu}^{(i)}(\mathbf{r}) = \mathcal{W}^{(i)\lambda}(R, r) Y_{\lambda-\mu}(\hat{\mathbf{r}}), \quad (\text{A20})$$

and scalar products in the intrinsic space are integrals over $r^2 dr d\Omega_{\hat{\mathbf{r}}}$.

Then, adopting the notation of Ref. [64], Eq. (A17) becomes

$$F_{\alpha\alpha'}^{(i)\lambda}(R) = \mathcal{F}_{\alpha\alpha'}^{(i)\lambda}(R) \langle j_\alpha || \mathbf{Y}_\lambda(\hat{\mathbf{r}}) || j_{\alpha'} \rangle, \quad (\text{A21})$$

where $\mathcal{F}_{\alpha\alpha'}^{(i)\lambda}(R)$ is the form factor,

$$\mathcal{F}_{\alpha\alpha'}^{(i)\lambda}(R) = \int dr u_{\alpha j_\alpha}^*(r) \mathcal{W}^{(i)\lambda}(R, r) u_{\alpha' j_{\alpha'}}(r). \quad (\text{A22})$$

The reduced matrix element of Eq. (A21) can be evaluated with help of Eq. (5.10) of Ref. [64], and one gets

$$\begin{aligned} \langle j_\alpha || \mathbf{Y}_\lambda(\hat{\mathbf{r}}) || j_{\alpha'} \rangle &= (-)^{j_\alpha - \lambda - s + l_{\alpha'}} \hat{l}_\alpha \hat{j}_{\alpha'} W(l_\alpha l_{\alpha'} j_\alpha j_{\alpha'}; \lambda s) \\ &\times \langle l_\alpha || \mathbf{Y}_\lambda(\hat{\mathbf{r}}) || l_{\alpha'} \rangle. \end{aligned}$$

Finally, evaluating $\langle l_\alpha || \mathbf{Y}_\lambda(\hat{\mathbf{r}}) || l_{\alpha'} \rangle$ as in Eq. (A13), the above equation becomes

$$\begin{aligned} \langle j_\alpha || \mathbf{Y}_\lambda(\hat{\mathbf{r}}) || j_{\alpha'} \rangle &= (-)^{j_\alpha - s} \frac{\hat{l}_{\alpha'} \hat{j}_{\alpha'} \hat{l}_\alpha \hat{\lambda}}{\sqrt{4\pi}} \begin{pmatrix} \lambda & l_{\alpha'} & l_\alpha \\ 0 & 0 & 0 \end{pmatrix} \\ &\times W(l_\alpha l_{\alpha'} j_\alpha j_{\alpha'}; \lambda s). \end{aligned} \quad (\text{A23})$$

Using the above equation in Eq. (A21) and inserting the result into Eq. (A15), the fusion probability becomes

$$\mathcal{P}^{(i)}(J) = \frac{4K}{E \hat{j}_0^2} \sum_{L_0} \sum_{\lambda} \sum_{\alpha L \alpha' L'} B_{\alpha L, \alpha' L'}^{\lambda}(J) \mathcal{M}_{\alpha L, \alpha' L'}^{(i)\lambda}(L_0, J). \quad (\text{A24})$$

Above, $B_{\alpha L, \alpha' L'}^{\lambda}(J)$ is the geometric factor,

$$B_{\alpha L, \alpha' L'}^{\lambda}(J) = \mathcal{A}_{\alpha L, \alpha' L'}^{J\lambda} \langle j_\alpha || \mathbf{Y}_\lambda(\hat{\mathbf{r}}) || j_{\alpha'} \rangle, \quad (\text{A25})$$

or explicitly,

$$\begin{aligned} B_{\alpha L, \alpha' L'}^{\lambda}(J) &= (-)^{\mathcal{N}} \hat{\lambda}^2 \hat{L} \hat{L}' \hat{l}_\alpha \hat{l}_{\alpha'} \hat{j}_\alpha \hat{j}_{\alpha'} \\ &\times W(LL' j_\alpha j_{\alpha'}; \lambda J) W(l_\alpha l_{\alpha'} j_\alpha j_{\alpha'}; \lambda s) \\ &\times \begin{pmatrix} \lambda & L' & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda & l_{\alpha'} & l_\alpha \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (\text{A26})$$

with

$$\mathcal{N} = J - s + j_\alpha - j_{\alpha'} - L - L' + \lambda, \quad (\text{A27})$$

and $\mathcal{M}_{\alpha L, \alpha' L'}^{(i)\lambda}(L_0, J)$ is the radial integral,

$$\begin{aligned} \mathcal{M}_{\alpha L, \alpha' L'}^{(i)\lambda}(L_0, J) &= i^{L'-L} \int dR \mathcal{F}_{\alpha\alpha'}^{(i)\lambda}(R) \\ &\times \mathcal{U}_{\alpha L, 0 L_0}^{J*}(K_\alpha, R) \mathcal{U}_{\alpha' L', 0 L_0}^J(K_{\alpha'}, R). \end{aligned} \quad (\text{A28})$$

Although the radial integrals are complex functions, the probabilities of Eq. (A24) are real. Using symmetry properties of the 3J and Racah coefficients (see, e.g., Ref. [64]) one can easily show that

$$B_{\alpha L, \alpha' L'}^{\lambda}(J) = B_{\alpha' L', \alpha L}^{\lambda}(J). \quad (\text{A29})$$

On the other hand, the radial integrals have the property,

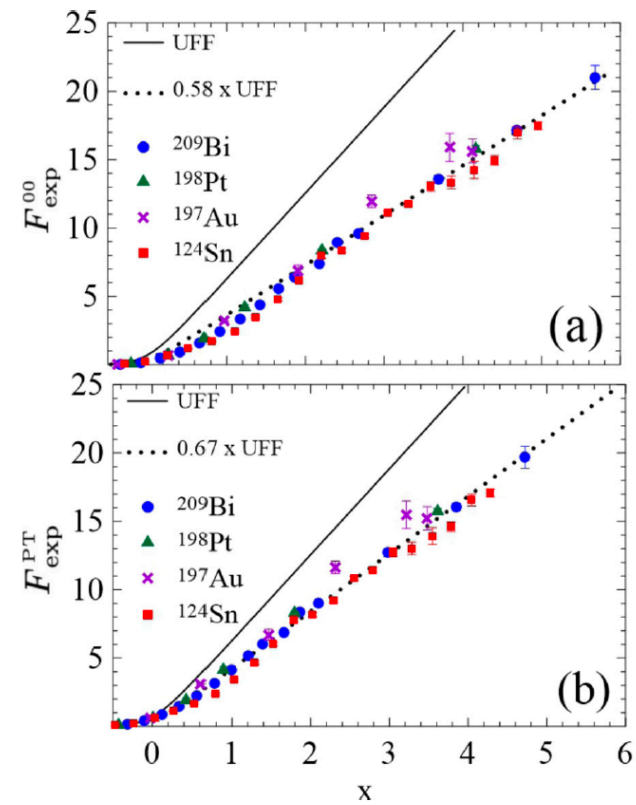
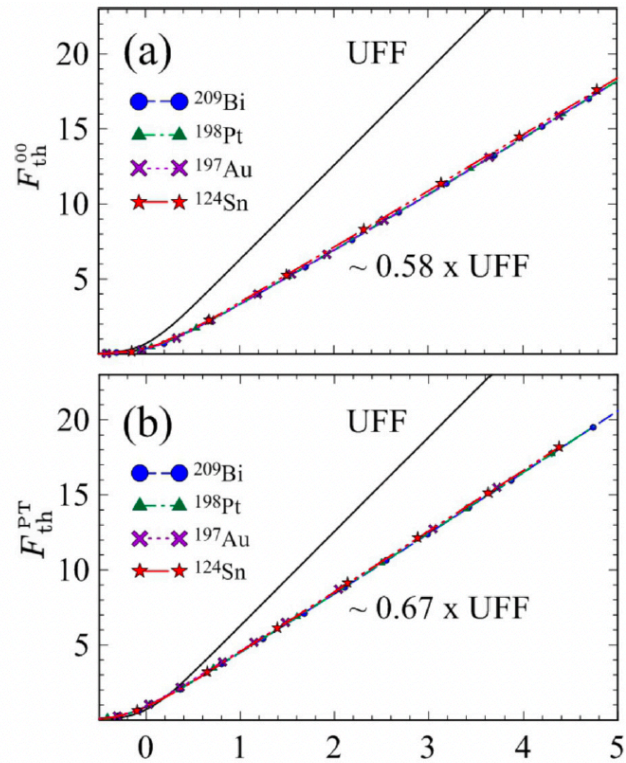
$$\mathcal{M}_{\alpha L, \alpha' L'}^{(i)\lambda}(L_0, J) = \mathcal{M}_{\alpha' L', \alpha L}^{(i)\lambda*}(L_0, J). \quad (\text{A30})$$

Since α, L, α', L' are dummy indices running over the same ranges, the fusion probability of Eq. (A24) does not change if one interchanges $\{\alpha, L\} \rightleftharpoons \{\alpha', L'\}$. Then, using Eqs. (A29) and (A30), one obtains the explicitly real expression for the absorption probabilities,

$$\mathcal{P}^{(i)}(J) = \frac{4K}{E \hat{j}_0^2} \sum_{\lambda L_0} \sum_{\alpha L \alpha' L'} B_{\alpha L, \alpha' L'}^{\lambda}(J) \text{Re} \{ \mathcal{M}_{\alpha L, \alpha' L'}^{(i)\lambda}(L_0, J) \}. \quad (\text{A31})$$

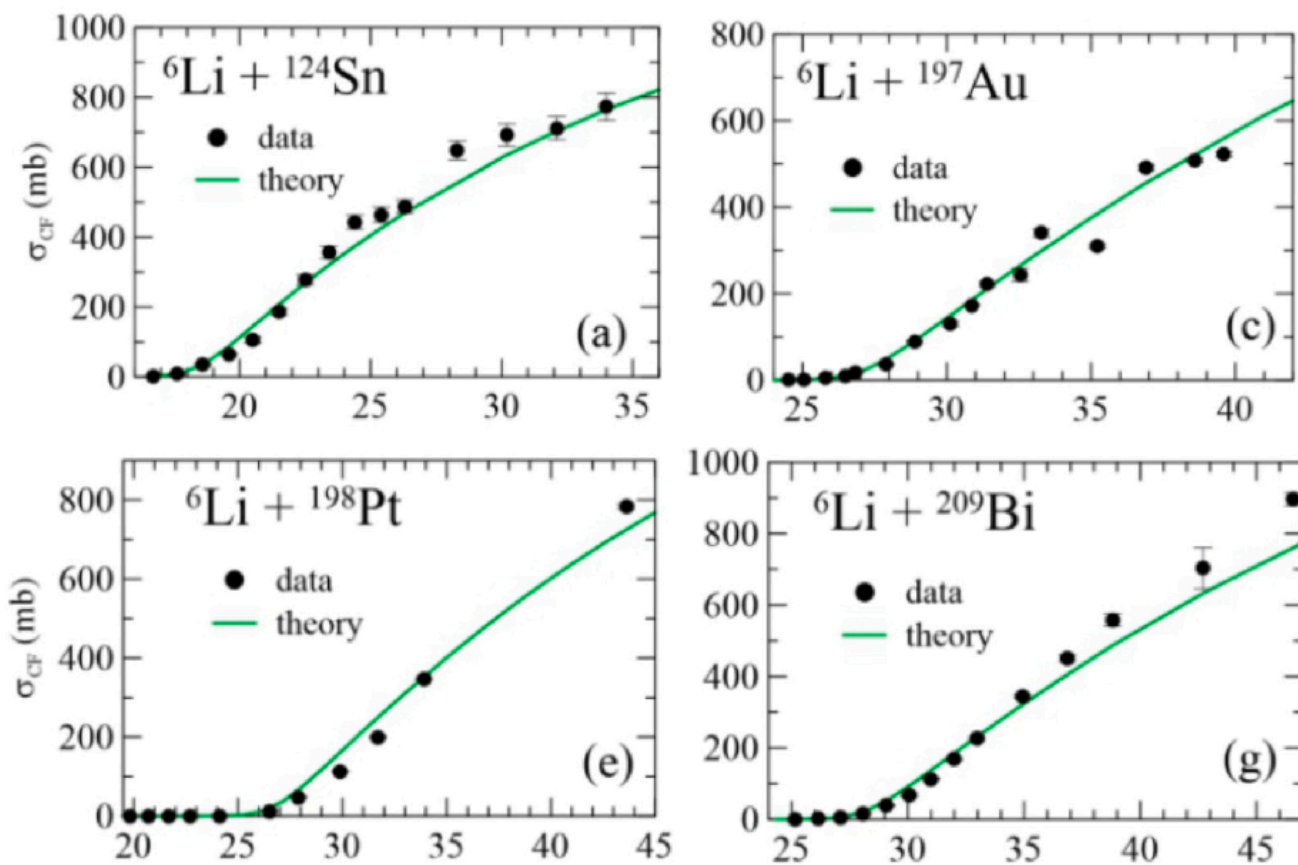
Finally, the probabilities $\mathcal{P}_B^{(i)}(J)$ and $\mathcal{P}_C^{(i)}(J)$ are given by the above expression, restricting the sum over channels to $\{\alpha, \alpha'\} \in B$ and to $\{\alpha, \alpha'\} \in C$, respectively.

CF reductions



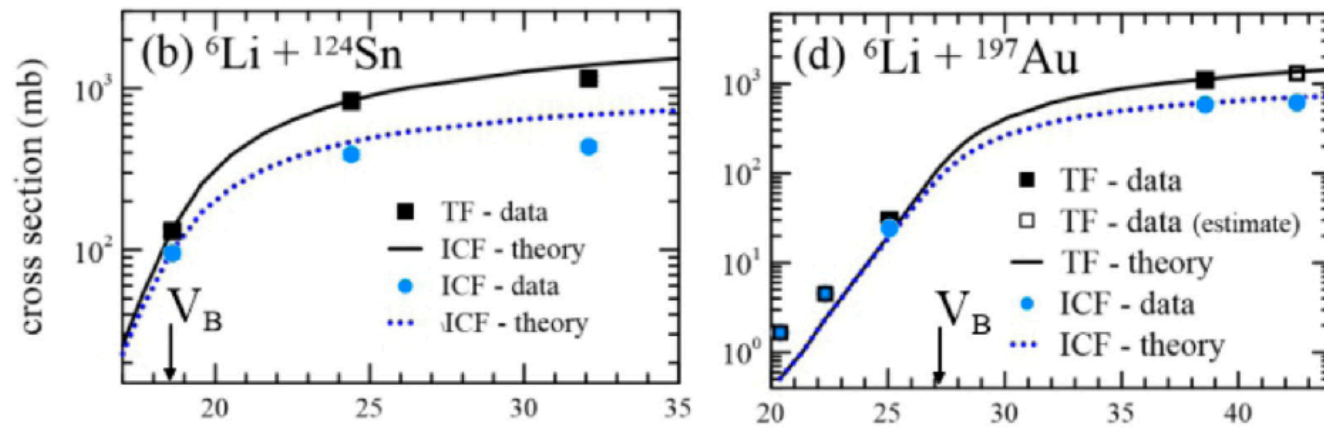
${}^6\text{Li}$

PRC **105**, 054601 (2022)



${}^6\text{Li}$

PRC **105**, 054601 (2022)



Indirect estimations using “Fusion” code

${}^6\text{Li} + {}^{90}\text{Zr}$

$E_{\text{c.m.}}$ (MeV)	$\sigma_{\alpha}^{\text{inc}}$ (mb)	$\sigma_{\alpha}^{\text{CF}}$ (mb)	σ^{EBU} (mb)	σ_{α}^{1n} (mb)	σ_{α}^{1p} (mb)
13.9	32 ± 19	0.01	8.5	2.0	0.3
15.8	122 ± 10	1.3	12.9	14.1	0.7
17.7	274 ± 15	16.2	17.9	27.5	1.1
19.6	360 ± 20	37.6	24.4	30.9	1.3
23.3	460 ± 30	94.2	35	29.5	1.5
28.0	540 ± 86	172	45.3	26.9	1.5



Inclusive Alpha data: Phys.
Rev. C, 81, 054601, (2010)

$\sigma_{\alpha}^{\text{EBU}}$ = elastic breakup cross section
 $({}^6\text{Li} + {}^{90}\text{Zr} \rightarrow {}^2\text{H} + {}^4\text{He} + {}^{90}\text{Zr}),$

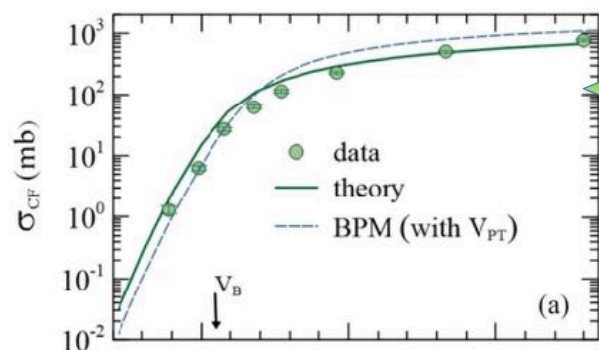
σ_{α}^{1n} = neutron stripping cross section
 $({}^6\text{Li} + {}^{90}\text{Zr} \rightarrow {}^5\text{Li} + {}^{91}\text{Zr} \rightarrow p + {}^4\text{He} + {}^{91}\text{Zr}),$

σ_{α}^{1p} = proton stripping cross section
 $({}^6\text{Li} + {}^{90}\text{Zr} \rightarrow {}^5\text{He} + {}^{91}\text{Nb} \rightarrow n + {}^4\text{He} + {}^{91}\text{Nb}),$

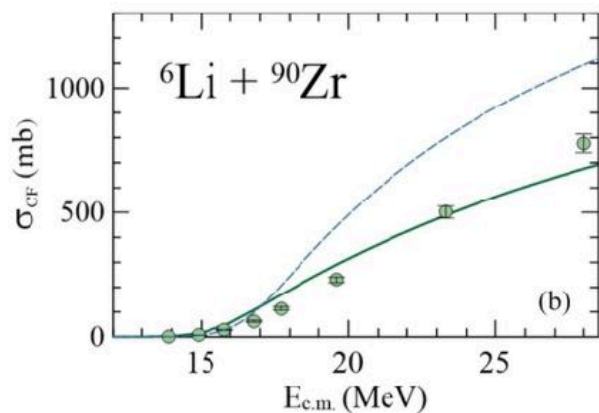
$\sigma_{\alpha}^{\text{CF}}$ = CF + evaporation of an α particle
 $({}^6\text{Li} + {}^{90}\text{Zr} \rightarrow {}^{96}\text{Nd} \rightarrow (1n, 2n, 3n), \alpha).$

$$\sigma_{\alpha}^{\text{inc}} = \sigma_{\alpha}^{\text{EBU}} + \sigma_{\alpha}^{1n} + \sigma_{\alpha}^{1p} + \sigma_{\alpha}^{\text{CF}} + \sigma_{\text{ICF}d}.$$

Indirect estimations using “Fusion” code

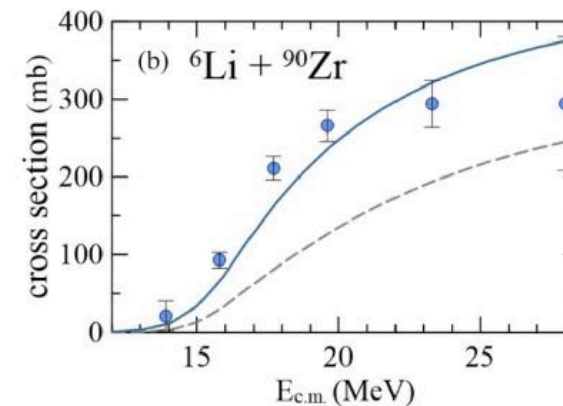
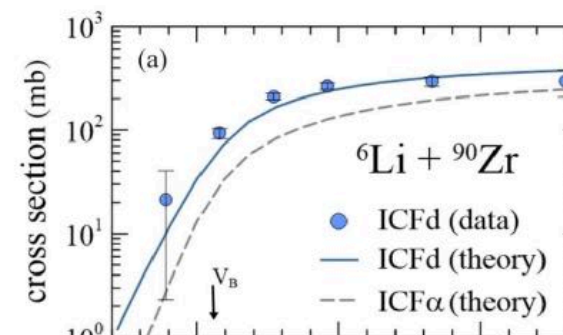


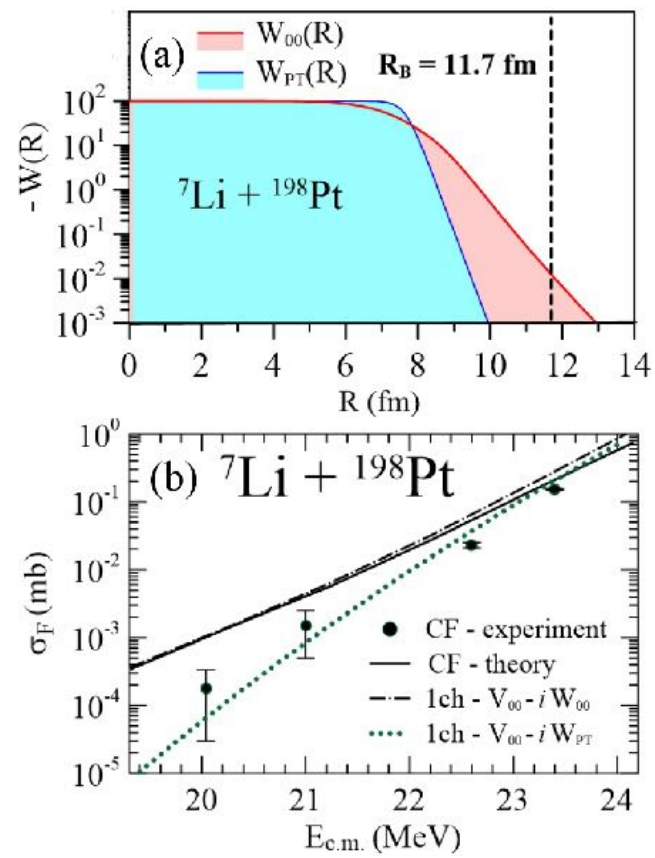
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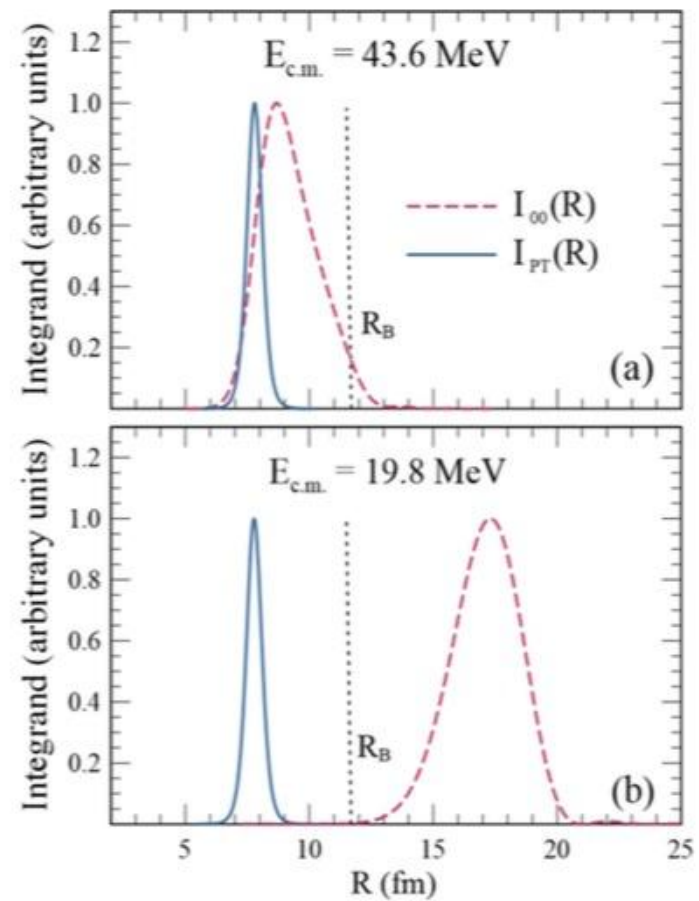
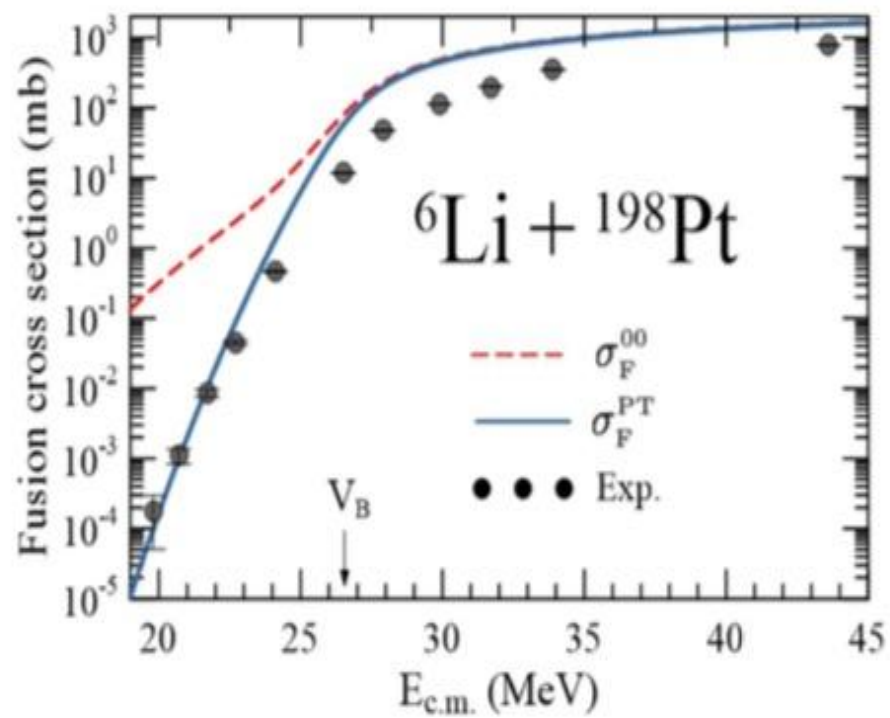


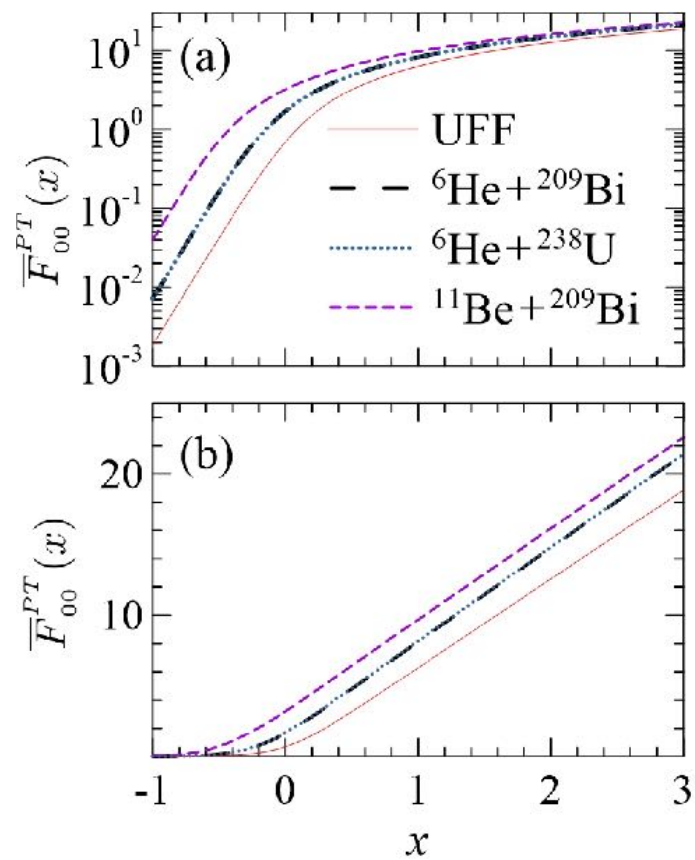
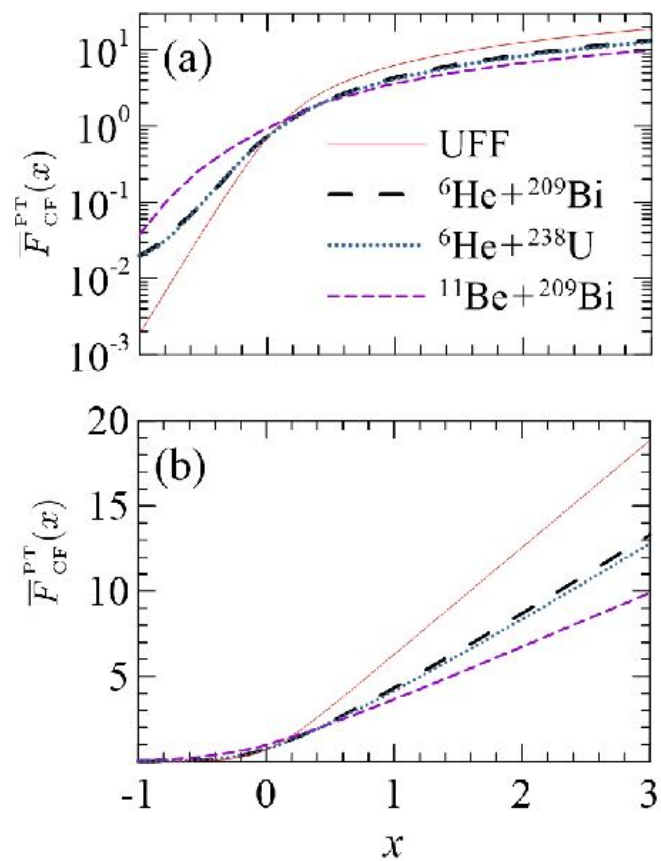
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81, 054601 (2010).

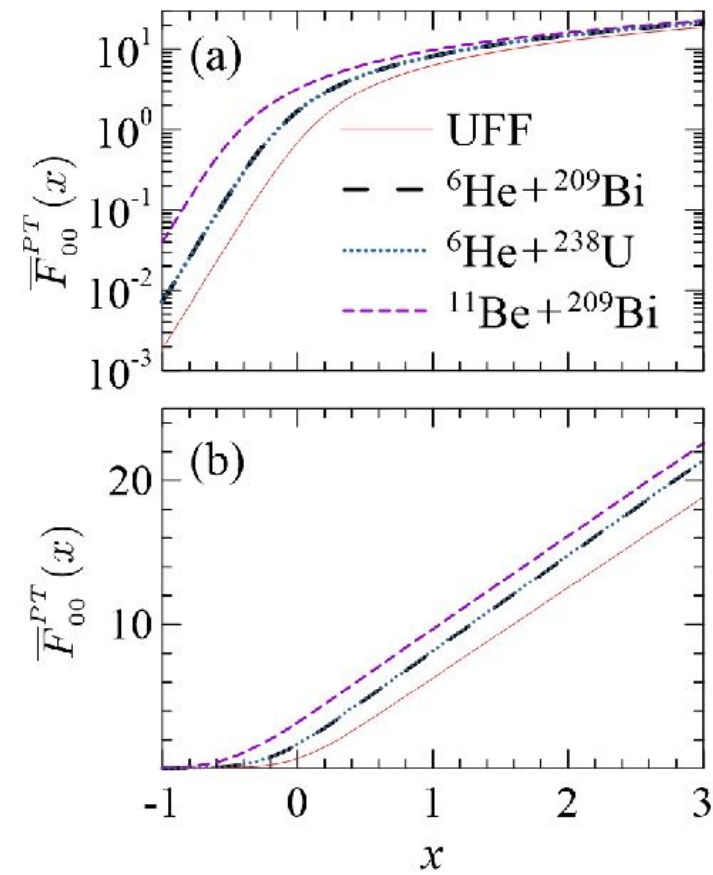
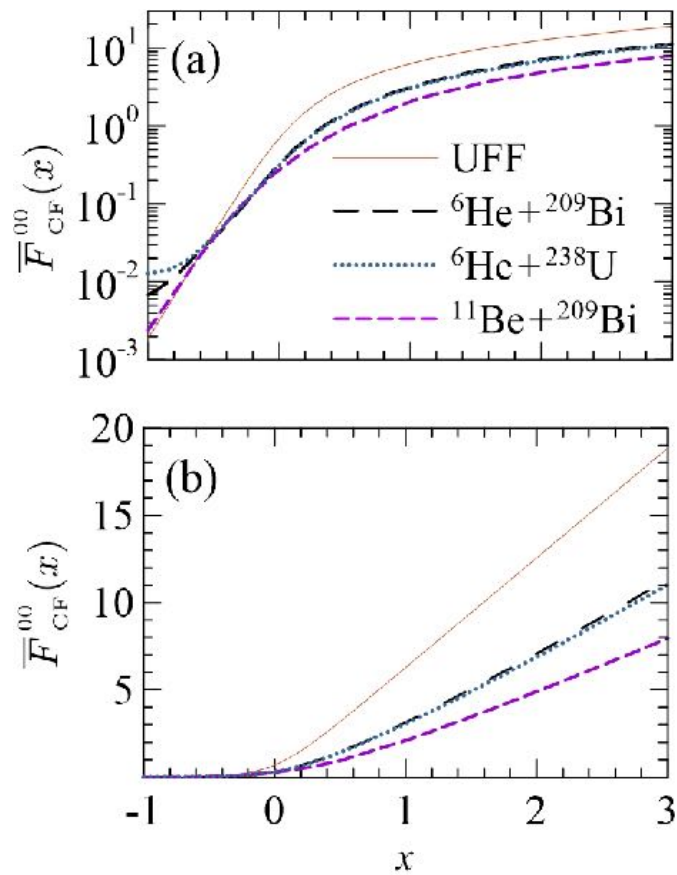
Phys. Rev. C, 108,
054601, (2023)











Potential Scattering approach

Fusion is not a channel

$S_{\alpha,\alpha'}^J \longrightarrow$ elastic and inelastic

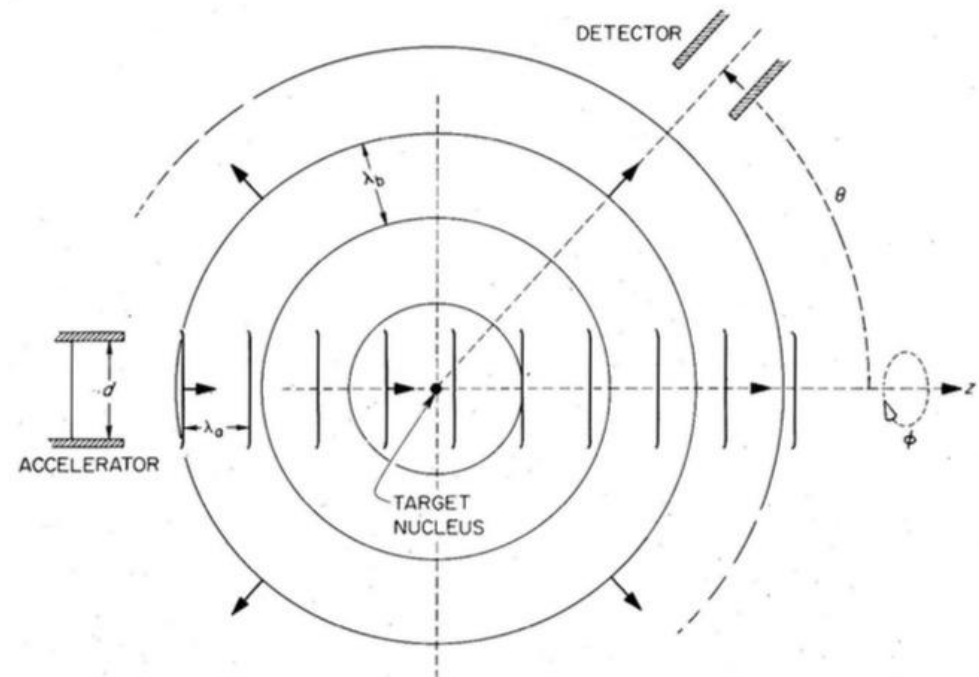
$$\sigma_R = \frac{\pi}{k^2} \sum (2J+1)(1 - |S_J|^2)$$

$\sigma_R =$ everything but elastic scattering

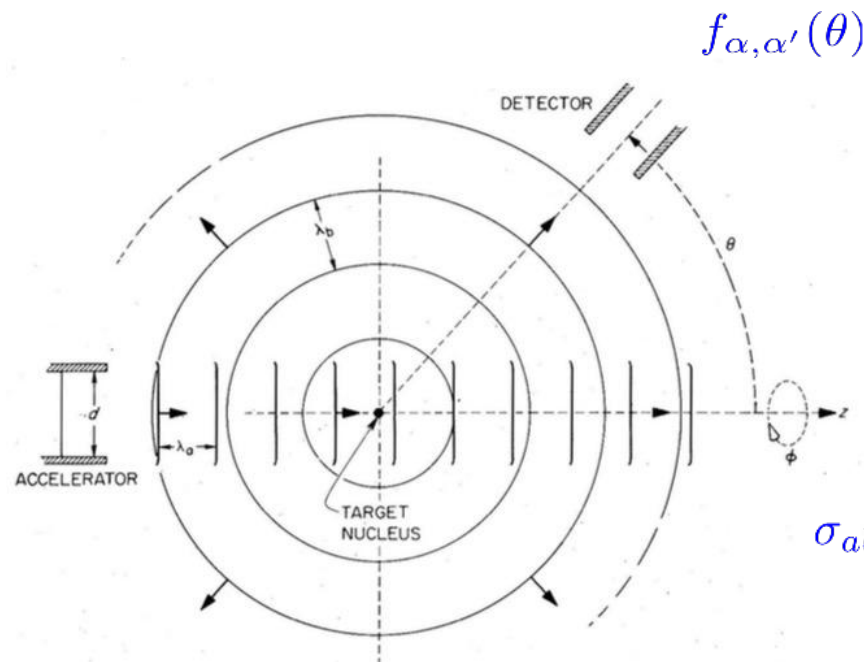
$$\sigma_R - \sum \sigma_{ine} = \sigma_F$$

the consistency of the work:

$$\sigma_R - \sum \sigma_{ine} = \frac{K}{E^2} \sum (2J+1) \langle \psi_{\alpha}^+ | W(R) | \psi_{\alpha'}^+ \rangle$$



- What means $\Psi^+(R)$?



Cross

section:

$$\sigma_{\alpha} = \frac{1}{(2I_{p_i} + 1)(2I_{t_i} + 1)} \sum |f_{\alpha, \alpha'}(\theta)|^2$$

Optical

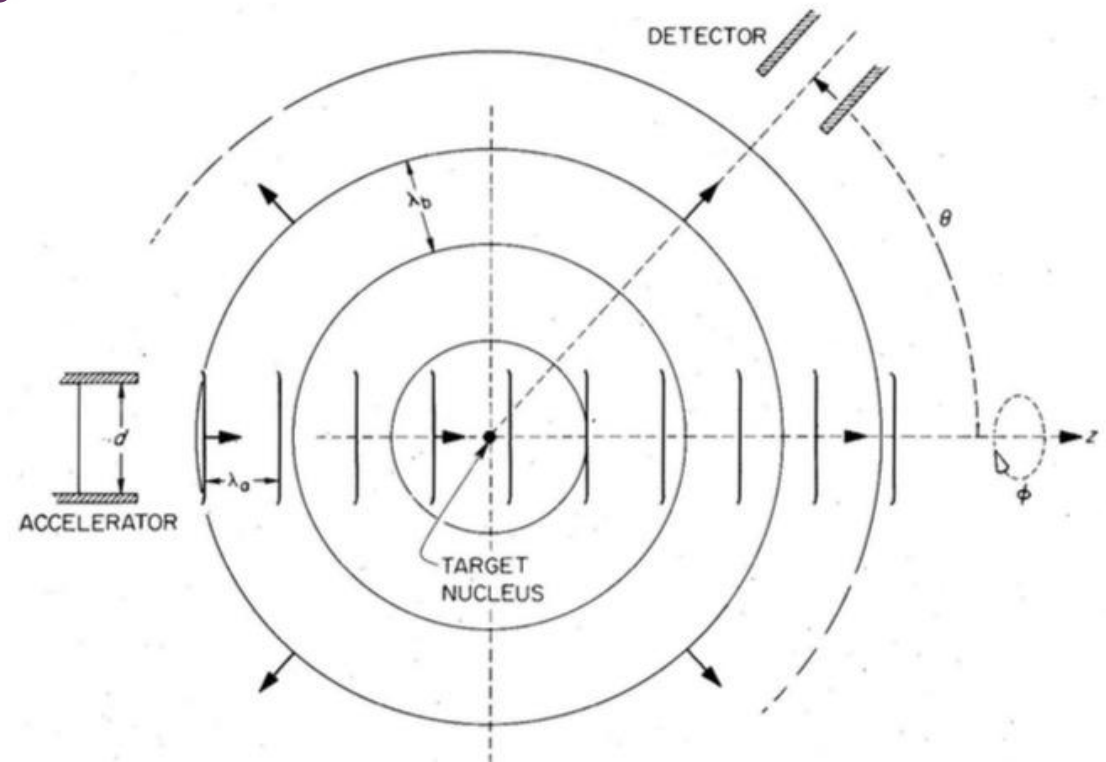
model:

$$\sigma_{abs} = \frac{E}{k^2} \sum \langle \Psi^+(R) | W(R) | \Psi^+(R) \rangle$$

$$\sigma_F = \frac{K}{E} \sum_{\alpha, \alpha'} \langle \psi_\alpha | W_{\alpha, \alpha'}^1(r_1) + W_{\alpha, \alpha'}^2(r_2) | \psi'_\alpha \rangle$$

$$W_{\alpha', \alpha}(R) = \int d\mathbf{r}(r) \phi_{\alpha'}^*(r) [\mathbb{W}_{vT}(r_1) + \mathbb{W}_{cT}(r_1)] \phi_\alpha(r)$$

Potential Scattering approach'
Fusion is not a channel



$$\sigma_F(E) = \frac{\pi}{K^2} \sum_{l=0} (2l+1) P_l^F(E) \quad \text{With} \quad P_l^F(E) = 1 - |S_l(E)|^2$$

Indirect determination of CF using spectator model*(IAV)

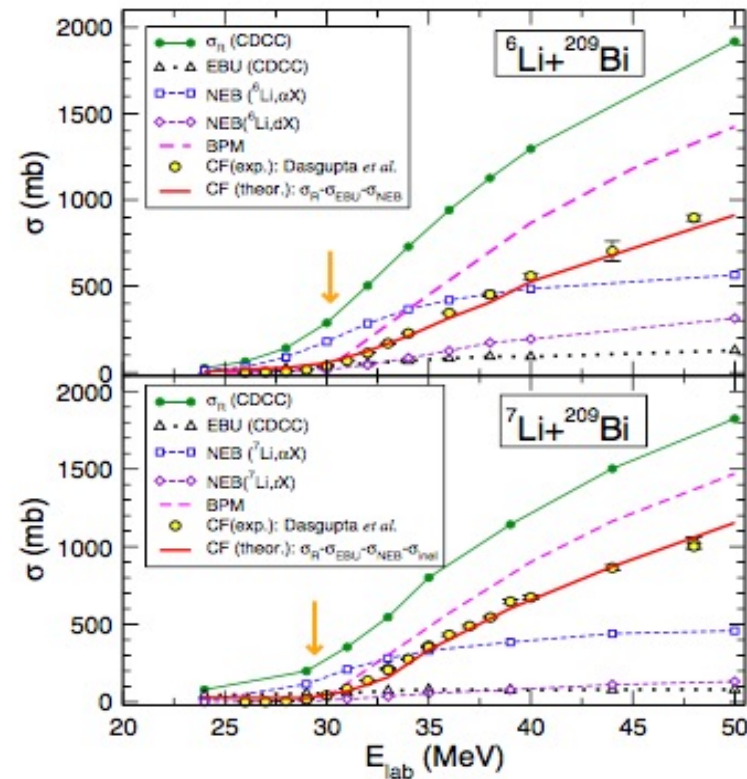
* **Lei and Moro PRL 122, 042503 (2019)**

• **Extract σ_{CF} from the relation**

$$\sigma_R = \sigma_{CF} + \sigma_{inel} + \sigma_{EBU} + \sigma_{NBU}^{(1)} + \sigma_{NBU}^{(2)}$$

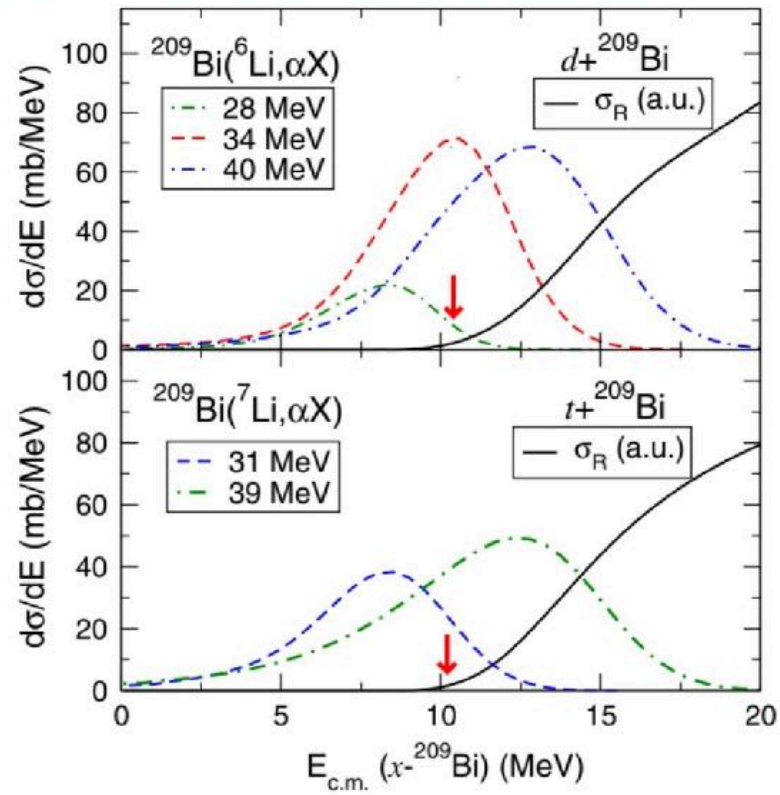
- σ_R : from CDCC calculation or optical model analysis
- σ_{ine} : from standard CC calculation (only bound channels)
- σ_{ebu} : from CDCC calculation:
- σ_{NEB1} , σ_{NEB2} : from inclusive spectator- participant model (IAV)

Indirect determination of CF using spectator model*(IAV)



* Lei and Moro, PRL 122, 042503 (2019)

- Verification of the Trojan Horse mechanism

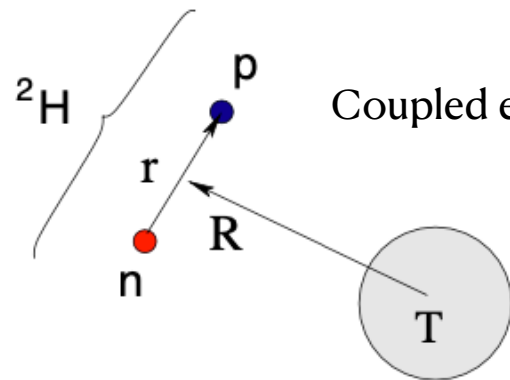


CDCC approach

Full numerical implementation by Kyushu group (Sakuragi, Yahiro, Kamimura, and co.):
Prog. Theor. Phys.(Kyoto) 68, 322 (1982)

Hamiltonian $H = T_R + h_r + U_{p-T} + U_{n-T}$

Model wave function $\Psi^+(\mathbf{R}, \mathbf{r}) = \phi_{gs}(\mathbf{r})\chi_o(\mathbf{R}) + \sum_{n>0} \phi_n(\mathbf{r})\chi_n(\mathbf{R})$



Coupled equations: $H\Psi^+(\mathbf{R}, \mathbf{r}) = E\Psi^+(\mathbf{R}, \mathbf{r})$

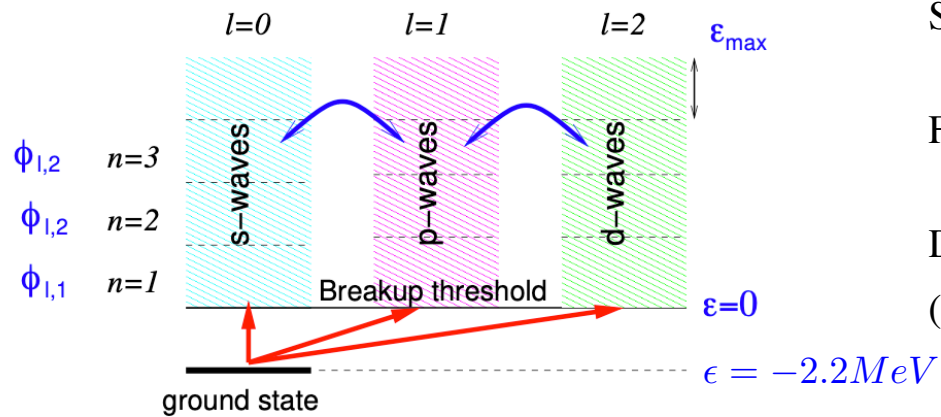
$$[E - \epsilon_n - T_R + U_{n,n}(R)]\chi(R) = \sum_{n' \neq n} U_{n',n}(R)\chi(R)$$

$$U_{n',n}(R) = \int d\mathbf{r}(r) \phi_n^*(r) [U_{pT}(r_1) + U_{nT}(r_1)] \phi_n(r)$$

CDCC approach

Model wave function

$$\Psi^+(\mathbf{R}, \mathbf{r}) = \phi_{gs}(\mathbf{r})\chi_o(\mathbf{R}) + \sum_{n>0} \phi_n(\mathbf{r})\chi_n(\mathbf{R})$$



Select a number of angular momenta ($l=0, \dots, l_{\text{max}}$).

For each l , set a maximum excitation energy ϵ_{max} .

Divide the interval $\epsilon = 0 - \epsilon_{\text{max}}$ in a set of sub-intervals (*bins*).

For each *bin*, calculate a representative wave function.

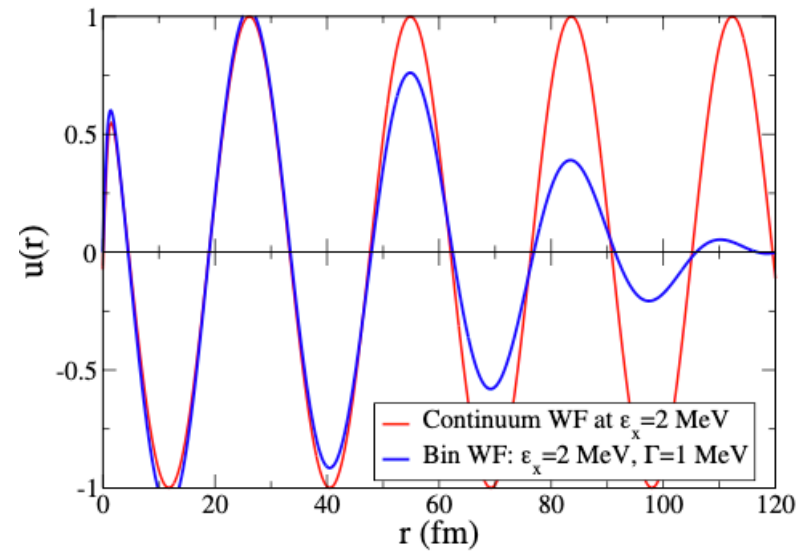
Courtesy from Antonio Moro

CDCC approach

Model wave function $\Psi^+(\mathbf{R}, \mathbf{r}) = \phi_{gs}(\mathbf{r})\chi_o(\mathbf{R}) + \sum_{n>0} \phi_n(\mathbf{r})\chi_n(\mathbf{R})$

Bins: $\phi_{lsjm}^{[k_1, k_2]}(r) = \frac{u_{lj}^{[k_1, k_2]}(r)}{r} [Y_l(\hat{r}) \otimes \chi_s]_{jm}$

$$u_{lj}^{[k_1, k_2]}(r) = \sqrt{\frac{2}{\pi N}} \int_{k_1}^{k_2} w(k) u_{lj,k}(r) dk$$



Courtesy from Antonio Moro



Faculdade de
Tecnologia
UERJ-Resende



A new QM method to evaluate CF and ICF*

Contribution from the B-space

$$\sigma_B = \sigma_{DCF}$$

$$\sigma_B = \frac{K}{E} \sum_{\alpha \in B} \langle \psi_\alpha | W^1(r_1) + W^2(r_2) | \psi_\alpha \rangle$$



$$\sigma_{DCF} = \frac{\pi}{k^2} \sum_J (2J+1) [P^1(J) + P^2(J)]$$



* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

A new QM method to evaluate CF and ICF*

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