ECT*
EUROPEAN CENTRE FOR THEORETICAL STUDIES
IN NUCLEAR PHYSICS AND RELATED AREAS

Multi-canonical methods and lattice field theory

The smeared R-ratio in isoQCD from first-principles lattice simulations

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December 2, 2025 ECT* Villa Tambosi, Villazzano

TOR VERGATA
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Extended Twisted Mass Collaboration

INFN
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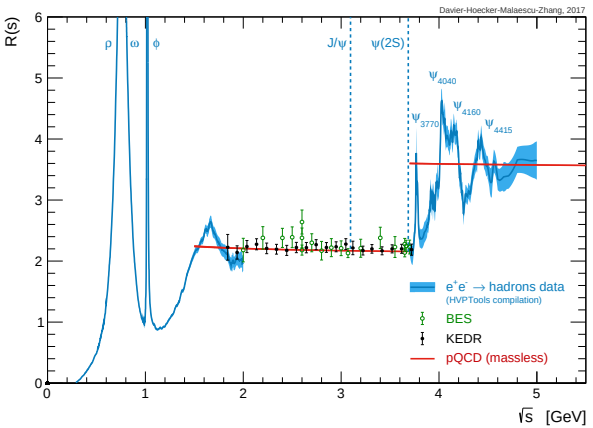
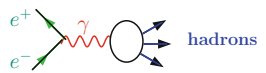
SEZIONE DI ROMA TOR VERGATA

The R-ratio

The R-ratio plays a fundamental role in particle physics since its introduction

N. Cabibbo, G. Parisi and M. Testa, *Hadron Production in e+ e− Collisions*, Lett. Nuovo Cim. **4S1**, 35 (1970).





$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

As any other cross-section, it is an energy-dependent probe of the theory and contains an infinite amount of information

In recent years, the importance of the R-ratio has been mainly associated with muon $g-2$ experiments ...

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The Dirac equation predicts a muon magnetic moment, $\mathbf{M} = g_\mu \dfrac{e}{2m_\mu}\mathbf{S}$, with $\boxed{g_\mu = 2}$.

Quantum loop effects lead to a small calculable deviation, parameterized by the magnetic anomaly

$$\boxed{a_\mu \equiv \frac{g_\mu - 2}{2}}$$

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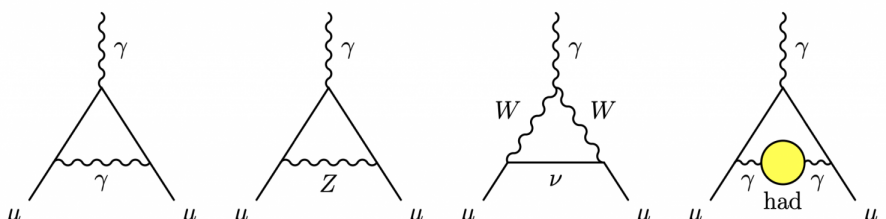
The SM prediction $a_\mu^{\rm SM}$ is generally divided into three parts

$$a_\mu^{\rm SM} = a_\mu^{\rm QED} + a_\mu^{\rm EW} + \boxed{a_\mu^{\rm Had}}$$



Dispersive approach

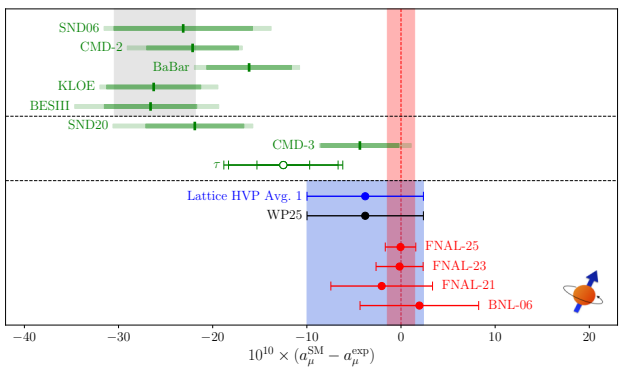
$$a_\mu^{\mathrm{HVP-LO}} = \int_{m_\pi}^{\infty} \mathrm{d}E\ R(E) \qquad \underbrace{\tilde{K}(E)}_{\text{analytic function}}$$

- The idea is to replace $R(E) \to R^{\mathrm{exp}}(E)$ and use previous formula.

Lattice QCD

$$a_\mu^{\mathrm{HVP-LO}} = \int_{0}^{\infty} \mathrm{d}t\ C(t) \qquad \underbrace{K(t)}_{\text{analytic function}}$$

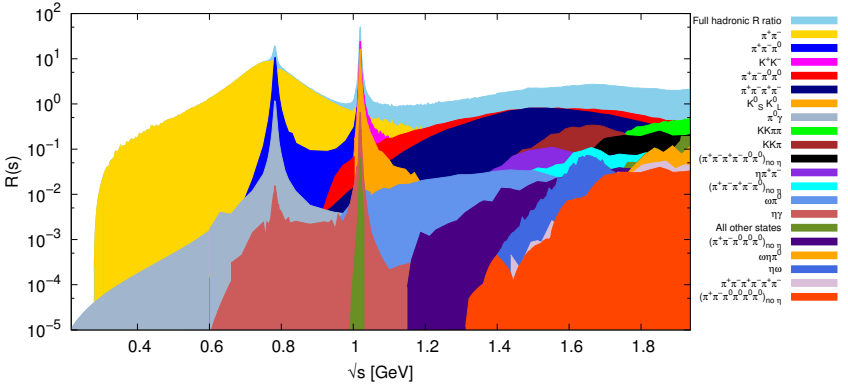
- **ETM Collaboration** effort in computing light-quark contribution to $a_\mu^{\mathrm{HVP-LO}}$ at subpercent accuracy.



[White Paper (2025) - Muon g-2 Theory Initiative]

Dispersive approach

- Inclusive measurement of $R^{\exp}(E)$



KNT [Keshavarzi et al., 1802.06229].

- The main groups involved in the analysis: DHMZ = Davier-Hoecker-Malaescu-Zhang, KNTW = Keshavarzi, Nomura, Teubner, Wright, CHKLS = Colangelo, Hoferichter, Kubis, Leplumery, Stoffer

- R-ratio from exclusive channels: source of systematic uncertainty

What about computing R-ratio directly on the lattice?

In PRL 130 (2023), the Hansen-Lupo-Tantalo (**HLT**) method is used to evaluate on the lattice:

$$C(t) = -\frac{1}{3}\sum_{i=1}^{3}\int \mathrm{d}^3x \mathrm{T}\langle 0|J_i(x)J_i(0)|0\rangle = \frac{1}{12\pi^2}\int_{2m_\pi}^{\infty} \mathrm{d}\omega\, e^{-\omega t}\,\omega^2 R(\omega)\ ,$$

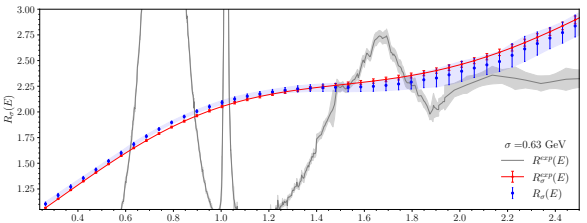
Theoretically $R(\omega)$ is a **distribution**, **the spectral density of** $C(t)$ [Bernecker, Meyer 1107.4388], and has to be probed by using suitable **smearing kernels**:

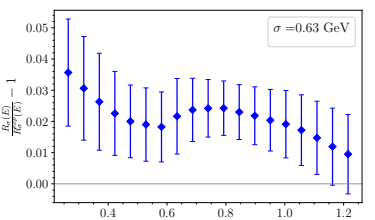
$$R_\sigma(E) = \int_0^{+\infty} d\omega\, R(\omega)\ \underbrace{G_\sigma(E-\omega)}_{\text{Gaussian kernels}}\ ; \qquad G_\sigma(E-\omega) = \frac{e^{-\frac{(E-\omega)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\ .$$

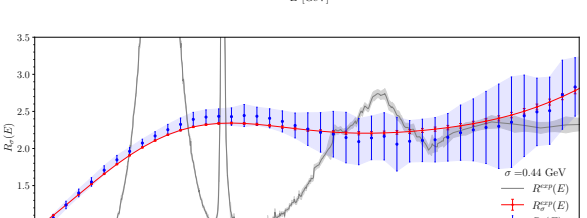
How the R-ratio can be probed on the lattice: **spectral reconstruction**

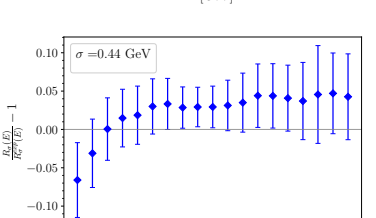
See A. De Santis's talk at Charming Lattice QCD @ CERN 2025

- Methods that provide regularization against noise and systematic error estimation
  - HLT Hansen, Lupo, Tantalo (2019) based on Backus, Gilbert (1968) - Pijpers, Thompson (1992)
  - Chebyshev polynomial approximation Bailas, Hashimoto, Ishikawa (2020) - Barata, Fredenhagen (1991)
- Other methods that address the extraction of smeared/unsmeared spectral functions
  - Bayesian inference Asakawa, Hatsuda, Nakahara (2000) - Burniern, Rothkopf (2013)
  - Machine Learning Fournier et al. (2018) - Kades et al. (2019) - Buzzicotti, De Santis, Tantalo (2023)
  - Gaussian processes Horak et al. (2021) (equivalent to HLT Del Debbio et. al (2024))
  - Others Bergamaschi, Jay, Oare (2023) - Bruno, Giusti, Saccardi (2024)









At $E \simeq m_\rho$:

- $\sigma \simeq 0.6$ GeV : the total relative error $\sim 1\%$, we observe a $\sim 3\sigma$ (or $2.5 - 3$ %) deviation w.r.t. e^+e^- experimental data.
- $\sigma \simeq 0.4$ GeV : the total relative error $\sim 3.5\%$, the larger errors do not allow us to observe significant deviations from experimental data.

Reconstruction of spectral density information from lattice field theory

- Composite Higgs framework: Masses extracted with the **HLT** method

  E. Bennett et al., *Chimera baryons and mesons on the lattice: A spectral density analysis*, Phys.Rev.D 112 (2025) 7

- The first non-perturbative determination of the QCD Sphaleron Rate with 2+1 quark flavors at the physical point using **HLT** method

  Talk by R. Dionisio today at 11:40

  C. Bonanno et al., *Sphaleron Rate of $N_f = 2 + 1$ QCD*, Phys. Rev. Lett. 132 (2024) 051903

- Study of the thermal modification of the isovector vector spectral function using the **Backus-Gilbert** method

  A. Krasniqi et al., *Hot QCD matter around the chiral crossover: A lattice study with O(a)-improved Wilson fermions*, Phys. Rev. D 110 (2024) 114506

- Inclusive hadronic decay rate of the τ lepton via **HLT** method

  A. Evangelista et al., *Inclusive hadronic decay rate of the τ lepton from lattice QCD*, Phys. Rev. D 108 (2023) 074513

- Inclusive semileptonic decays of the D_s meson via **HLT** method

  A. De Santis et al., *Inclusive semileptonic decays of the Ds meson: A first-principles lattice QCD calculation*, Phys. Rev. D 112 (2025) 054503

Precision and resolution to be improved with more statistics and by increasing the statistical precision of our lattice correlators.

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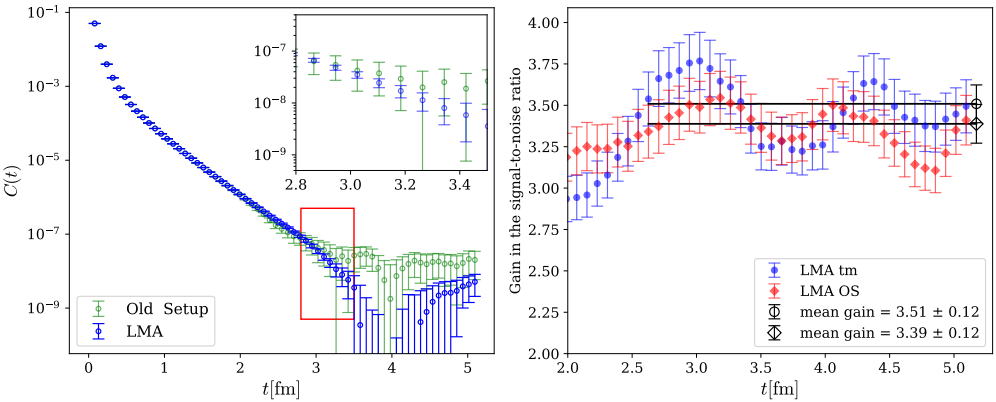


We employ noise-reduction techniques:

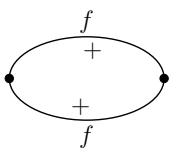
by computing a relatively small number of low modes of the Dirac operator exactly, which we refer to as low-mode averaging (LMA).

ETMC Lattice setup

ensemble	$(L^3 \times T)/a^4$	a [fm]	L [fm]
B64	$64^3 \times 128$	0.0795	5.09
B96	$96^3 \times 192$	0.0795	7.64
C80	$80^3 \times 160$	0.0682	5.46
D96	$96^3 \times 192$	0.0569	5.46
E112	$112^3 \times 224$	0.0489	5.46

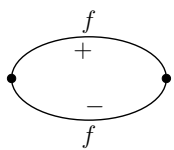
Table: Four physical-point $N_f = 2 + 1 + 1$ ensembles.

$J_\mu^{f,\mathrm{OS}} \propto \bar{\psi}_f^+ \gamma_\mu \psi_f^+$



$R_\sigma^{ff,C,\mathrm{OS}}(a) = A + B^{\mathrm{OS}}\, a^2$

$J_\mu^{f,\mathrm{TM}} \propto \bar{\psi}_f^+ \gamma_\mu \psi_f^-$



$R_\sigma^{ff,C,\mathrm{TM}}(a) = A + B^{\mathrm{TM}}\, a^2$

$$R_\sigma(E) = \boxed{R_\sigma^{\ell\ell,C}(E)} + R_\sigma^{ss,C}(E) + R_\sigma^{cc,C}(E) + R_\sigma^{disco}(E)$$

$$R_\sigma(E) = \int_{2m_\pi}^{\infty} d\omega\, R(\omega)\ \underbrace{G_\sigma(E-\omega)}_{\text{Gaussian kernels}}\ ; \qquad \underbrace{K(\omega;\mathbf{g})}_{\text{Approximated smearing kernels}} = \sum_{\tau=1}^{T/a} g_\tau\, e^{-a\omega\tau}\,.$$

which allows to evaluate $R_\sigma(E)$ from $C(t)$ using

$$\sum_{\tau=1}^{\tau_{\max}} g_\tau(E,\sigma)\, C(\tau a) = \int_{2m_\pi}^{\infty} d\omega \left(\frac{1}{12\pi^2} \sum_{\tau=1}^{\tau_{\max}} g_\tau(E,\sigma) e^{-\omega a\tau} \right) R(\omega)\, \omega^2 \simeq R_\sigma(E)$$

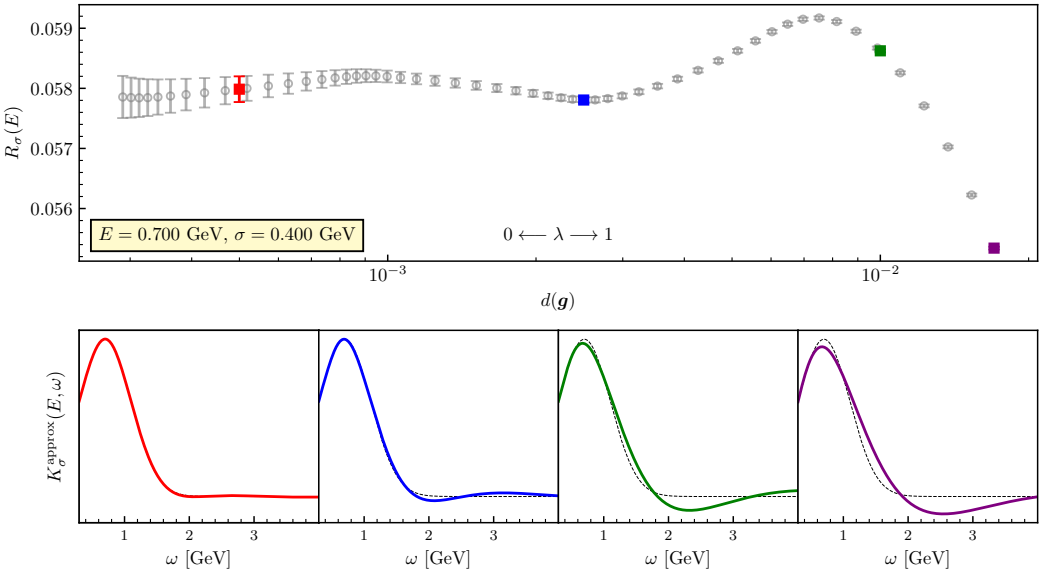
Accuracy of the approximated kernel

$$A[\mathbf{g}] = \int_{2m_\pi}^{\infty} \mathrm{d}\omega\ \left\{ K(\omega;\mathbf{g}) - \frac{12\pi^2 G_\sigma(E-\omega)}{\omega^2} \right\}^2 \cdot e^{\alpha\omega}\ , \qquad \alpha = 2^-$$

The coefficients $\mathbf{g}$ are obtained by minimizing

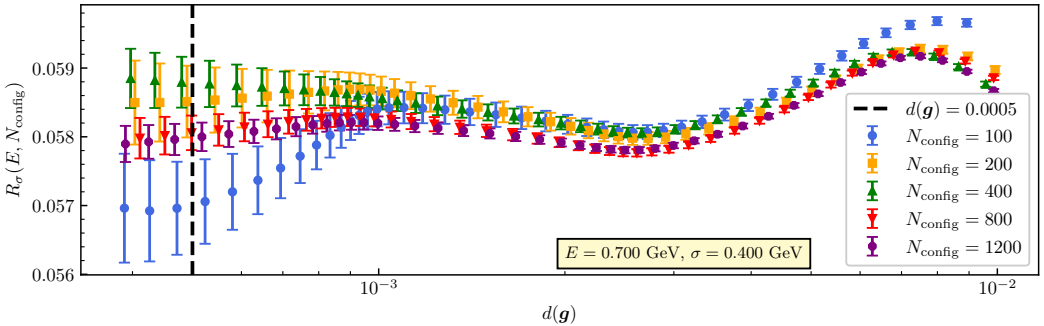
$$W[\lambda,\mathbf{g}] = (1-\lambda)\, \underbrace{\frac{A[\mathbf{g}]}{A[0]}}_{d^2[\mathbf{g}]} + \lambda B[\mathbf{g}], \qquad B[\mathbf{g}] = \sum_{\tau_1,\tau_2=1}^{T/a} g_{\tau_1} g_{\tau_2}\, \mathrm{Cov}_{\tau_1\tau_2}.$$

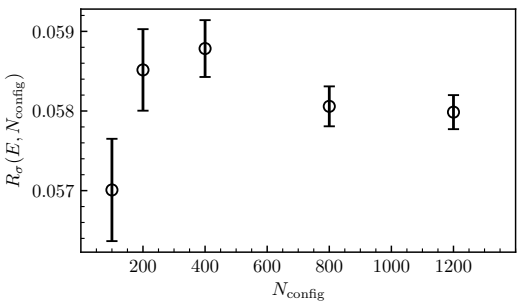
Effect of varying λ

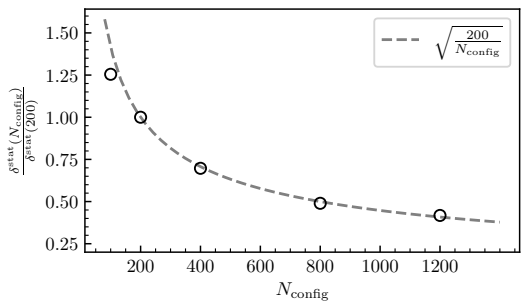


HLT stability analysis: algorithmic parameters are tuned by identifying a *stability region* where systematic fluctuations lie within statistical errors.

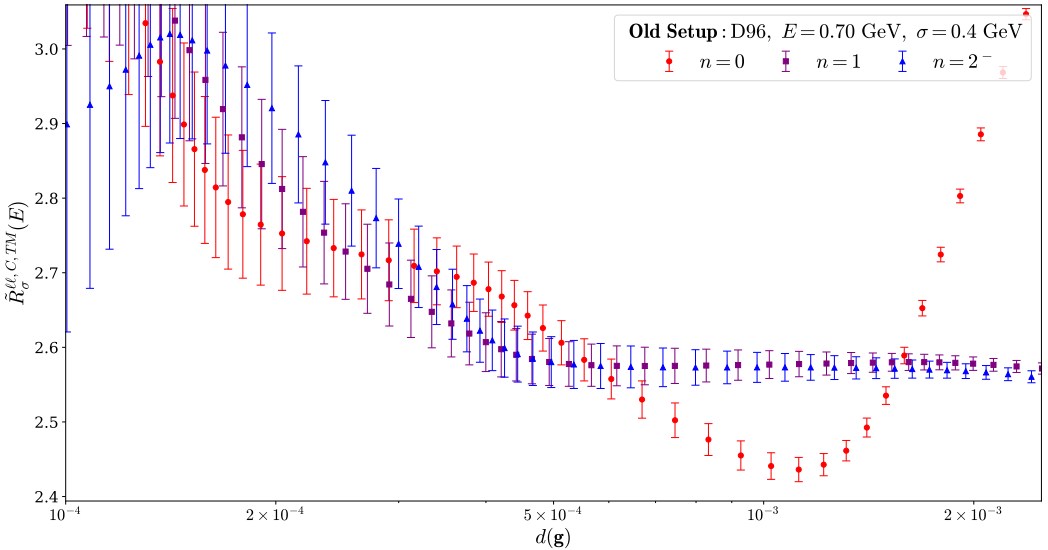
What happens if the statistics is increased?







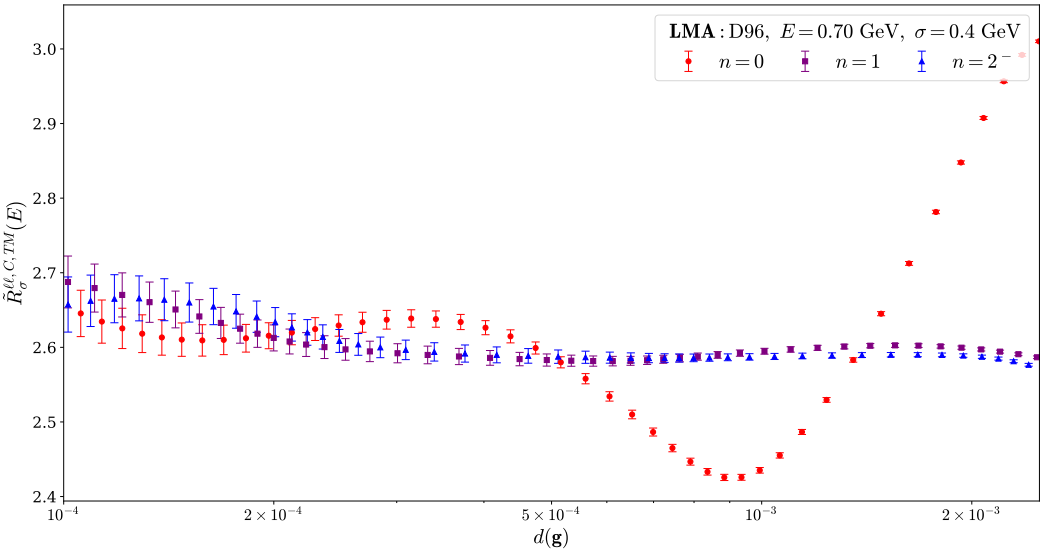
Previously, it was . . .



Blinding data: $R_\sigma^{\ell\ell,C,TM}(E) \to \tilde{R}_\sigma^{\ell\ell,C,TM}(E)$.

HLT stability analysis: algorithmic parameters are tuned by identifying a *stability region* where systematic fluctuations lie within statistical errors and results from different norms are compatible.

Now, it is ...



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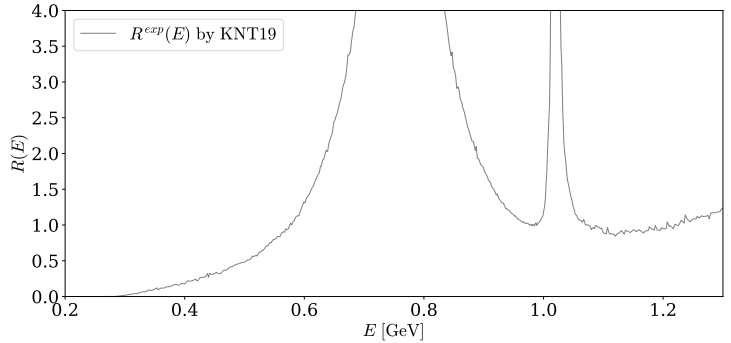
$R_\sigma^{\rm exp}(E)$ based on KNT19 compilation



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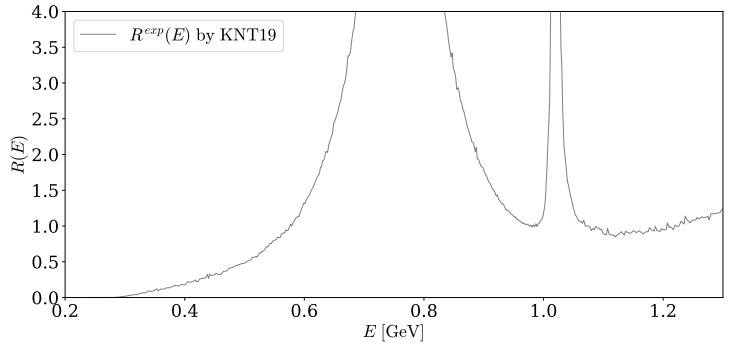

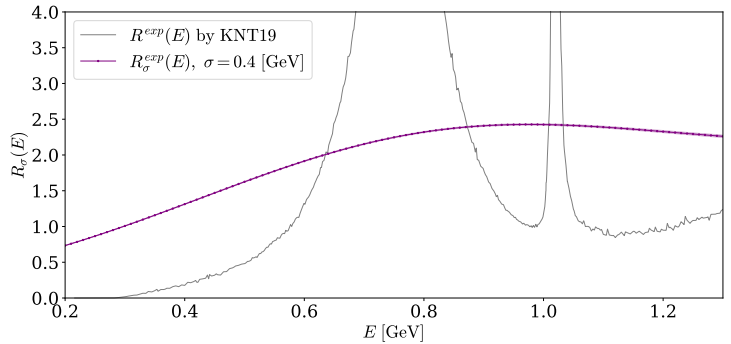

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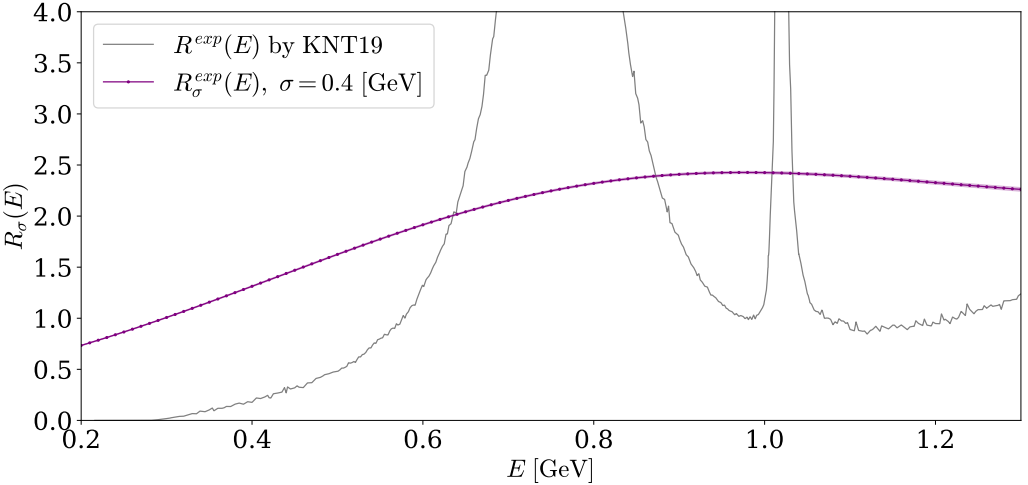

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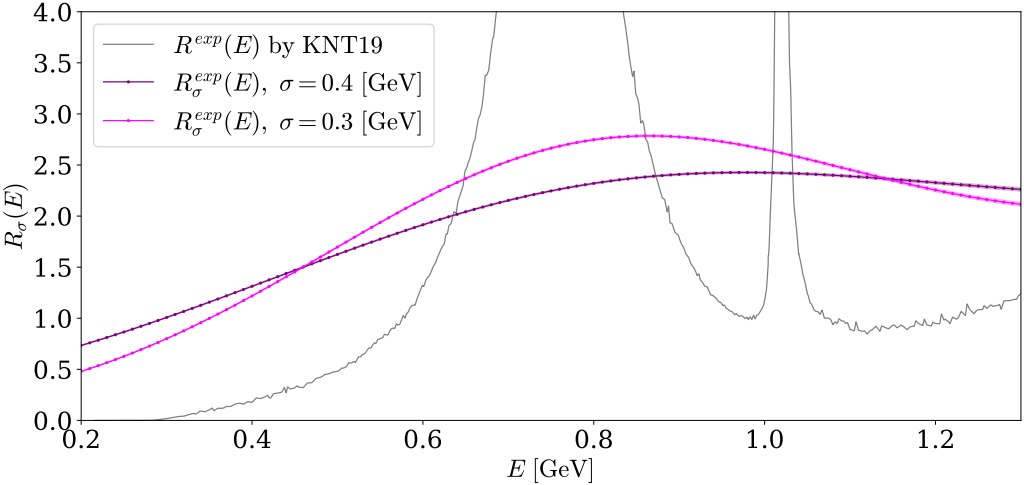

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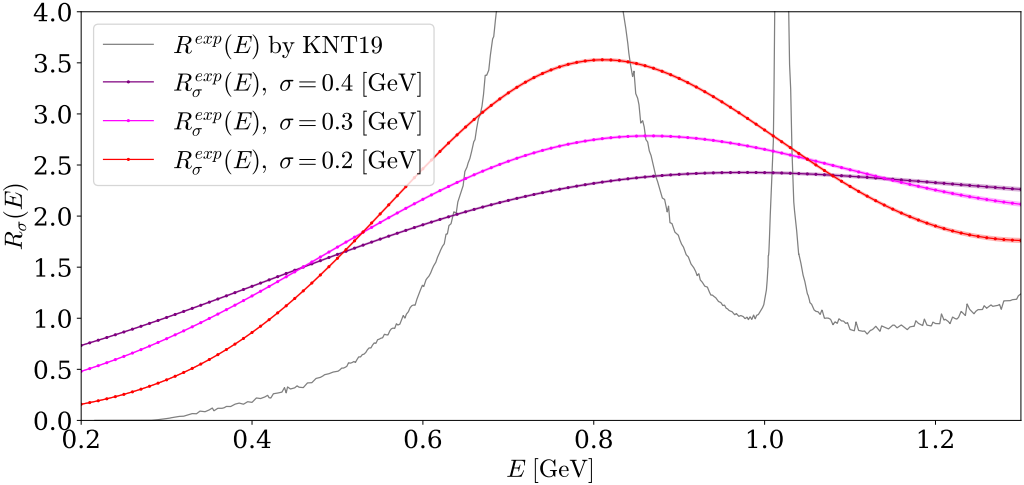
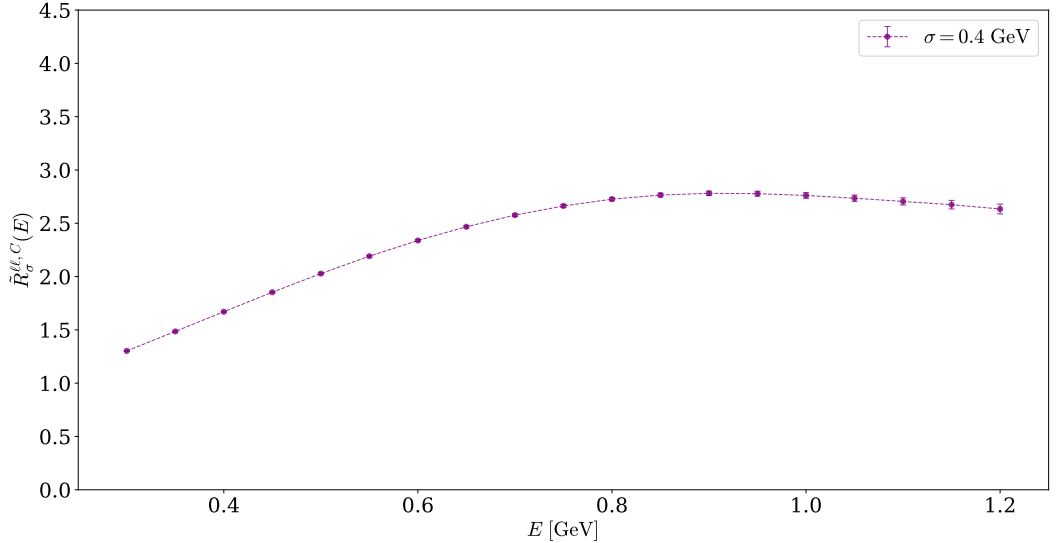


Light-light connected contribution



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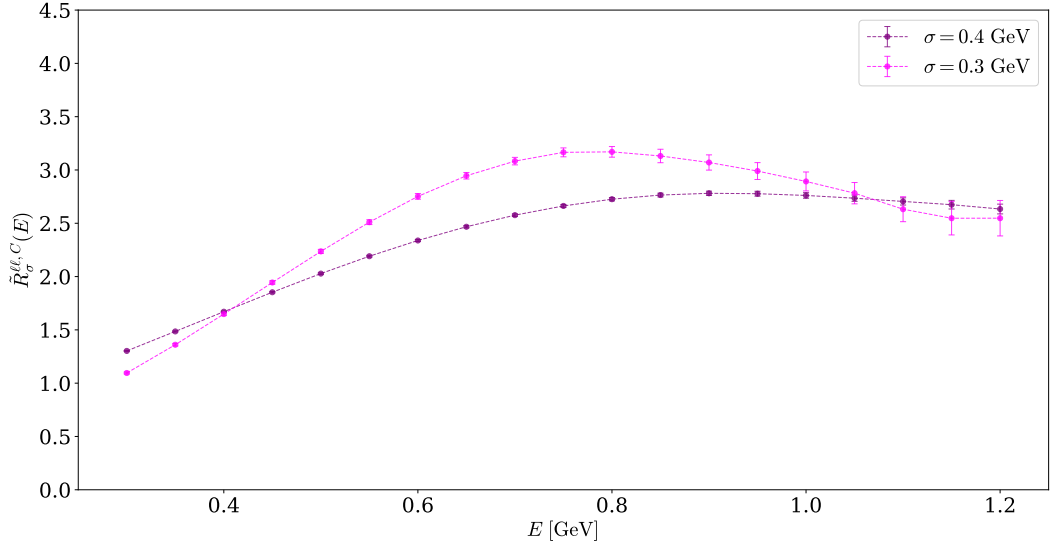

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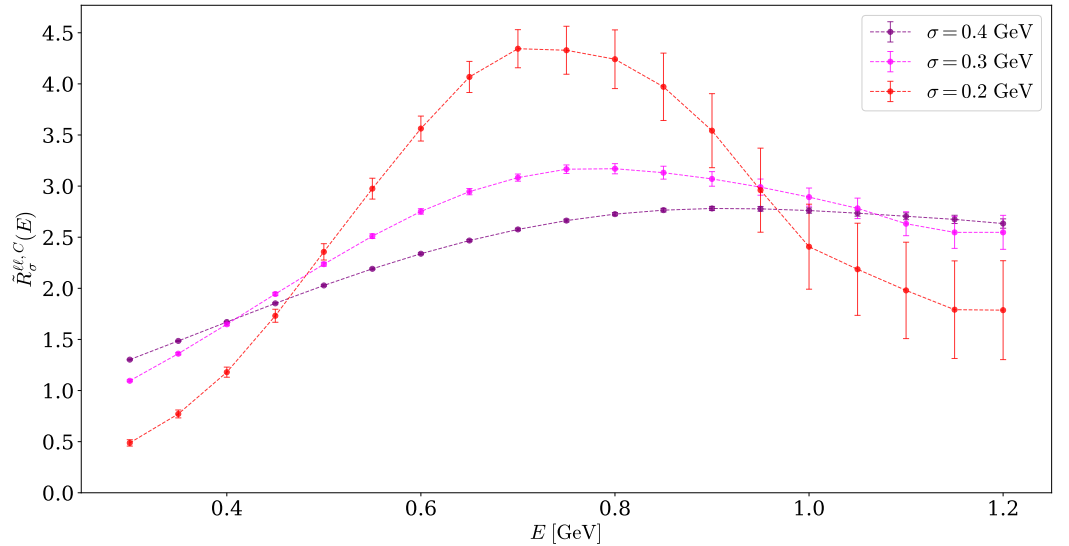

How we proceed so far

$$R_\sigma(E) = R_\sigma^{\ell\ell,C}(E) + R_\sigma^{ss,C}(E) + R_\sigma^{cc,C}(E) + R_\sigma^{disco}(E)$$

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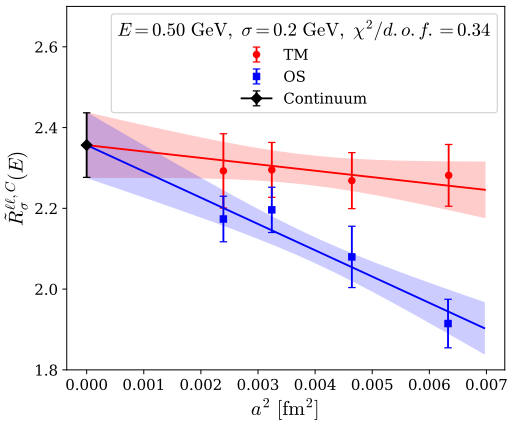

- *Continuum extrapolations*: constrained extrapolation, we fit TM and OS data by performing a correlated χ^2-minimization at fixed E, σ at $L \sim 5$ fm.
- Then,

$$\lim_{L\to\infty} R_\sigma(E)$$

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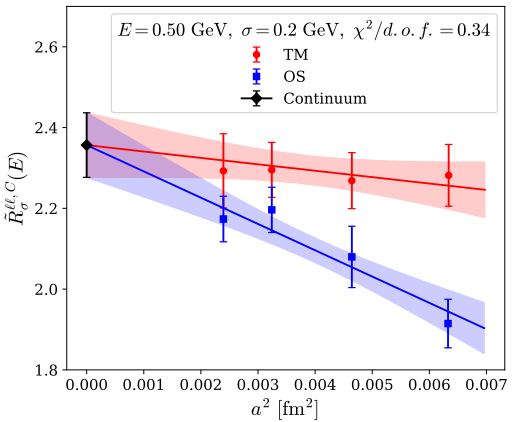

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- Then,

$$\lim_{L\to\infty} R_\sigma(E)$$

- We do not take the $\sigma \to 0$ limit, since we compare our results with $R_\sigma^{\rm exp}(E)$.
- If one were to carry it out, then

$$\lim_{\sigma\to 0} \lim_{L\to\infty} R_\sigma(E)$$ IN THIS PRECISE ORDER!

[M. T. Hansen, H. B. Meyer, D. Robaina 2017]
[M. Hansen, A. Lupo, N. Tantalo 2019]

Finite-volume effects: O(3) non-linear σ-model

In JHEP 07 (2022) 034, the authors addressed the question of **finite-volume effects in a model smeared spectral density**.

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$$\rho_{\infty,\sigma}(E) - \rho_{L,\sigma}(E) = \mathcal{O}(L^{-\infty})$$

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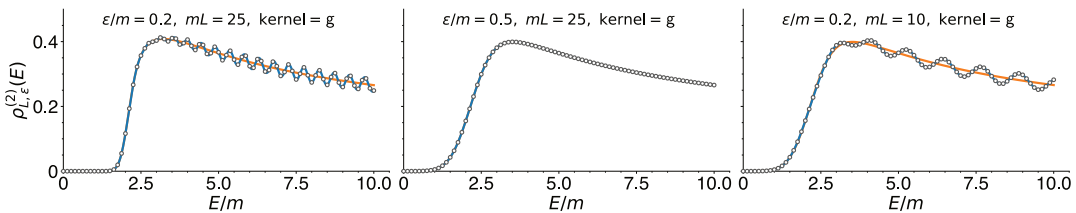

J. Bulava, M. T. Hansen, M. W. Hansen, A. Patella, and N. Tantalo, *Inclusive rates from smeared spectral densities in the two-dimensional O(3) non-linear σ-model*, JHEP **07**, 034 (2022).

Finite-volume effects: coming back to $R_\sigma(E)$ ...

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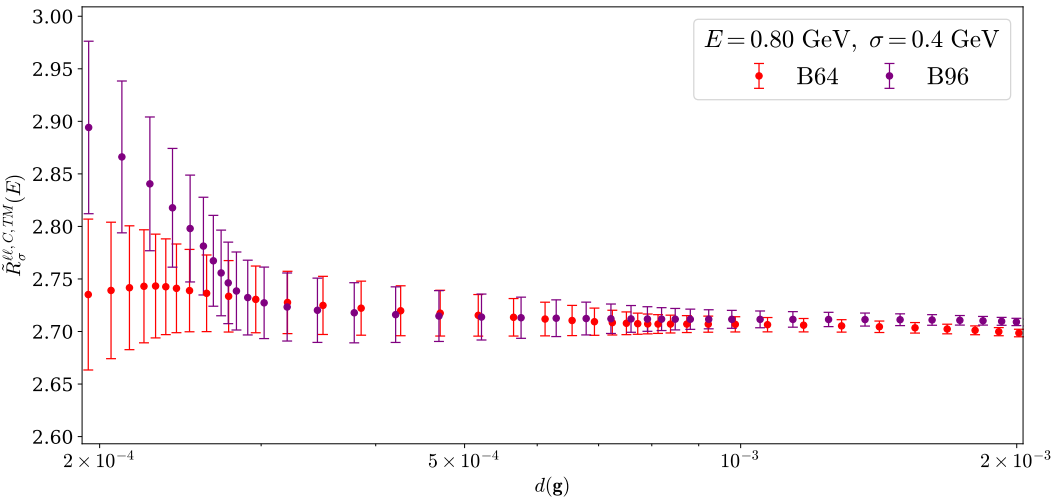

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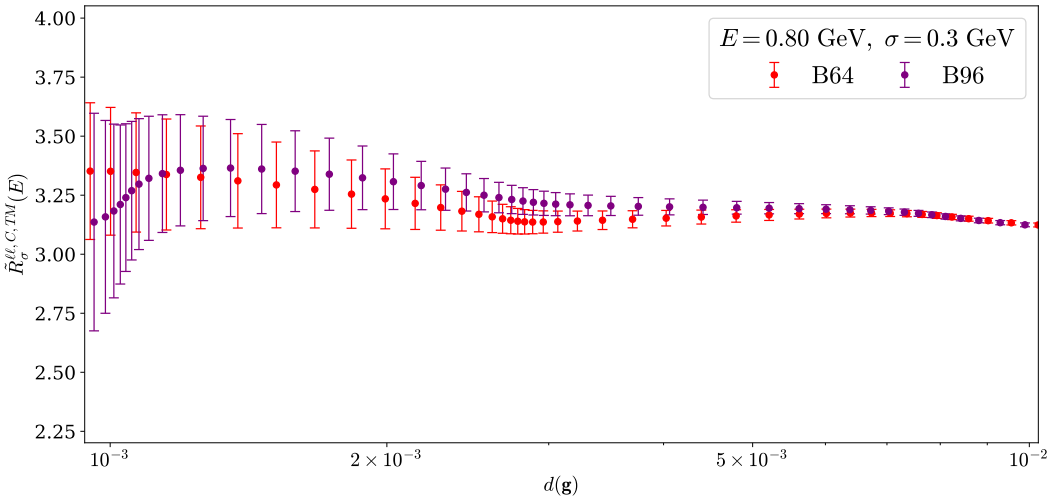

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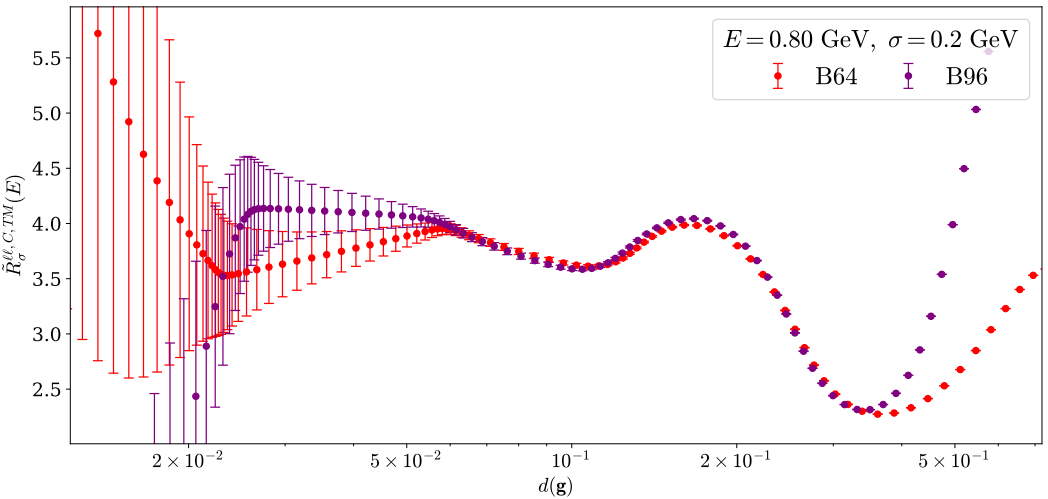

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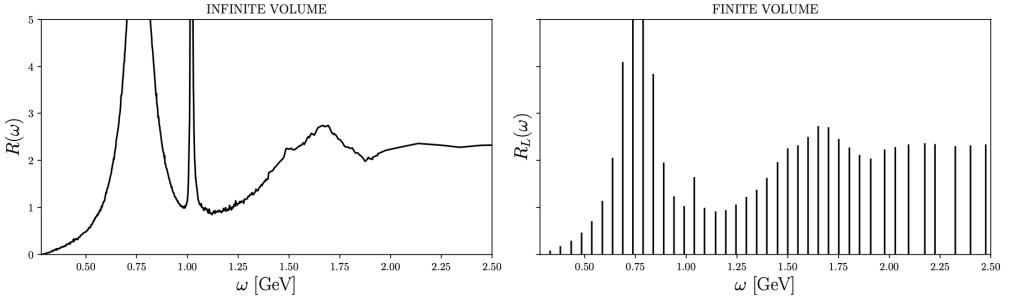
- We plan to account **QED+IB effects** from first principles.

Thank you!

Backup slides

Spectral density in a finite volume



$$R_\sigma(E) = \int_{2m_\pi}^{+\infty} \mathrm{d}\omega\, R(\omega)\ \underbrace{G_\sigma(E-\omega)}_{\text{Gaussian kernels}}\ ; \qquad G_\sigma(E-\omega) = \frac{e^{-\frac{(E-\omega)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\,.$$