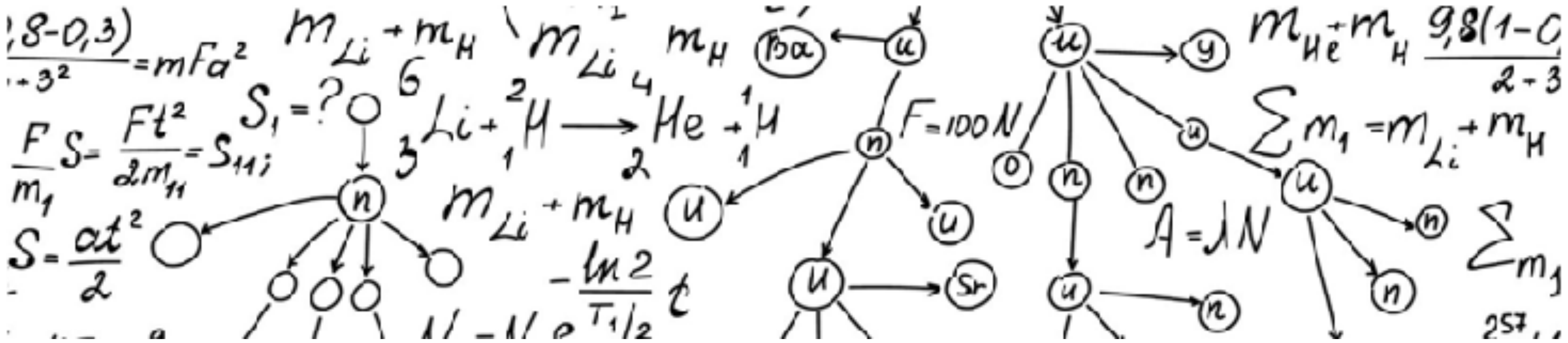
PAN-AMERICAN FEW-BODY PHYSICS BOOT CAMP: FOSTERING COLLABORATION



Fusion reactions in collisions of weakly bound nuclei with heavy targets

Jeannie Rangel Borges

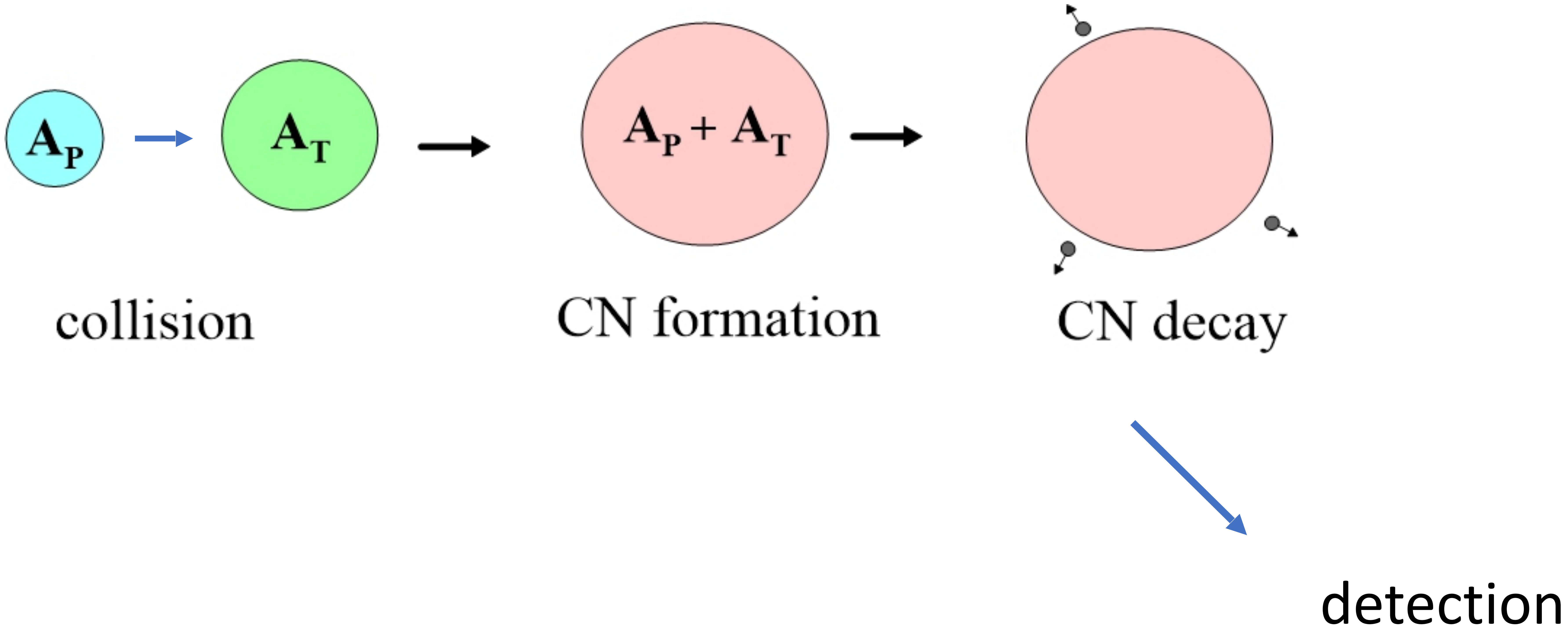
15/10/2025

Faculdade de Tecnologia UERJ-Resende

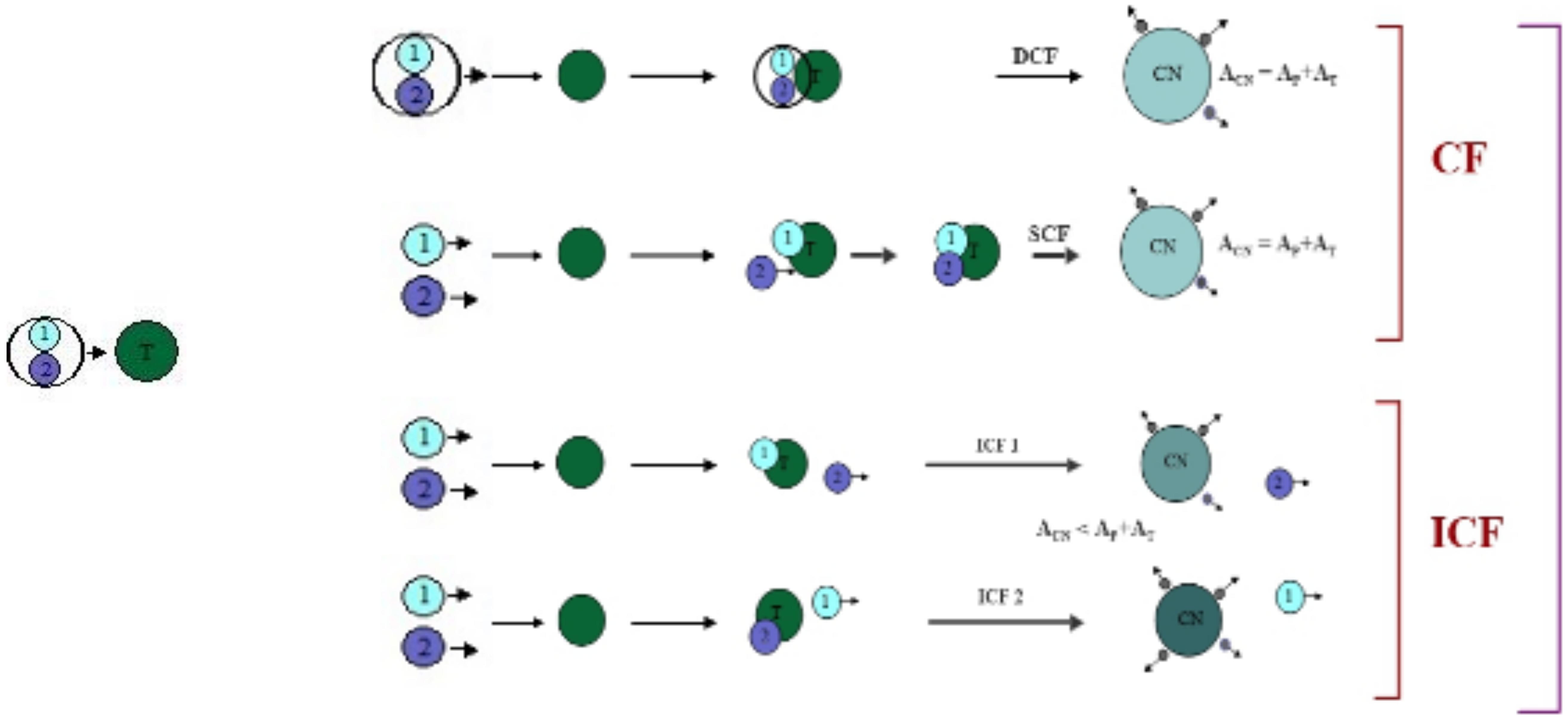
Outline

- Motivation
- Bref review of the theory
  - ✓ Classical and semiclassical model
  - ✓ Quantum mechanics models
- Our approach
- Some ilustrations
- conclusions

Fusion of tightly bound nuclei



Collisions of weakly bound nuclei (different fusion processes)



$$\sigma_{\mathrm{CF}} = \sigma_{\mathrm{DCF}} + \sigma_{\mathrm{SCF}}$$

$$\sigma_{\mathrm{ICF}} = \sigma_{\mathrm{ICF}_1} + \sigma_{\mathrm{ICF}_2}$$

$$\boxed{\sigma_{\mathrm{TF}} = \sigma_{\mathrm{CF}} + \sigma_{\mathrm{ICF}}}$$

Finding CF and ICF cross section is a great challenge

(both for experimentalists and theorists)

Experiment:

- CF absorption of all projectile charge ($^{11}Be = {}^{10}Be + n$)
- Most experiments determine only TF
- Individual CF and/or ICF have been measured for some particular weakly bound nuclei:

^{6}Li : BE= 1.47 MeV ($\alpha + d$)
^{7}Li : BE= 2.47 MeV ($\alpha + t$)
^{11}Be: BE = 0.500 MeV ($^{10}Be + n$)
^{6}He : BE =0.600 MeV ($^{4}He + n+n$)

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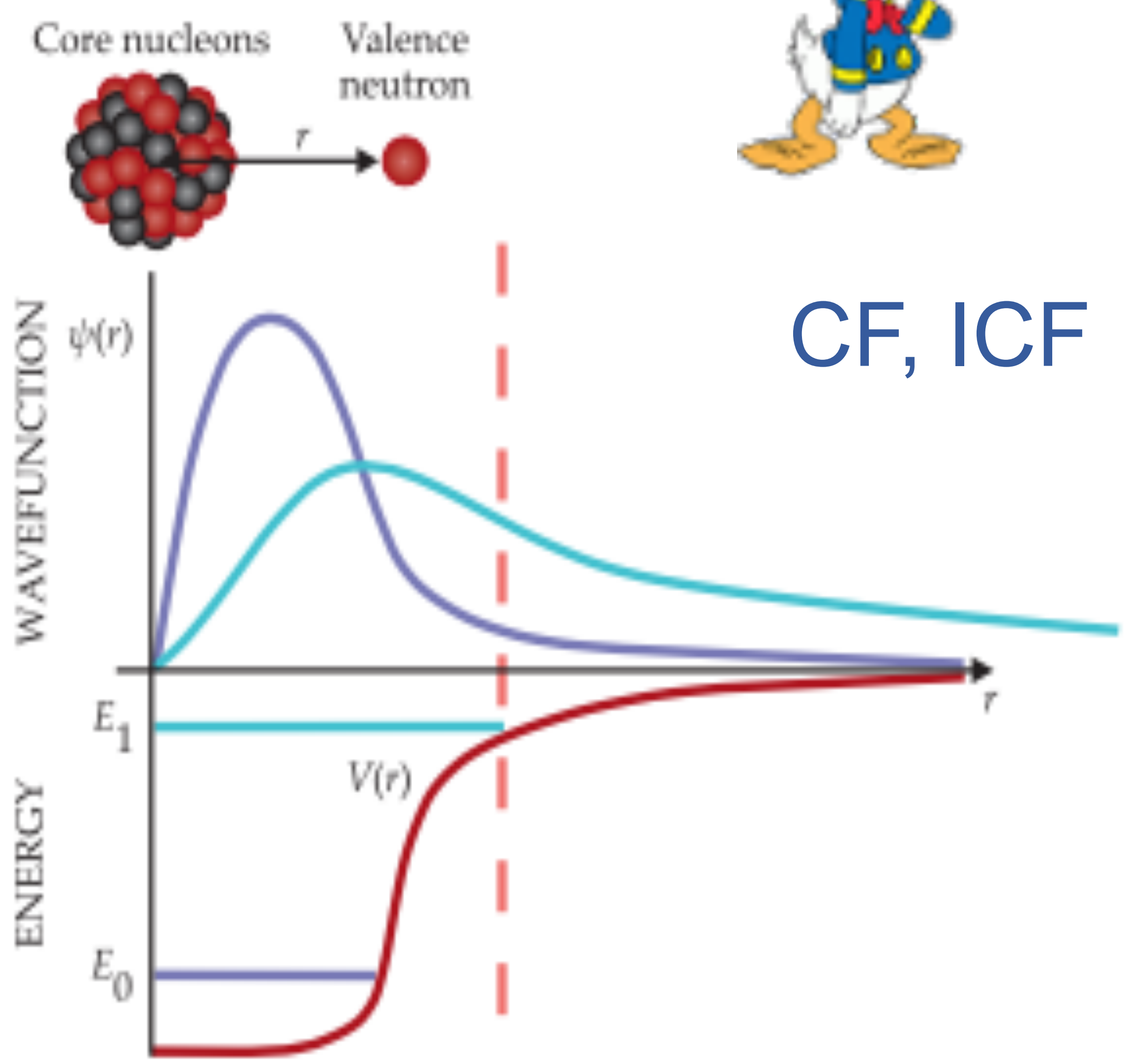

CF, ICF

^{6}Li : BE= 1.47 MeV (α + d)
^{7}Li : BE= 2.47 MeV (α + t)
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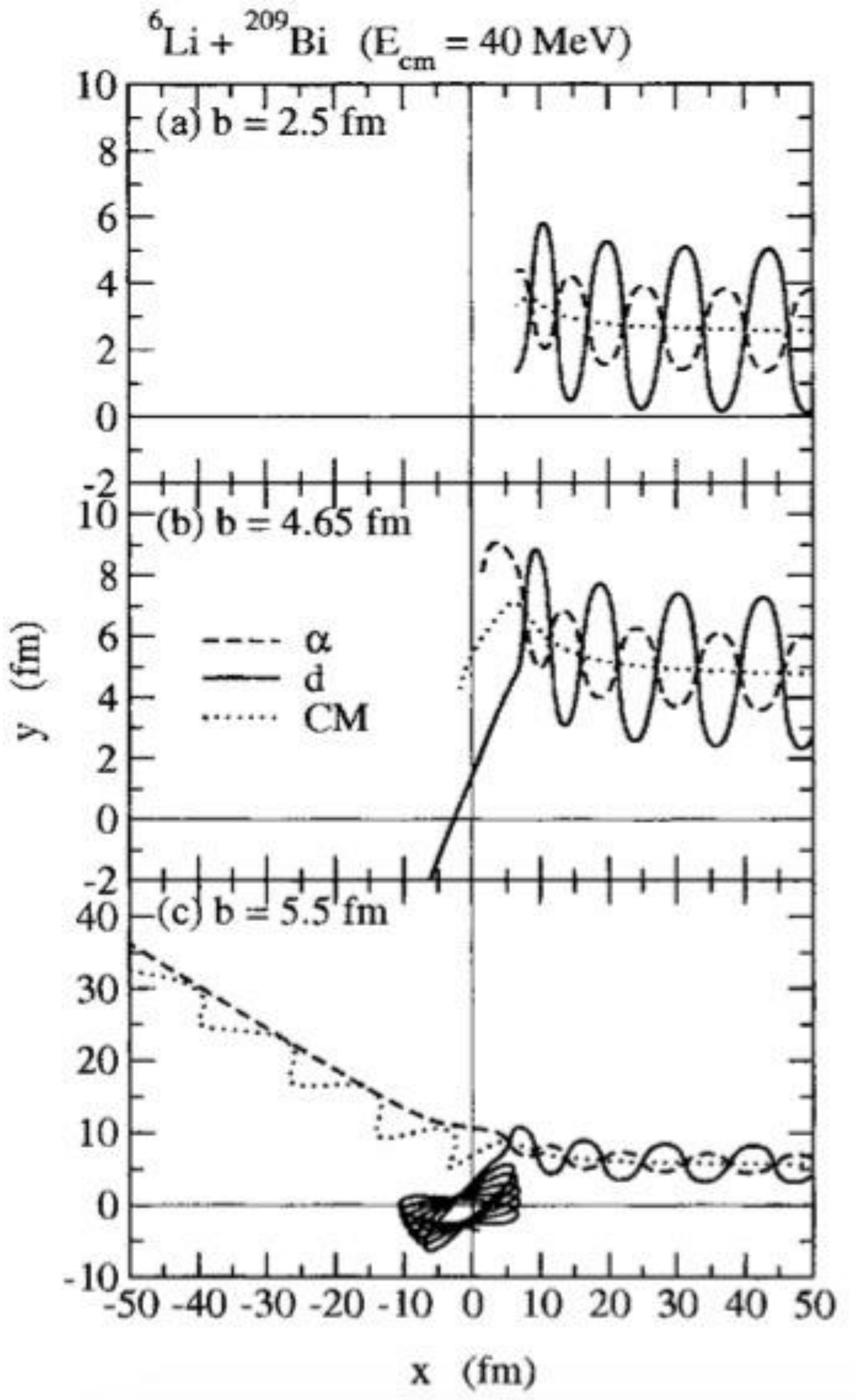
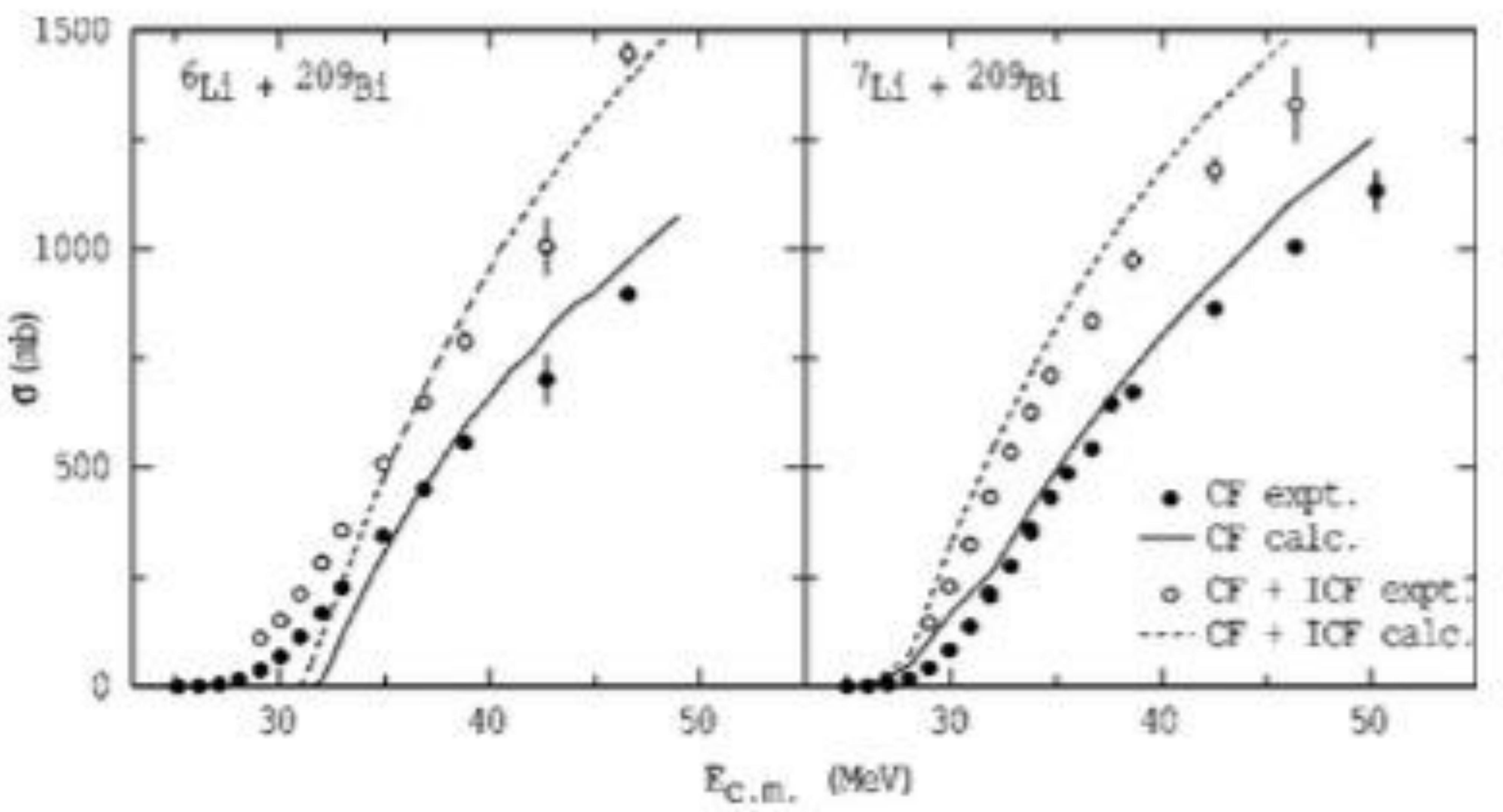
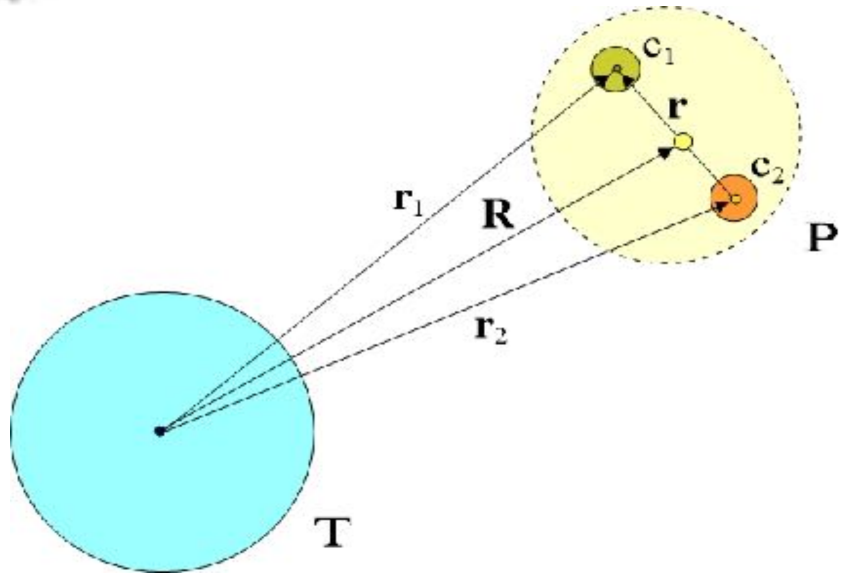
Fusion Estimations: classical picture



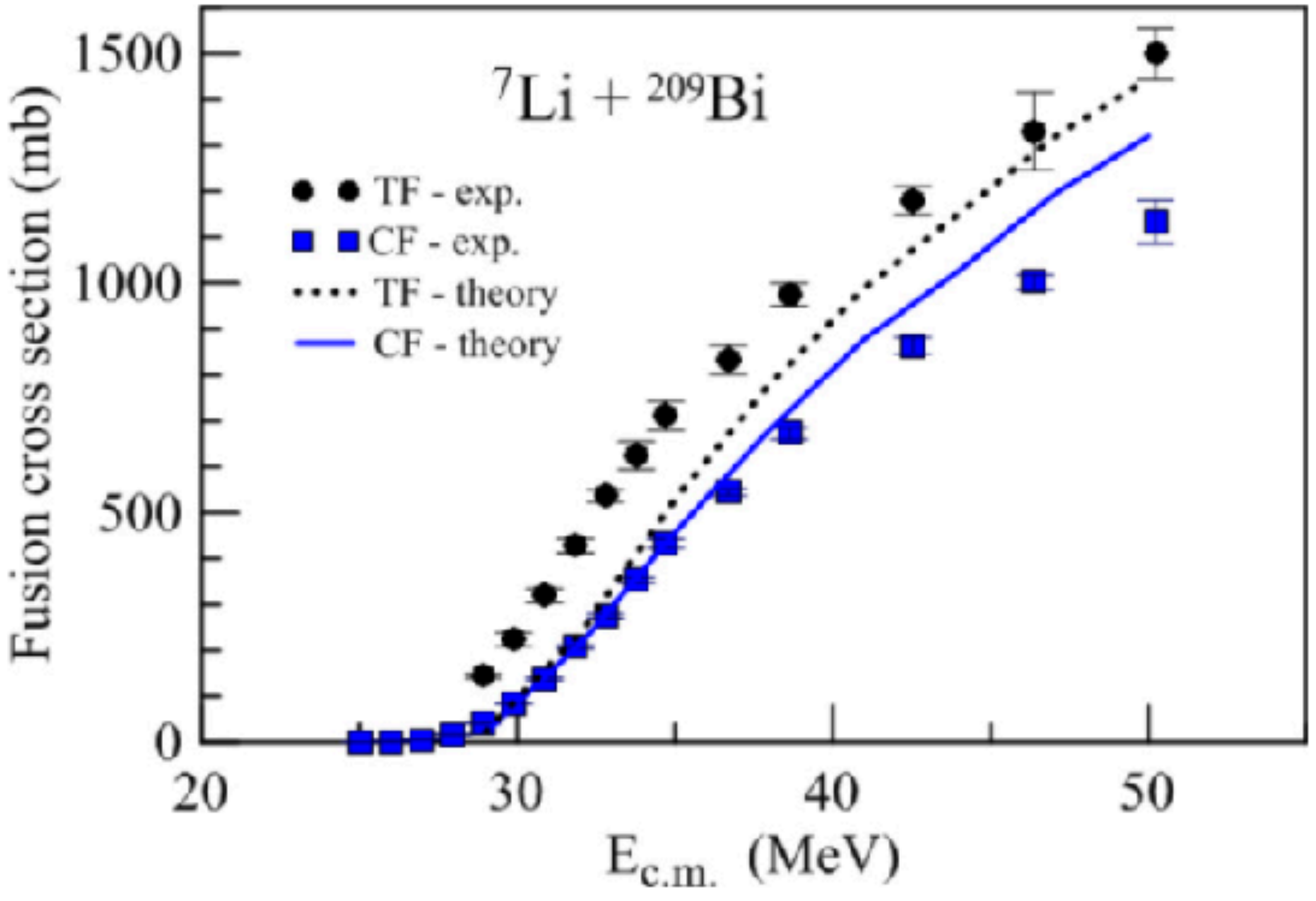


- Classical picture with stochastic parameters.



Hagino et al., NPA **238**, 475 (2004), Dasgupta et al., PRC **66**, 041602 (2002),

Fusion Estimations: semi classical models



- Classical trajectory
- Intrinsic dynamic: time dependent Schrödinger equation

$$[h_0 + V(b,\vec{r},t)]\Psi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t)$$

$$\Psi(b,t) = \sum c_i(b,t)\psi_i e^{-i\epsilon_i t/\hbar}$$

- Fusion: tunelling through the barrier

* Marta et al., PRC **89**, 034625 (2014), Kolinger et al., PRC **98**, 044604 (2018)

The method of Hagino, Vitturi, Dasso and Lenzi (HVDL)

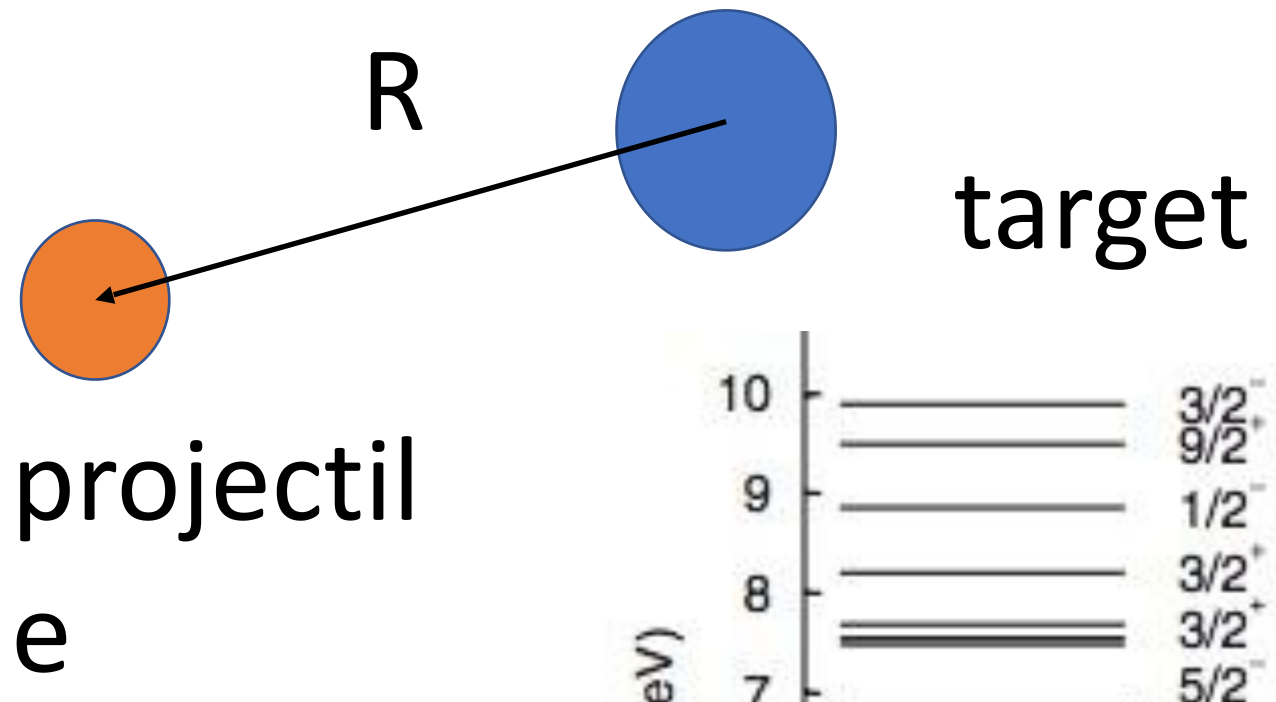
- To solve the Shrödinger

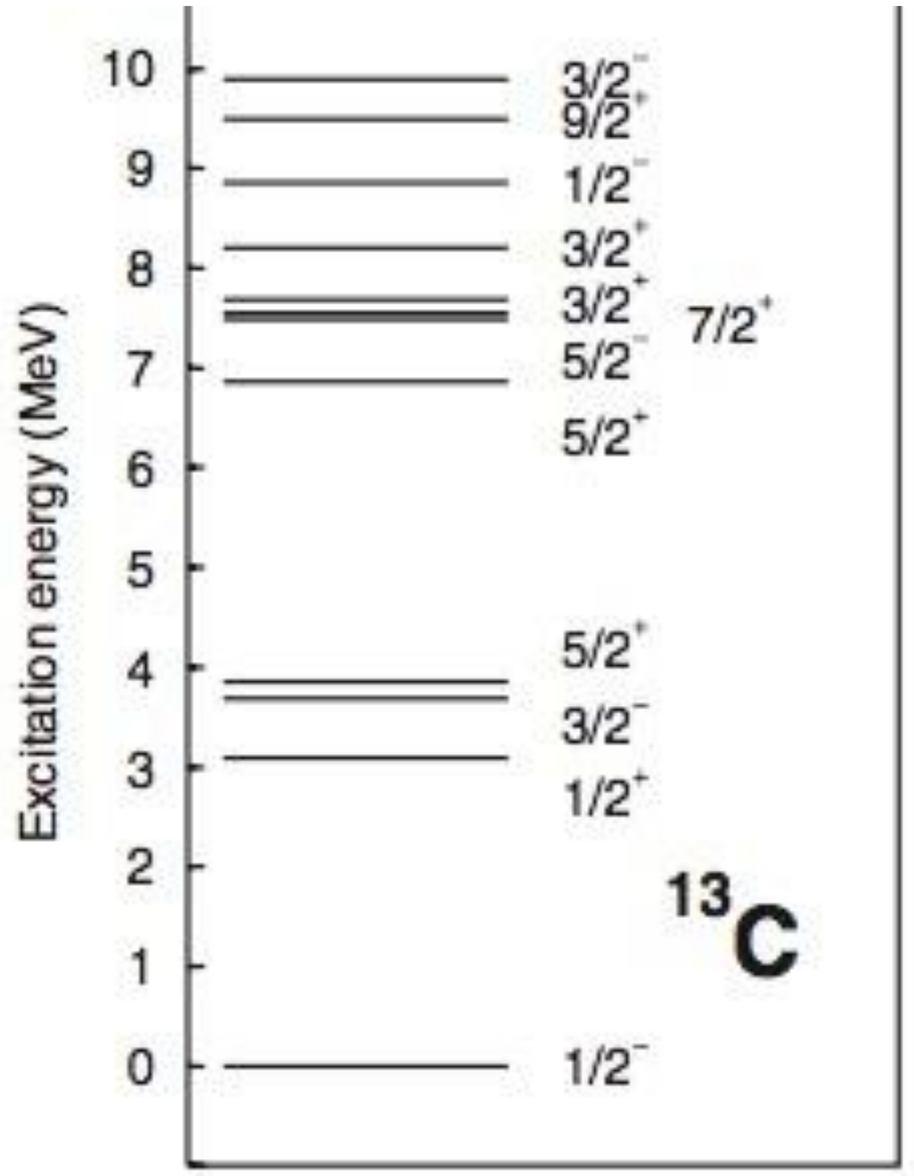
$$\mathbb{H}\Psi^{+} = E\Psi^{+}$$

- Coupled Channels equations:

$$\Psi^{+}(R,\xi) = \sum \psi_{\alpha}^{+}(R)\phi(\xi)$$

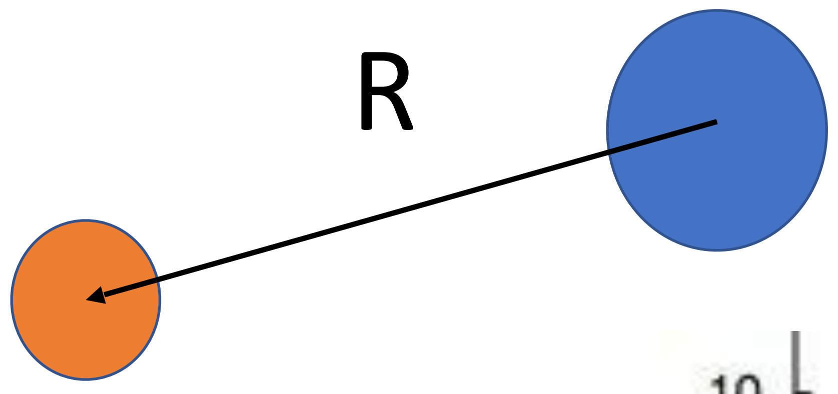
$$[E - \epsilon_{\alpha} - T_R - U_{\alpha,\alpha}(R)]\psi_{\alpha}(R) = \sum_{\alpha' \neq \alpha} U_{\alpha',\alpha}(R)\psi_{\alpha'}(R) \ , \alpha, \alpha' = 1, N$$





The method of Hagino, Vitturi, Dasso and Lenzi (HVDL)

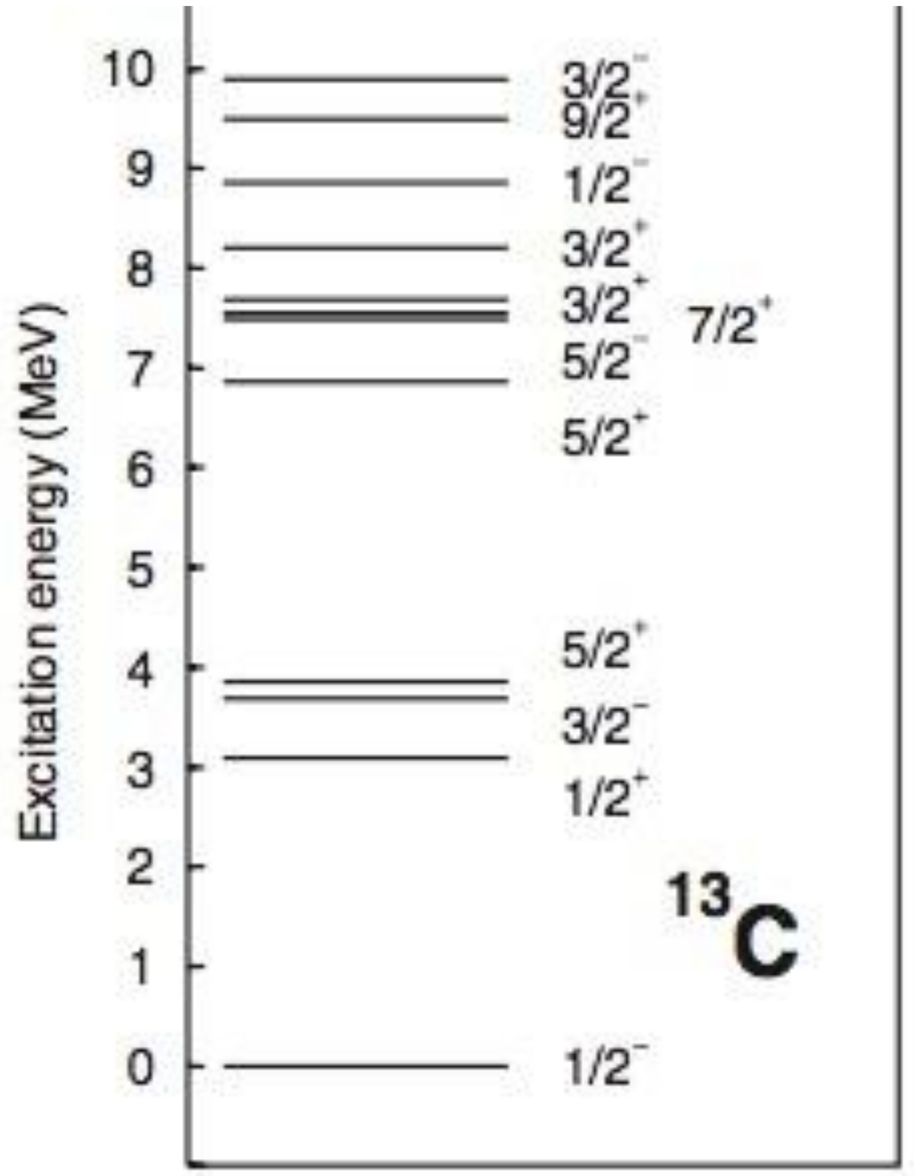
- **Set of coupled equations**



$$[E - \epsilon_0 - T_R - U_{0,0}(R)]\psi_0(R) = \sum_{\alpha' \neq 0} U_{\alpha',0}(R)\psi_{\alpha'}(R)$$

$$\vdots$$

$$[E - \epsilon_N - T_R - U_{N,N}(R)]\psi_N(R) = \sum_{\alpha' \neq N} U_{\alpha',N}(R)\psi_{\alpha'}(R)$$



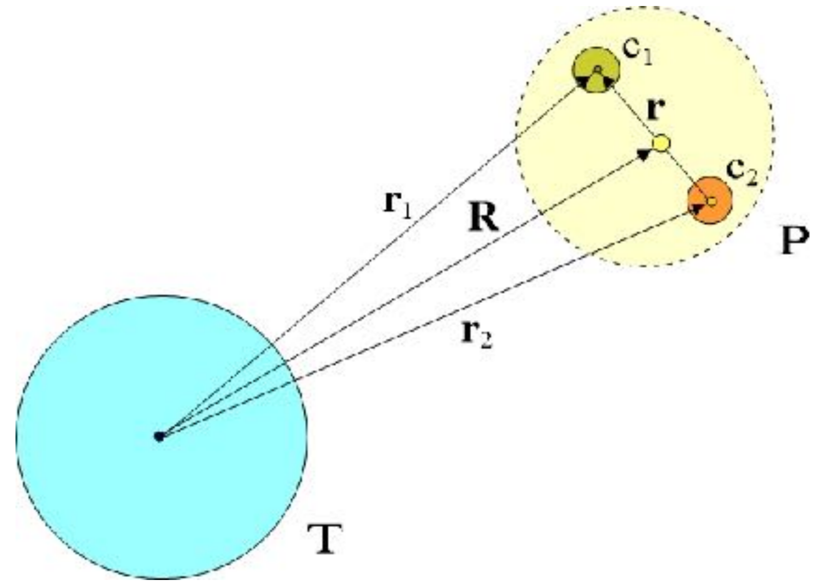
- **Coupled potential**

$$U_{\alpha',\alpha}(R) = \int d\xi \phi^*_{\alpha'}(\xi) \mathbb{U}_{PT}(R,\xi)\phi_\alpha(\xi)$$

The method of Hagino, Vitturi, Dasso and Lenzi (HVDL)

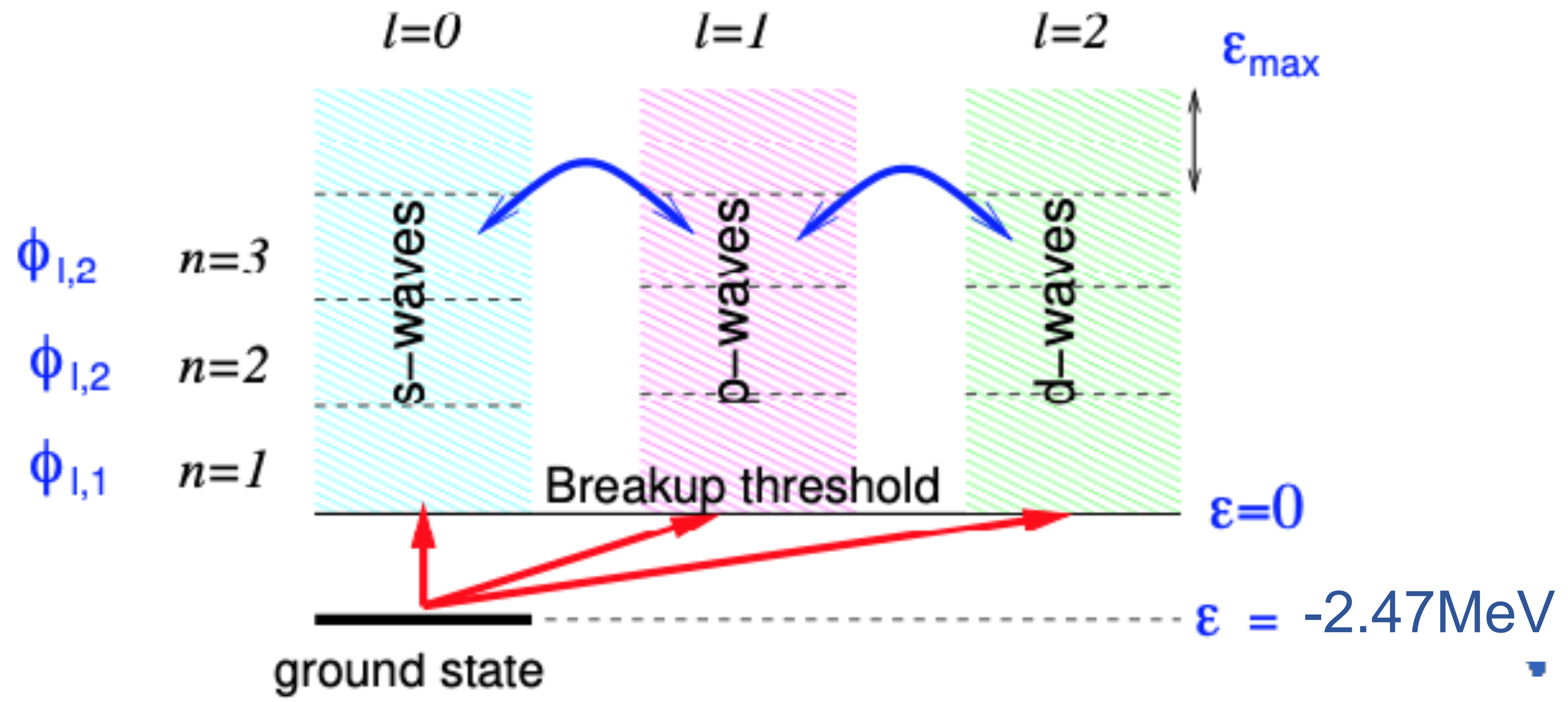
CDCC :

$$U(r_1, r_2) = U_1(r_1) + U(r_2)$$



- **Continuum Discretized Coupled Channels equations:**

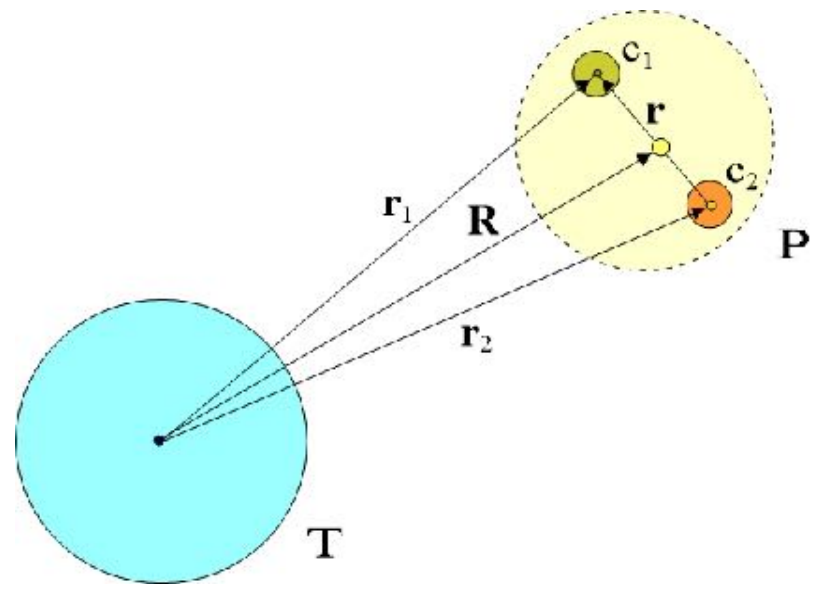
Bins



The method of Hagino, Vitturi, Dasso and Lenzi (HVDL)

CDCC :

$$U_{\alpha',\alpha}(R) = \int d\mathbf{r}(r)\phi^*_{\alpha'}(r)[\mathbb{U}_{vT}(r_1) + \mathbb{U}_{cT}(r_1)]\phi_\alpha(r)$$



- **Continuum Discretized Coupled Channels equations:**

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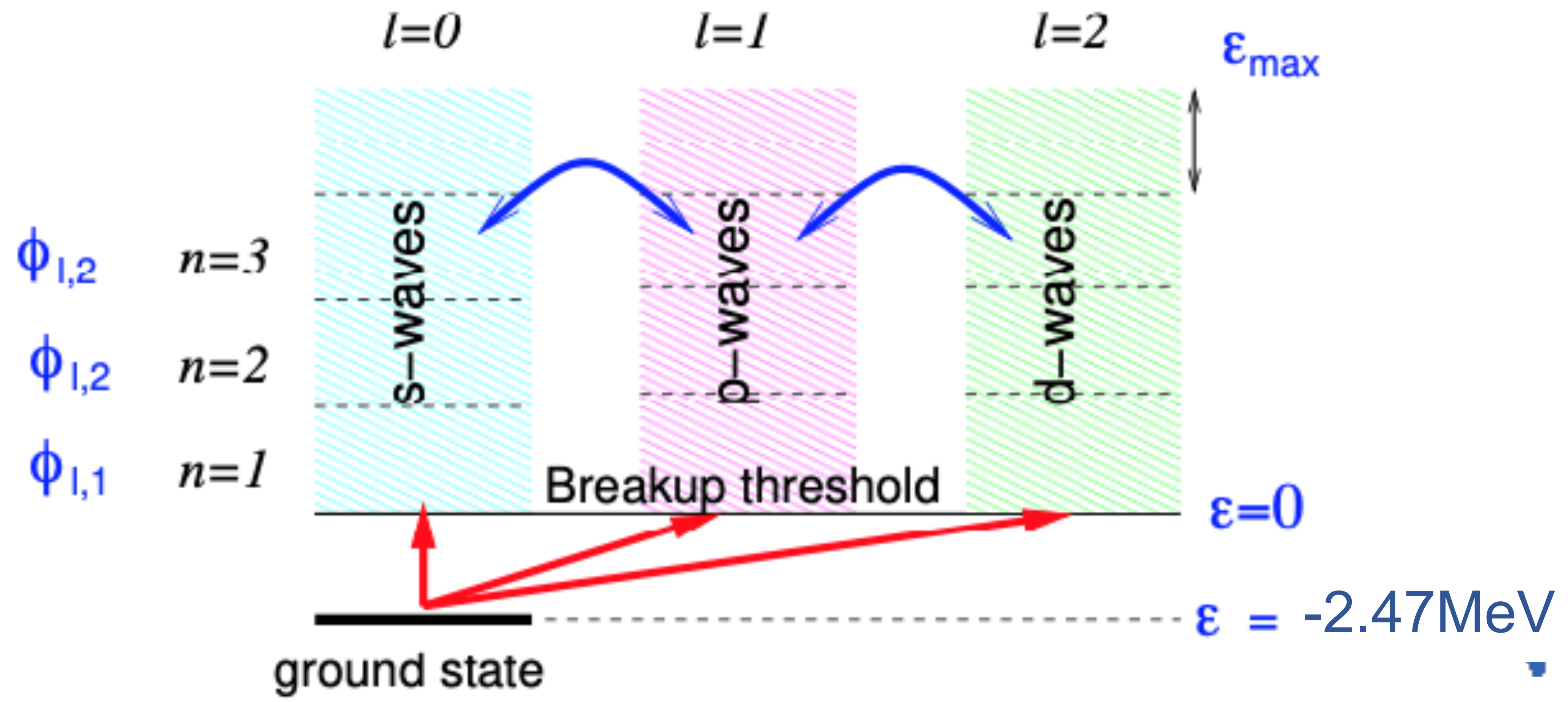


The method of Hagino, Vitturi, Dasso and Lenzi (HVDL)

- **P-T imaginary potential (instead of W(1) + W(2))**

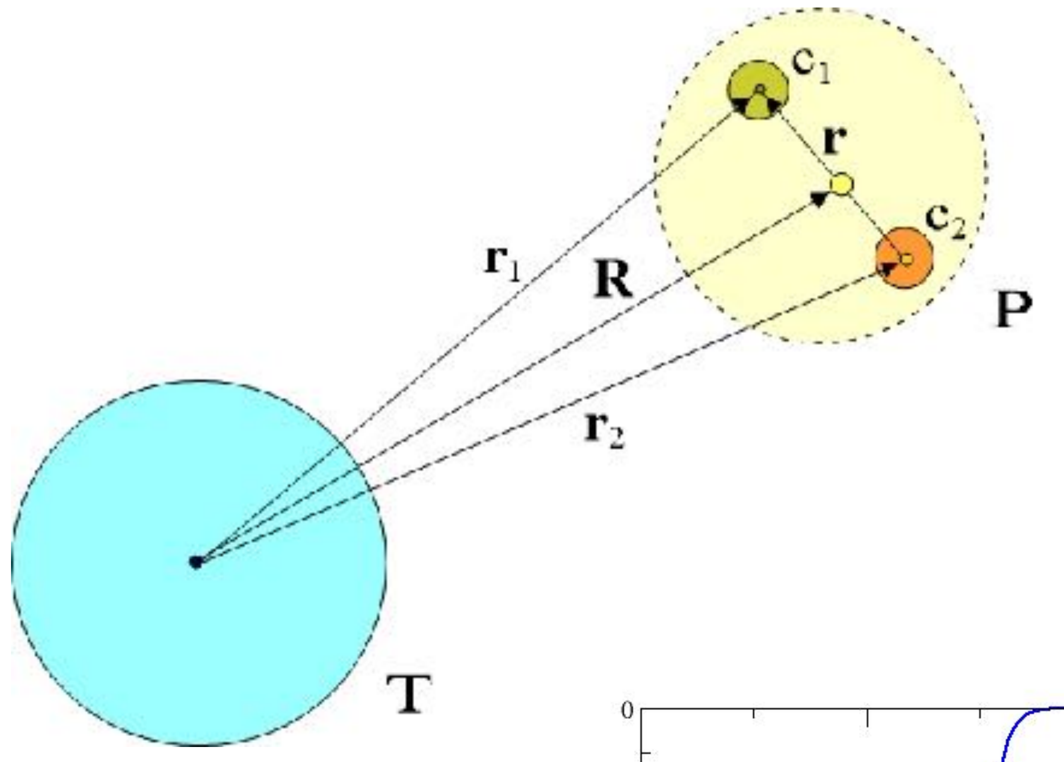
$$\sigma_F = \frac{K}{E}\sum_{\alpha=1} < \psi_\alpha | W(R) | \psi_\alpha >= \sum_{\alpha=1} \sigma_\alpha$$

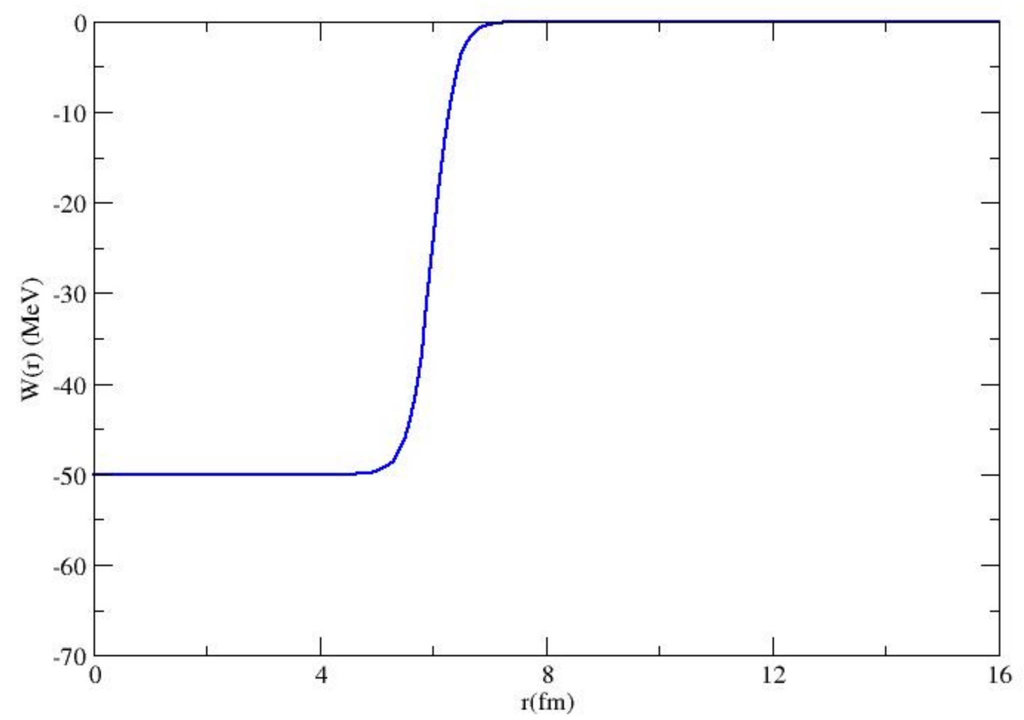
- **Or**

$$\sigma_F = \sigma_C + \sigma_B$$

$$\boxed{\sigma_B = \frac{K}{E} \sum_{\alpha \in bound} < \psi_\alpha | W(R) | \psi_\alpha >}$$

$$\boxed{\sigma_C = \frac{K}{E} \sum_{\alpha \in continnum} < \psi_\alpha | W(R) | \psi_\alpha >}$$

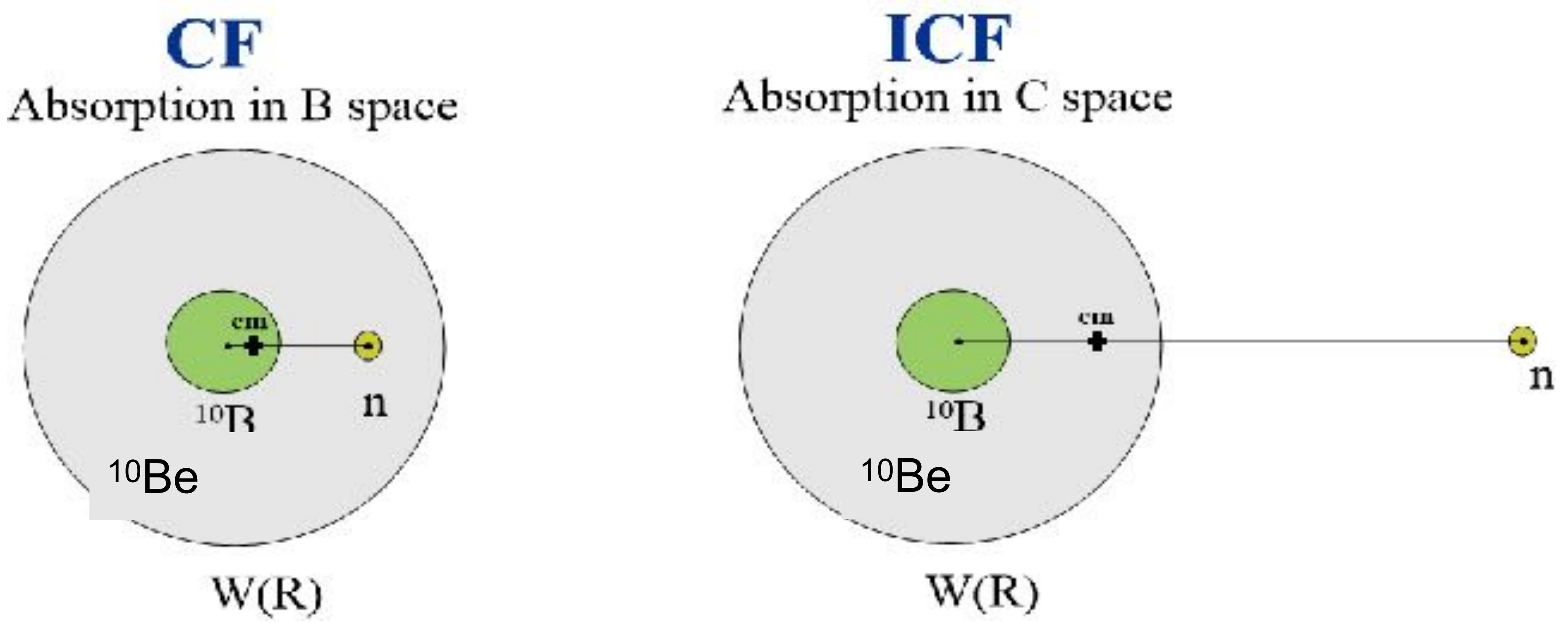




Basic Assumption: $\sigma_{\mathrm{CF}} = \sigma_{\mathrm{B}}$, $\sigma_{\mathrm{ICF}} = \sigma_{\mathrm{C}}$

Limitation: works for a fragment much heavier than the other

^{11}Be (^{10}Be - n) + ^{208}Pb

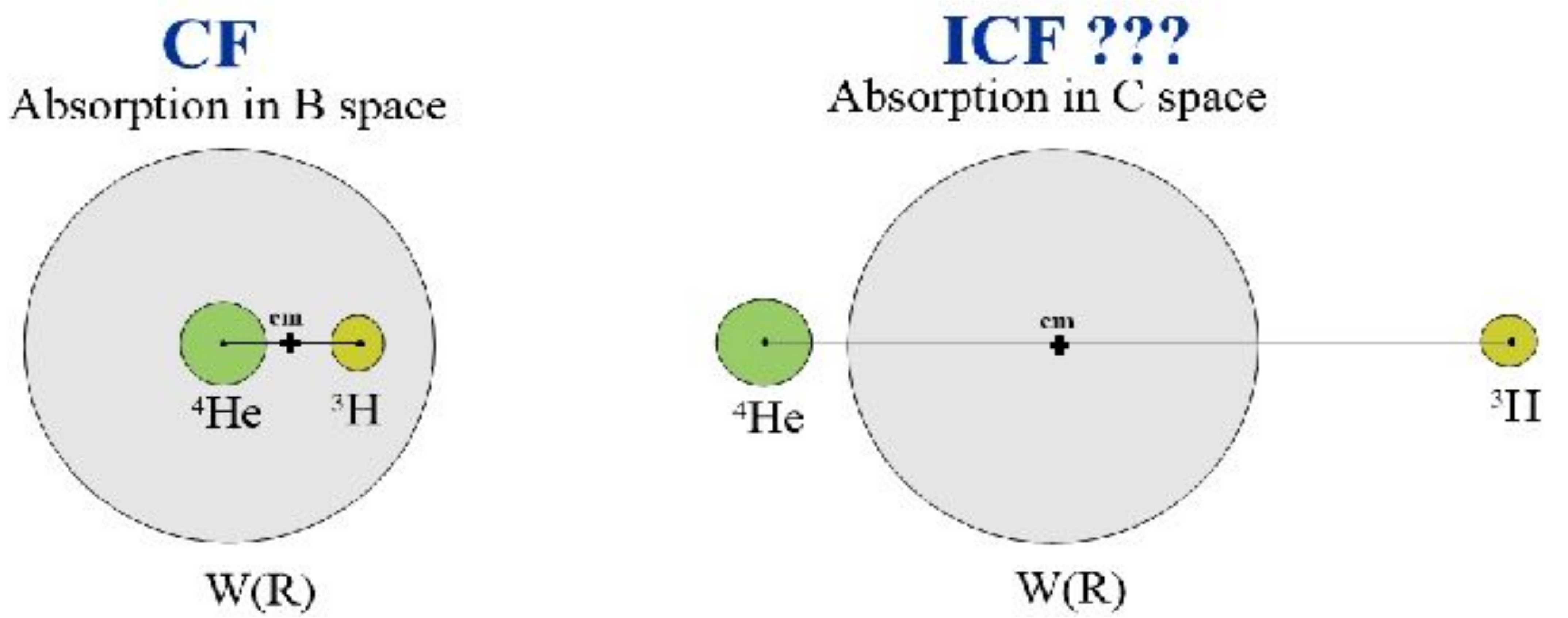


Works fine !

Basic Assumption: $\sigma_{\mathrm{CF}} = \sigma_{\mathrm{B}} \quad , \quad \sigma_{\mathrm{ICF}} = \sigma_{\mathrm{C}}$

Limitation: works for a fragment much heavier than the other

^{7}Li (^{4}He - ^{3}H) + ^{209}Bi



Does not work !

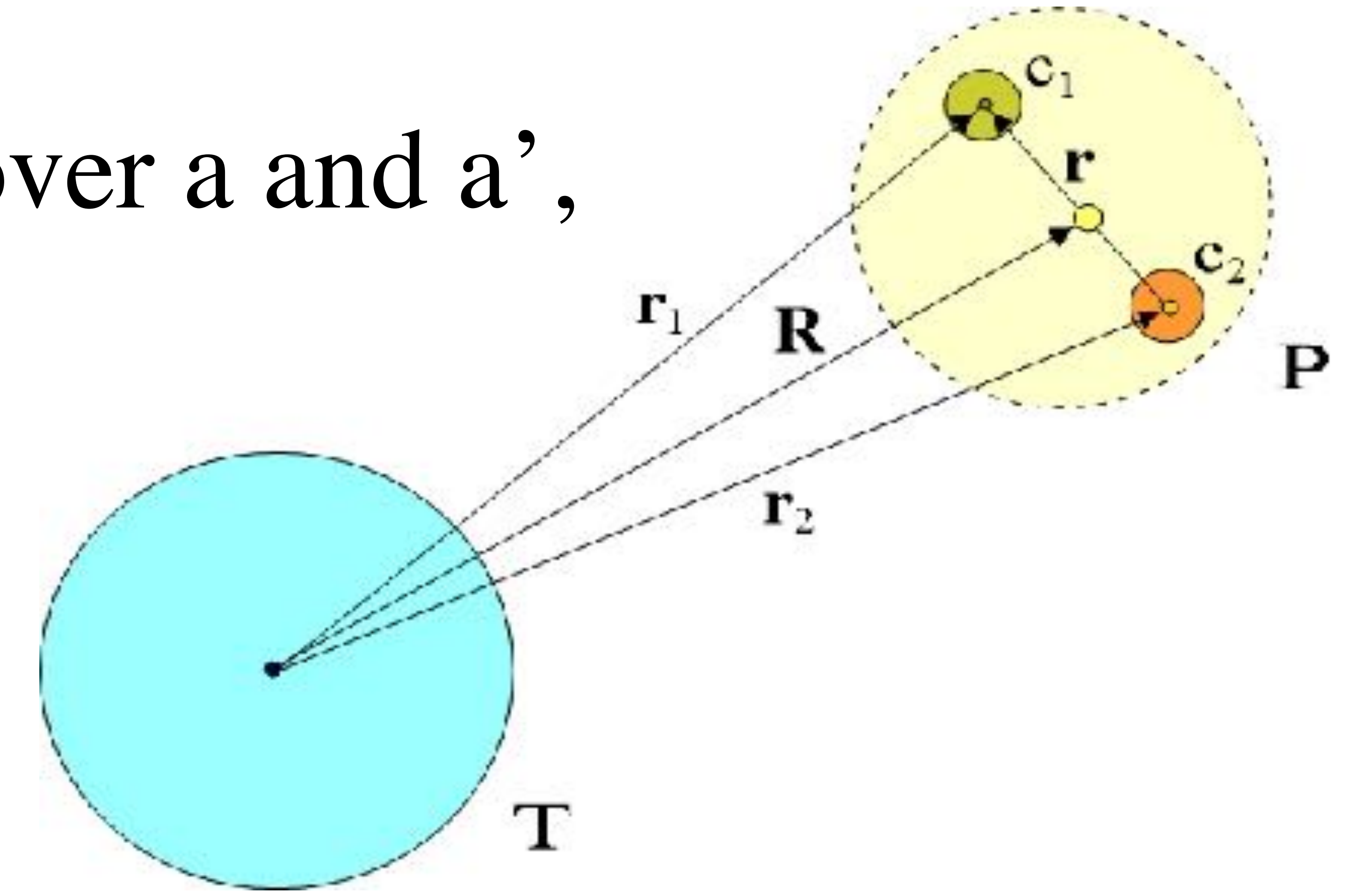
Theory: based on HVDL *

- Direct calculation using radial wave functions

$$\sigma_F = \frac{K}{E} \sum_{\alpha,\alpha'} < \psi_\alpha | W^1_{\alpha,\alpha'}(r_1) + W^2_{\alpha,\alpha'}(r_2) | \psi'_\alpha >$$

Performing ang. mom. projection and the summing over a and a',

$$\sigma_F = \frac{\pi}{k^2} \sum_J (2J+1)[P^1(J) + P^2(J)]$$



* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

Theory: based on HVDL *

- Individual probability

$$\mathcal{P}^{(i)}(J) = \frac{4K}{E\,\hat{j}_0^2} \sum_{L_0} \sum_{\lambda} \sum_{\alpha L \alpha' L'} \mathcal{B}^{\lambda}_{\alpha L,\alpha' L'}(J) \mathcal{M}^{(i)\lambda}_{\alpha L,\alpha' L'}(L_0, J).$$

where

$$\mathcal{B}^{\lambda}_{\alpha L,\alpha' L'}(J) = (-)^{J-s+j_\alpha-j_{\alpha'}-L-L'+\lambda} \;\; \hat{\lambda}^2\,\hat{L}\,\hat{L}'\,\hat{l}_\alpha\,\hat{l}_{\alpha'}\,\hat{j}_\alpha\,\hat{j}_{\alpha'} \\ W\left(LL'j_\alpha j_{\alpha'};\lambda J\right)\; W\left(l_\alpha l_{\alpha'} j_\alpha j_{\alpha'};\lambda s\right) \begin{pmatrix} \lambda & L' & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda & l_{\alpha'} & l_\alpha \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{M}^{(i)\lambda}_{\alpha L,\alpha' L'}(L_0, J) = \; i^{L'-L} \int dR\; u^{J*}_{\alpha L,0L_0}(k_\alpha, R)\;\; \mathcal{F}^{(i)\lambda}_{l_\alpha j_\alpha, l_{\alpha'} j_{\alpha'}}(R)\;\; u^{J}_{\alpha' L',0L_0}(k'_\alpha, R).$$

* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

A new QM method to evaluate CF and ICF*

Contribution from the B-space: $\sigma_B = \sigma_{\mathrm{DCF}}$

$$\sigma_B = \frac{K}{E} \sum_{\alpha \in B} < \psi_\alpha | W(R) | \psi_\alpha >$$

$$\sigma_{\mathrm{DCF}} = \frac{\pi}{k^2} \sum_J (2J+1) P(J)$$

* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

Contribution from channels in the continuum to TF*

$$\sigma_C = \frac{K}{E} \sum_{\alpha,\alpha' \in C} < \psi_\alpha | W^1_{\alpha,\alpha'}(r_1) + W^2_{\alpha,\alpha'}(r_2) | \psi'_\alpha >$$

$$\sigma_C = \frac{\pi}{k^2} \sum_J (2J+1)[P^1(J) + P^2(J)]$$

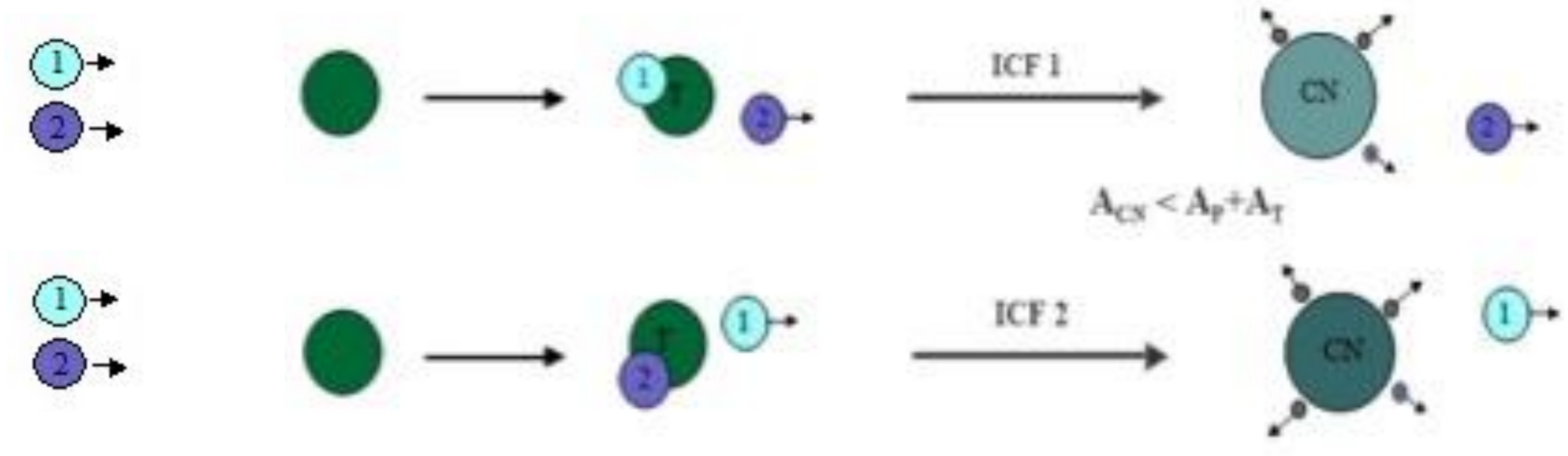
$P(i)$ (J) = abs. probability of fragment i in the C-space

* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

ICF (ICF1, ICF2), SCF cross sections*

$$\sigma_{\mathrm{ICF1}} = \frac{\pi}{k^2} \sum_J (2J+1) P^1(J)[1 - P^2(J)]$$

$$\sigma_{\mathrm{ICF2}} = \frac{\pi}{k^2} \sum_J (2J+1) P^2(J)[1 - P^1(J)]$$

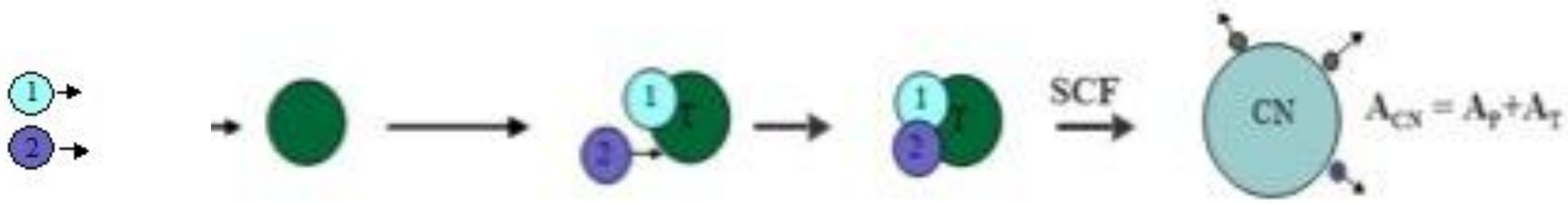


$$\sigma_{\mathrm{ICF}} = \sigma_{\mathrm{ICF1}} + \sigma_{\mathrm{ICF2}}$$

* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

ICF (ICF1, ICF2), SCF cross sections*

$$\sigma_{\mathrm{SCF}} = \frac{\pi}{k^2}\sum_J (2J+1)P^1(J) \times P^2(J)$$

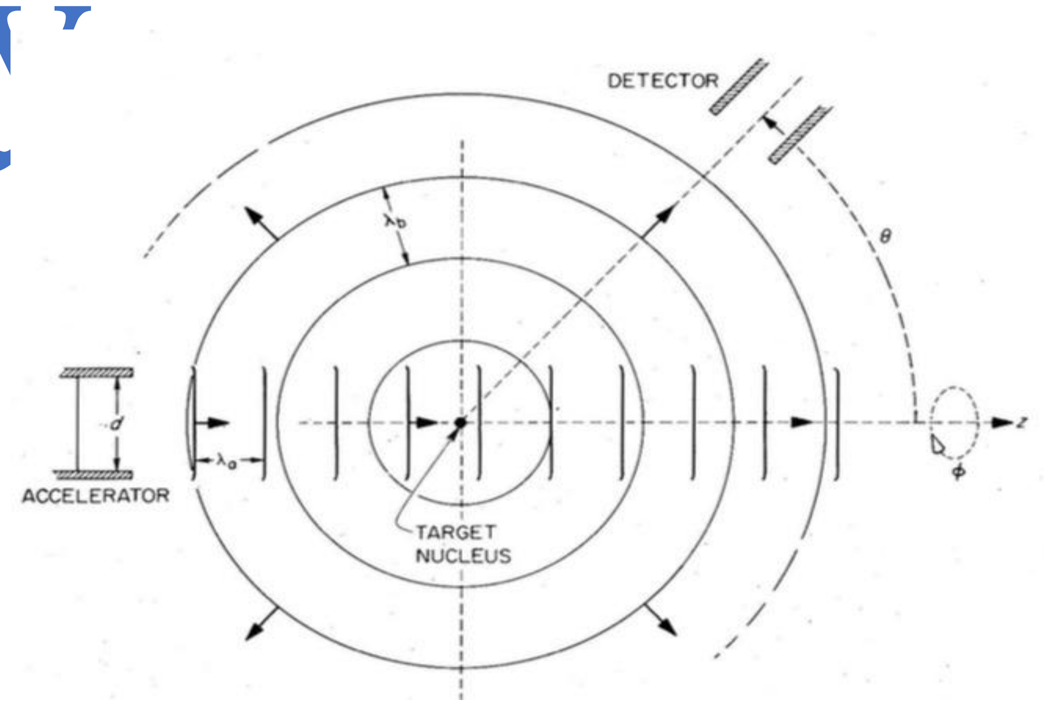


$$\sigma_{\mathrm{CF}} = \sigma_{\mathrm{DCF}} + \sigma_{\mathrm{SCF}}$$

- J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)
- Further applications: PRC, **102**, 064628 (2020) and PRC **105**, 054601 (2022)

Procedure Summary

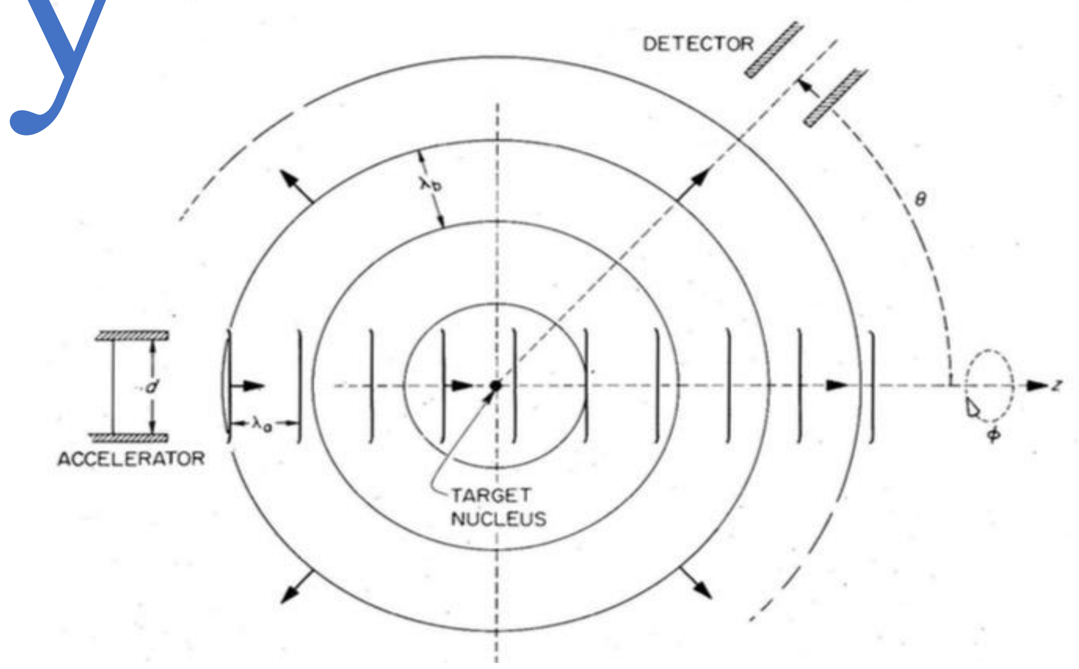
- Run FRESCO code: CDCC Method



$$\sigma_{\alpha} = \frac{1}{(2Ip_i + 1)(2It_i + 1)} \sum |f_{\alpha,\alpha'}(\theta)|^2$$

* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

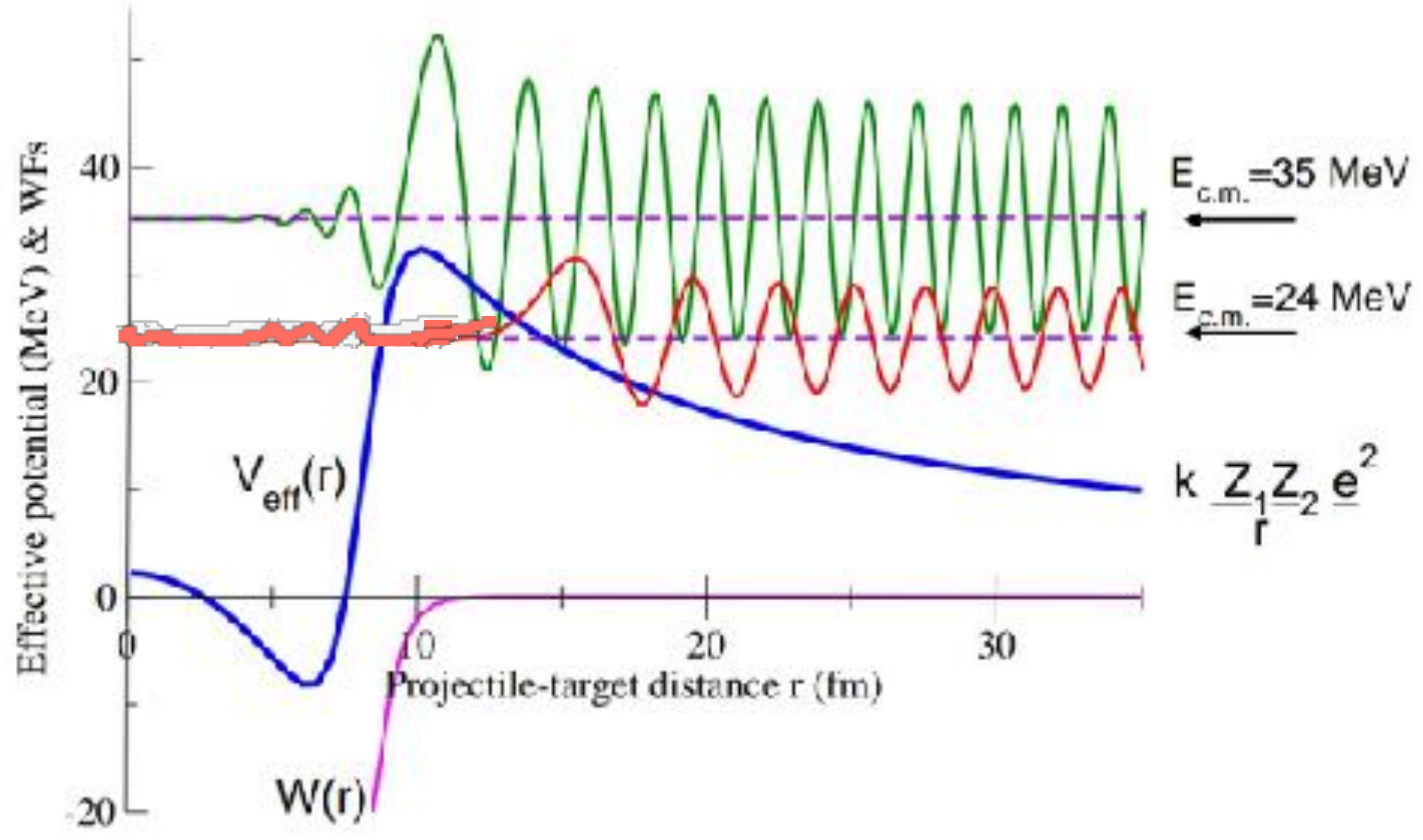
Procedure Summary



- Run FRESCO code: CDCC Method

$$\sigma_\alpha = \frac{1}{(2Ip_i+1)(2It_i+1)} \sum |f_{\alpha,\alpha'}(\theta)|^2$$

- Extract scattering wave function



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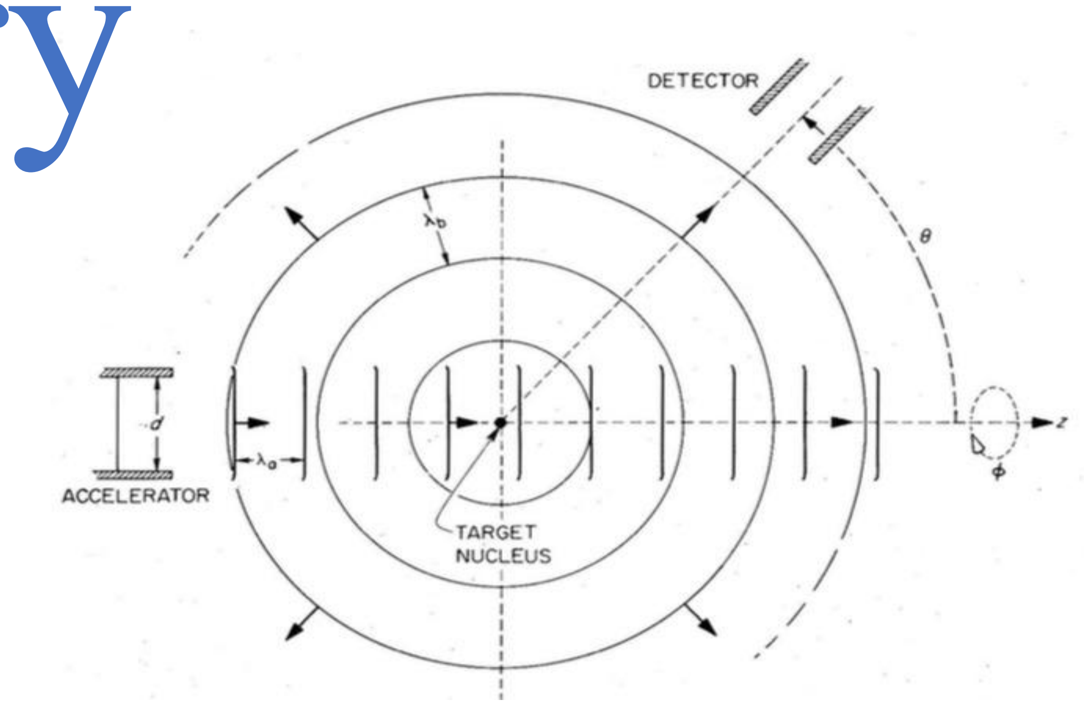

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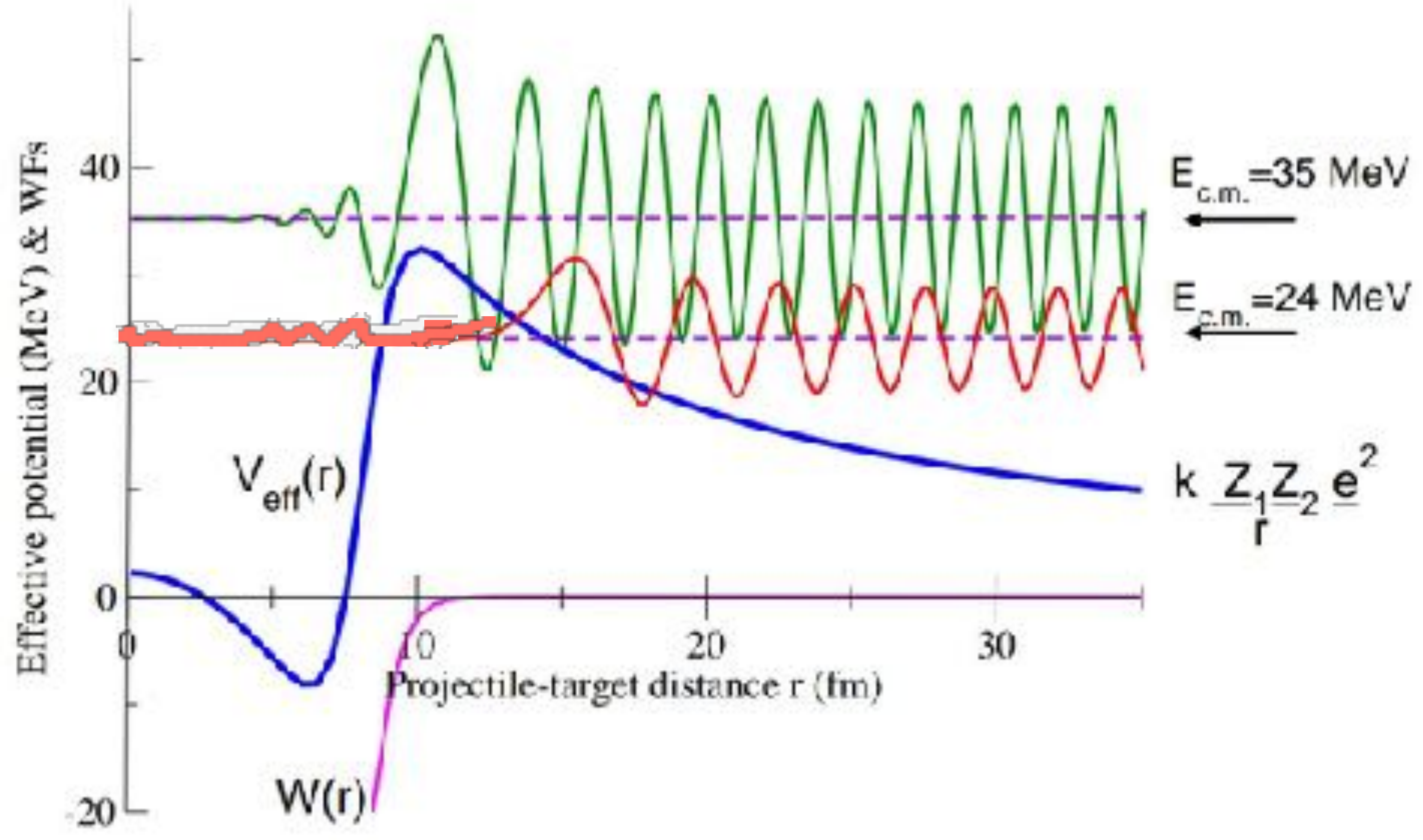
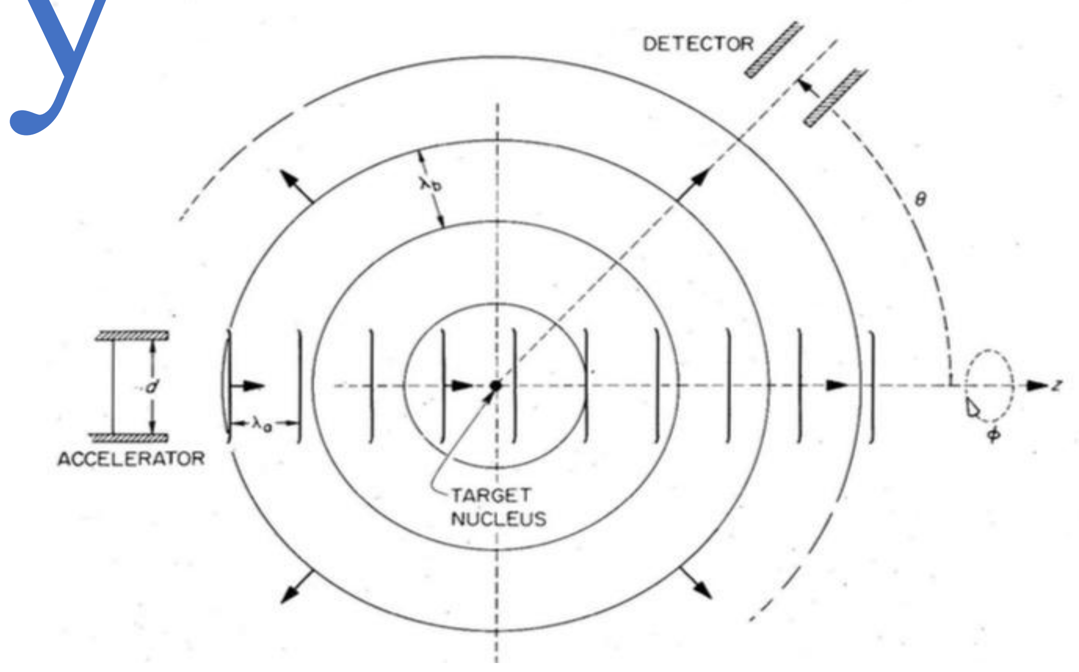


- Estimate probabilities

$$\sigma_F = \frac{\pi}{k^2} \sum_J (2J+1)[P^1(J)+P^2(J)]$$

Procedure Summary



- Run FRESCO code: CDCC Method

$$\sigma_\alpha = \frac{1}{(2Ip_i+1)(2It_i+1)} \sum |f_{\alpha,\alpha'}(\theta)|^2$$

- Extract scattering wave function

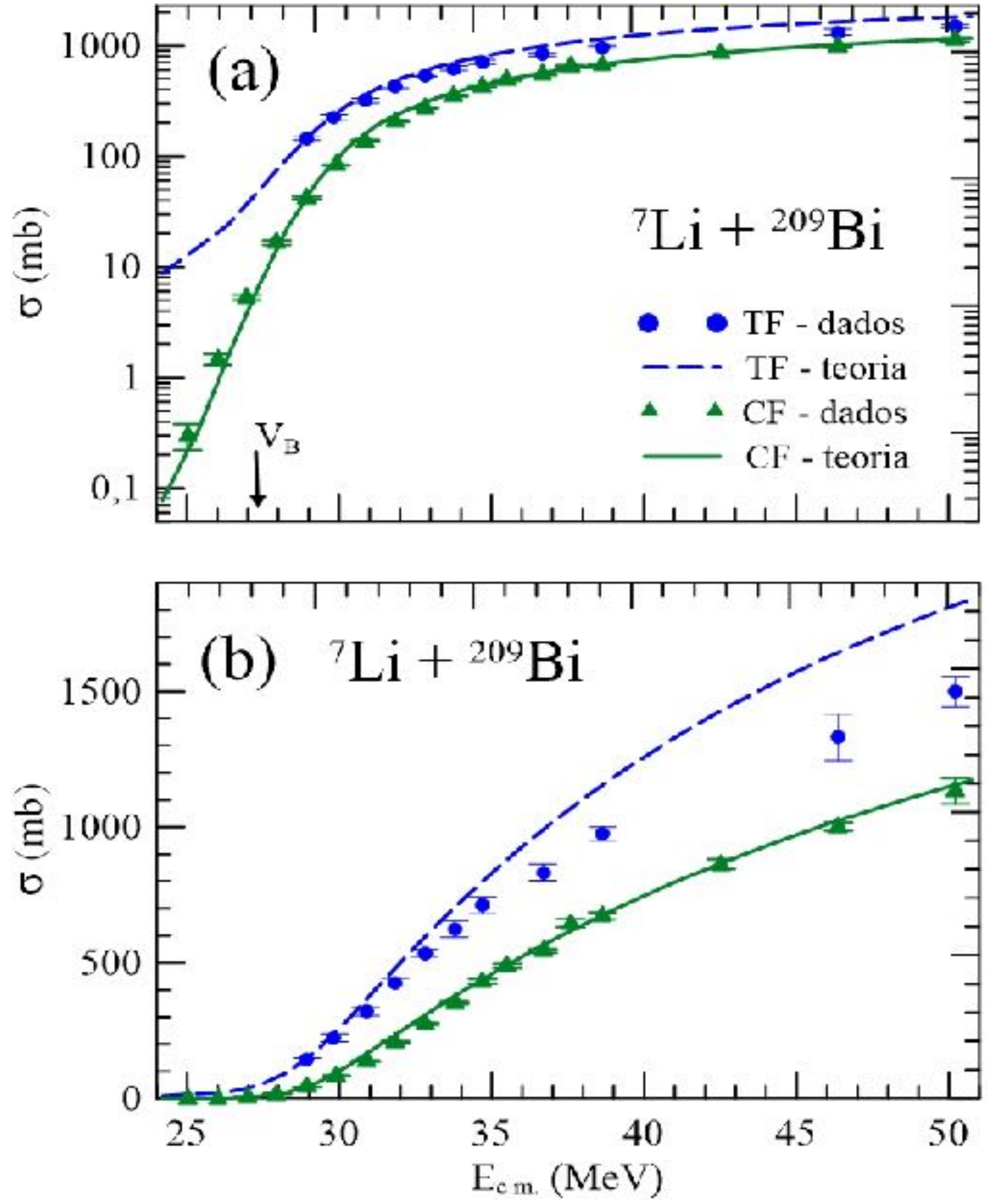
- Estimate probabilities

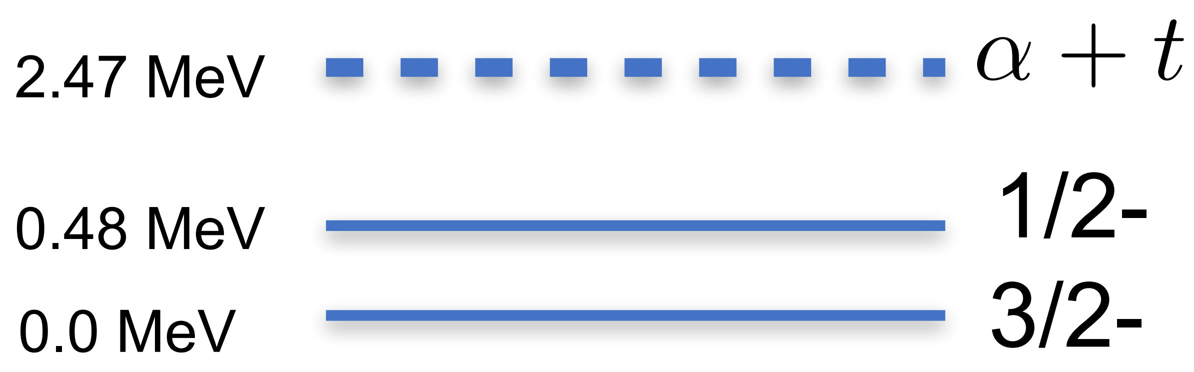
$$\sigma_F = \frac{\pi}{k^2} \sum_J (2J+1)[P^1(J) + P^2(J)]$$

- Separate fusion processes:

$$\boxed{\begin{aligned} \sigma_{\mathrm{ICF}} &= \sigma_{\mathrm{ICF1}} + \sigma_{\mathrm{ICF2}} \\ \sigma_{\mathrm{CF}} &= \sigma_{\mathrm{DCF}} + \sigma_{\mathrm{SCF}} \end{aligned}}$$

^{7}Li + ^{209}Bi fusion – theory vs. experiment*



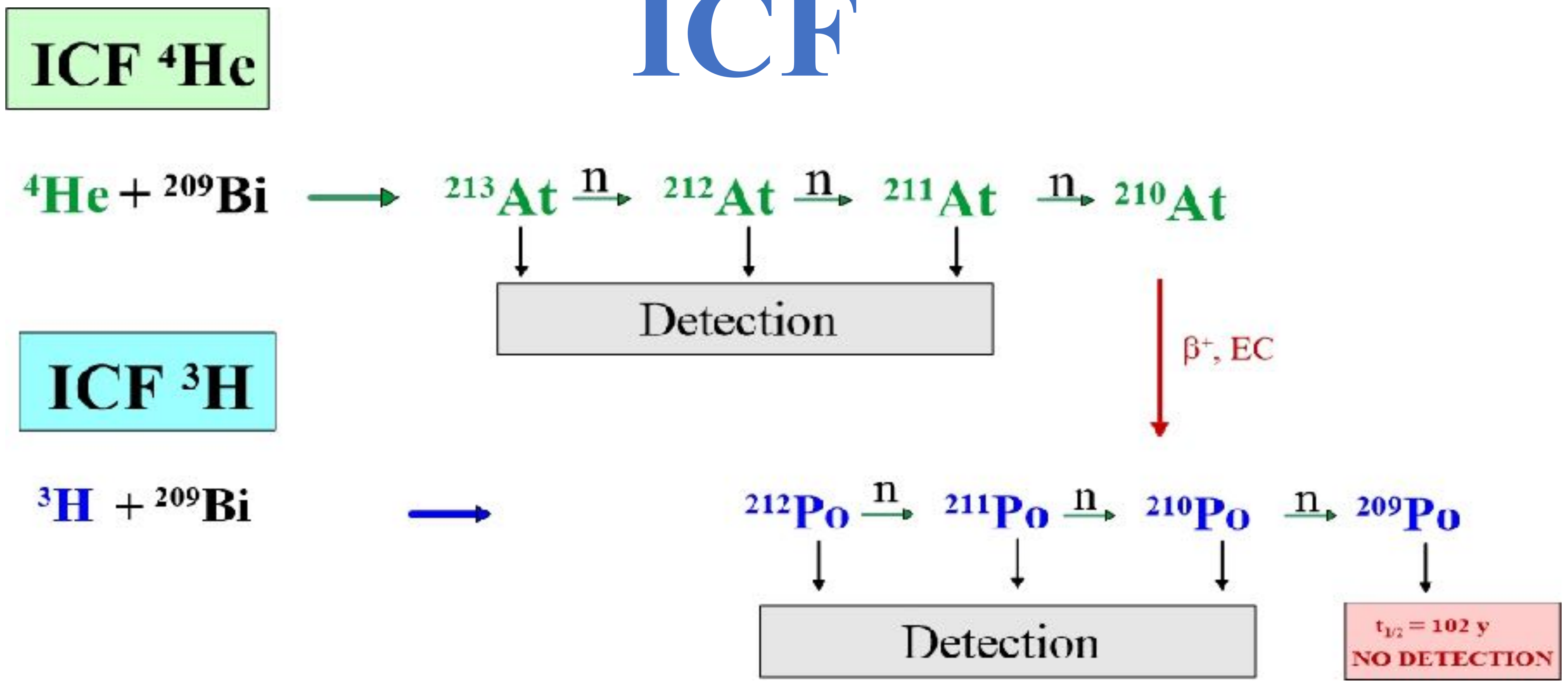


- TF and CF predictions are in excellent agreement with data below V_B, and above up to E ~ 36 MeV

- Predictions for TF above ~ 36 MeV overestimate the experiment. But…

* Dasgupta *et al*., PRC **66**, 041602(R) (2002); PRC **70**, 024606 (2004)

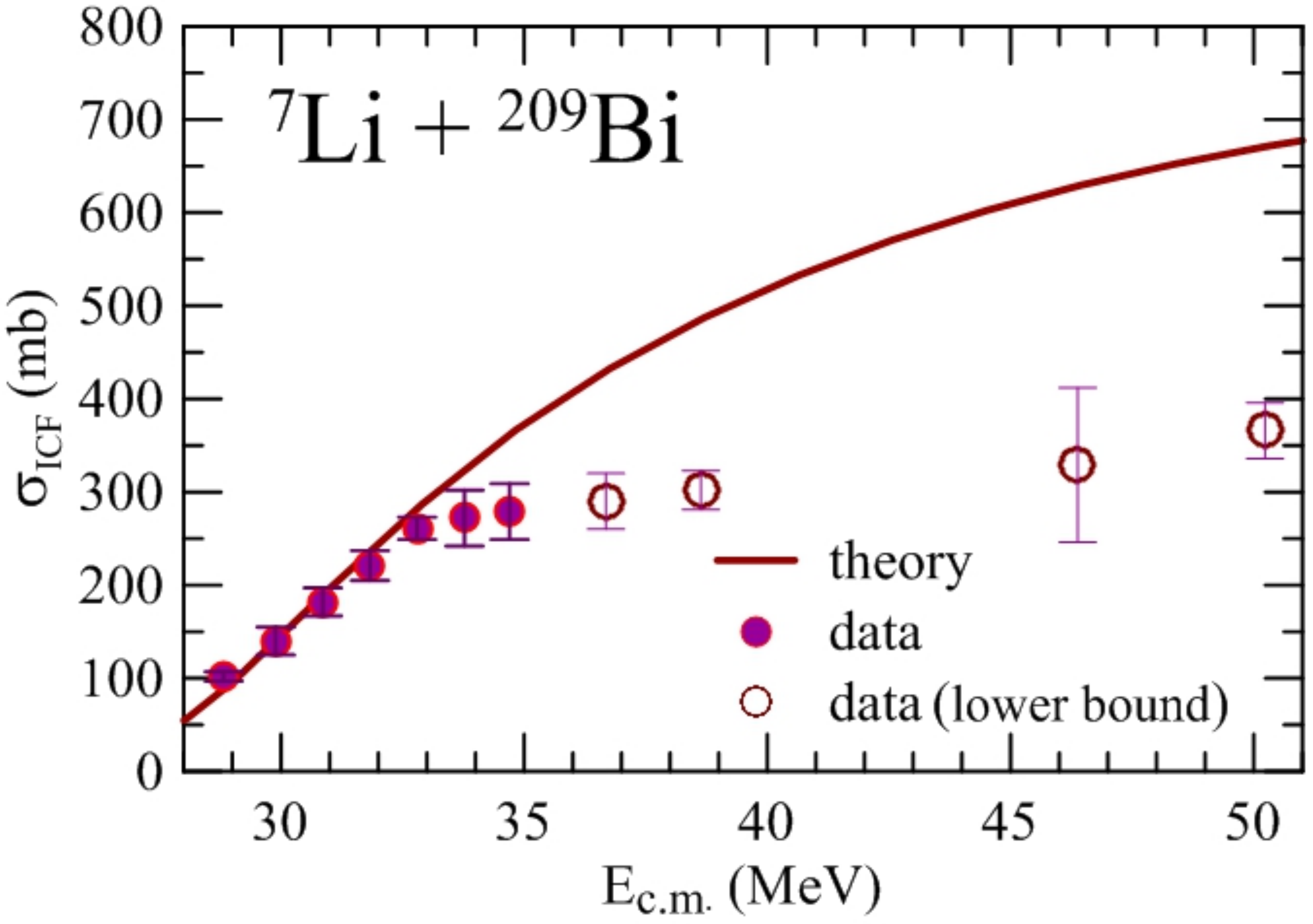
Decay schemes of nuclei produced by ICF



- ^{209}Po is the decay chain of both ICF processes
- Contribution from ^{209}Po is not detected
- Estimates with PACE: ^{209}Po is important above 36 MeV.
- **Above $E_{c.m.}$ ~ 36 MeV, data is only a lower bound**

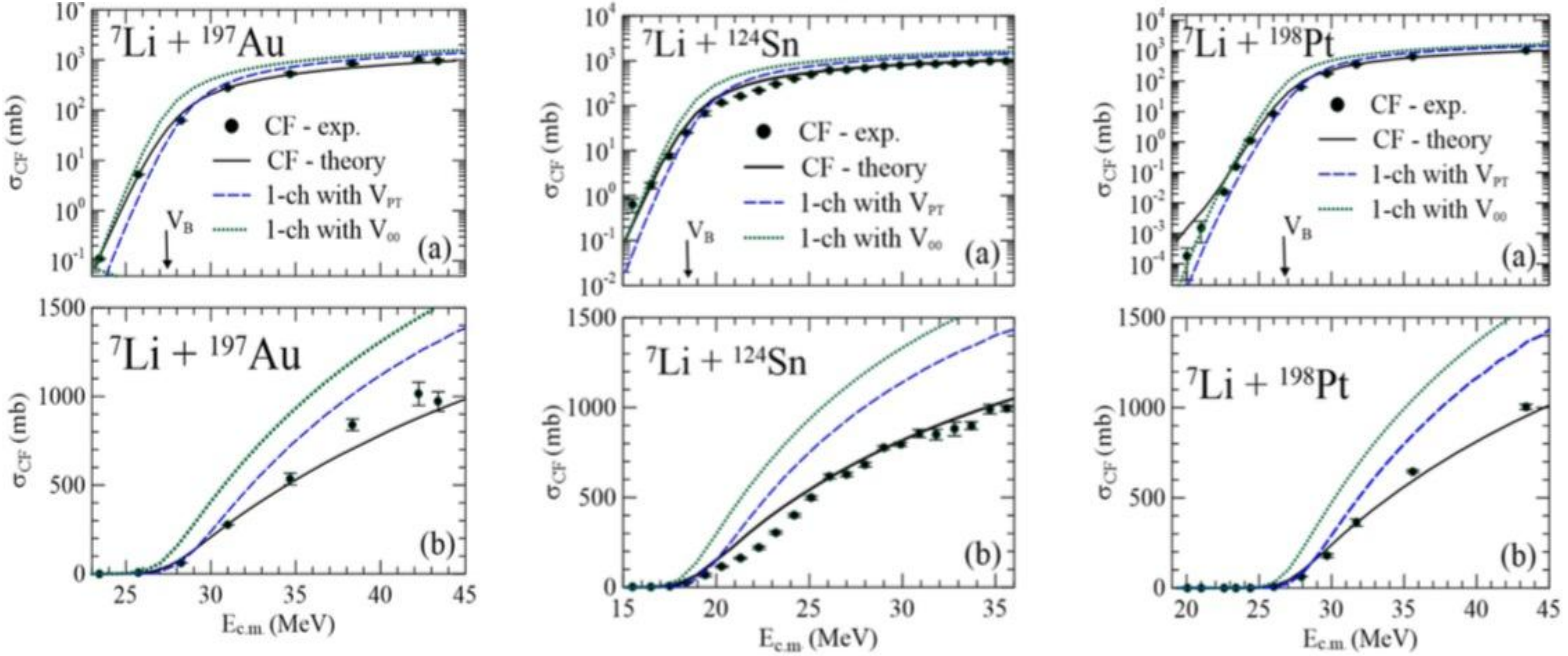
Authors' conclusions of the experiment

ICF



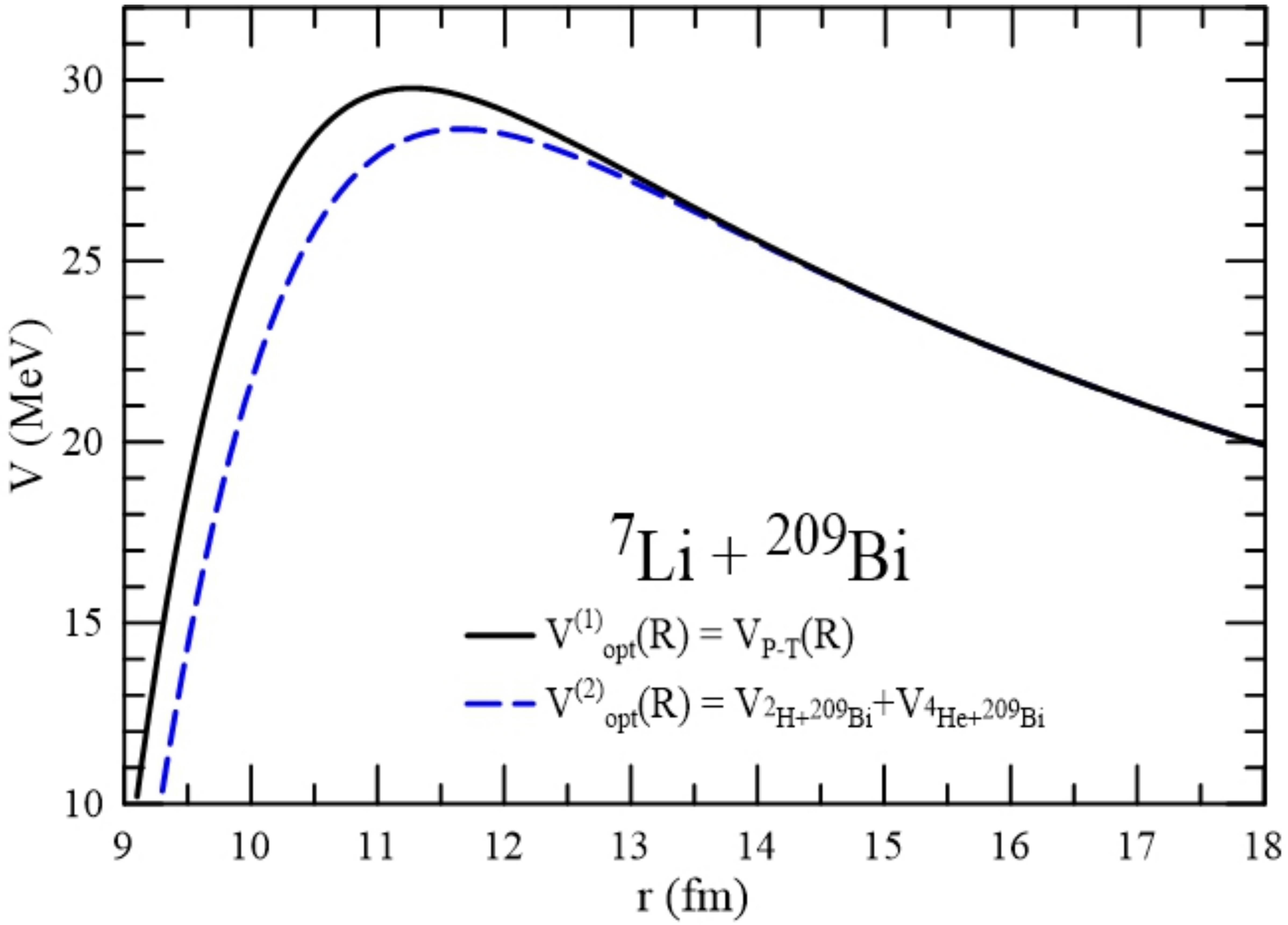
- Excellent agreement where all relevant decay channels are measured
- Consistent with data where they give a lower bound

CF – ^{7}Li + targets



* M. Cortes, J. Rangel, J. L. Ferreira,J. Lubian, LFC (Phys. Rev. C, 00, 004600- 2020)

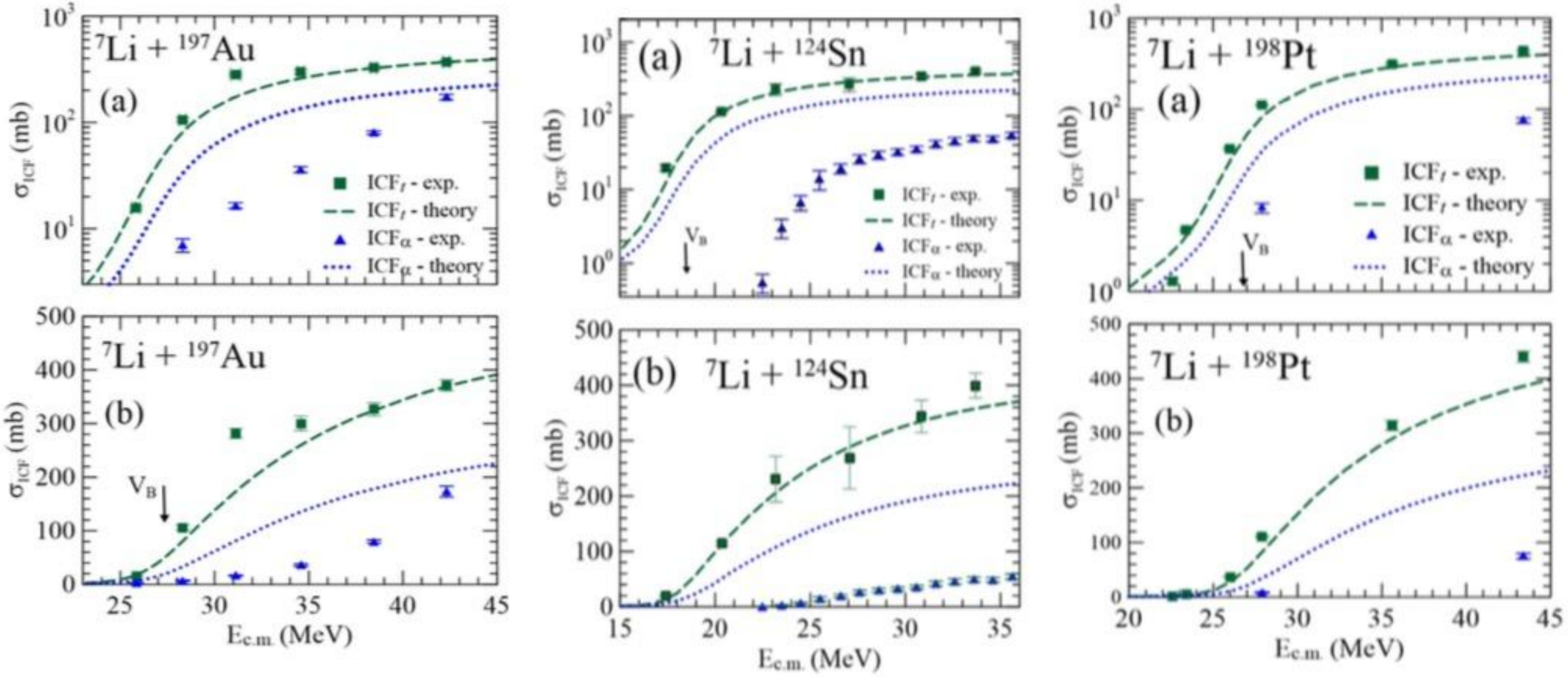
CF – ^{7}Li + targets



projectile's effects
over Coulomb barrier

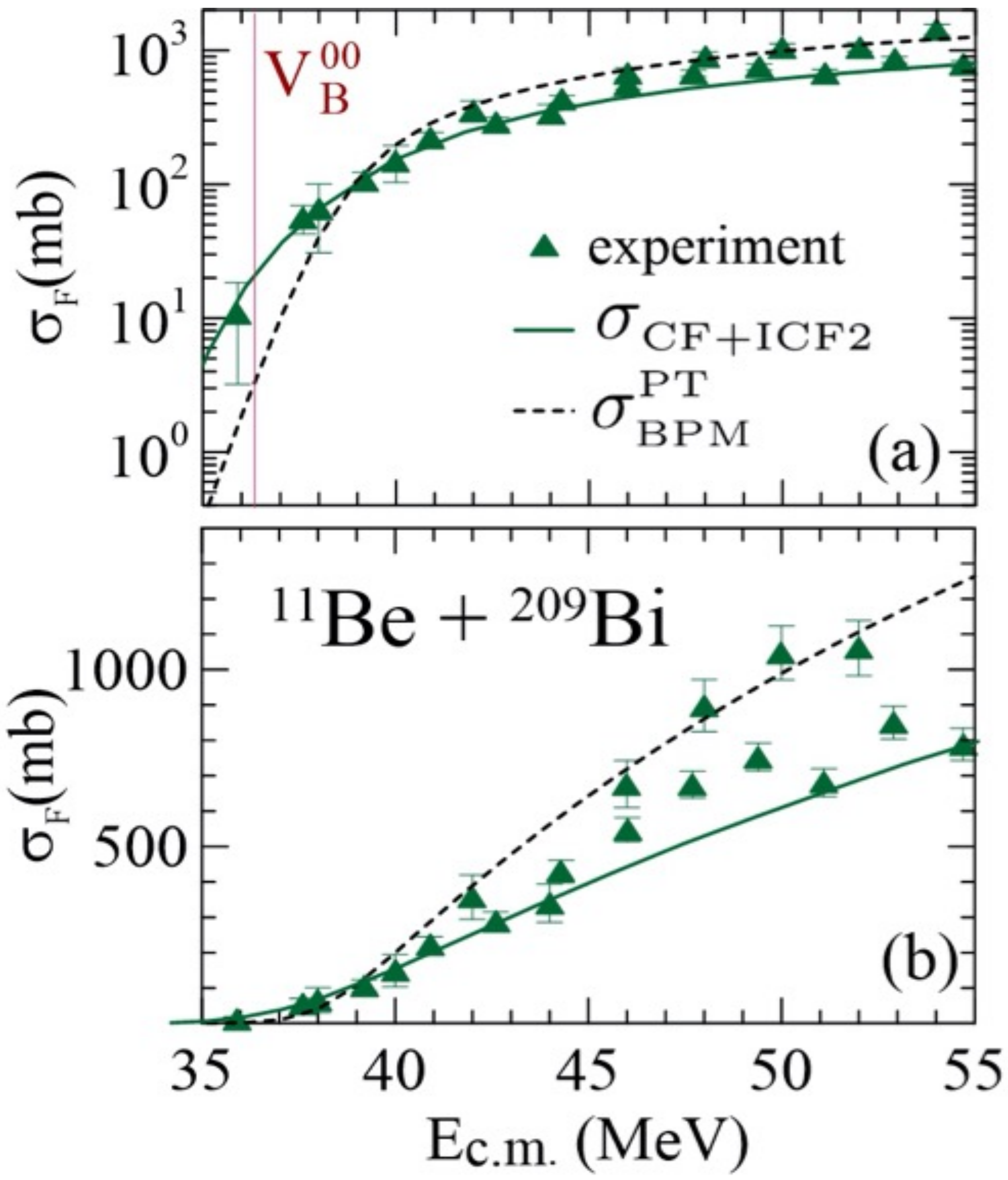
* M. Cortes, J. Rangel, J. L. Ferreira,J. Lubian, LFC (Phys. Rev. C, 00, 004600- 2020)

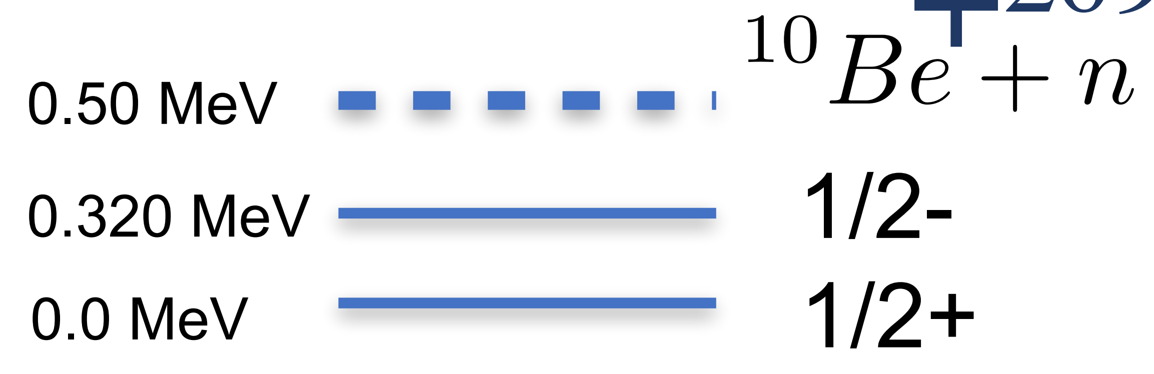
ICF – ^{7}Li + targets



* M. Cortes, J. Rangel, J. L. Ferreira,J. Lubian, LFC (Phys. Rev. C, 00, 004600- 2020)

Complete fusion: neutron valence $^{11}Be + {}^{209}Bi$





- The same evaporation residues

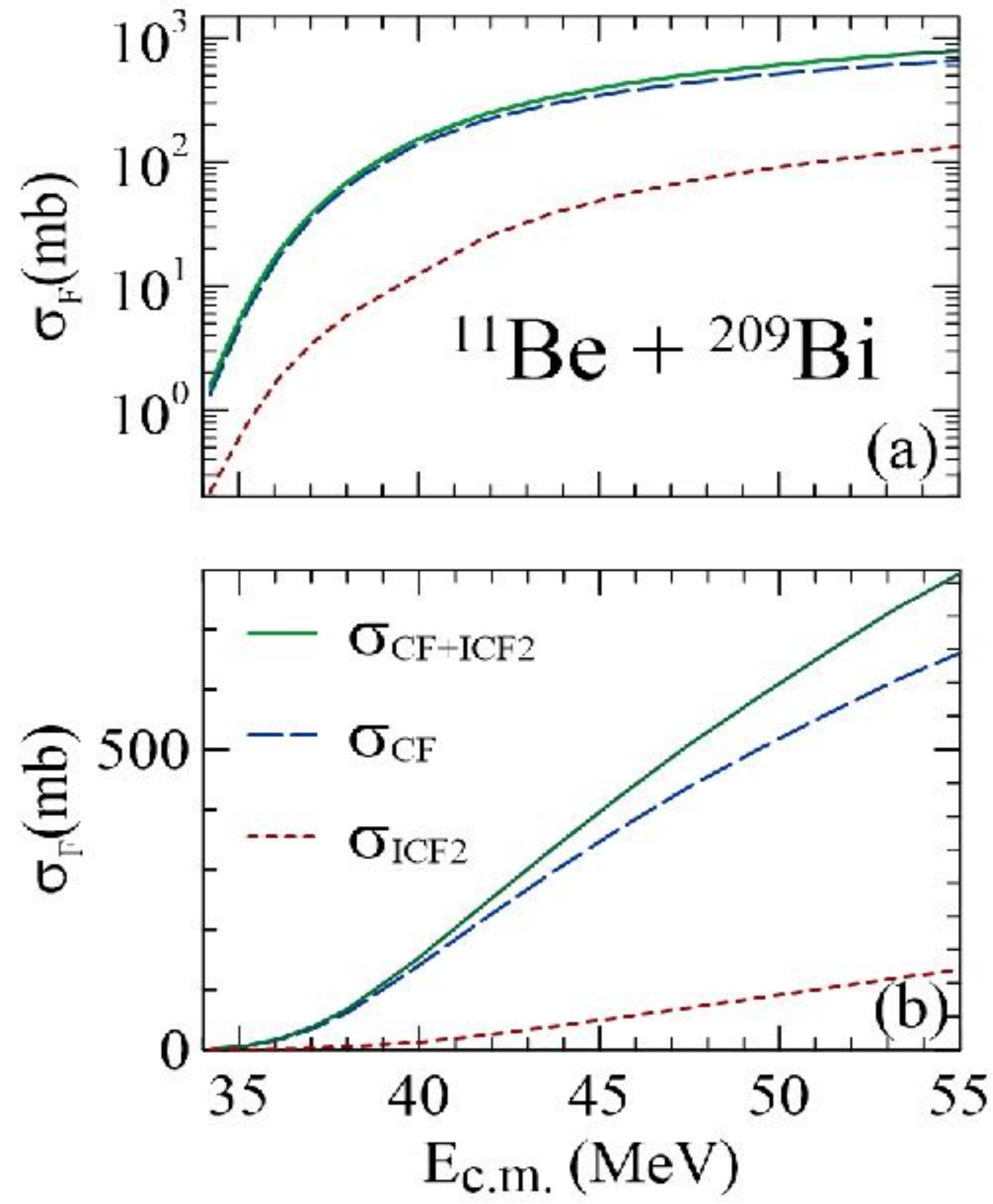
are produced by the fusion of ^{11}Be and ^{10}Be

Data : Nuc. Phys. A 735, 329 (2004)

J. Ferreira , J. Rangel, J. Lubian, L.F. Canto

Phys. Rev. C **107**, 034603 (2023)

Complete fusion: neutron valence ^{11}Be + ^{209}Bi



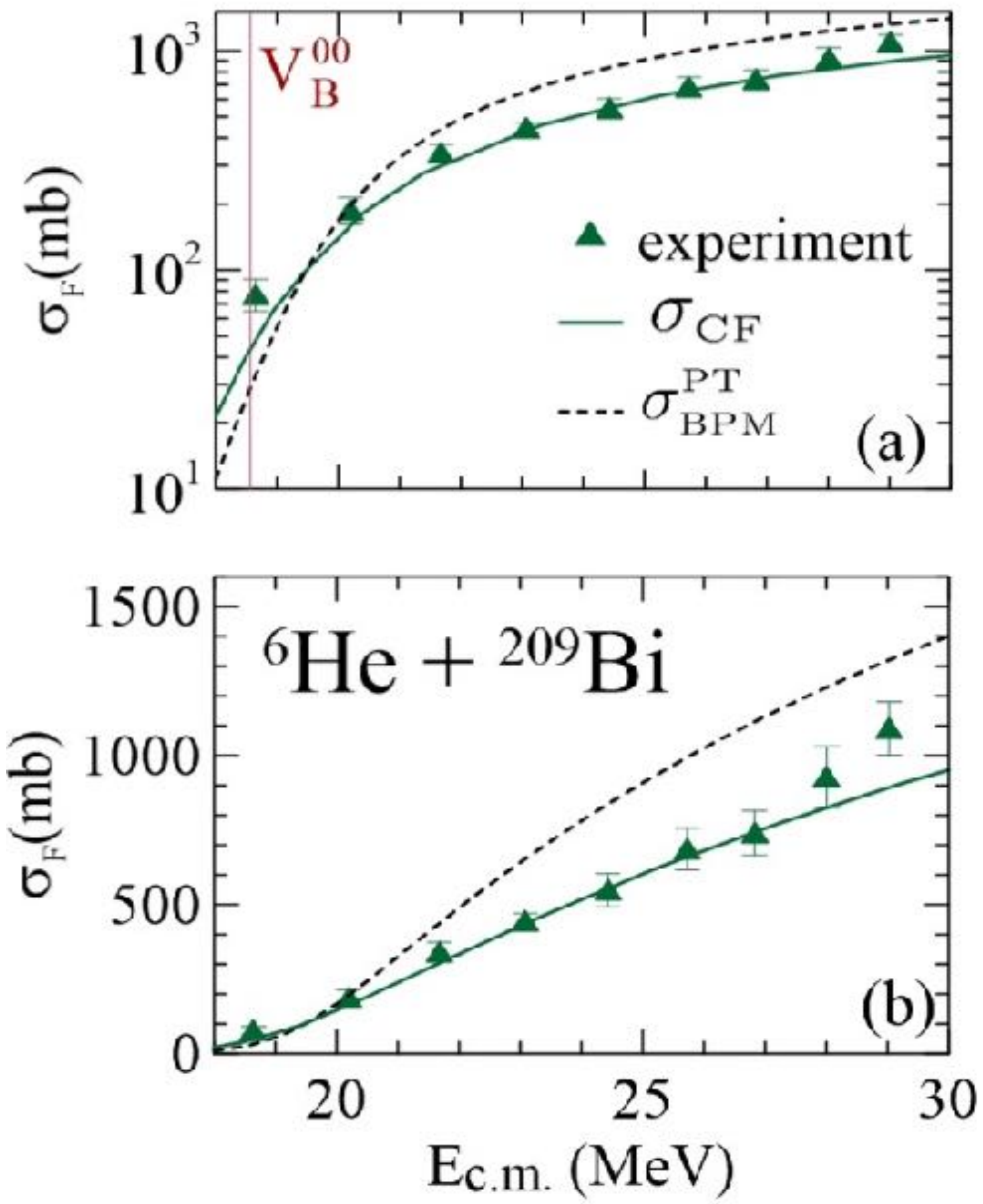
- The contribution of the absorption of ^{10}Be is too small (~20 %)

Data : Nuc. Phys. A 735, 329 (2004)

J. Ferreira , J. Rangel, J. Lubian, L.F. Canto

Phys. Rev. C **107**, 034603 (2023)

Complete fusion: neutron valence $^{6}He + ^{209}Bi$

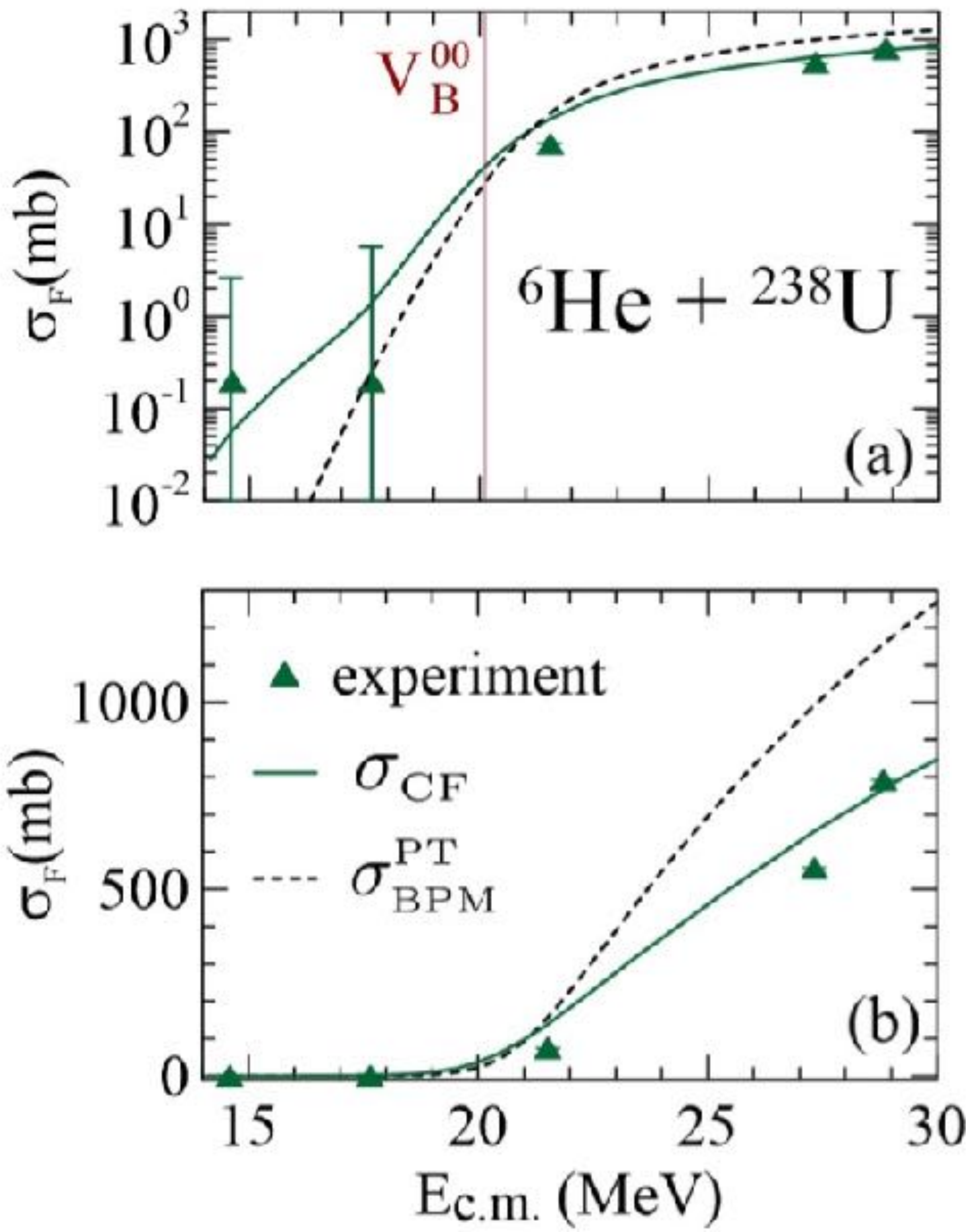


Data: Phys. Rev. C 58, 3442 (1998)

J. Ferreira , J. Rangel, J. Lubian, L.F. Canto
Phys. Rev. C **107**, 034603 (2023)

Complete fusion

$^{6}He + ^{238}U$



Data: Phys. Rev. Lett. 84, 2342 (2004)

J. Ferreira , J. Rangel, J. Lubian, L.F. Canto
Phys. Rev. C **107**, 034603 (2023)

Conclusions

- We have proposed a quantum mechanical method to evaluate CF and ICF in collisions of weakly bound nuclei
- The method was applied to ^{7}Li and neutron-valence projectiles and the results were compared with very good agreement to CF / CF + ICF experimental data
- Calculations of other litght systems are under way.

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APPENDIX: CALCULATION OF THE ABSORPTION PROBABILITY

In this Appendix we evaluate the probabilities $P_{\mathrm{B}}^{(\mathrm{i})}(J)$ and $P_{\mathrm{C}}^{(\mathrm{i})}(J)$ of Sec. II. We consider the collision of a projectile formed by two fragments, one with spin zero and the other with s on a spinless target. In this case, the contribution from the absorption of fragment c_i to the TF cross section is given by the expression,

$$\sigma_{\mathrm{TF}}^{(i)} = \frac{K}{E}\frac{(2\pi)^3}{(2j_0+1)}\sum_{\nu_0}\langle\Psi_{\mathbf{k}j_0\nu_0}^{(+)}|\mathbb{W}^{(i)}|\Psi_{\mathbf{k}j_0\nu_0}^{(+)}\rangle, \tag{A1}$$

where $\Psi_{\mathbf{k}j_o\nu_o}^{(+)}$ is the scattering wave function for a collision with wave-vector $\mathbf{k}$, initiated with intrinsic angular momentum j_0 and z-component ν_0. In this equation, the normalization constant of Eq. (5) was set as $A = (2\pi)^{-3/2}$.

The angular momentum projected scattering wave function is obtained coupling the intrinsic angular momentum ($\mathbf{j}_\alpha$) with the orbital angular momentum of the projectile-target motion ($\mathbf{L}$). It is given by [40]

$$\Psi_{\mathbf{k}j_o\nu_o}^{(+)}(\mathbf{R},\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\sum_{\alpha JLL_0}\frac{\mathcal{U}_{\alpha L,0L_0}^{J}(K_\alpha,R)}{KR}\,e^{i\sigma_{L_0}} \times\sqrt{4\pi\,(2L_0+1)}\langle J\nu_0|L_0 0\,j_0\nu_0\rangle\mathcal{Y}_{\alpha L}^{J\nu_0}(\hat{\mathbf{R}},\mathbf{r}), \tag{A2}$$

where $\mathcal{U}_{\alpha L,0L_0}^{J}(k_\alpha,R)$ are the solutions of the radial equation and $\mathcal{Y}_{\alpha L}^{JM}(\hat{\mathbf{R}},\mathbf{r})$ are the spin-channel wave functions (in the present case, the intrinsic coordinates $\mathbf{r}$ are simply the components of vector $\mathbf{r}$),

$$\mathcal{Y}_{\alpha L}^{J\nu_0}(\hat{\mathbf{R}},\mathbf{r}) = i^L\sum_{\nu M_L}\langle LM_L\,j_\alpha\nu|J\nu_0\rangle Y_{LM_L}(\hat{\mathbf{R}})\phi_{\alpha j_\alpha\nu}(\mathbf{r}). \tag{A3}$$

Above, $\phi_{\alpha j_\alpha\nu}(\mathbf{r})$ is the eigenstate of the intrinsic Hamiltonian of the projectile with energy ε_α, angular momentum j_α, and projection ν (the explicit form of these states will be discussed later).

Next, we carry out the multipole expansion of the imaginary potential,

$$\mathbb{W}^{(i)}(\mathbf{R},\mathbf{r}_i') = 4\pi\sum_{\lambda,\mu}(-)^\mu\,Y_{\lambda\mu}(\hat{\mathbf{R}})Q_{\lambda-\mu}^{(i)}(\mathbf{r}_i'). \tag{A4}$$

where $Q_{\lambda-\mu}^{(i)}(\mathbf{r}_{\mathrm{i}}')$ is the spherical tensor operator,

$$Q_{\lambda-\mu}^{(i)}(\mathbf{r}_i') = \mathcal{W}^{(i)\lambda}(R,r_i')Y_{\lambda-\mu}(\hat{\mathbf{r}}_i'). \tag{A5}$$

Using Eqs. (A2)–(A4) in Eq. (A1), $\sigma_{\mathrm{TF}}^{(i)}$ can be put in the form

$$\sigma_{\mathrm{TF}}^{(i)} = \frac{\pi}{K^2}\sum_J(2J+1)\mathcal{P}^{(i)}(J), \tag{A6}$$

where $P^{(i)}(J)$ is the probability of absorption of fragment c_i by the target in a collision with angular momentum J, given by

$$P^{(i)}(J) = \frac{4K}{E\,\hat{j}_0^2\hat{J}^2}\sum_\lambda\sum_{\alpha LL_0}\sum_{\alpha'L'L_0'}(i)^{L'-L}\hat{L}_0\hat{L}_0'e^{i(\sigma_{L_0'}-\sigma_{L_0})} \times\sum_{\nu_0}\langle L_0 0\,j_0\nu_0|J\nu_0\rangle\langle J\nu_0|L_0' 0\,j_0\nu_0\rangle \times\int dR\,\mathcal{U}_{\alpha L,0L_0}^{J*}(K_\alpha R)\mathcal{U}_{\alpha' L,0L_0'}^{J}(K_{\alpha'}R)X_{\alpha L,\alpha' L'}^{J(\lambda)}(R). \tag{A7}$$

Above, we denote: $\hat{j}_0 = \sqrt{2j_0+1}$ and use an analogous notation for other angular momentum quantum numbers, and

$$X_{\alpha L,\alpha' L'}^{J(\lambda)}(R) = 4\pi\left(\mathcal{Y}_{\alpha L}^{J\nu_0}\middle|\mathbf{Y}_\lambda\cdot\mathbf{Q}_\lambda\middle|\mathcal{Y}_{\alpha' L'}^{J\nu_0}\right). \tag{A8}$$

The above quantity seems to depend on ν_0, but it actually does not. It cannot depend on orientation because $\mathbf{Y}_\lambda \cdot \mathbf{Q}_\lambda$ is a scalar. Thus, the ν_0 dependence is restricted to the Clebsh-Gordan coefficients. Then, carrying out the sum over ν_0, we get [64]

$$\sum_{\nu_0} \langle L_0 0\, j_0 \nu_0 | J \nu_0 \rangle \langle J \nu_0 | L_0' 0\, j_0 \nu_0 \rangle$$
$$= \hat{J}^2 \sum_{\nu_0} \begin{pmatrix} L_0 & j_0 & J \\ 0 & \nu_0 & -\nu_0 \end{pmatrix} \begin{pmatrix} L_0' & j_0 & J \\ 0 & \nu_0 & -\nu_0 \end{pmatrix}$$
$$= \frac{\hat{J}^2}{\hat{L}_0^2} \delta_{L_0 L_0'}. \tag{A9}$$

Using this result, Eq. (A7) takes the form

$$P^{(i)}(J) = \frac{4K}{E\,\hat{j}_0^2} \sum_\lambda \sum_{\alpha L \alpha' L' L_0} (i)^{L'-L} \int dR\, X^{(i)\lambda}_{\alpha L, \alpha' L'}(R)$$
$$\times\, \mathcal{U}^{J*}_{\alpha L, 0 L_0}(K_\alpha R) \mathcal{U}^{J}_{\alpha' L, 0 L_0'}(K_{\alpha'} R). \tag{A10}$$

1. Evaluation of $X^{(i)\lambda}_{\alpha L, \alpha' L'}(R)$

Using the notation of Ref. [64] for the wave functions: $\mathcal{Y}^{J\nu_0}_{\alpha L} \to |\alpha(L j_\alpha) J\rangle$, Eq. (A8) reads

$$X^{(i)\lambda}_{\alpha L, \alpha' L'}(R) = 4\pi \langle \alpha(L j_\alpha) J || \mathbf{Y}_\lambda(\hat{\mathbf{R}}) \cdot \mathbf{Q}_\lambda(R, \mathbf{r}) || \alpha'(L' j_{\alpha'}) J \rangle, \tag{A11}$$

or (Eq. (5.13) of Ref. [64])

$$X^{(i)\lambda}_{\alpha L, \alpha' L'}(R) = 4\pi\, (-)^{J-L-j_{\alpha'}} \hat{L} \hat{j}_\alpha W(L L' j_\alpha j_{\alpha'}; \lambda J)$$
$$\times\, \langle L || \mathbf{Y}_\lambda(\hat{\mathbf{R}}) || L' \rangle \langle \alpha j_\alpha || \mathbf{Q}_\lambda(R, \mathbf{r}) || \alpha' j_{\alpha'} \rangle. \tag{A12}$$

The first reduced matrix element is (Eq. (4.17) of Ref. [64])

$$\langle L || \mathbf{Y}_\lambda(\hat{\mathbf{R}}) || L' \rangle = (-)^{\lambda - L'} \frac{\hat{\lambda} \hat{L}'}{\sqrt{4\pi}} \begin{pmatrix} \lambda & L' & L \\ 0 & 0 & 0 \end{pmatrix}. \tag{A13}$$

Using this result, Eq. (A12) can be put in the form

$$X^{(i)\lambda}_{\alpha L, \alpha' L'}(R) = \mathcal{A}^{J\lambda}_{\alpha L; \alpha' L'} F^{(i)\lambda}_{\alpha\alpha'}(R), \tag{A14}$$

and Eq. (A10) becomes

$$P^{(i)}(J) = \frac{4K}{E\,\hat{j}_0^2} \sum_\lambda \sum_{\alpha L \alpha' L' L_0} \mathcal{A}^{J\lambda}_{\alpha L; \alpha' L'} (i)^{L'-L}$$
$$\times \int dR\, \mathcal{U}^{J*}_{\alpha L, 0 L_0}(K_\alpha R) \mathcal{U}^{J}_{\alpha' L, 0 L_0'}(K_{\alpha'} R) F^{(i)\lambda}_{\alpha\alpha'}(R), \tag{A15}$$

with

$$\mathcal{A}^{J\lambda}_{\alpha L; \alpha' L'} = \sqrt{4\pi}\, (-)^{J+\lambda-L-L'-j_{\alpha'}} \hat{L} \hat{L}' \hat{\lambda}\, \hat{j}_\alpha$$
$$\times\, W(L L' j_\alpha j_{\alpha'}; \lambda J) \begin{pmatrix} \lambda & L' & L \\ 0 & 0 & 0 \end{pmatrix}, \tag{A16}$$

and

$$F^{(i)\lambda}_{\alpha\alpha'}(R) = \langle \alpha j_\alpha || \mathbf{Q}_\lambda(R, \mathbf{r}) || \alpha' j_{\alpha'} \rangle. \tag{A17}$$

2. Calculation of $F^{(i)\lambda}_{\alpha\alpha'}(R)$ for a two-fragment projectile

Now we consider the situation where the projectile is formed by two fragments, one with spin zero and the other with spin s. In this case the intrinsic coordinates are $\mathbf{r} \equiv \{r, \hat{\mathbf{r}}\}$. The angular momentum-projected intrinsic states are then given by

$$\phi_{\alpha j_\alpha \nu}(\mathbf{r}) = \frac{u_{\alpha l_\alpha j_\alpha}(r)}{r} \mathcal{J}_{l_\alpha j_\alpha \nu}(\hat{\mathbf{r}}), \tag{A18}$$

with

$$\mathcal{J}_{l_\alpha j_\alpha \nu}(\hat{\mathbf{r}}) = \sum_{m_l, m_s} \langle l_\alpha m_l s m_s | j_\alpha \nu \rangle\, Y_{l_\alpha m_l}(\hat{\mathbf{r}}) | s m_s \rangle, \tag{A19}$$

where $|s m_s\rangle$ are states in the spin space and $\langle l_\alpha m_l s m_s | j_\alpha \nu \rangle$ are Clebsh-Gordan coefficients. In Eq. (A18), $u_{\alpha l_\alpha j_\alpha}(r)$ stands for the radial wave functions of the projectile. They are either bound states or bins generated by scattering states of the fragments $u_{\varepsilon l_\alpha j_\alpha}(r)$, where ε is the collision energy.

The tensor of Eq. (A5) then becomes

$$Q^{(i)}_{\lambda-\mu}(\mathbf{r}'_i) \rightarrow Q^{(i)}_{\lambda-\mu}(\mathbf{r}) = \mathcal{W}^{(i)\lambda}(R,r)\,Y_{\lambda-\mu}(\hat{\mathbf{r}}), \tag{A20}$$

and scalar products in the intrinsic space are integrals over $r^2 dr\, d\Omega_{\hat{\mathbf{r}}}$.

Then, adopting the notation of Ref. [64], Eq. (A17) becomes

$$F^{(i)\lambda}_{\alpha\alpha'}(R) = \mathcal{F}^{(i)\lambda}_{\alpha\alpha'}(R)\langle j_\alpha||\mathbf{Y}_\lambda(\hat{\mathbf{r}})||j_{\alpha'}\rangle, \tag{A21}$$

where $\mathcal{F}^{(i)\lambda}_{\alpha\alpha'}(R)$ is the form factor,

$$\mathcal{F}^{(i)\lambda}_{\alpha\alpha'}(R) = \int dr\, u^*_{\alpha l_\alpha j_\alpha}(r)\,\mathcal{W}^{(i)\lambda}(R,r)\,u_{\alpha' l_{\alpha'} j_{\alpha'}}(r). \tag{A22}$$

The reduced matrix element of Eq. (A21) can be evaluated with help of Eq. (5.10) of Ref. [64], and one gets

$$\langle j_\alpha||\mathbf{Y}_\lambda(\hat{\mathbf{r}})||j_{\alpha'}\rangle = (-)^{j_\alpha-\lambda-s+l_{\alpha'}}\hat{l}_\alpha \hat{j}_{\alpha'}\, W(l_\alpha l_{\alpha'} j_\alpha j_{\alpha'};\lambda s) \times \langle l_\alpha||\mathbf{Y}_\lambda(\hat{\mathbf{r}})||l_{\alpha'}\rangle.$$

Finally, evaluating $\langle l_\alpha||\mathbf{Y}_\lambda(\hat{\mathbf{r}})||l_{\alpha'}\rangle$ as in Eq. (A13), the above equation becomes

$$\langle j_\alpha||\mathbf{Y}_\lambda(\hat{\mathbf{r}})||j_{\alpha'}\rangle = (-)^{j_\alpha-s}\frac{\hat{l}_{\alpha'}\hat{j}_{\alpha'}\hat{l}_\alpha\hat{\lambda}}{\sqrt{4\pi}}\begin{pmatrix}\lambda & l_{\alpha'} & l_\alpha\\ 0 & 0 & 0\end{pmatrix} \times W(l_\alpha l_{\alpha'} j_\alpha j_{\alpha'};\lambda s). \tag{A23}$$

Using the above equation in Eq. (A21) and inserting the result into Eq. (A15), the fusion probability becomes

$$\mathcal{P}^{(i)}(J) = \frac{4K}{E\hat{j}_0^2}\sum_{L_0}\sum_{\lambda}\sum_{\alpha L\alpha' L'}\mathcal{B}^\lambda_{\alpha L,\alpha' L'}(J)\,\mathcal{M}^{(i)\lambda}_{\alpha L,\alpha' L'}(L_0,J). \tag{A24}$$

Above, $\mathcal{B}^\lambda_{\alpha L,\alpha' L'}(J)$ is the geometric factor,

$$\mathcal{B}^\lambda_{\alpha L,\alpha' L'}(J) = \mathcal{A}^{J\lambda}_{\alpha L;\alpha' L'}\langle j_\alpha||\mathbf{Y}_\lambda(\hat{\mathbf{r}})||j_{\alpha'}\rangle, \tag{A25}$$

or explicitly,

$$\mathcal{B}^\lambda_{\alpha L,\alpha' L'}(J) = (-)^{\mathcal{N}}\hat{\lambda}^2\hat{L}\hat{L}'\hat{l}_\alpha\hat{l}_{\alpha'}\hat{j}_\alpha\hat{j}_{\alpha'} \times W(LL' j_\alpha j_{\alpha'};\lambda J)\, W(l_\alpha l_{\alpha'} j_\alpha j_{\alpha'};\lambda s) \times \begin{pmatrix}\lambda & L' & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\lambda & l_{\alpha'} & l_\alpha\\ 0 & 0 & 0\end{pmatrix}, \tag{A26}$$

with

$$\mathcal{N} = J - s + j_\alpha - j_{\alpha'} - L - L' + \lambda, \tag{A27}$$

and $\mathcal{M}^{(i)\lambda}_{\alpha L,\alpha' L'}(L_0,J)$ is the radial integral,

$$\mathcal{M}^{(i)\lambda}_{\alpha L,\alpha' L'}(L_0,J) = i^{L'-L}\int dR\, \mathcal{F}^{(i)\lambda}_{\alpha\alpha'}(R) \times \mathcal{U}^{J*}_{\alpha L,0L_0}(K_\alpha,R)\,\mathcal{U}^{J}_{\alpha' L',0L_0}(K'_\alpha,R). \tag{A28}$$

Although the radial integrals are complex functions, the probabilities of Eq. (A24) are real. Using symmetry properties of the $3J$ and Racah coefficients (see, e.g., Ref. [64]) one can easily show that

$$\mathcal{B}^\lambda_{\alpha L,\alpha' L'}(J) = \mathcal{B}^\lambda_{\alpha' L',\alpha L}(J). \tag{A29}$$

On the other hand, the radial integrals have the property,

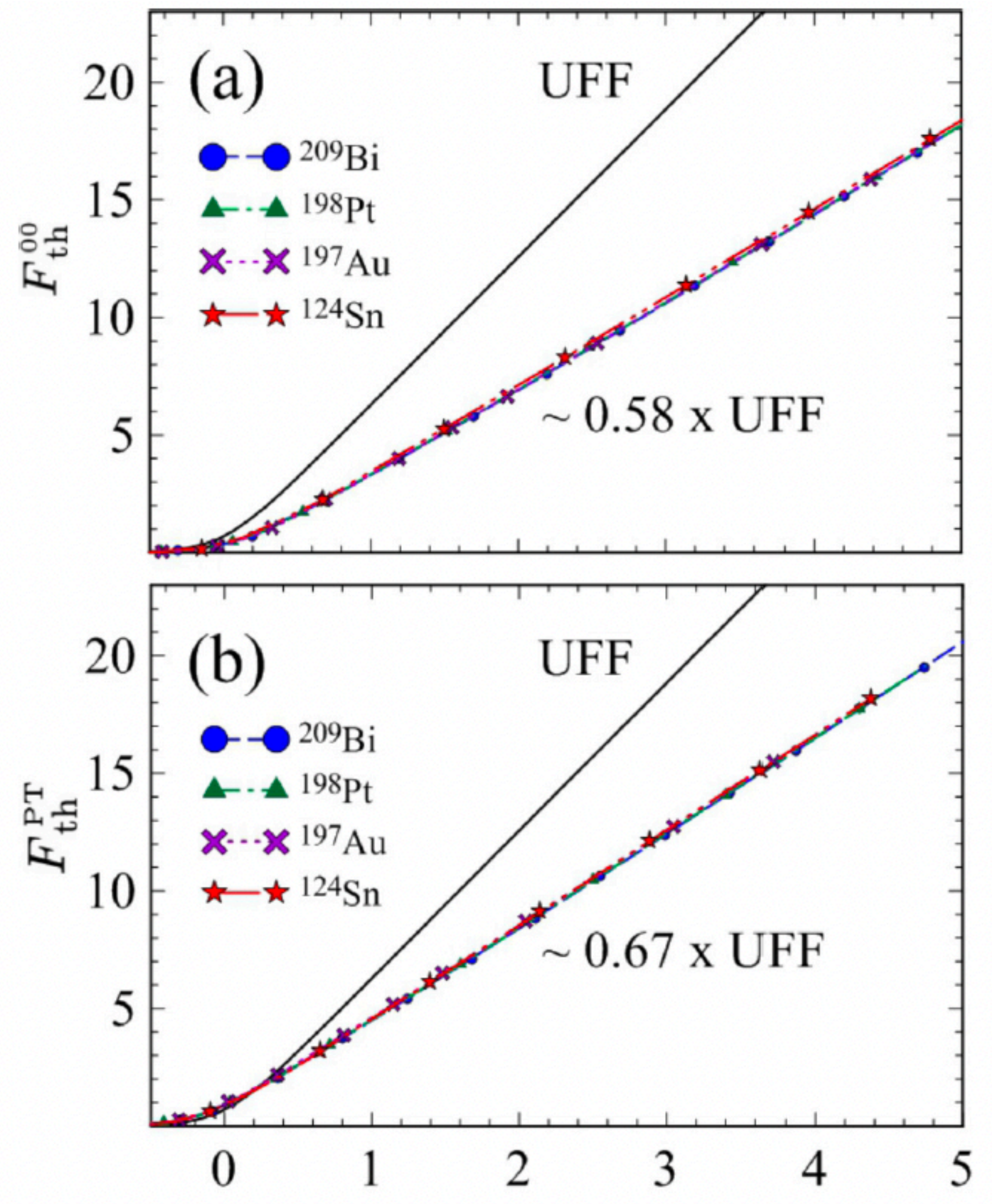
$$\mathcal{M}^{(i)\lambda}_{\alpha L,\alpha' L'}(L_0,J) = \mathcal{M}^{(i)\lambda *}_{\alpha' L',\alpha L}(L_0,J). \tag{A30}$$

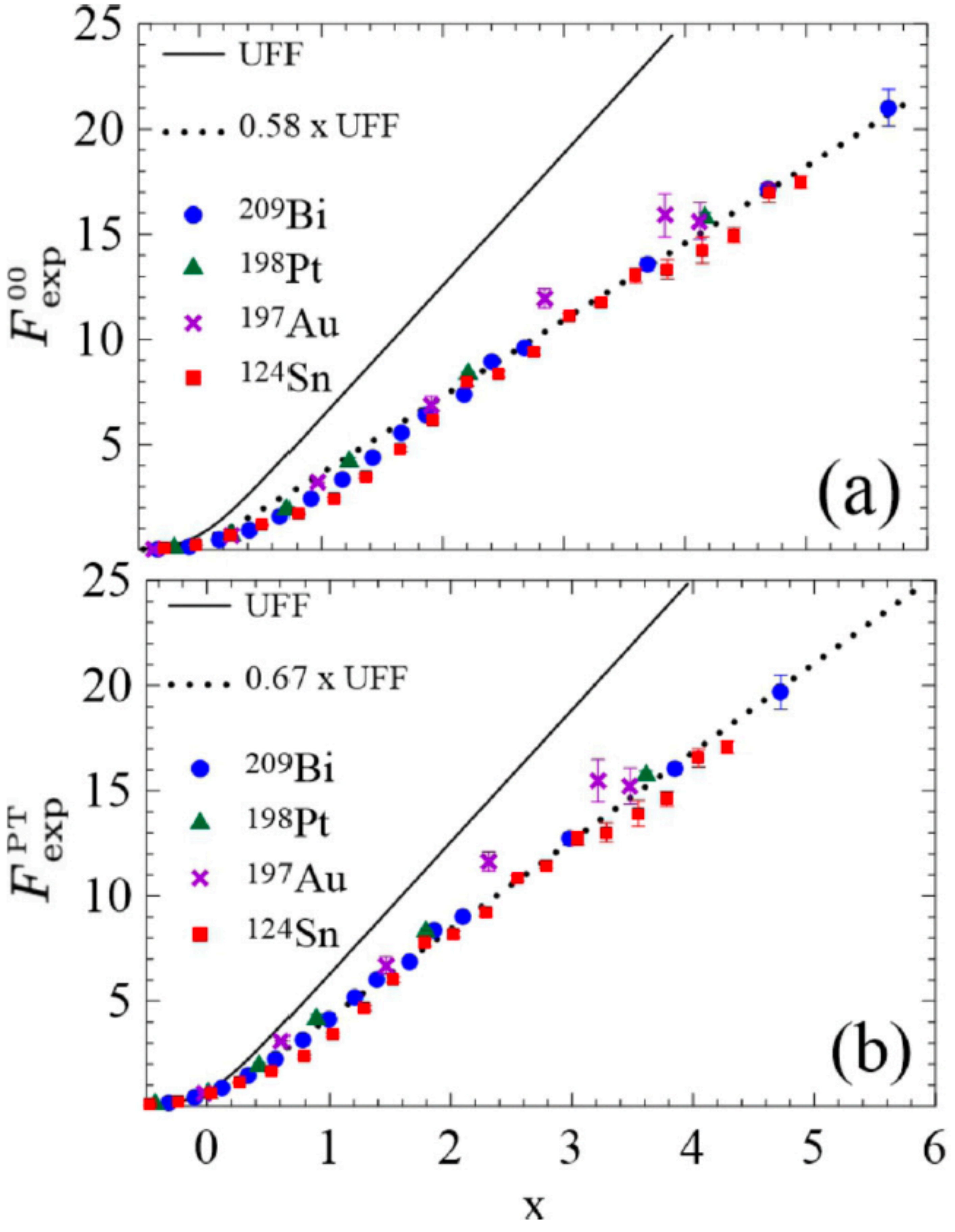
Since $\alpha, L, \alpha' L'$ are dummy indices running over the same ranges, the fusion probability of Eq. (A24) does not change if one interchanges $\{\alpha, L\} \rightleftarrows \{\alpha', L'\}$. Then, using Eqs. (A29) and (A30), one obtains the explicitly real expression for the absorption probabilities,

$$\mathcal{P}^{(i)}(J) = \frac{4K}{E\hat{j}_0^2}\sum_{\lambda L_0}\sum_{\alpha L\alpha' L'}\mathcal{B}^\lambda_{\alpha L,\alpha' L'}(J)\,\mathrm{Re}\big\{\mathcal{M}^{(i)\lambda}_{\alpha L,\alpha' L'}(L_0,J)\big\}. \tag{A31}$$

Finally, the probabilities $\mathcal{P}^{(i)}_{\mathrm{B}}(J)$ and $\mathcal{P}^{(i)}_{\mathrm{C}}(J)$ are given by the above expression, restricting the sum over channels to $\{\alpha, \alpha'\} \in B$ and to $\{\alpha, \alpha'\} \in C$, respectively.

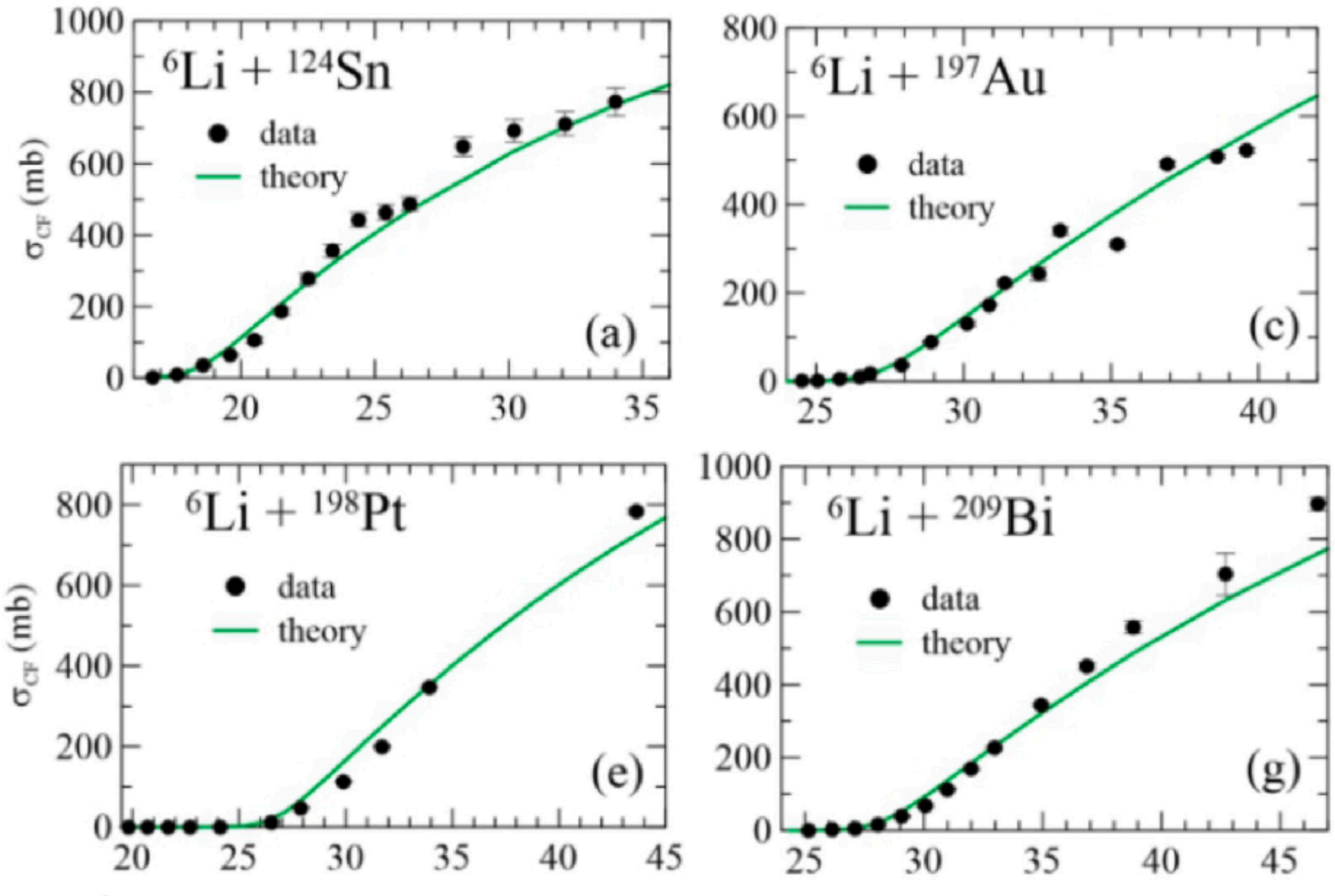
CF reductions





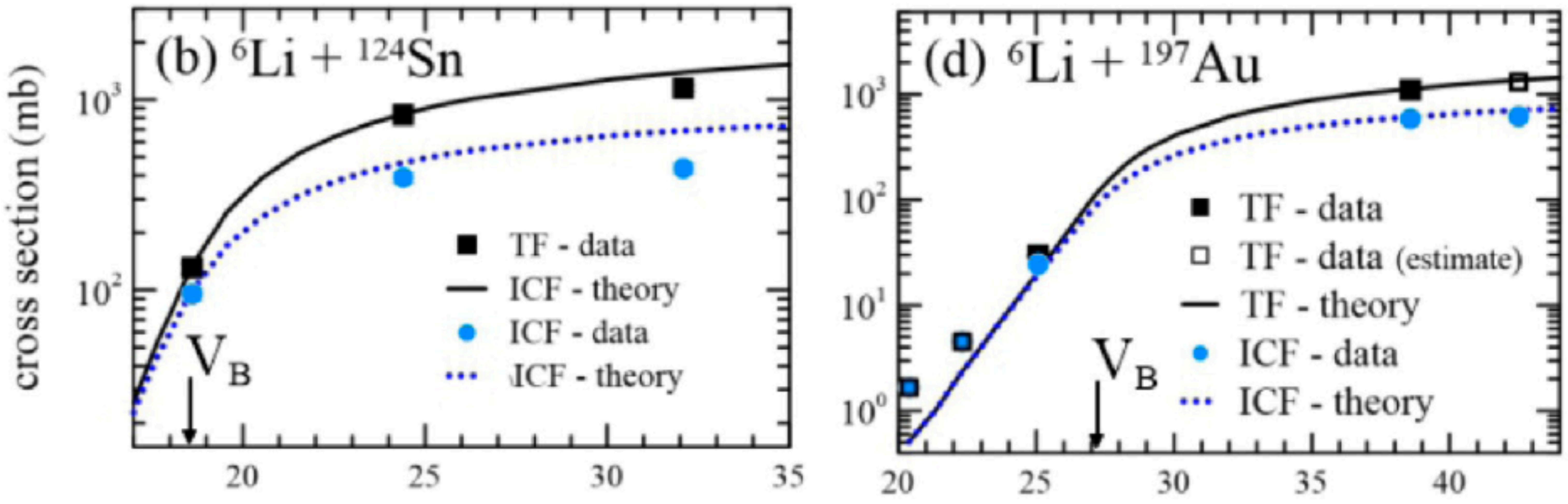
^{6}Li

PRC **105**, 054601 (2022)



^{6}Li

PRC **105**, 054601 (2022)



Indirect estimations using "Fusion" code

$^{6}Li + ^{90}Zr$

$E_{c.m.}$ (MeV)	σ_α^{inc} (mb)	σ_α^{CF} (mb)	σ^{EBU} (mb)	σ_α^{1n} (mb)	σ_α^{1p} (mb)
13.9	32 ± 19	0.01	8.5	2.0	0.3
15.8	122 ± 10	1.3	12.9	14.1	0.7
17.7	274 ± 15	16.2	17.9	27.5	1.1
19.6	360 ± 20	37.6	24.4	30.9	1.3
23.3	460 ± 30	94.2	35	29.5	1.5
28.0	540 ± 86	172	45.3	26.9	1.5

Inclusive Alpha data: Phys. Rev. C, 81, 054601, (2010)

σ_α^{EBU} = elastic breakup cross section ($^{6}Li + ^{90}Zr \rightarrow ^{2}H + ^{4}He + ^{90}Zr$),

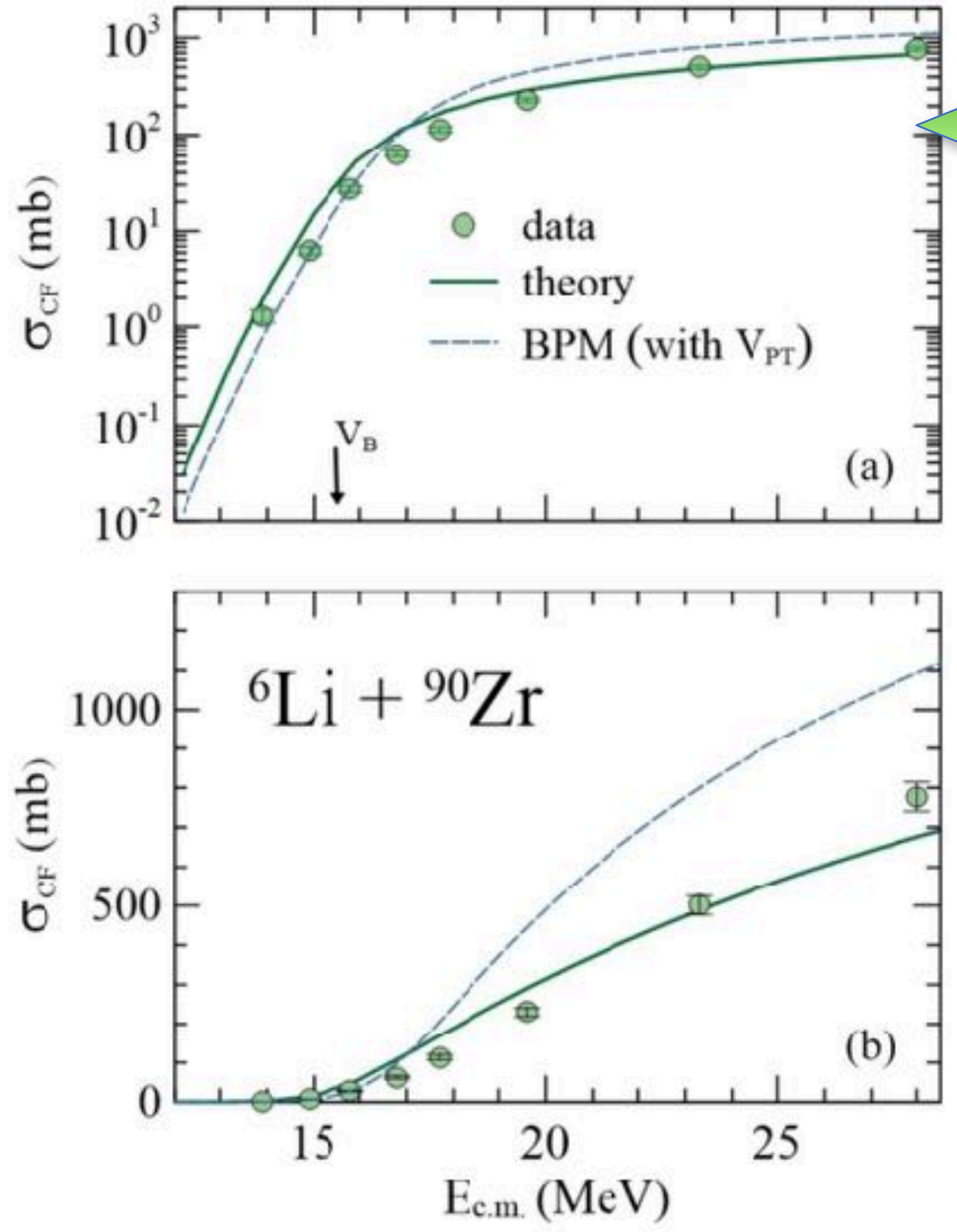
σ_α^{1n} = neutron stripping cross section ($^{6}Li + ^{90}Zr \rightarrow ^{5}Li + ^{91}Zr \rightarrow p + ^{4}He + ^{91}Zr$),

σ_α^{1p} = proton stripping cross section ($^{6}Li + ^{90}Zr \rightarrow ^{5}He + ^{91}Nb \rightarrow n + ^{4}He + ^{91}Nb$),

σ_α^{CF} = CF + evaporation of an α particle ($^{6}Li + ^{90}Zr \rightarrow ^{96}Nd \rightarrow (1n, 2n, 3n), \alpha$).

$$\sigma_\alpha^{inc} = \sigma_\alpha^{EBU} + \sigma_\alpha^{1n} + \sigma_\alpha^{1p} + \sigma_\alpha^{CF} + \sigma_{ICFd}.$$

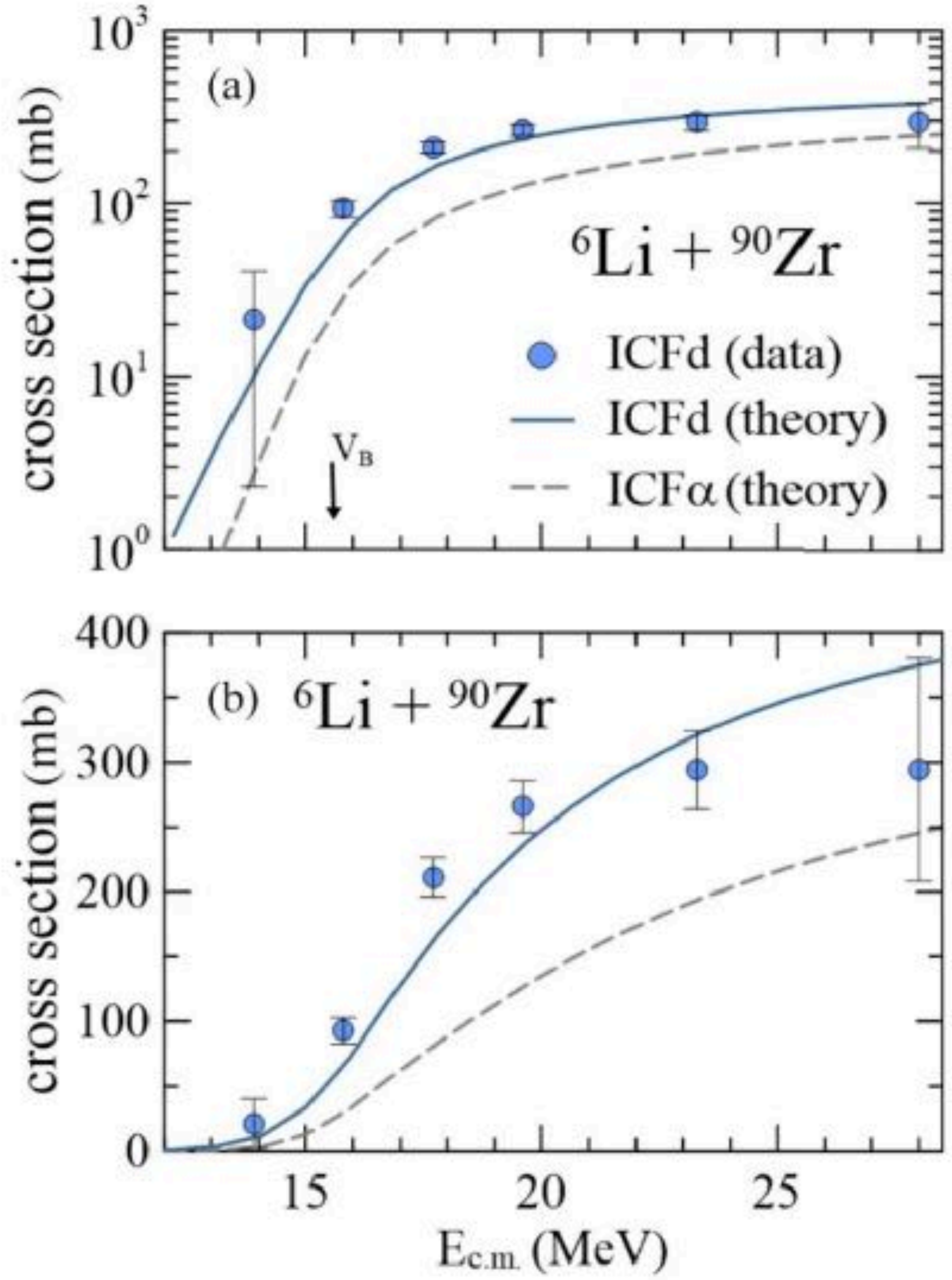
Indirect estimations using “Fusion” code

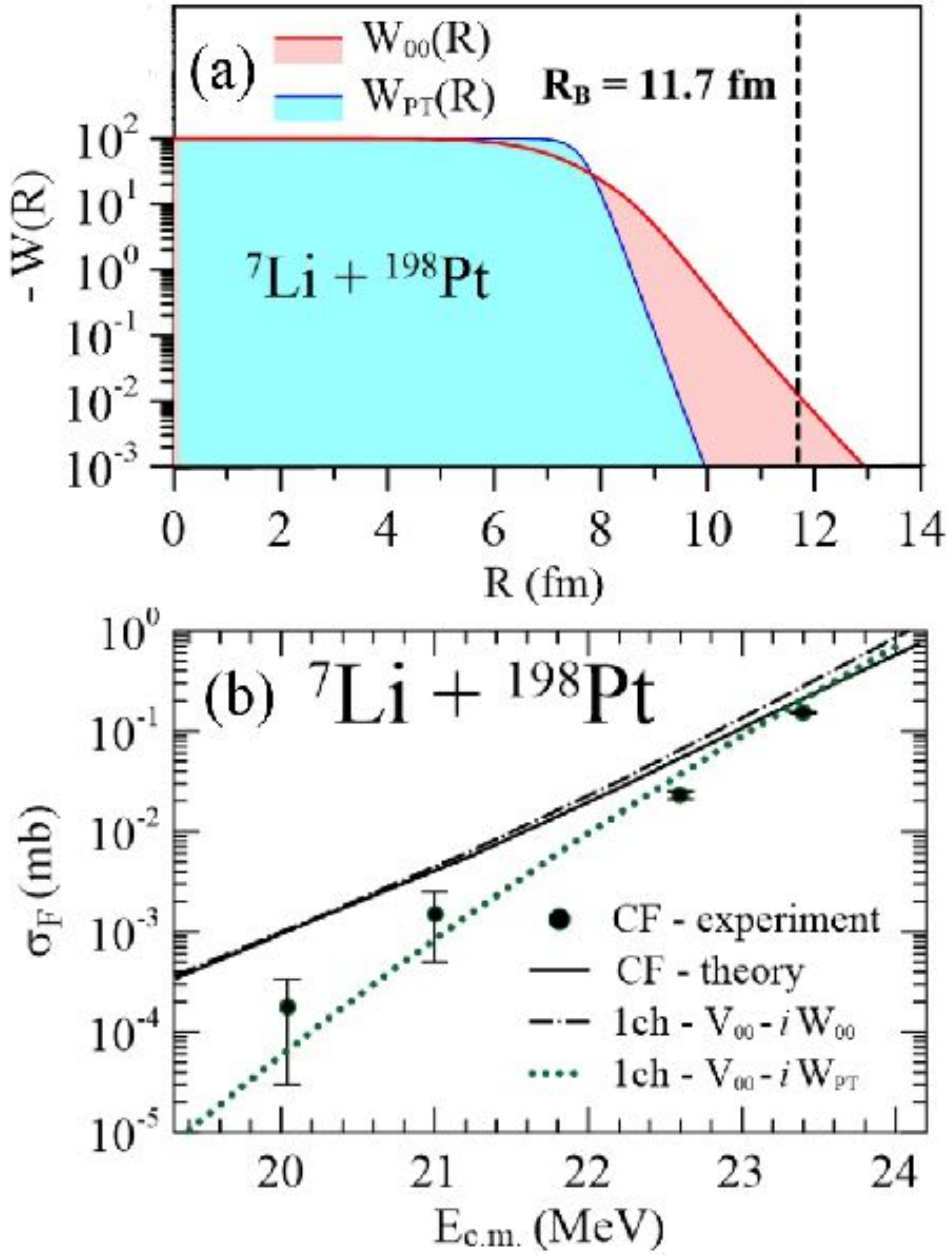


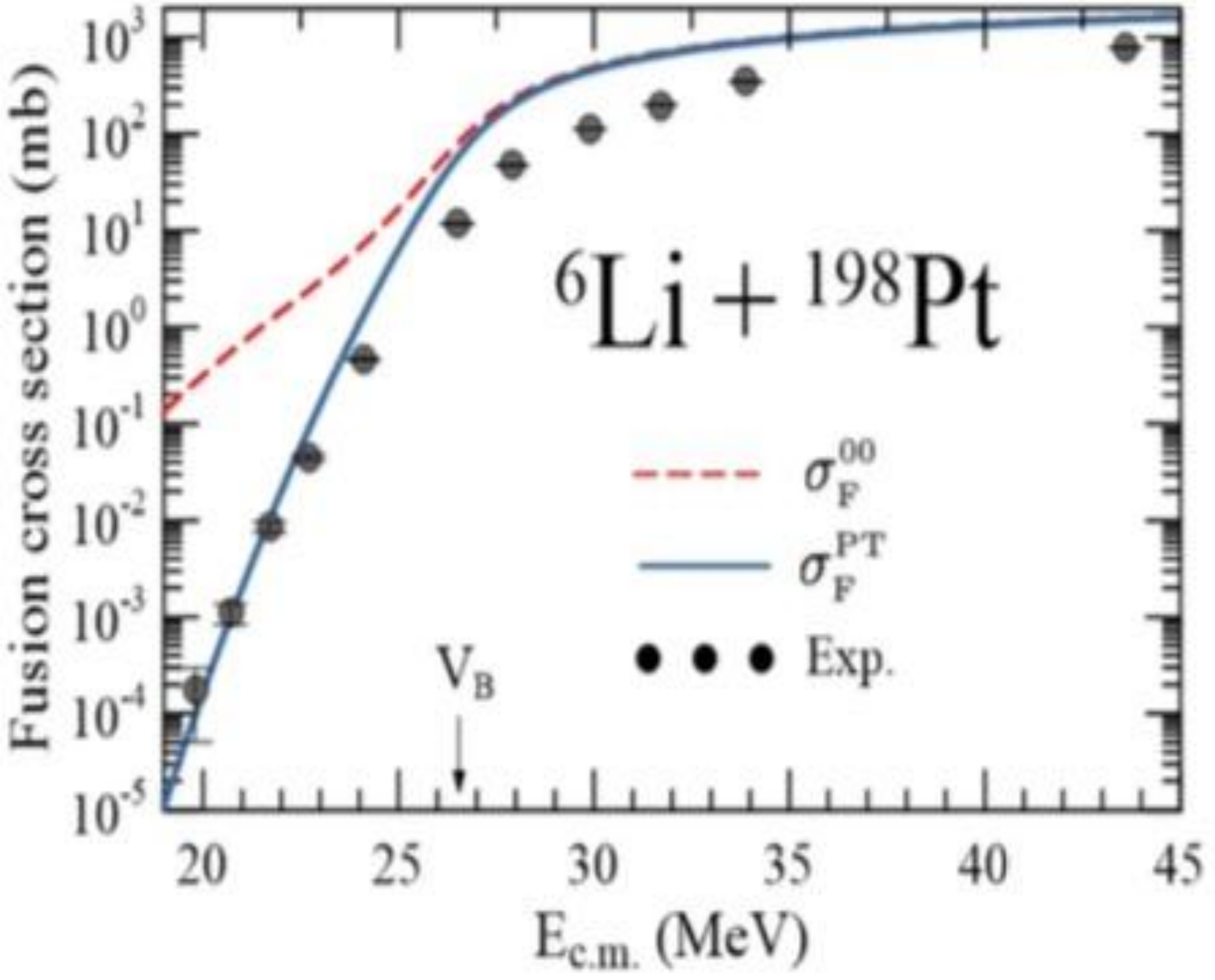
Data: Phys. Rev. C 86, 024607 (2012).

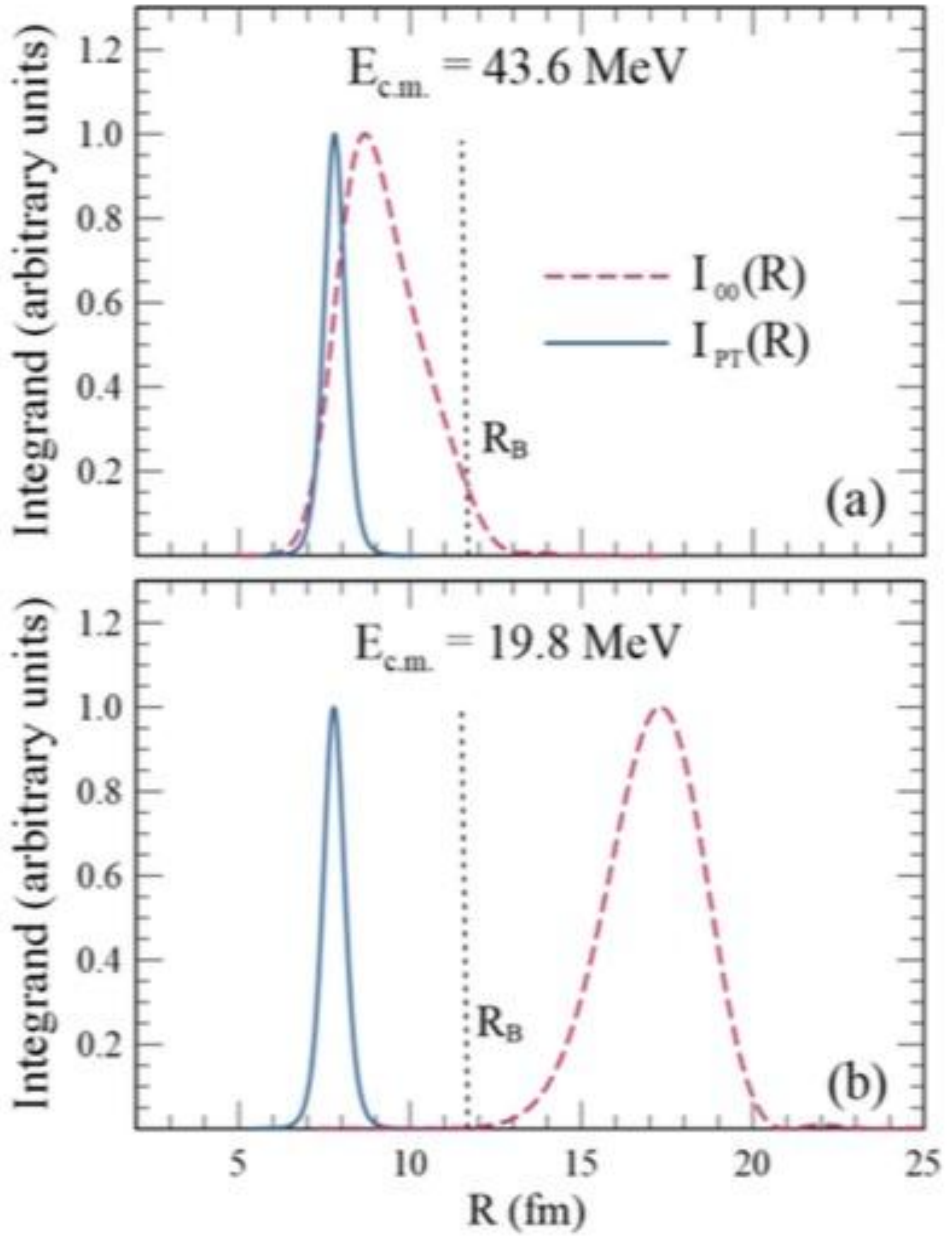
"Data": Phys. Rev. C 81, 054601 (2010).

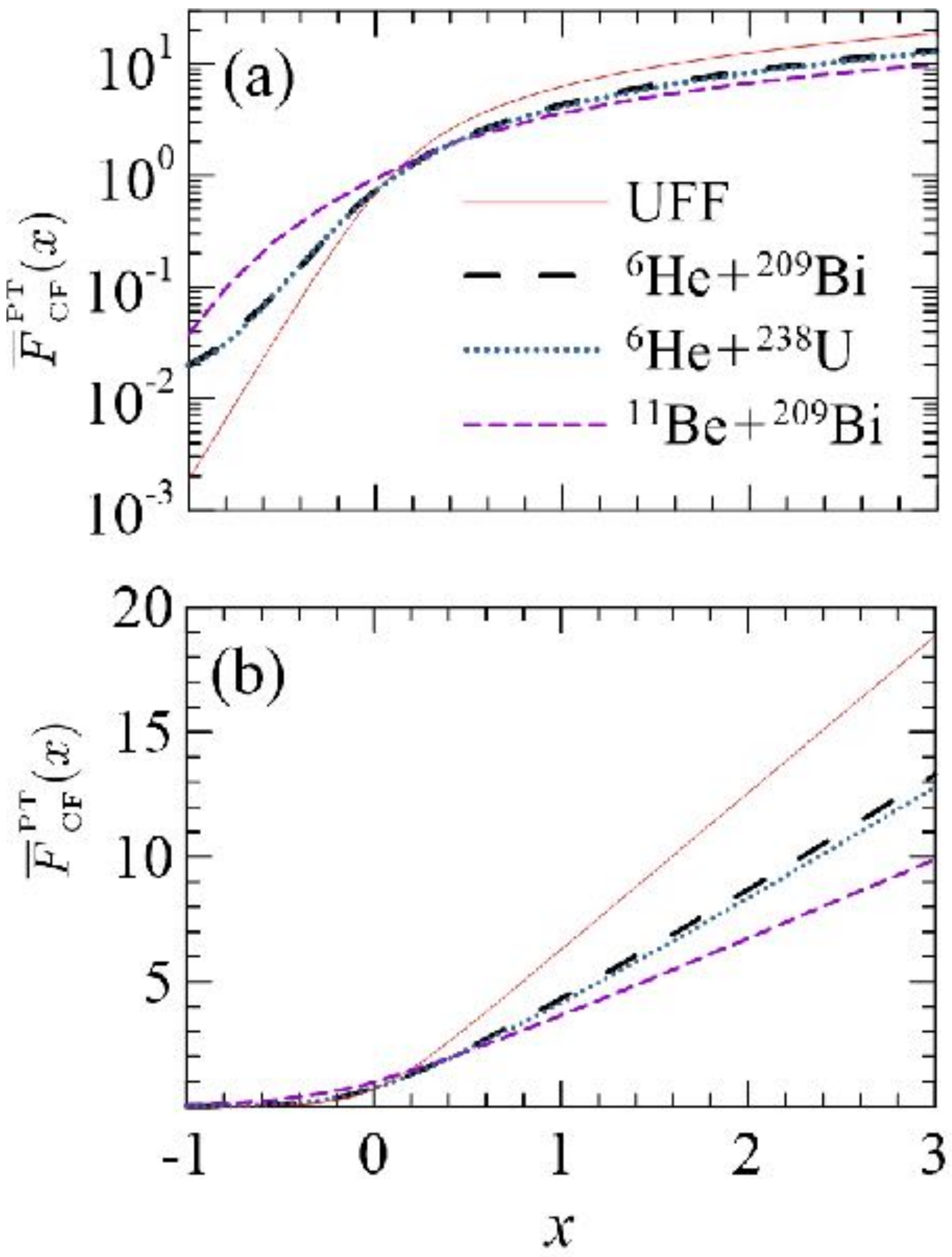
Phys. Rev. C, 108, 054601, (2023)

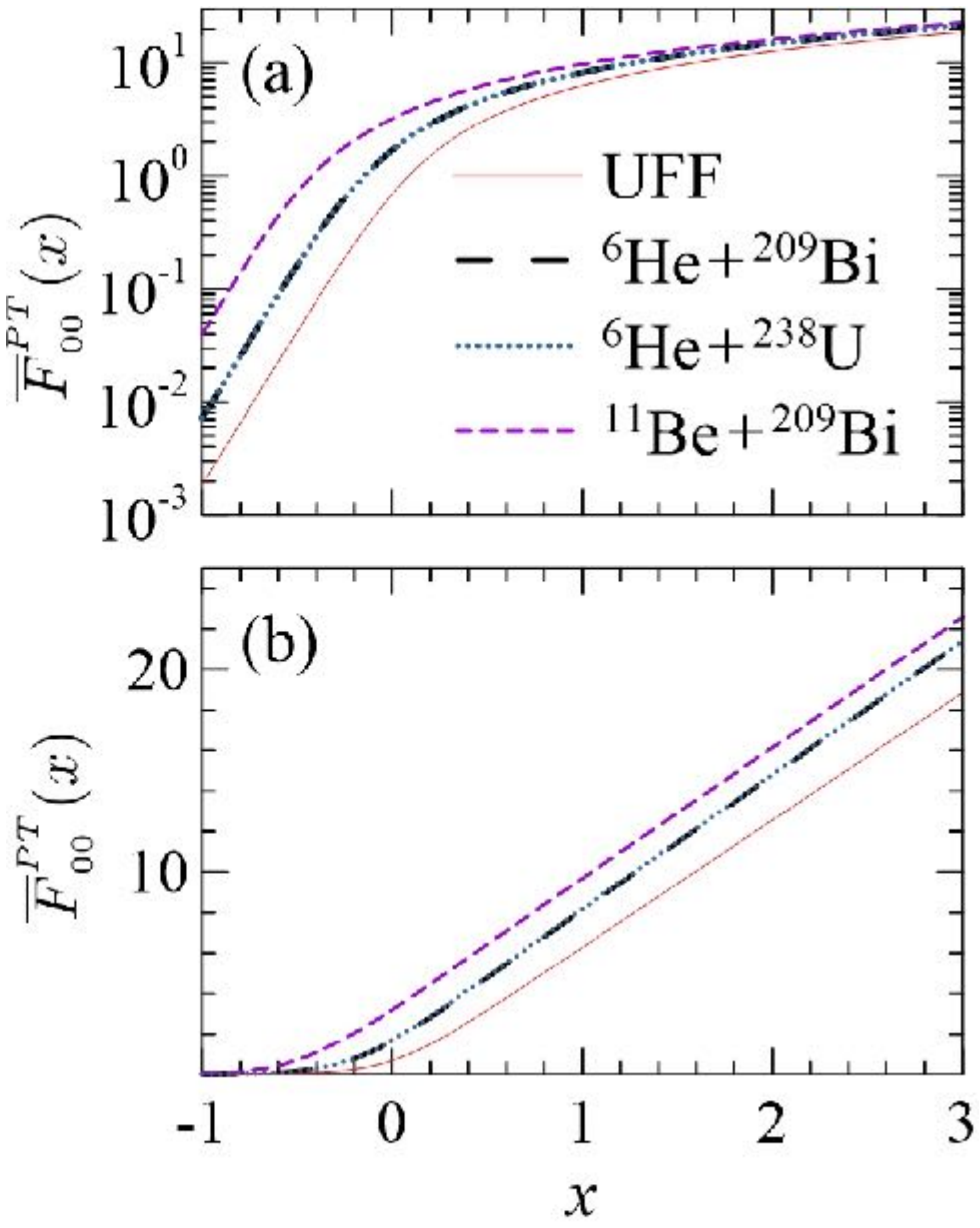


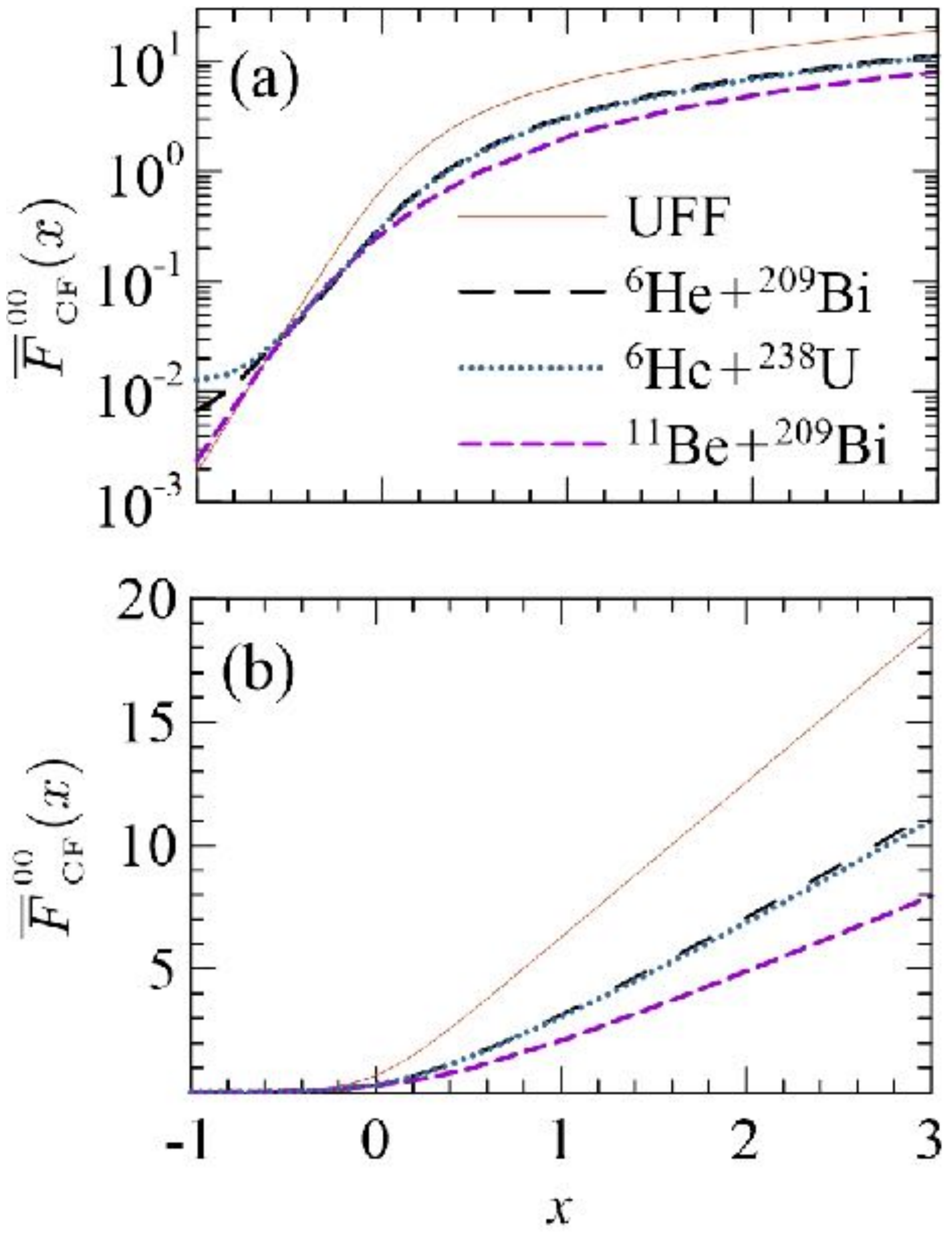

(a)
$W_{00}(R)$
$W_{PT}(R)$
R_B = 11.7 fm
-W(R)
^{7}Li + ^{198}Pt
R (fm)
(b) ^{7}Li + ^{198}Pt
σ_F (mb)
CF - experiment
CF - theory
1ch - V_{00} - $i\,W_{00}$
1ch - V_{00} - $i\,W_{PT}$
$E_{c.m.}$ (MeV)


Fusion cross section (mb)
$^{6}Li + ^{198}Pt$
σ_F^{00}
σ_F^{PT}
Exp.
V_B
$E_{c.m.}$ (MeV)

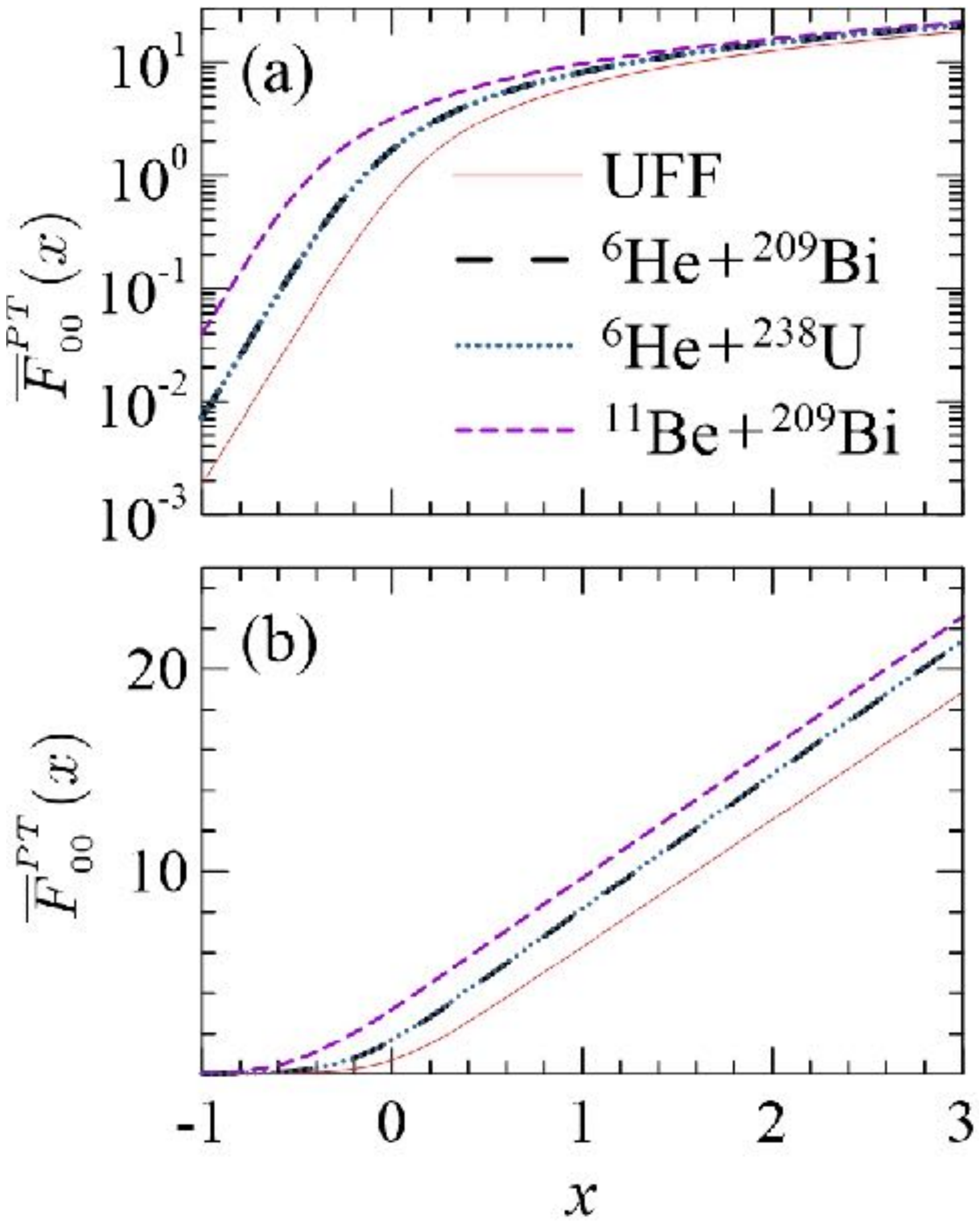

Integrand (arbitrary units)
$E_{c.m.}$ = 43.6 MeV
$I_{00}(R)$
$I_{PT}(R)$
R_B
(a)
$E_{c.m.}$ = 19.8 MeV
(b)
R (fm)


(a)
UFF
^{6}He+^{209}Bi
^{6}He+^{238}U
^{11}Be+^{209}Bi
$\overline{F}^{\mathrm{PT}}_{\mathrm{CF}}(x)$
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
(b)
$\overline{F}^{\mathrm{PT}}_{\mathrm{CF}}(x)$
20
15
10
5
-1
0
1
2
3
x


(a)
UFF
^{6}He+^{209}Bi
^{6}He+^{238}U
^{11}Be+^{209}Bi
$\overline{F}^{PT}_{00}(x)$
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
(b)
$\overline{F}^{PT}_{00}(x)$
20
10
-1
0
1
2
3
x



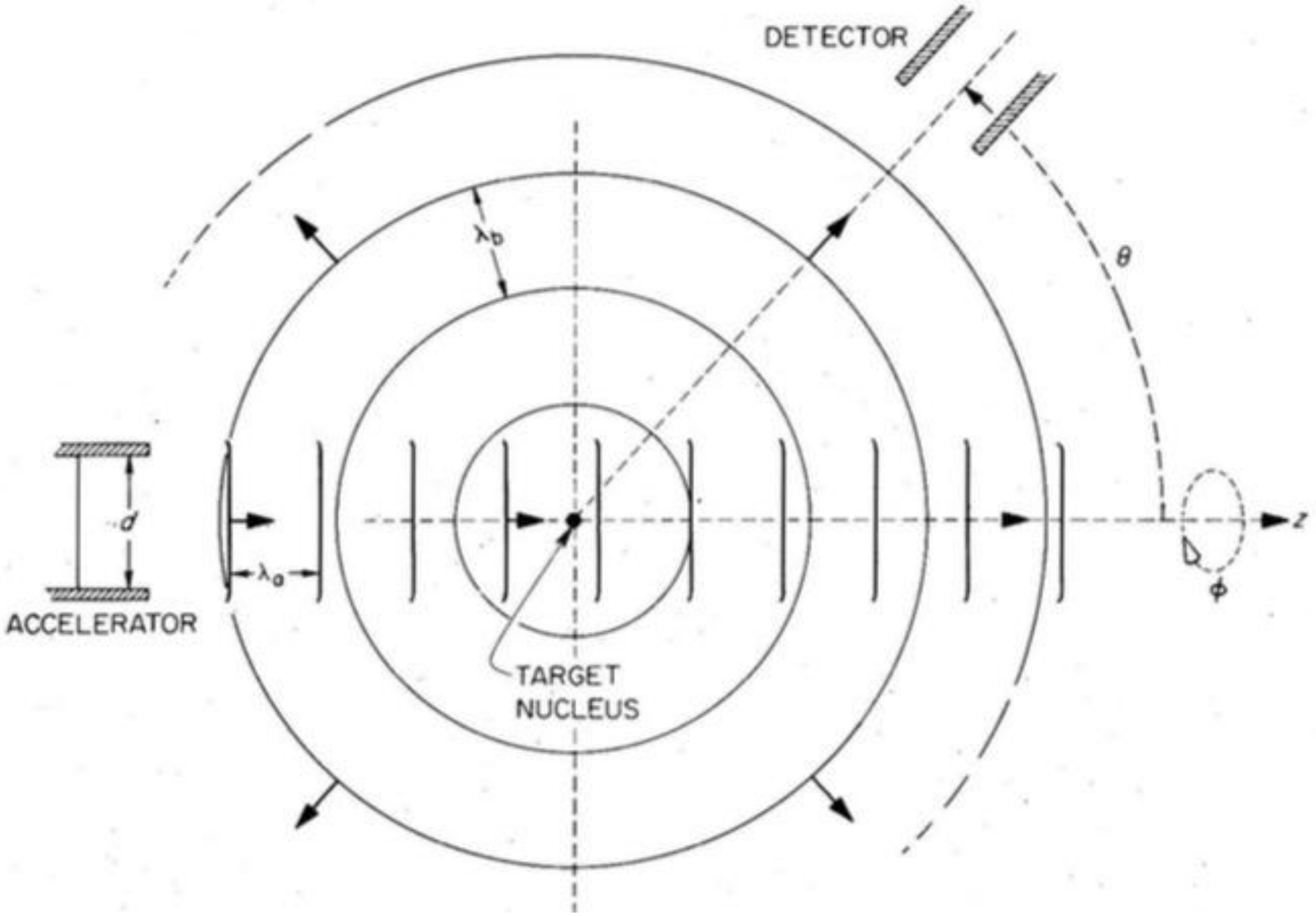
(a)
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
UFF
^{6}He+^{209}Bi
^{6}Hc+^{238}U
^{11}Be+^{209}Bi
$\overline{F}^{00}_{\mathrm{CF}}(x)$
(b)
20
15
10
5
-1
0
1
2
3
x



(a)
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
UFF
^{6}He+^{209}Bi
^{6}He+^{238}U
^{11}Be+^{209}Bi
$\overline{F}^{PT}_{00}(x)$
(b)
20
10
-1
0
1
2
3
x

Potential Scattering approac'

Fusion is not a channel



$$\sigma_{\mathrm{F}}(E) = \frac{\pi}{K^2} \sum_{l=0} (2l+1) \; P_l^{\mathrm{F}}(E)$$ With $$P_l^{\mathrm{F}}(E) = 1 - |S_l(E)|^2$$

Indirect determination of CF using spectator model*(IAV)

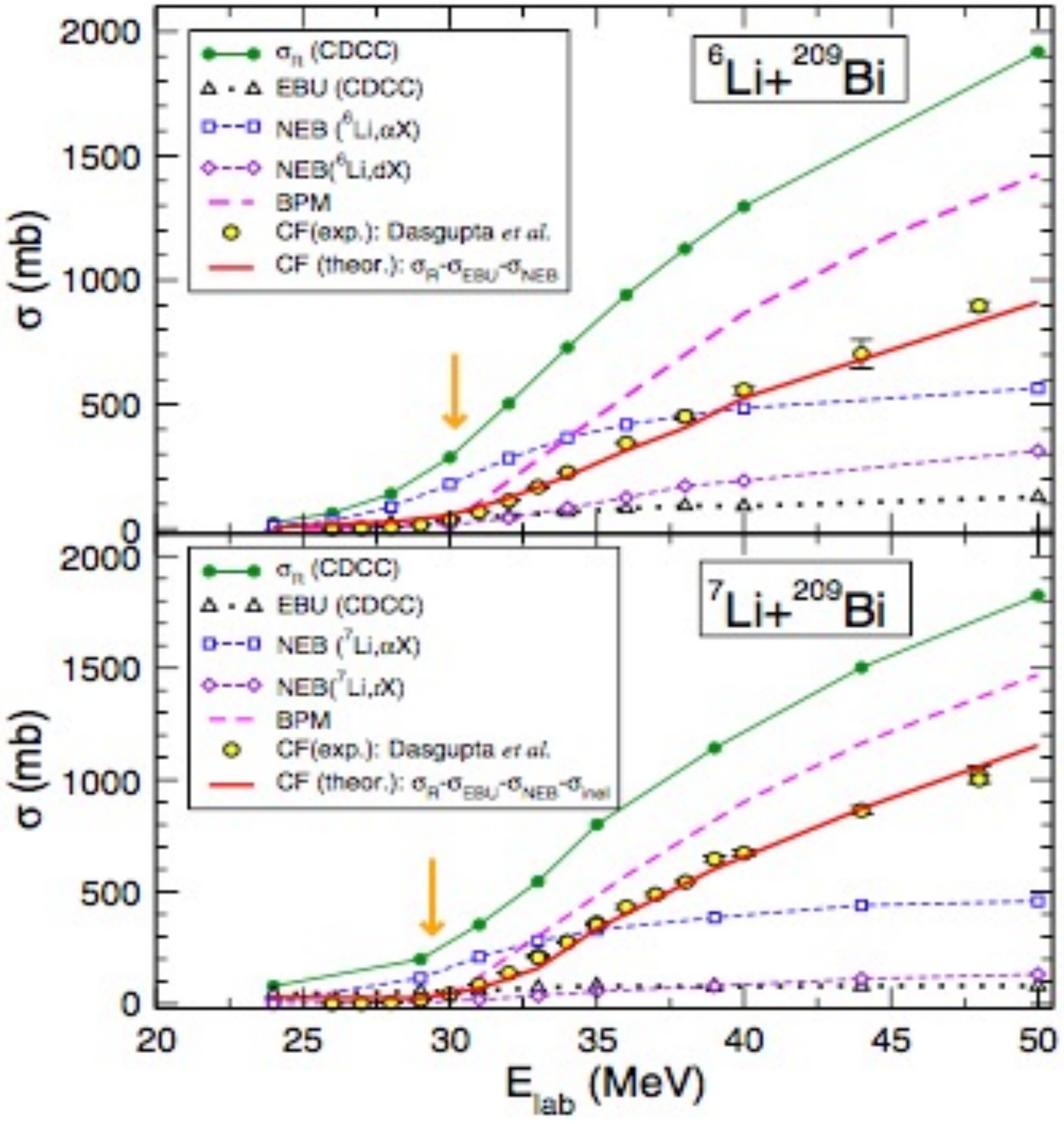
*** Lei and Moro PRL 122, 042503 (2019)**

- **Extract σ_{CF} from the relation**

$$\sigma_{\mathrm{R}} = \sigma_{\mathrm{CF}} + \sigma_{\mathrm{inel}} + \sigma_{\mathrm{EBU}} + \sigma^{(1)}_{\mathrm{NBU}} + \sigma^{(2)}_{\mathrm{NBU}}$$

- $\sigma_{R:}$ from CDCC calculation or optical model analysis
- σ_{ine}:from standard CC calculation (only bound channels)
- σ_{ebu}: from CDCC calculation:
- σ_{NEB1} , σ_{NEB2} :from inclusive spectator- participant model (IAV)

Indirect determination of CF using spectator model*(IAV)

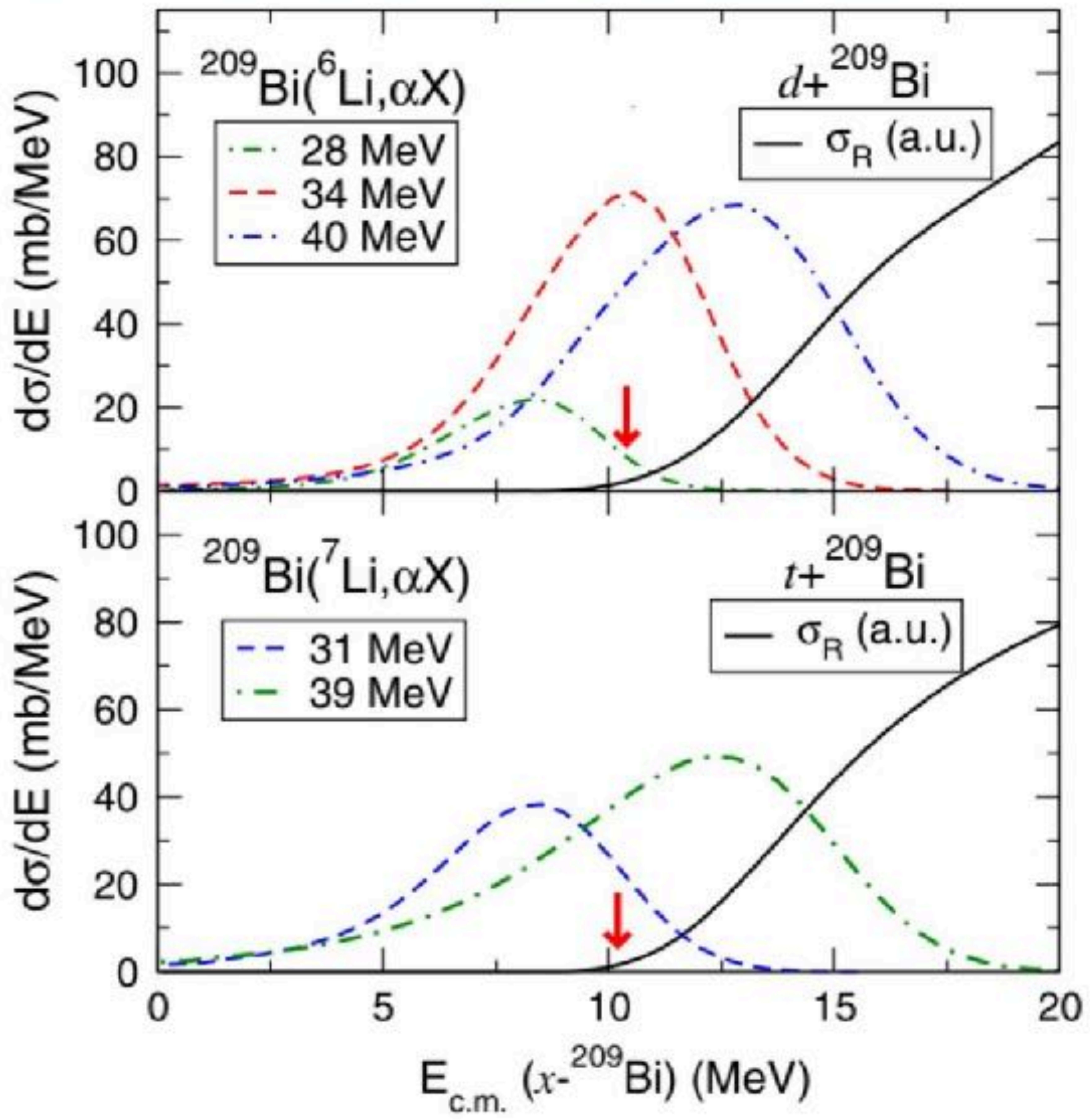


* Lei and Moro, PRL 122, 042503 (2019)

- Verification of the Trojan Horse mechanism

IAV

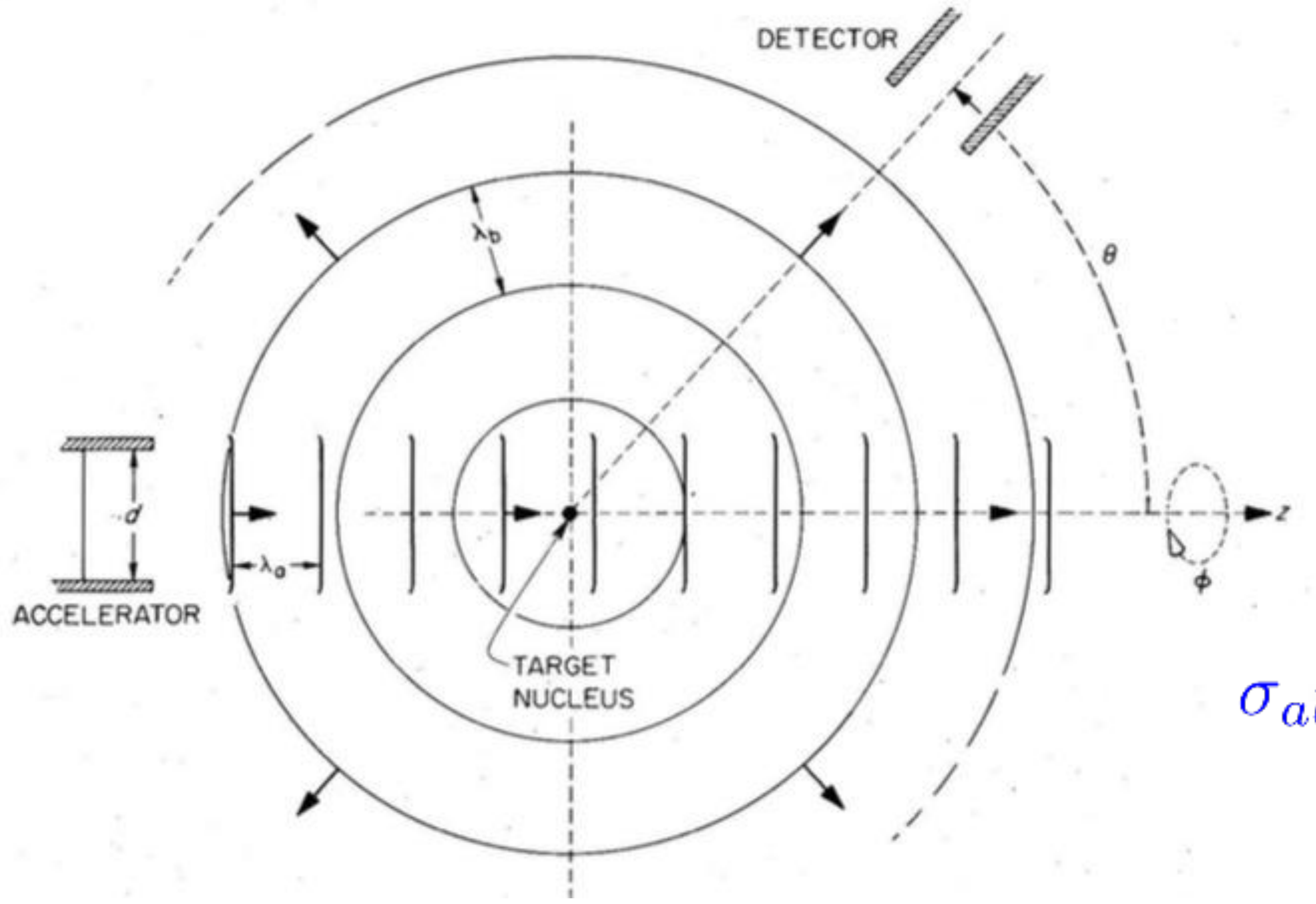
PHYSICAL REVIEW LETTERS **122,** 042503 (2019)



The method of Hagino, Vitturi, Dasso and Lenzi (HVDL)

- **What means** $\Psi^+(R)$ **?**

$f_{\alpha,\alpha'}(\theta)$



Cross section:

$$\sigma_\alpha = \frac{1}{(2Ip_i+1)(2It_i+1)} \sum |f_{\alpha,\alpha'}(\theta)|^2$$

Optical model:

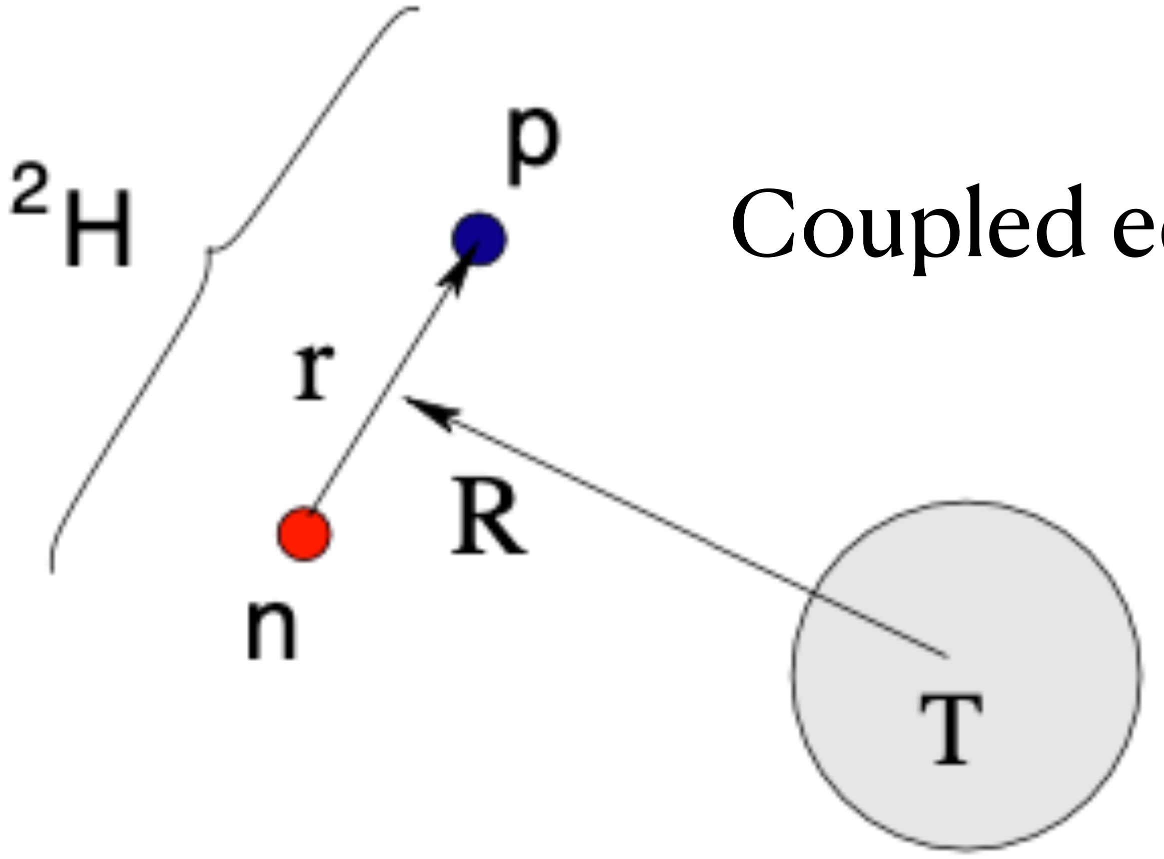
$$\sigma_{abs} = \frac{E}{k^2} \sum < \Psi^+(R) \mid W(R) \mid \Psi^+(R) >$$

CDCC approach

Full numerical implementation by Kyushu group (Sakuragi, Yahiro, Kamimura, and co.):
Prog. Theor. Phys.(Kyoto) 68, 322 (1982)

Hamiltonian $$H = T_R + h_r + U_{p-T} + U_{n-T}$$

Model wave function $$\Psi^+(\mathbf{R},\mathbf{r}) = \phi_{gs}(\mathbf{r})\chi_o(\mathbf{R}) + \sum_{n>0} \phi_n(\mathbf{r})\chi_n(\mathbf{R})$$



Coupled equations: $$H\Psi^+(\mathbf{R},\mathbf{r}) = E\Psi^+(\mathbf{R},\mathbf{r})$$

$$[E - \epsilon_n - T_R + U_{n,n}(R)]\chi(R) = \sum_{n'\neq n} U_{n',n}(R)\chi(R)$$

$$U_{n',n}(R) = \int d\mathbf{r}(r)\phi_n^*(r)[U_{pT}(r_1) + U_{nT}(r_1)]\phi_n(r)$$

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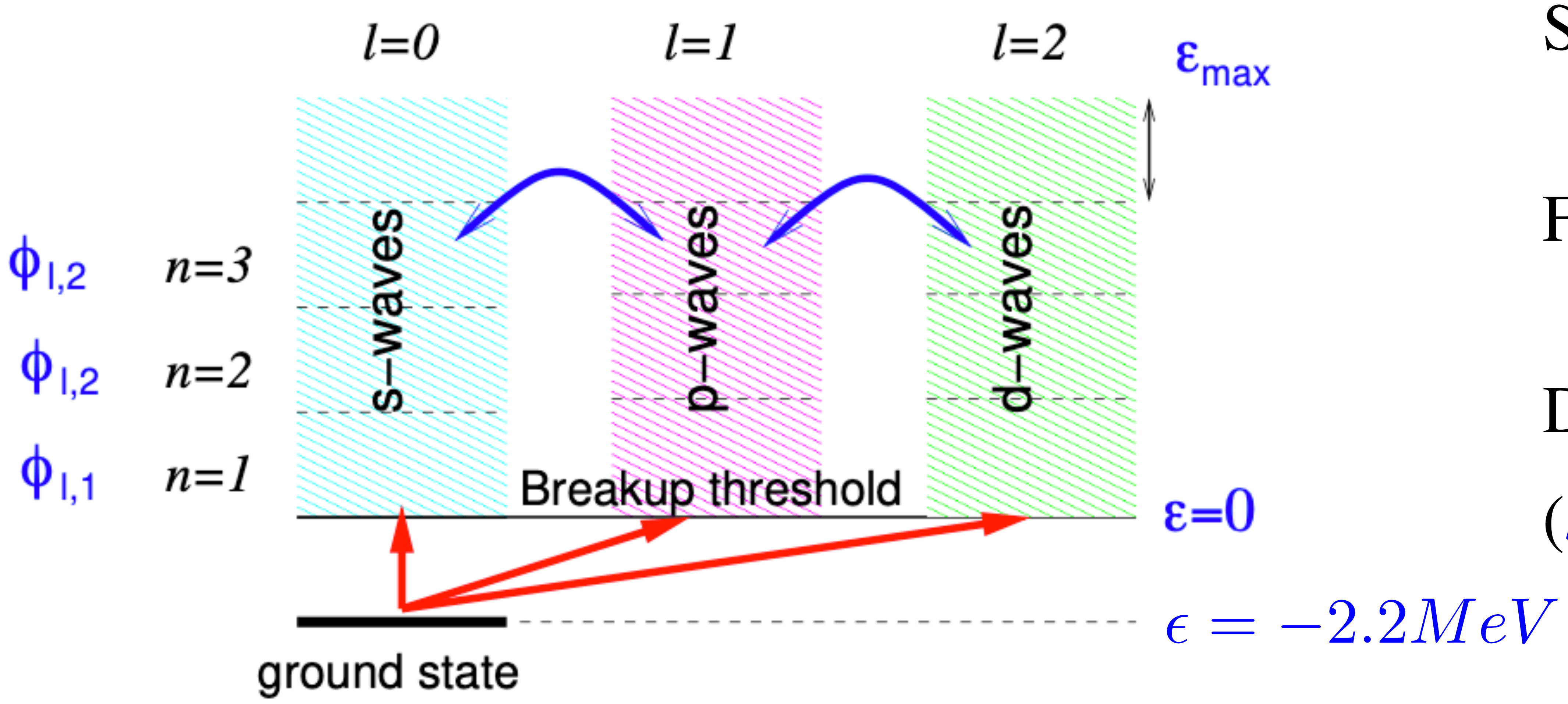


Courtesy from Antonio Moro

Select a number of angular momenta (l=0,…,lmax).

For each l, set a maximum excitation energy εmax.

Divide the interval ε = 0 – εmax in a set of sub-intervals (*bins*).

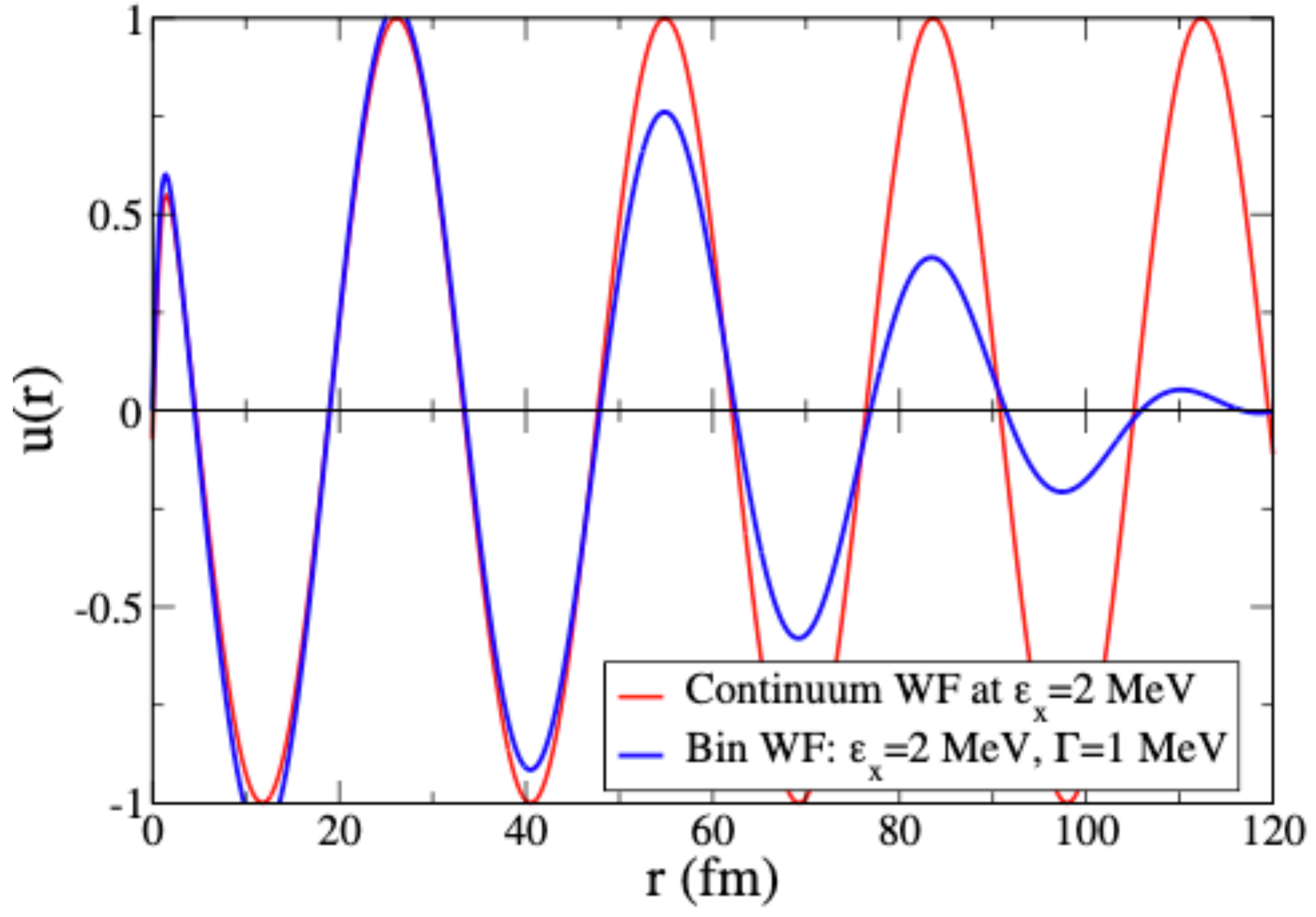
For each bin, calculate a representative wave function.

CDCC approach

Model wave function $$\Psi^+(\mathbf{R},\mathbf{r}) = \phi_{gs}(\mathbf{r})\chi_o(\mathbf{R}) + \sum_{n>0}\phi_n(\mathbf{r})\chi_n(\mathbf{R})$$

Bins: $$\phi_{lsjm}^{[k_1,k_2]}(r) = \frac{u_{lj}^{[k_1,k_2]}(r)}{r}[Y_l(\hat{r})\otimes\chi_s]_{jm}$$

$$\boxed{u_{lj}^{[k_1,k_2]}(r) = \sqrt{\frac{2}{\pi N}}\int_{k_1}^{k_2} w(k)u_{lj,k}(r)dk}$$



Courtesy from Antonio Moro

A new QM method to evaluate CF and ICF*

Contribution from the B-spa

$$\sigma_B = \sigma_{DCF}$$

$$\sigma_B = \frac{K}{E} \sum_{\alpha \in B} < \psi_\alpha | \, W^1(r_1) + W^2(r_2) \, | \psi_\alpha >$$

$$\sigma_{DCF} = \frac{\pi}{k^2} \sum_J (2J+1)[P^1(J) + P^2(J)]$$

* J. Rangel, M. Cortes, J. Lubian, LFC (Phys. Let. B, 803- 2020)

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