Bogoliubov Theory of 1D Anyons in a Lattice

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Abstract

In a one-dimensional lattice anyons can be defined via generalized commutation relations containing a statistical parameter θ, which interpolates between the bosonic limit $\theta = 0$ and the pseudo-fermionic limit $\theta = \pi$ [1]. The corresponding anyon-Hubbard model is mapped to a Bose-Hubbard model via a fractional Jordan-Wigner transformation, yielding a complex hopping term with a density-dependent Peierls phase. Here we work out a corresponding Bogoliubov theory. To this end we start with the underlying mean-field theory, where we allow for the condensate a finite momentum, where we allow for a finite condensate momentum in order to incorporate inherent spatio-temporal asymmetry [2-5]. With this we calculate various physical properties and discuss their dependence on the statistical parameter. Among them are both the condensate as well as the equation of state and the compressibility. Furthermore, we analyze the resulting dispersion of Bogoliubov quasi-particles, which turns out to be in accordance with the Goldstone theorem. In particular, this leads to two different sound velocities for wave propagations to the left and the right, which originates from parity breaking. Furthermore, we determine the quantum depletion of the condensate and investigate the quasi-momentum distributions of anyonic as well as bosonic particles, as they are experimentally relevant [6,7].

The Model

- Anyon-Hubbard model in 1D lattice[1]:

$$\hat{H} = -J\sum_{j=1}^{L}\left(\hat{a}_j^\dagger \hat{a}_{j+1} + \text{h.c.}\right) - \mu\sum_j^L \hat{n}_j + \frac{U}{2}\sum_j^L \hat{n}_j(\hat{n}_j - 1)$$

- Generalized commutation relations[1]: $\hat{a}_i\hat{a}_j^\dagger - e^{i\theta\,\text{sgn}(i-j)}\hat{a}_j^\dagger\hat{a}_i = \delta_{i,j}$, $\theta \in [0,\pi]$
- Bosonic fractional Jordan-Wigner transformation[1]: $\hat{a}_j = \hat{b}_j\, e^{i\theta\sum_{l<j}\hat{n}_l}$
- AHM in bosonic representation with density-dependent Peierls phase:

$$\hat{H}^{(\mathrm{B})}_{\mathrm{AHM}} = -J\sum_j\left(\hat{b}_j^\dagger e^{i\theta\hat{n}_j}\hat{b}_{j+1} + \text{h.c.}\right) + \sum_j\left[\frac{U}{2}\hat{n}_j(\hat{n}_j-1) - \mu\hat{n}_j\right]$$

$$\Big\downarrow \text{ Normal ordering}^{[8]}$$

$$:\hat{H}^{(\mathrm{B})}_{\mathrm{AHM}}: = -J\sum_{j=1}^{L}\sum_{q=0}^{\infty}\sum_{m=0}^{q}\left[\frac{(i\theta)^q}{q!}\underbrace{S(q,m)}_{\substack{\text{Stirling number}\\ \text{of 2nd kind}}}(\hat{b}_j^\dagger)^{m+1}(\hat{b}_j)^m\hat{b}_{j+1} + \text{h.c.}\right]$$

$$-\mu\sum_{j=1}^{L}\hat{b}_j^\dagger\hat{b}_j + \frac{U}{2}\sum_{j=1}^{L}\hat{b}_j^\dagger\hat{b}_j^\dagger\hat{b}_j\hat{b}_j\,,$$

Bogoliubov Theory

- Modified Bogoliubov *ansatz*:

$$\hat{b}_j = \sqrt{n_{k_0}}\,e^{-ik_0ja} + \frac{1}{\sqrt{L}}\sum_{k\neq k_0}\hat{b}_k\,e^{-ikja}$$

  - Shift of quasi-condensate momentum k_0

- Decomposition of Hamiltonian:

$$:\hat{H}^{(\mathrm{B})}_{\mathrm{AHM}}: = E_0 + \hat{H}_1 + \hat{H}_2 + \mathcal{O}[(\delta\hat{b}_{k_0}^{(\dagger)})^2] + \mathcal{O}[(\hat{b}_{k\neq k_0}^{(\dagger)})^3]$$

$\hat{H}_2$ → Contributions from bi-linear operators

$\hat{H}_1$ → First-order fluctuations of condenate

E_0 → Condensate energy

Ground-state Properties

$$E_0 = L\left\{-2Je^{-2n_{k_0}\sin^2\frac{\theta}{2}}\cos\left(n_{k_0}\sin\theta - k_0\right)n_{k_0} - \mu n_{k_0} + \frac{U}{2}n_{k_0}^2\right\}$$

- Determine the minimum w.r.t. order parameters k_0 & n_{k_0}
- Results from Hessian matrix:

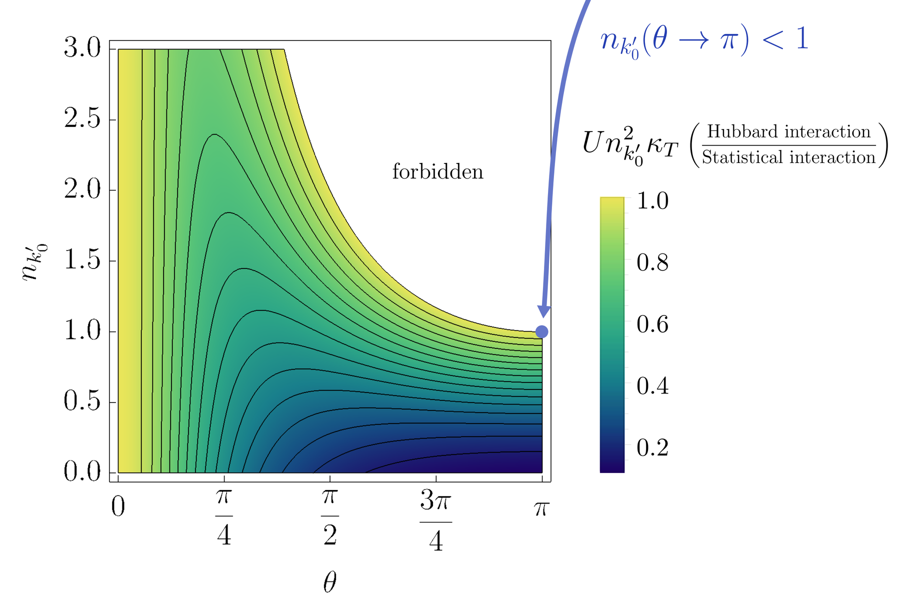
  ⇒ Condition for k_0: $k_0'\cdot a = n_{k_0'}\sin\theta + 2m\pi,\ m\in\mathbb{Z}$

  ⇒ Equation of state: $\mu = Un_{k_0'} - 2Je^{-2n_{k_0'}\sin^2\frac{\theta}{2}}\left(1 - 2n_{k_0}\sin^2\frac{\theta}{2}\right)$

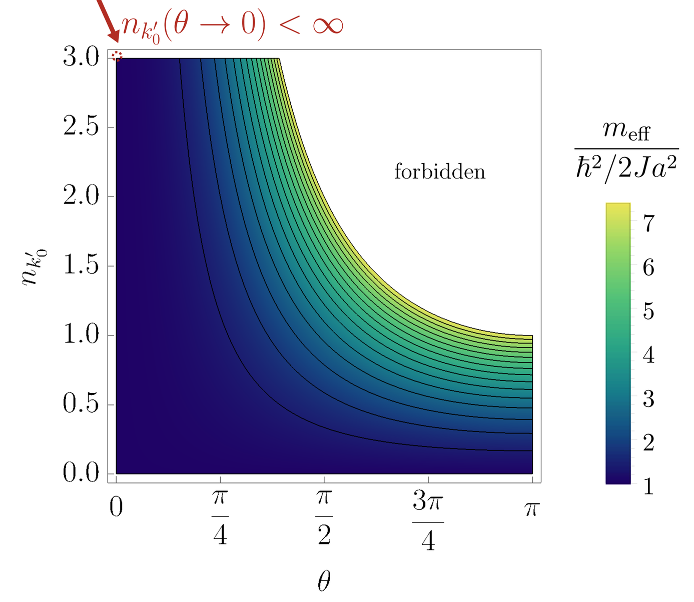
  ⇒ Generalized Pauli principle from mean-field stability:

$$\boxed{n_{k_0'} < \sin^{-2}\frac{\theta}{2}}$$

Isothermal compressibility: $n_{k_0'}(\theta\to\pi) < 1$



Effective mass: $n_{k_0'}(\theta\to 0) < \infty$



$$\kappa_T = \left[Un_{k_0'}^2 + 8n_{k_0'}^2Je^{-2n_{k_0'}\sin^2\frac{\theta}{2}}\sin^2\frac{\theta}{2}\left(1 - n_{k_0'}\sin^2\frac{\theta}{2}\right)\right]^{-1}$$

$$m_{\mathrm{eff}} = \frac{\hbar^2}{2Ja^2e^{-2n_{k_0'}\sin^2\frac{\theta}{2}}}$$

Elementary Excitations

- Relative momentum: $q = k - k_0$
- Bogoliubov transformation: $\begin{bmatrix}\hat{d}_{k_0+q}\\ \hat{d}_{k_0-q}^\dagger\end{bmatrix} = \begin{bmatrix}\cosh\alpha_q & \sinh\alpha_q\\ \sinh\alpha_q & \cosh\alpha_q\end{bmatrix}\begin{bmatrix}\hat{b}_{k_0+q}\\ \hat{b}_{k_0-q}^\dagger\end{bmatrix}$
  - Diagonalized Hamiltonian:

$$\hat{H}_{\mathrm{diag}} = E_0 + \frac{1}{2}\sum_{q\neq 0}\left[E_{-q} - A_{-q}\right] + \sum_{q\neq 0}E_q\,\hat{d}_{k_0+q}^\dagger\hat{d}_{k_0-q}$$

  - Coefficients: $A_q = -2Je^{-2n_{k_0}\sin^2\frac{\theta}{2}}\left[-4n_{k_0}^2\cos\theta\sin^2\frac{\theta}{2} - 2n_{k_0}\sin^2\frac{\theta}{2} + n_{k_0}\left(\cos\left(\theta - qa\right) - \cos qa\right) + \cos qa - 1\right] + Un_{k_0}$

$$B_q = -Je^{-2n_{k_0}\sin^2\frac{\theta}{2}}\left[-4n_{k_0}^2\cos\theta\sin^2\frac{\theta}{2} - 2n_{k_0}\sin^2\frac{\theta}{2} + i\cdot n_{k_0}\sin\theta + n_{k_0}(e^{-i\theta} - 1)\,e^{-iqa}\right] + \frac{U}{2}n_{k_0}$$

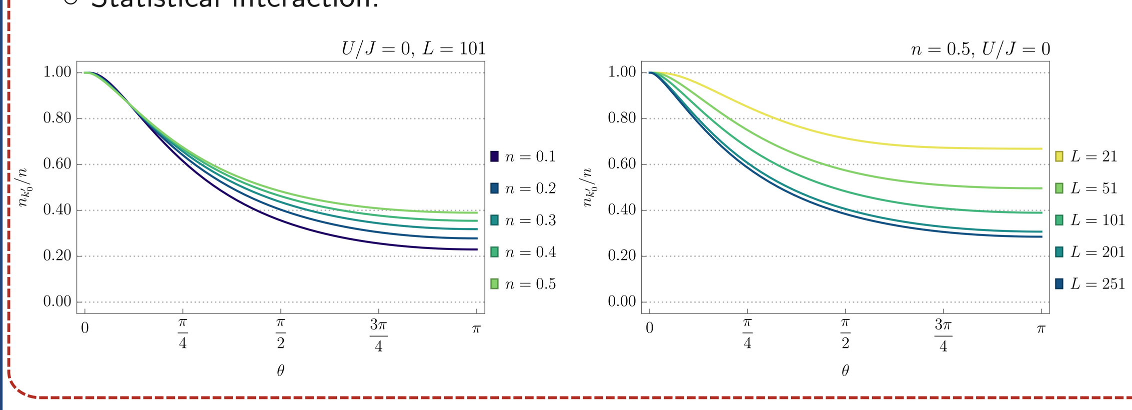
- Bogoliubov dispersion:

$$E_q = \underbrace{\frac{1}{2}\sqrt{(A_q + A_{-q})^2 - \left[2\mathrm{Re}\left(B_q + B_{-q}\right)\right]^2}}_{\text{Even in } q} + \underbrace{\frac{1}{2}(A_q - A_{-q})}_{\text{Odd in } q}$$

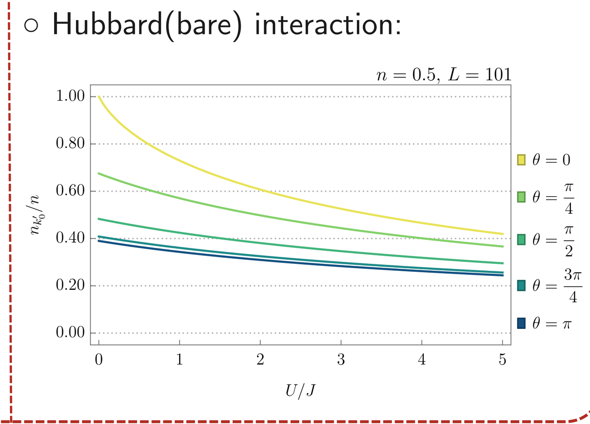
- Quantum Depletion (Popov approximation[9]):

$$n = n_{k_0} + \frac{1}{2L}\sum_{q\neq 0}\left\{\frac{A_q + A_{-q}}{2E_q - (A_q - A_{-q})} - 1\right\}$$

○ Statistical interaction:



○ Hubbard(bare) interaction:

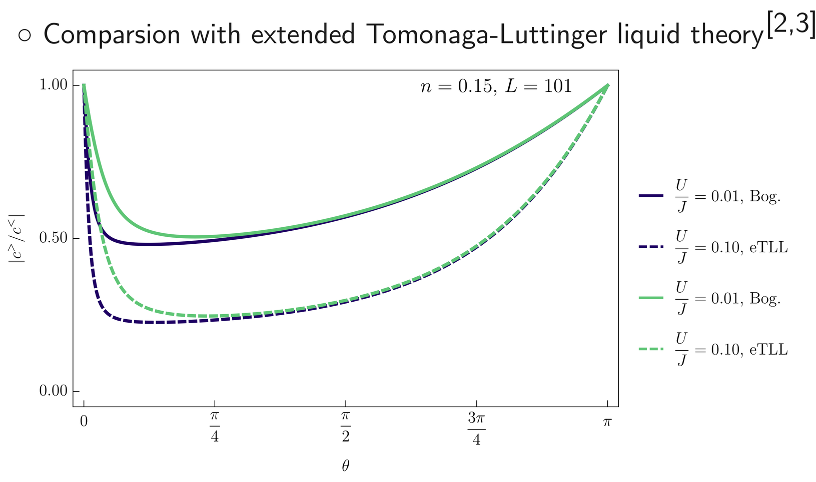


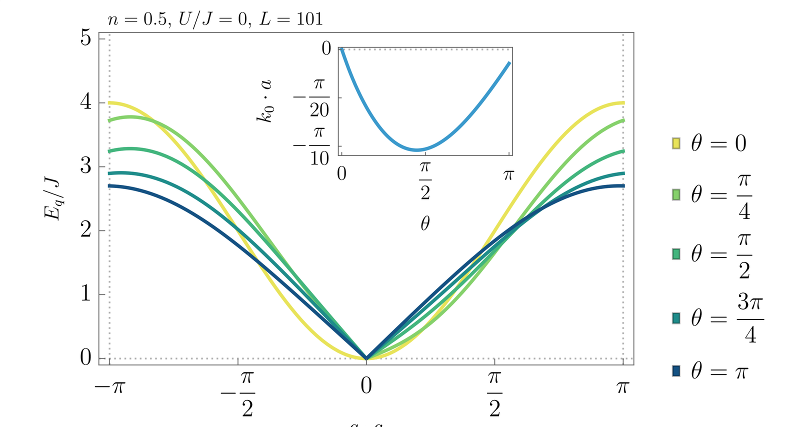
- Asymmetric dispersion around the condensate momentum

⇒ Parity-broken transport[6]

⇒ Two sound modes with different velocities $c^{>}$ & $c^{<}$

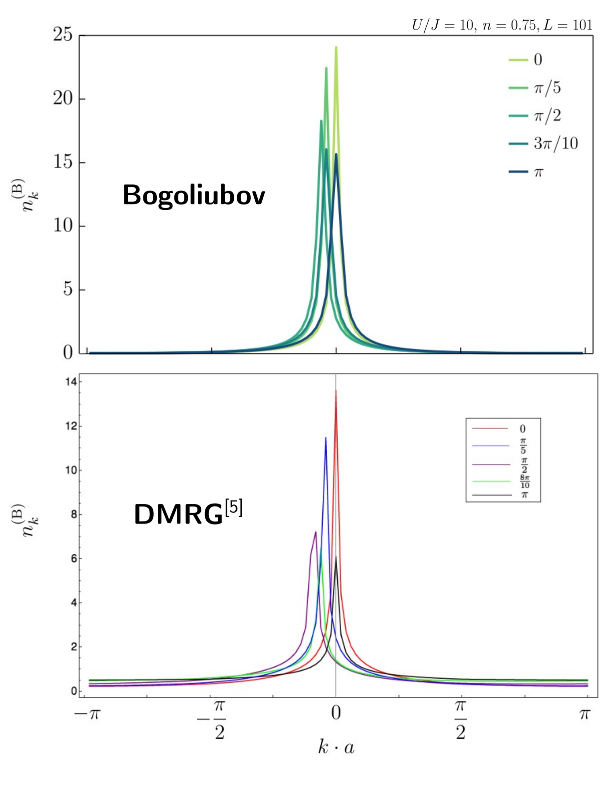
○ Comparsion with extended Tomonaga-Luttinger liquid theory[2,3]:





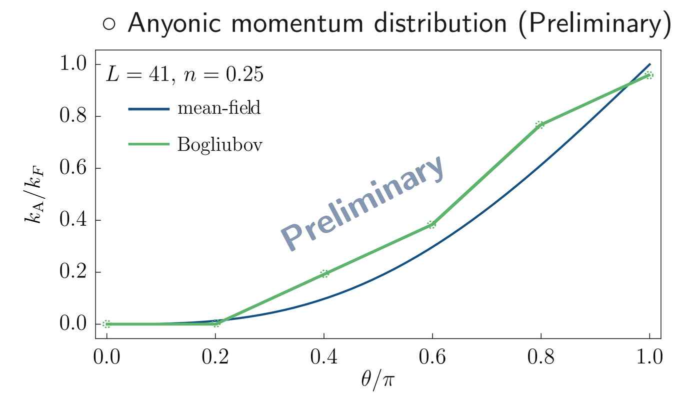
- Momentum distributions

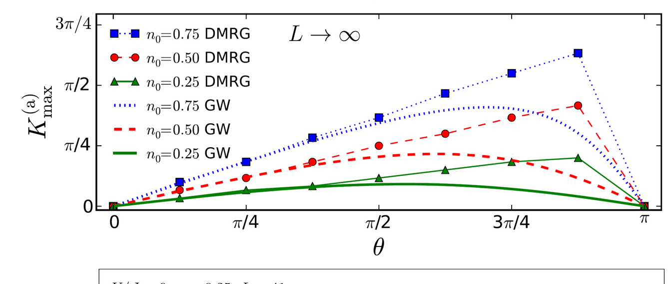
○ Bosonic momentum distribution



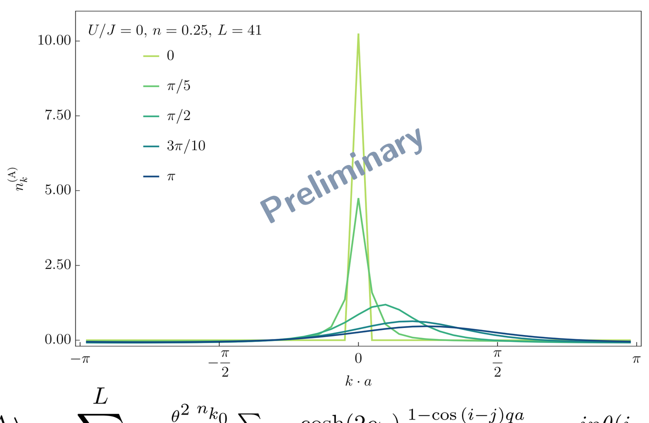
$$\langle\hat{n}_k^{\mathrm{B}}\rangle = \frac{1}{L}\sum_{i,j=1}^{L}e^{ik(i-j)a}\left\langle\hat{b}_i^\dagger\hat{b}_j\right\rangle = \sum_{i,j=1}^{L}\hat{n}_{k,ij}^{\mathrm{B}}$$

○ Anyonic momentum distribution (Preliminary)





from [2]



$$\langle\hat{n}_k^{\mathrm{A}}\rangle = \sum_{i,j=1}^{L}\underbrace{e^{-\frac{\theta^2}{2}\frac{n_{k_0}}{L}\sum_{q\neq 0}\cosh(2\alpha_q)\frac{1-\cos(i-j)qa}{1-\cos qa}}}_{\text{damping factor}}\ \underbrace{e^{-in\theta(i-j)}}_{\text{phase shift}}\hat{n}_{k,ij}^{\mathrm{B}}$$

Conclusion & Outlook

- Generalized Pauli exclusion principle on mean-field level
- Analytical calculation of anyonic Bogoliubov theory
- Asymmetric Bogoliubov dispersion, statisitcal depletion
- Correlation functions, bosonic and anyonic momentum distributions
- Further analysis: superfluidity, dynamical correlations, open system, ...

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