Exact duality at low energy in a Josephson junction coupled to a transmission line

Luca Giacomelli

ECT* workshop

8 October 2025

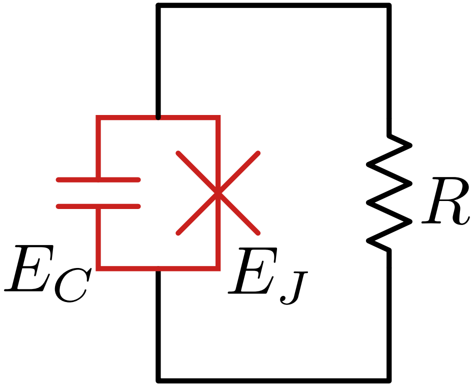
[LG,Ciuti - Nat. Comm. 15, 5455 (2024)]

[Paris,LG,...,Mora - PRB, 111(6), 064509 (2025)]

[LG,Devoret,Ciuti - arXiv:2504.14651]

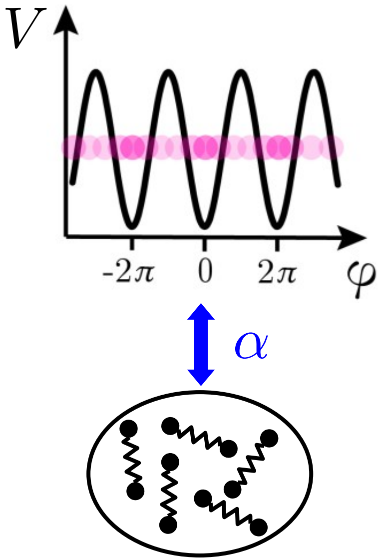
The Schmid-Bulgadaev transition

Resistively shunted junction



$$\hat{\mathcal{H}} = 4E_C\hat{N}^2 - E_J\cos\hat{\varphi}$$

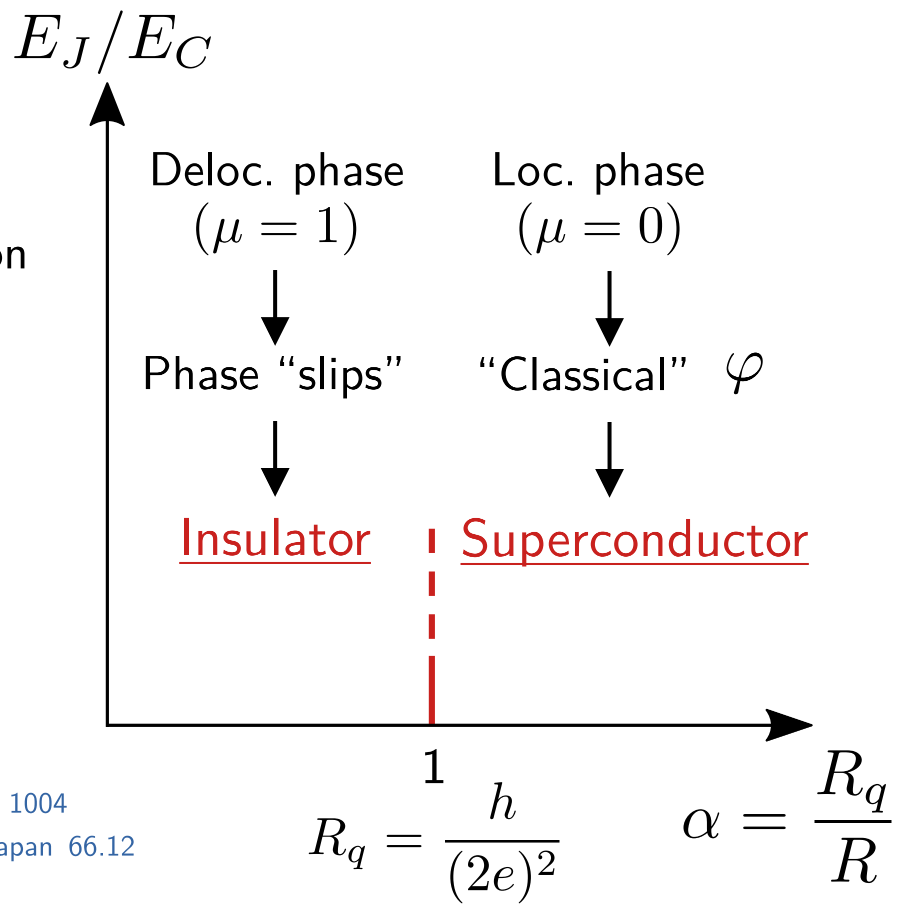
Caldeira-Leggett model (Ohmic bath) for Brownian particle in cos potential



The Schmid-Bulgadaev transition

[Schmid (1983) PRL, 51(17), 1506]
[Bulgadaev (1984) JETP Lett, 39(6), 264-267]

- Perturbatively in E_J/E_C
  - Localization of the particle above critical dissipation (superconducting state)
- Characterized with the DC mobility $\mu(\omega \to 0)$
  - Reflects in the IV curve at small bias (Kubo)
- Confirmed by many following studies



Theory

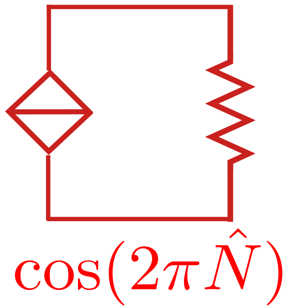
Aslangul et al. (1985) Phys.Lett.A, 111(4), 175-178
Fisher, Zwerger (1985) PRB, 32(10), 6190
Guinea et al. (1985) PRL, 54(4), 263
Schön, Zaikin (1990) Phys. Rep., 198(5-6), 237-412..
Herrero, Zaikin (2002) PRB, 65(10), 104516
Werner, Troyer (2005) PRL, 95(6), 060201
Lukyanov, Werner (2007) J. Stat. Mech., 2007(06), P06002.

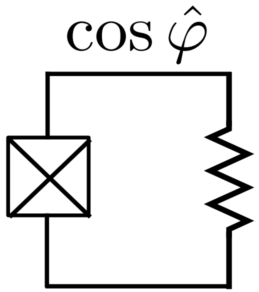
Experiments

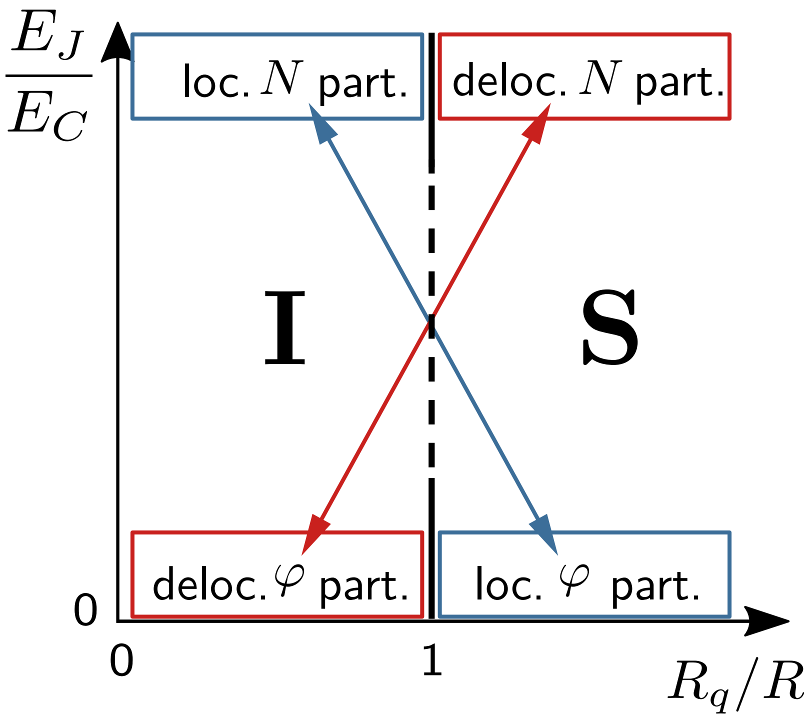
Penttilä et al. (1999) PRL, 82(5), 1004
Yagi et al. (1997) J. Phys. Soc. Japan 66.12

Schmid (approximate) phase-charge duality

Perturbative in phase slips (tightly bound particle)







Conjecture:

the transition line is vertical

Perturbative in Cooper pair tunneling (weakly bound particle)

A renewed debate

[Murani et al. (2020) PRX]
[Hakonen, Sonin, Comment (2021)]
[Murani et al., Reply (2021)]

- Finite frequency experiment (no phase transition)

[Masuki et al. (2022) PRL]
[Sepulcre et al. Comment. (2022)]
[Masuki.et al. Reply. (2022)]

- New RG calculations (heavily modified phase diagram)

[Subero et al. (2023) Nat.Comm.]

- Coulomb blockade but inductance-like for thermal transport

[Kuzmin et al. (2025) Nat.Phys.]

- Transmission line instead of resistor, probe the environment (transition yes)

[Subero et al. arXiv:2509.20480]

- New low-frequency measurements of IV curve (transition yes)

More theoretical papers [Houzet, Glazman (2020) PRL] [Daviet, Dupuis (2023) PRB] [Houzet et al. (2024) PRB] [Altimiras et al. arXiv:2312.14754] [Kashuba, Riwar (2024) PRB] [Kurilovich et al. (2025) Nat.Comm]

Our approach to check the theory

- Tricky problem because it involves several limits:
  - Infinite environmental size
  - Infinite ultraviolet cutoff
  - Zero temperature
  - Zero frequency

- Most studies restricted to perturbative regimes

- We solve by numerical diagonalization with finite number of modes
  - Limited in the environment size but exact in E_J
  - No limits are taken

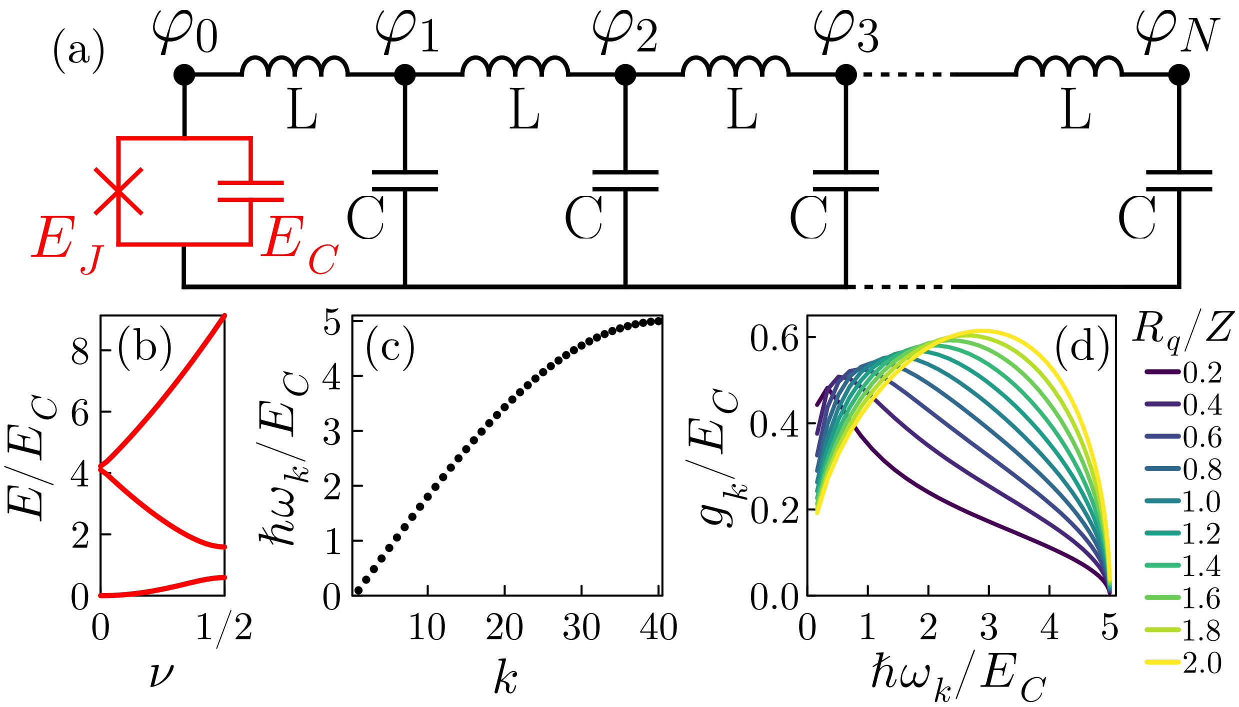
Key messages

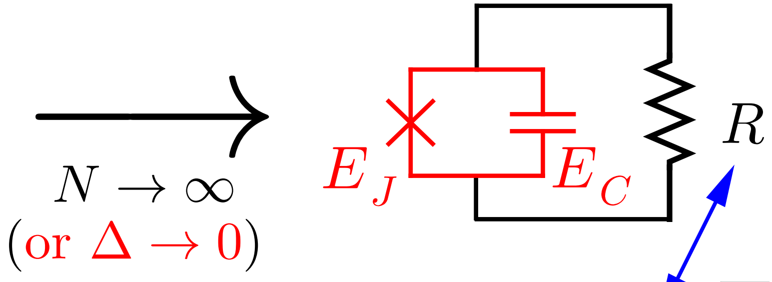
- There is a critical point in the Caldeira-Leggett model for a JJ
- We get the thermodynamic critical spectrum from finite systems
- The system is exactly self-dual: critical point independent of E_J/E_C

Circuit QED of the Schmid transition

[LG,C.Ciuti - Nat. Comm. 15, 5455 (2024)]

A JJ coupled to a transmission line





Characteristic impedance: $Z = \sqrt{\dfrac{L}{C}}$

Cutoff frequency: $\omega_c = \dfrac{2}{\sqrt{LC}}$

Free spectral range: $\Delta = \dfrac{\pi}{\sqrt{LC}}\dfrac{1}{N}$

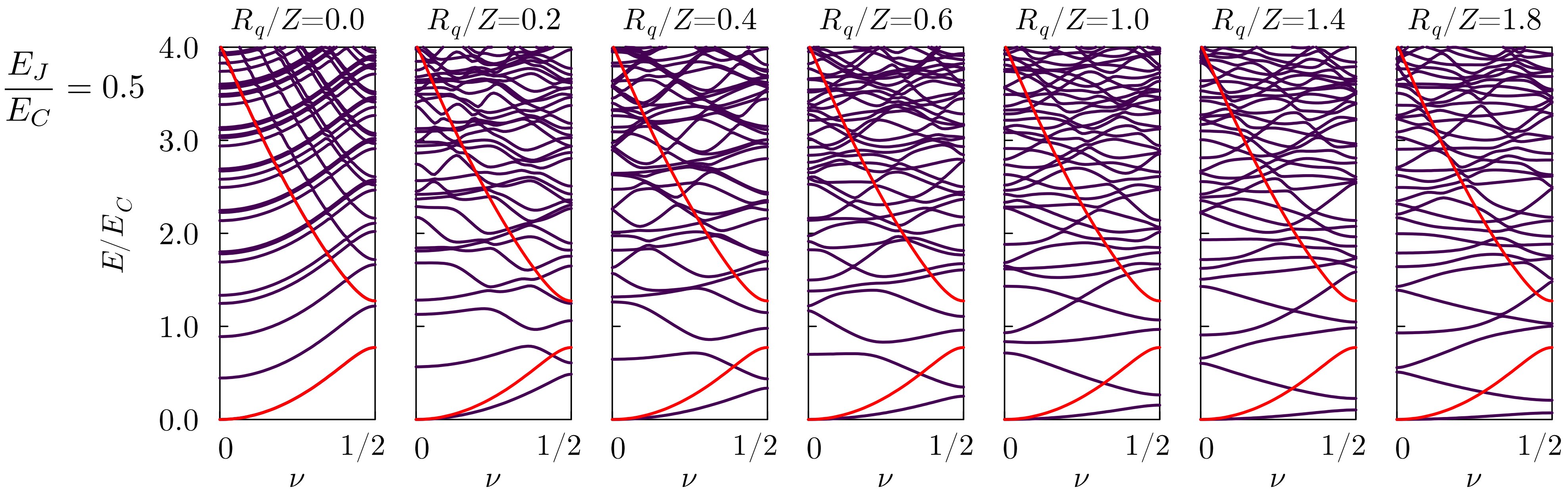
Charge gauge Hamiltonian

$$\hat{\mathcal{H}} = \underset{\hat{\mathcal{H}}_{JJ}}{\boxed{4E_C\hat{N}^2 - E_J\cos\hat{\varphi}}} + \underset{\hat{\mathcal{H}}_{modes}}{\boxed{\sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k}} + \underset{\hat{\mathcal{H}}_{int}}{\boxed{i\hat{N}\sum_k g_k(\hat{a}_k^\dagger - \hat{a}_k)}}$$

Bands for the interacting system

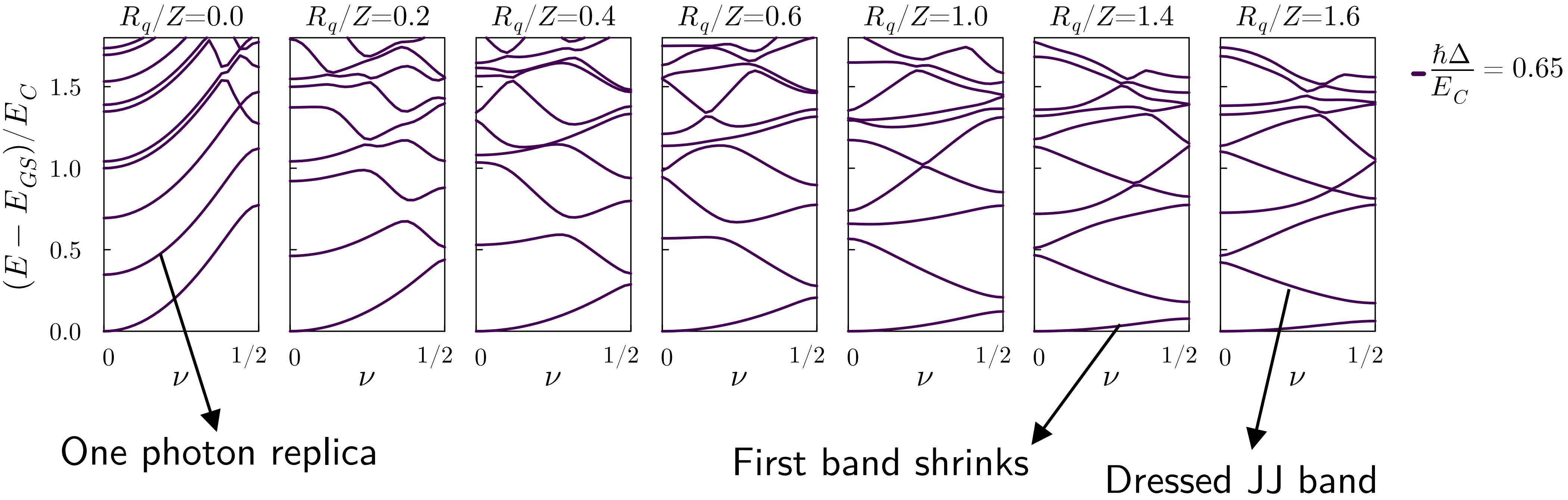
$$\hat{\mathcal{H}} = 4E_C\hat{N}^2 - E_J\cos\hat{\varphi} + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + i\hat{N}\sum_k g_k(\hat{a}_k^\dagger - \hat{a}_k)$$

- We can use Bloch's theorem: $|\nu, s\rangle = e^{i\nu\phi}u_\nu^s(\phi) \longrightarrow \nu$ "quasi-charge"



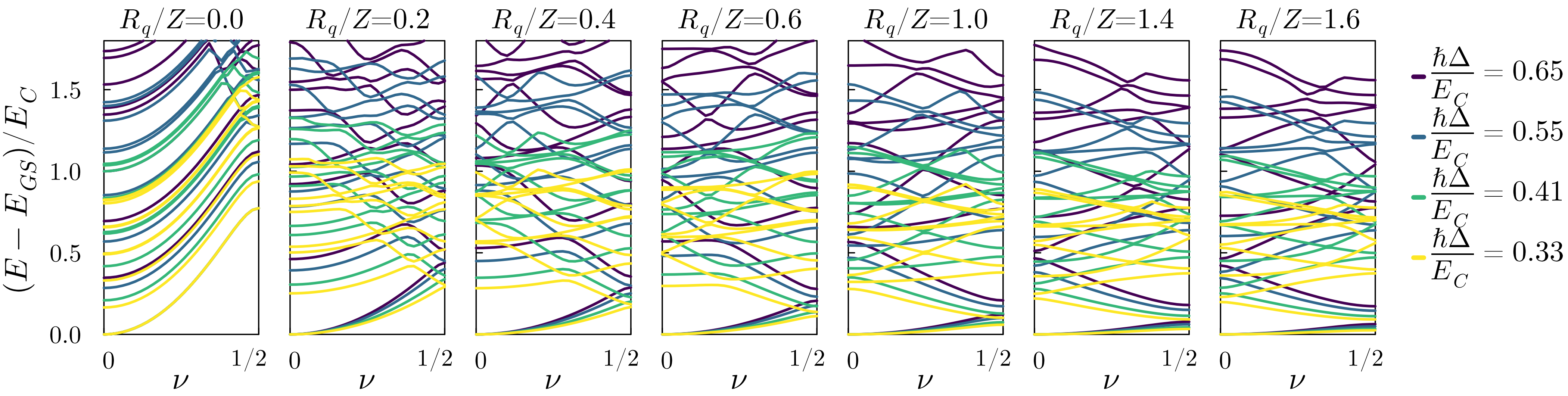
Low energy spectra

$$\frac{E_J}{E_C} = 0.5$$



Decrease Δ

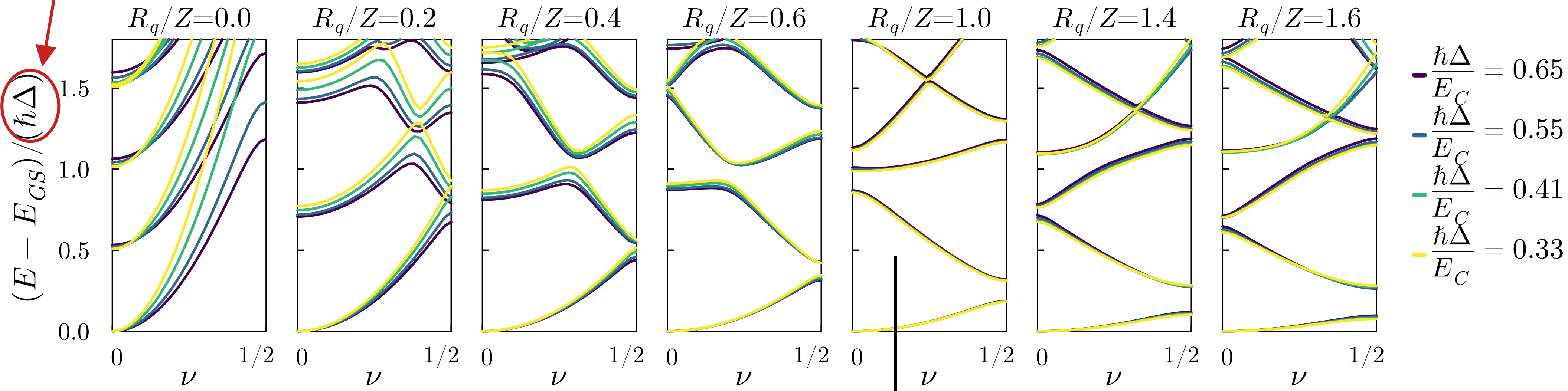
$$\frac{E_J}{E_C} = 0.5$$



Decrease Δ

Rescale by Δ

$$\frac{E_J}{E_C} = 0.5$$



Scale invariance

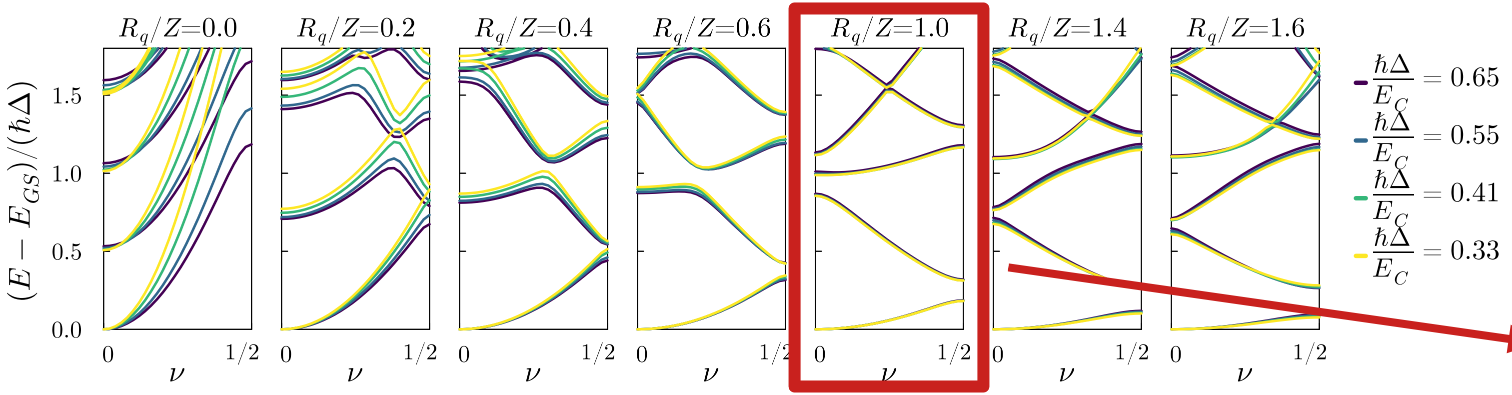
Critical point

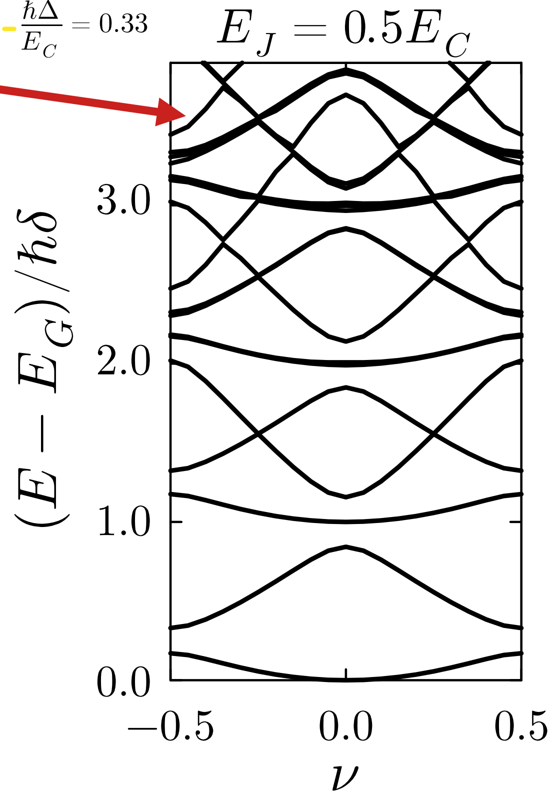
Spectrum on the critical line

[Paris, LG, Daviet, Ciuti, Dupuis, Mora - PRB, 111(6), 064509 (2025)]

(Editors' Suggestion)

The critical spectrum





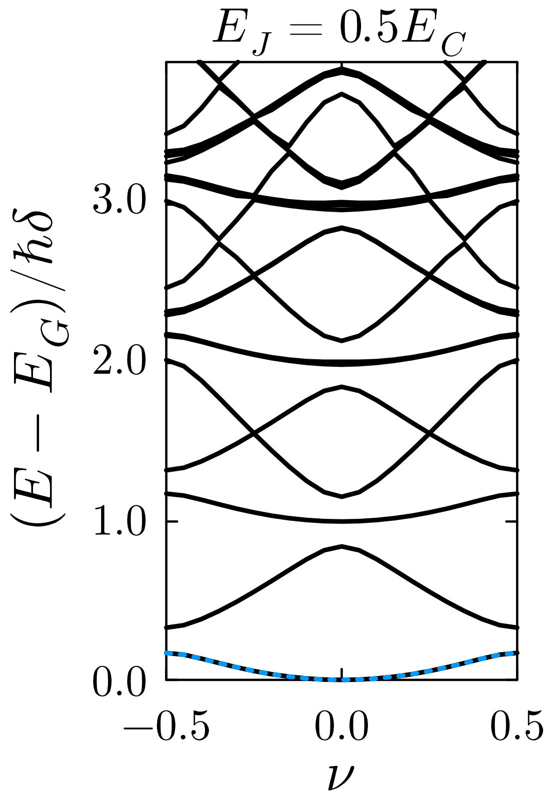
- Scale invariance of the spectrum
  - We are thermodynamic
- Simple structure with replicas
  - "Non-interacting" system
- Extends to finite frequencies
  - Visible on the photons

CFT spectrum at the critical point $Z = R_q$

- Prediction for the bands at the critical point [Hasselfield et al. Ann. Phys. 321,2849 (2006)]

$$\frac{E(\nu)}{\hbar\Delta} = (\underbrace{n}_{\text{“Band index” } n \in \mathbb{Z}} + \lambda(\mu, \nu))^2 + \underbrace{p}_{\text{“Replicas” } p \in \mathbb{N}} - \frac{1}{24} \qquad \underline{\text{with}} \quad \lambda(\mu,\nu) = \frac{1}{2\pi}\cos^{-1}\left(\sqrt{\mu}\cos(2\pi\nu)\right)$$

- The formula for $(n = 0,\ p = 0)$ fits the first numerical band well
  - We can extract μ for a given E_J



CFT spectrum at the critical point $Z = R_q$

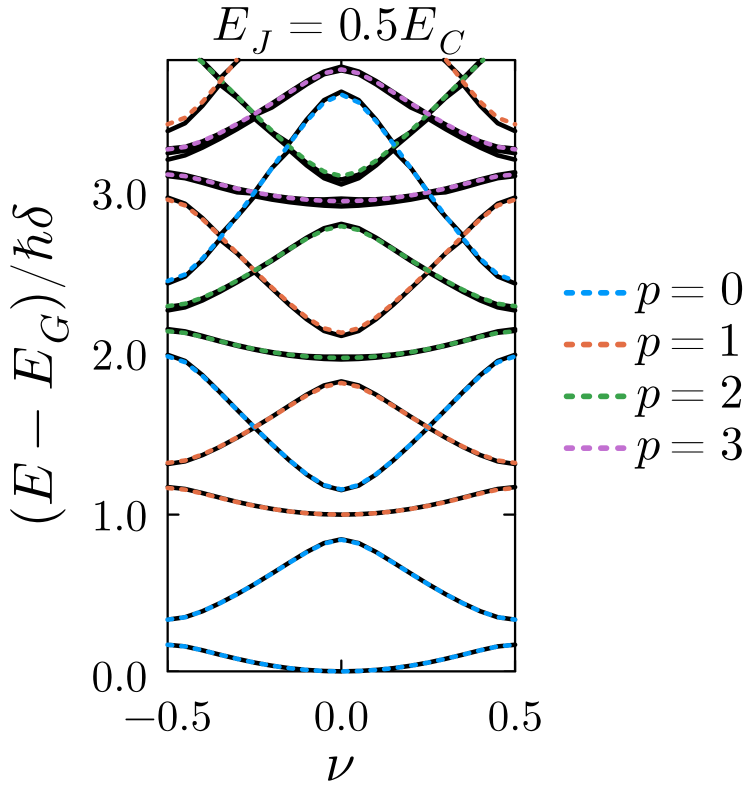
- Prediction for the bands at the critical point

[Hasselfield et al. Ann. Phys. 321,2849 (2006)]

$$\frac{E(\nu)}{\hbar\Delta} = (n + \lambda(\mu,\nu))^2 + p - \frac{1}{24} \qquad \underline{\text{with}} \quad \lambda(\mu,\nu) = \frac{1}{2\pi}\cos^{-1}\left(\sqrt{\mu}\cos(2\pi\nu)\right)$$

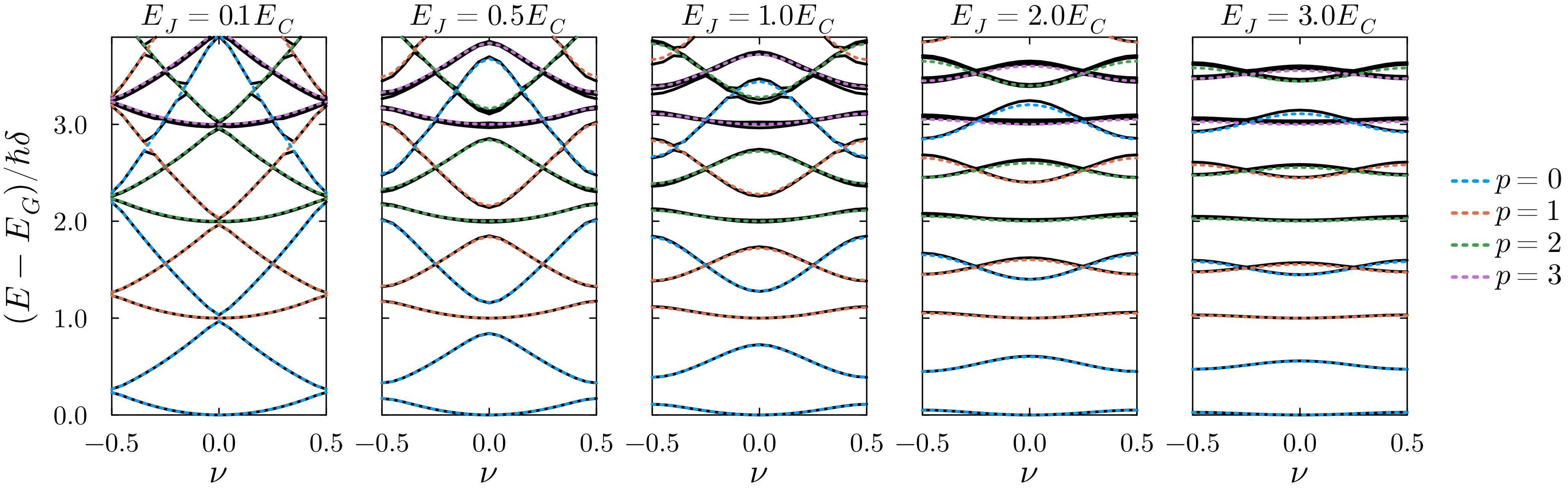
"Band index" $n \in \mathbb{Z}$ "Replicas" $p \in \mathbb{N}$

- The formula for $(n = 0,\ p = 0)$ fits the first numerical band well
  - We can extract μ for a given E_J
- From this value we can predict the excited bands
  - Remarkable agreement given the finite ω_p

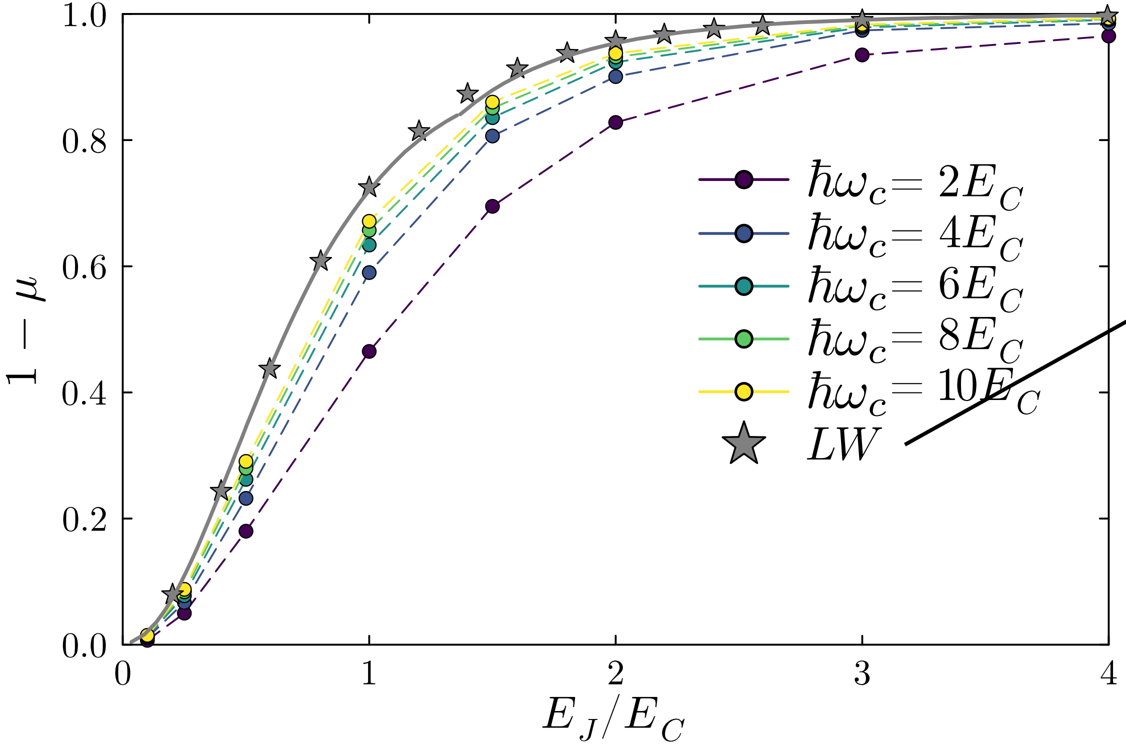


CFT spectrum at the critical point $Z = R_q$

(spectra computed in the polaron frame)



The fitted critical point mobility



- The mobility depends on the UV cutoff
- With increasing ω_p the curves approach Monte Carlo results for $\omega_c \to \infty$

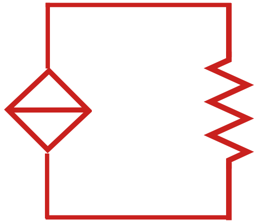
[Lukyanov, Werner - J.Stat.Mech. (2007)]

Exact duality

[LG, Devoret, Ciuti - arXiv:2504.14651]

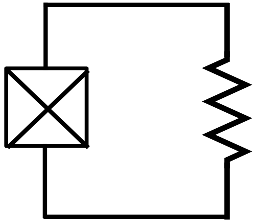
Schmid (approximate) phase-charge duality

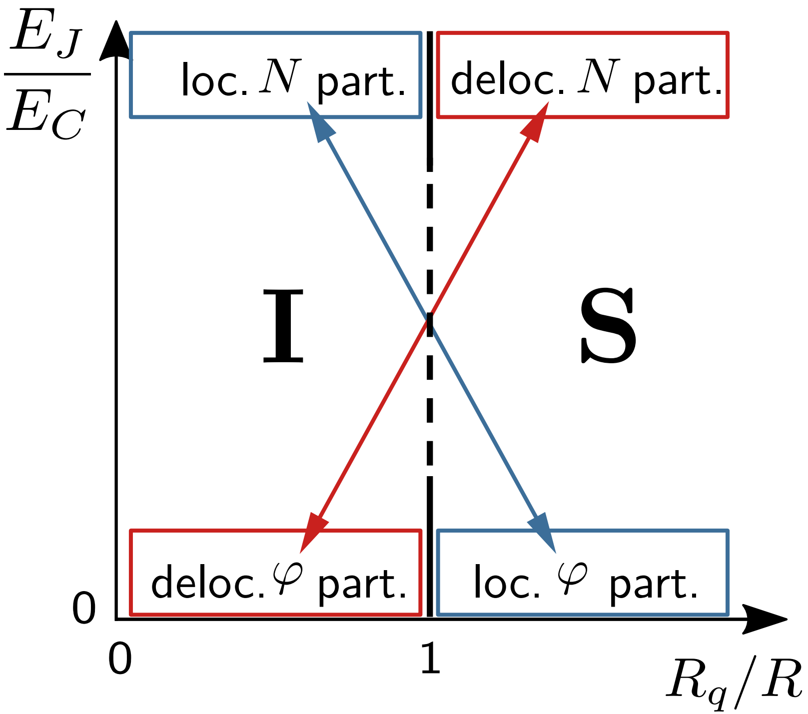
Perturbative in phase slips (tightly bound particle)



$\cos(2\pi\hat{N})$

$\cos\hat{\varphi}$





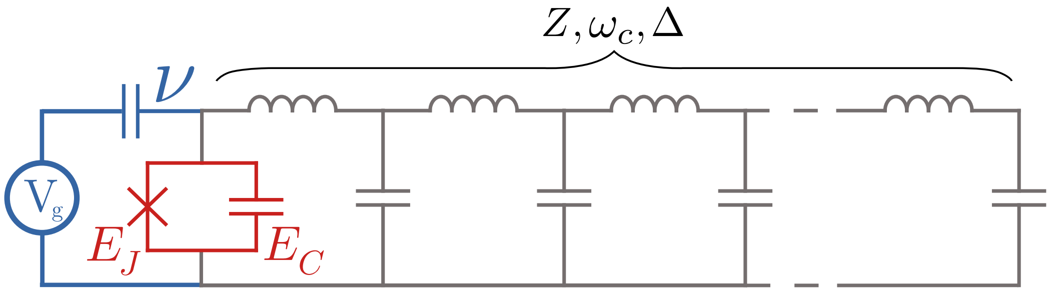
Perturbative in Cooper pair tunneling (weakly bound particle)

Conjecture:

the transition line is vertical

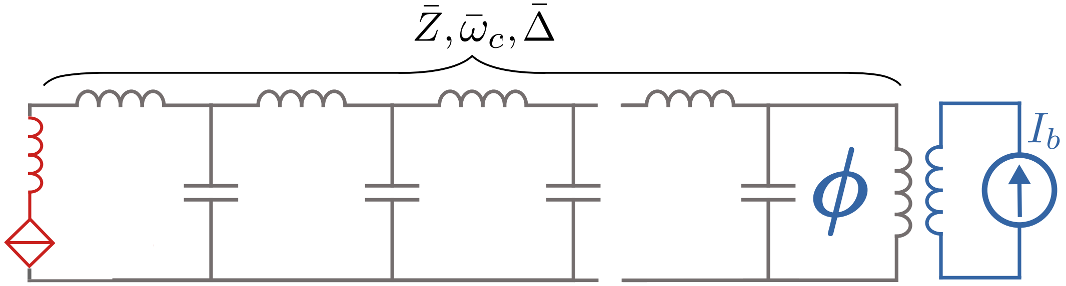
Dual (microscopic) circuits

Charge is conserved



Satisfy the same circuit equations with flux and charge interchanged with $\bar{Z} = \frac{R_q^2}{Z}$

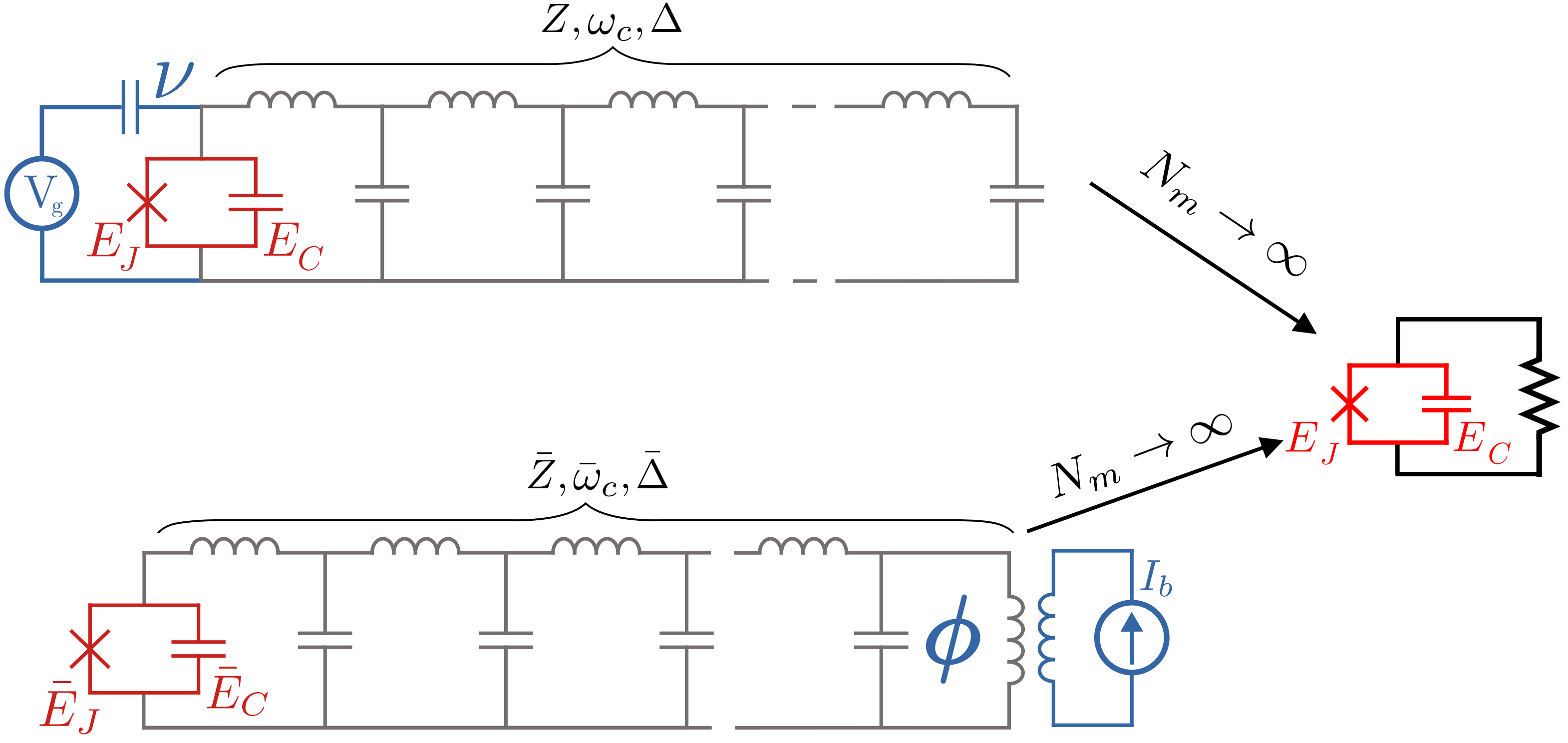
Flux is conserved



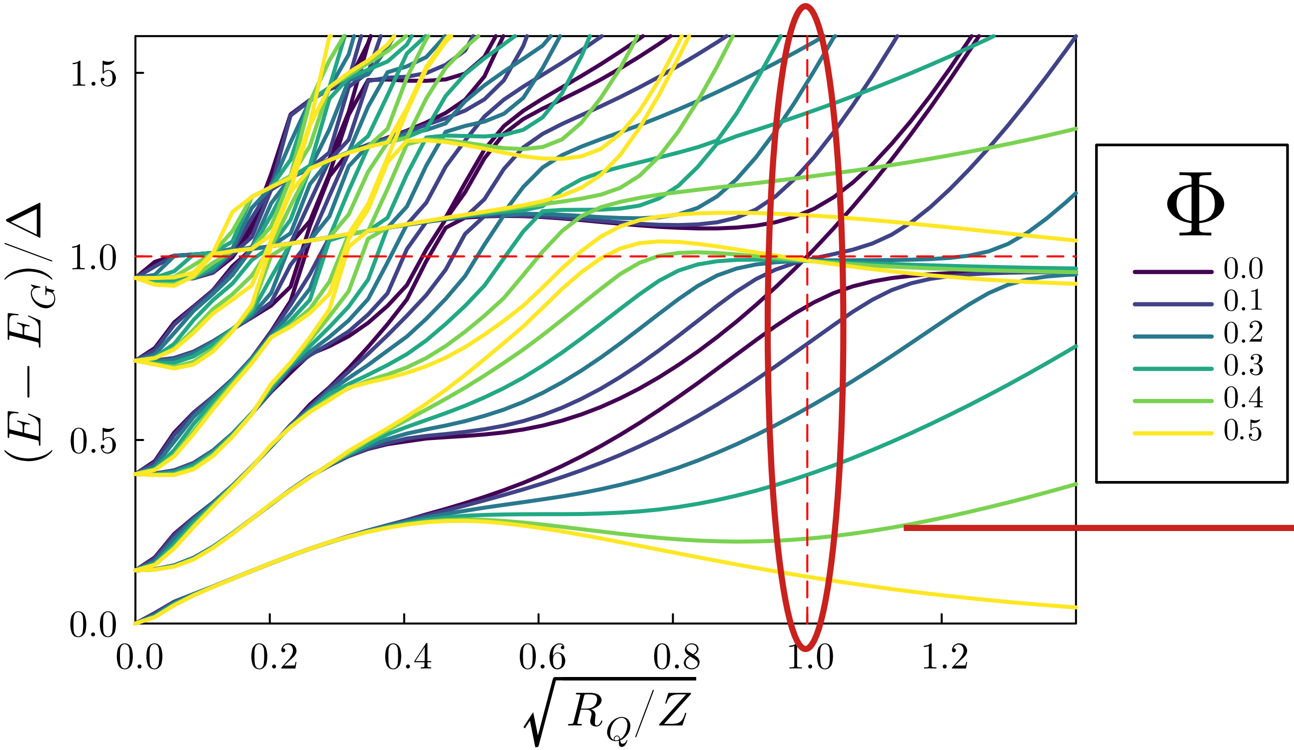
Dual (microscopic) circuits

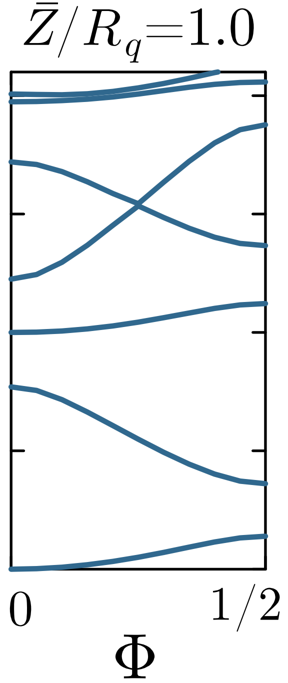
Charge is conserved

Flux is conserved



Dual circuit spectrum



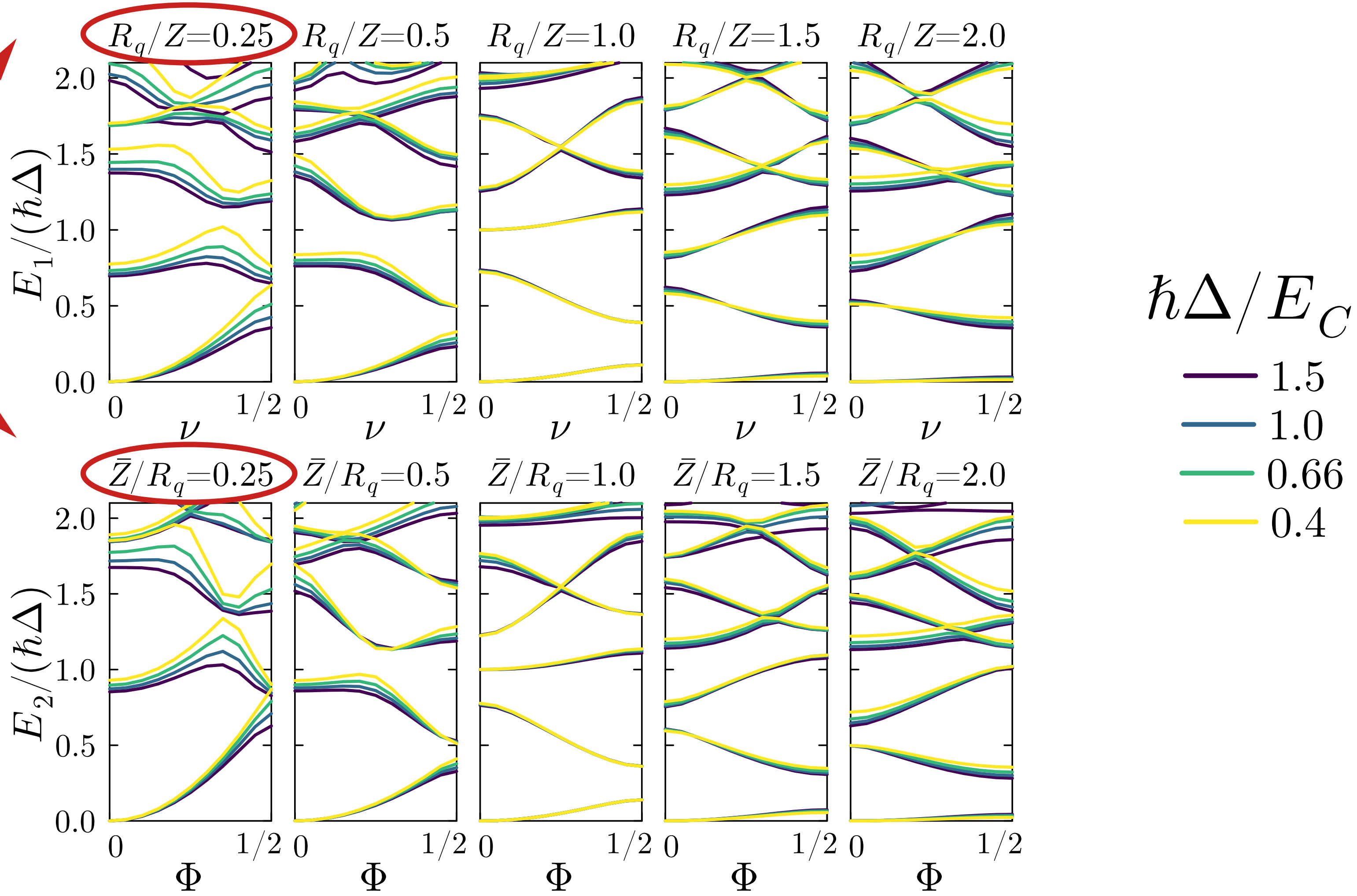


Dual Brillouin zone

Bands comparison

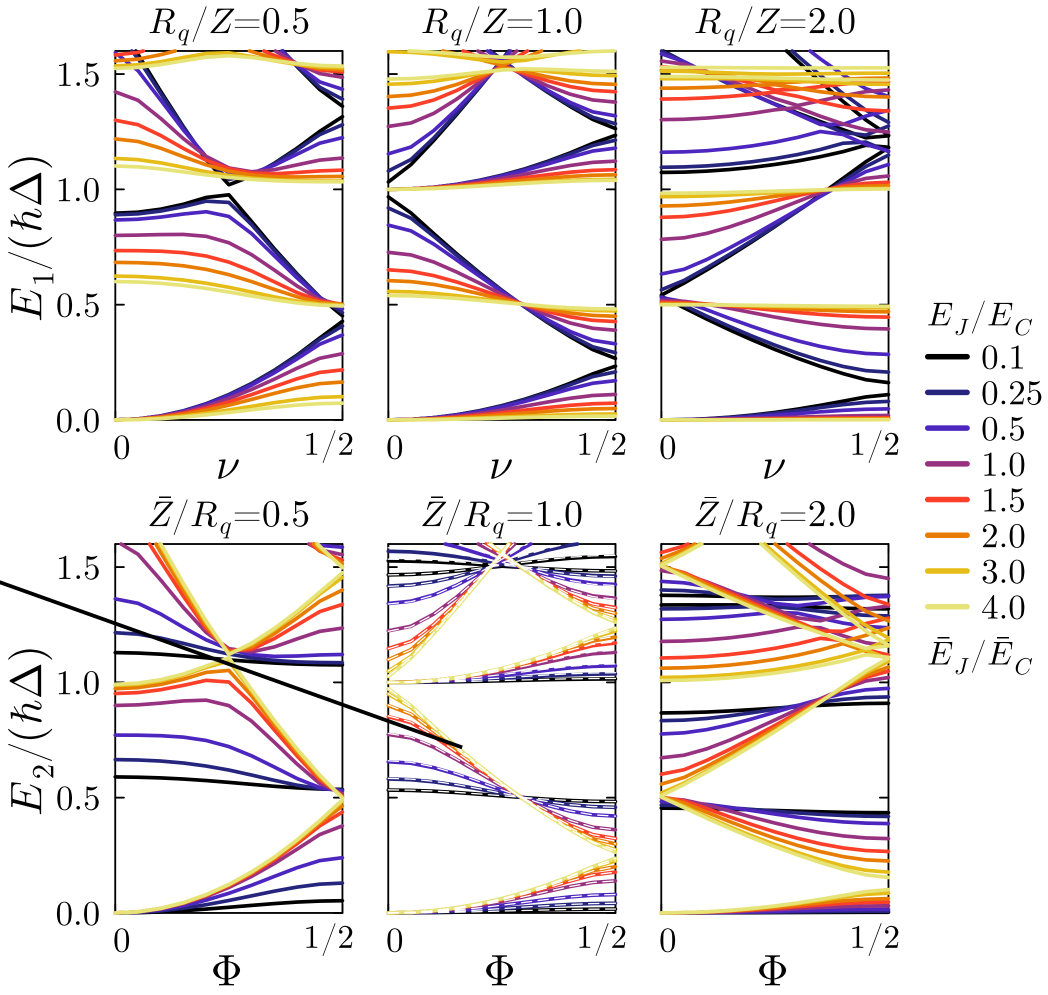
$E_J = E_C$

Dual impedances

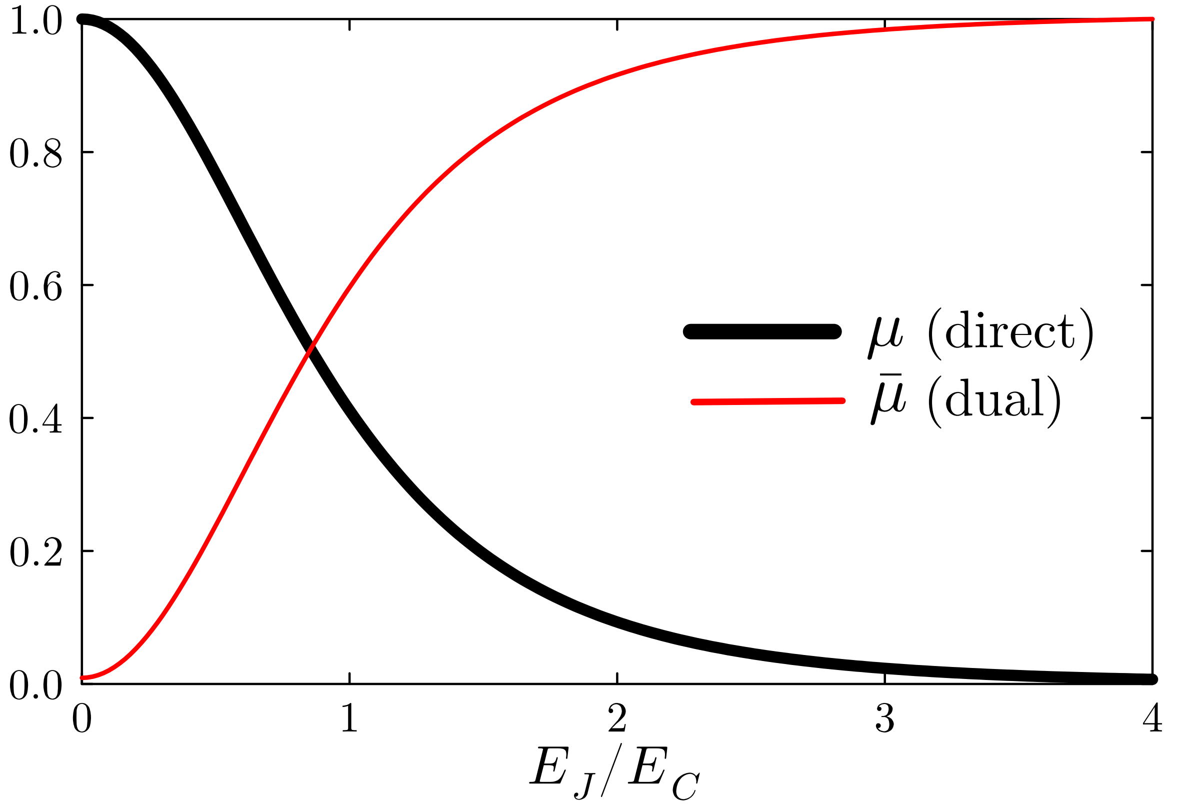


Opposite dependence on E_J/E_C

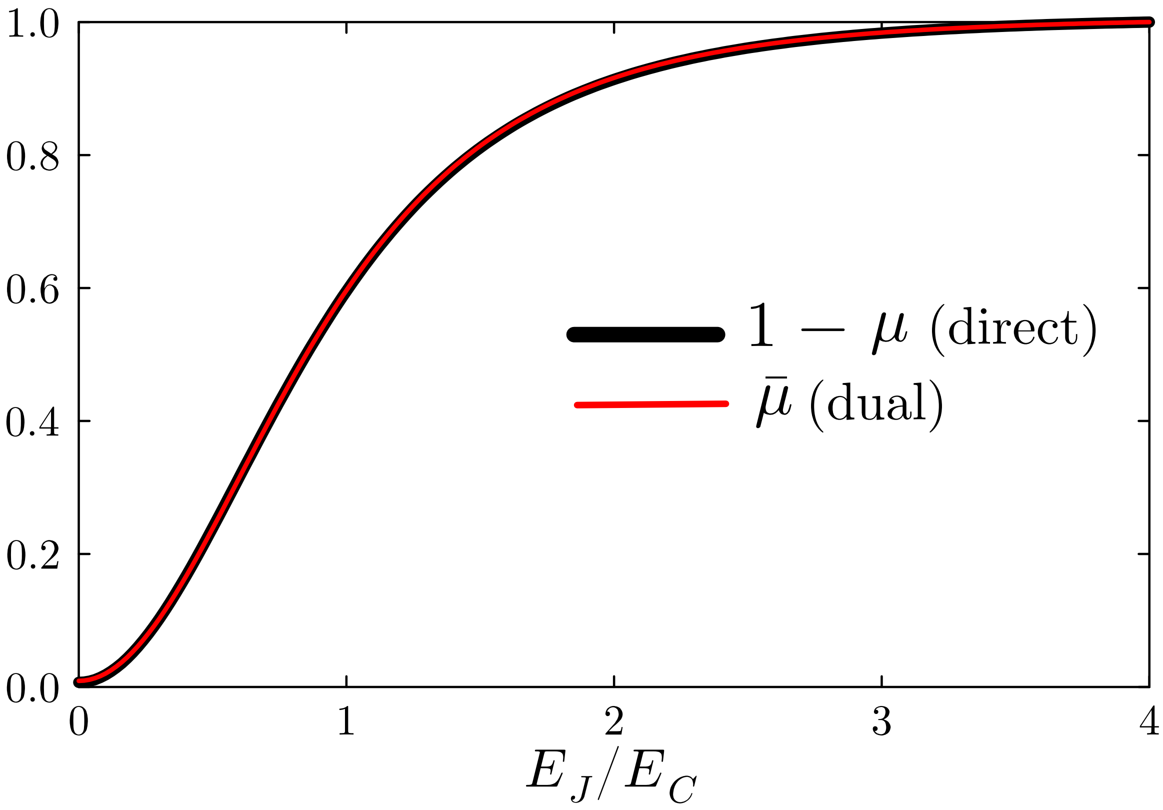
Fit with CFT spectrum again

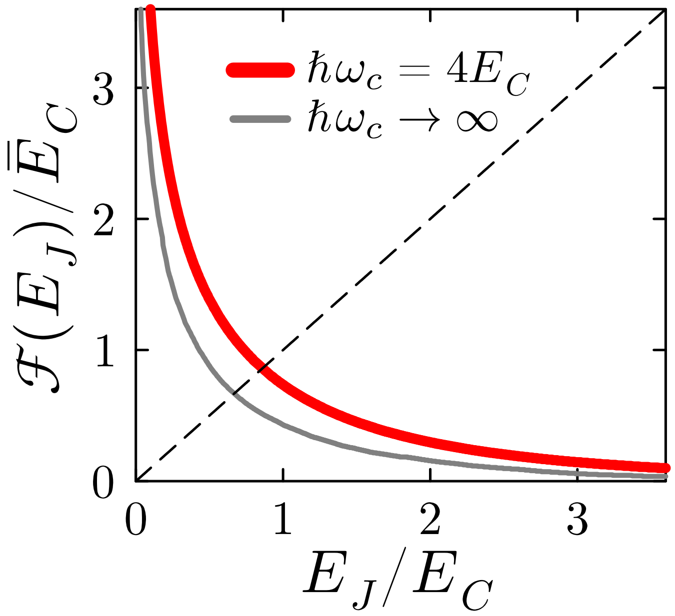


Critical line mapping

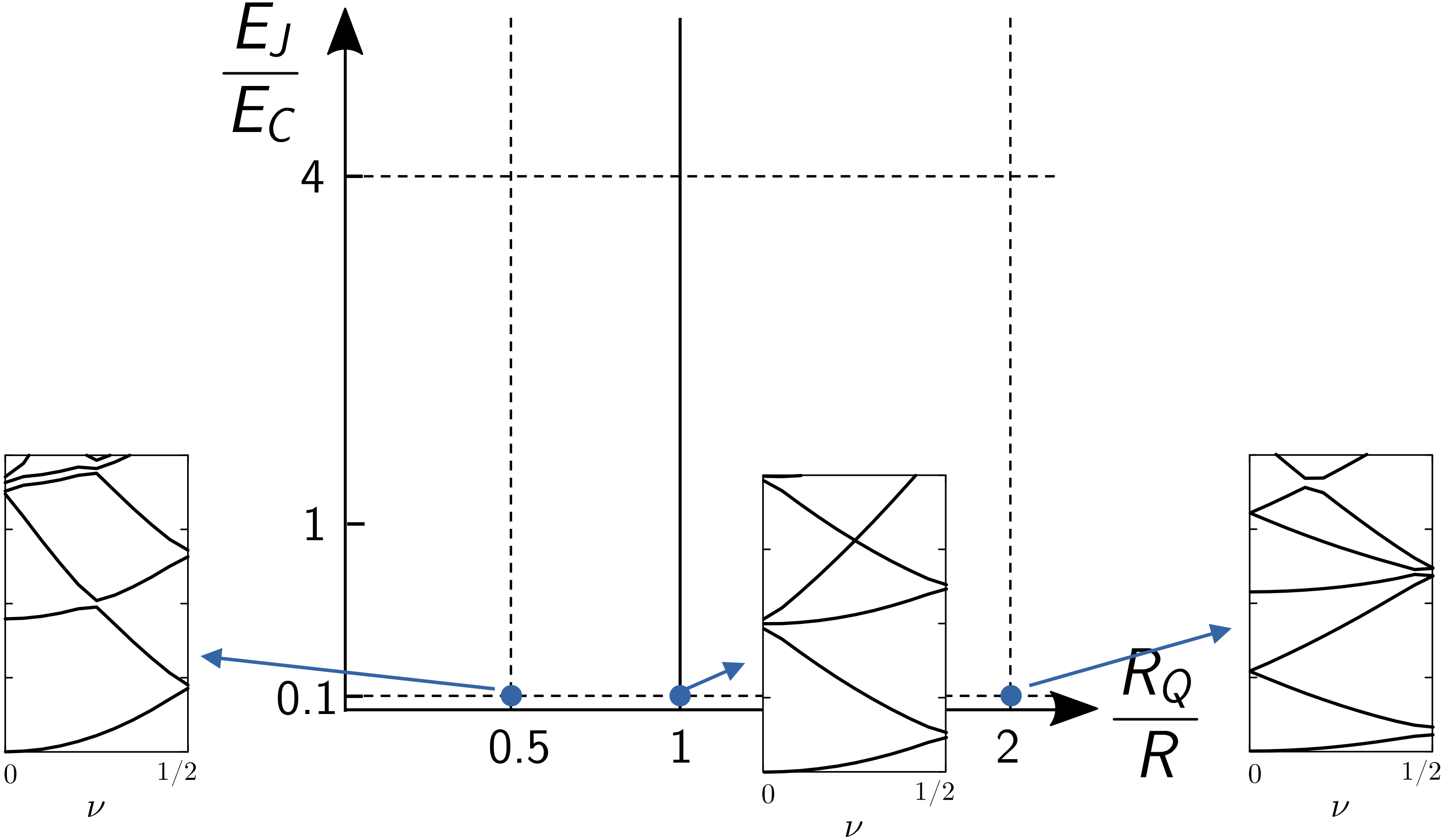


Critical line mapping





Exact self-duality



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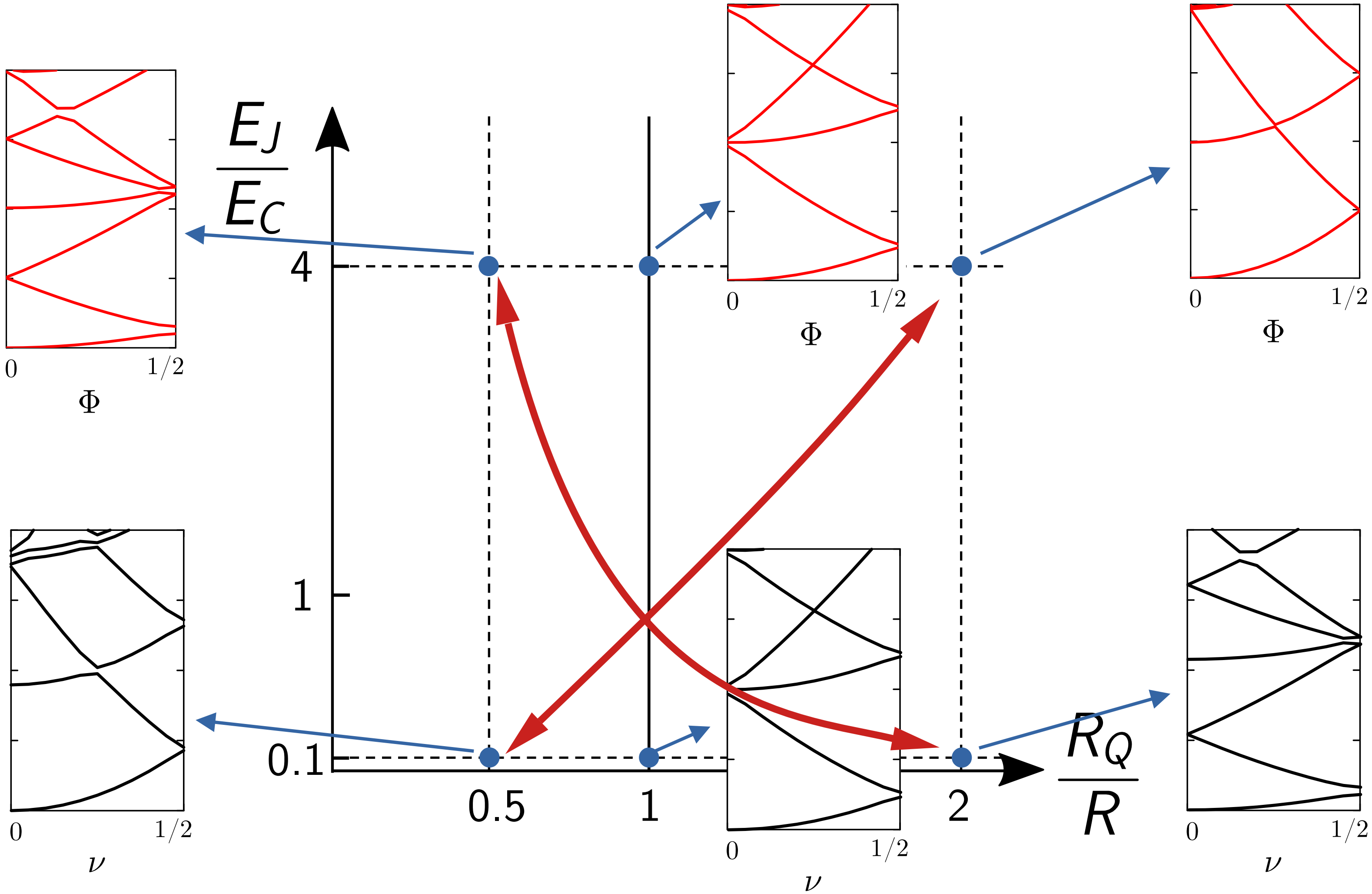
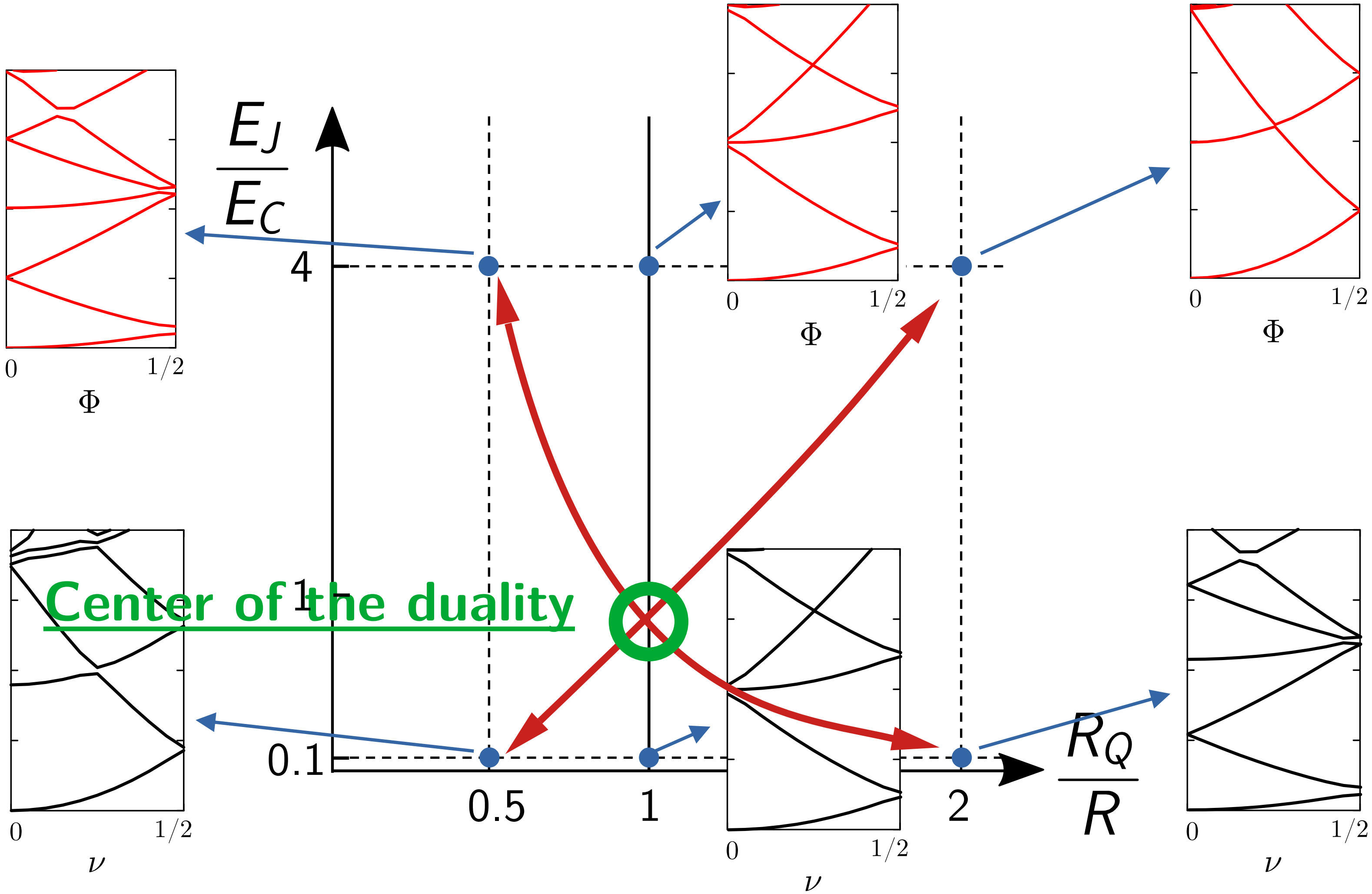


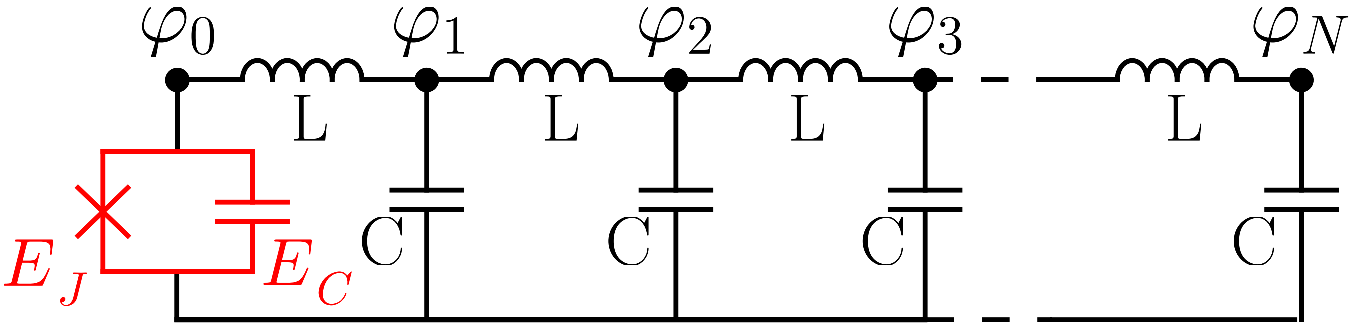
Exact self-duality



Key messages

- There is a critical point in the Caldeira-Leggett model for a JJ
- We get the thermodynamic critical spectrum from finite systems
- The system is exactly self-dual: critical point independent of E_J/E_C

A finite transmission line



$$Z = \sqrt{\frac{L}{C}}; \qquad \omega_p = \frac{2}{\sqrt{LC}};$$

$$\Delta = \frac{\pi}{\sqrt{LC}}\frac{1}{N}.$$

$$\mathbf{C} = \begin{pmatrix} C_J & 0 & \dots & & & \\ 0 & C & 0 & \dots & & \\ \vdots & 0 & C & 0 & \dots \\ & \vdots & \ddots & \ddots & \ddots \\ & & & 0 & C \end{pmatrix}$$

$$\mathbf{\Gamma} = \frac{1}{L}\begin{pmatrix} 1 & -1 & \dots & & \\ -1 & 2 & -1 & \dots & \\ \vdots & -1 & 2 & -1 & \dots \\ & \vdots & \ddots & \ddots & \ddots \\ & & & -1 & 1 \end{pmatrix}$$

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^T\mathbf{C}\dot{\varphi} - \frac{1}{2}\varphi\mathbf{\Gamma}\varphi + E_J\cos(\varphi_0)$$

$$\hat{\mathcal{H}} = 4E_C\left(\hat{N} + i\sum_k \widetilde{g}_k(\hat{a}_k^\dagger - \hat{a}_k)\right)^2 - E_J\cos\hat{\varphi} + \sum_k \hbar\omega_k\hat{a}_k^\dagger\hat{a}_k$$

$$\hat{\mathcal{H}} = 4E_C\hat{N}^2 - E_J\cos\hat{\varphi} + \sum_k \hbar\omega_k\hat{a}_k^\dagger\hat{a}_k + i\hat{N}\sum_k g_k(\hat{a}_k^\dagger - \hat{a}_k)$$

The "polaron frame"

$$\hat{\mathcal{H}} = 4E_C\hat{N}^2 - E_J\cos\hat{\varphi} + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + i\hat{N}\sum_k g_k(\hat{a}_k^\dagger - \hat{a}_k)$$

- Polaron transformation: $\hat{\mathcal{U}}_p = \exp\left(i\hat{N}\sum_k \frac{g_k}{\hbar\omega_k}(\hat{a}_k + \hat{a}_k^\dagger)\right).$

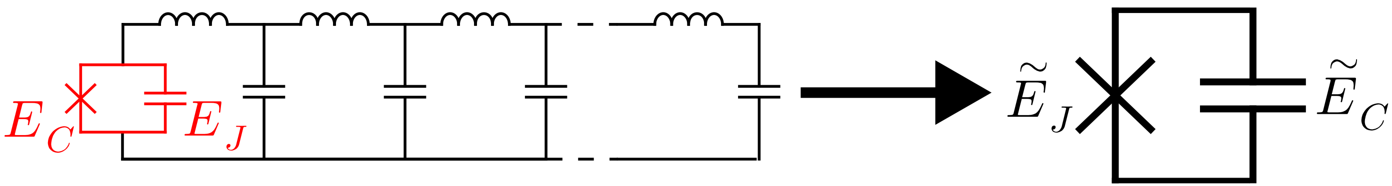
$$\hat{\mathcal{U}}_p\hat{\mathcal{H}}_1^{ch}\hat{\mathcal{U}}_p^\dagger = \underbrace{\left(4E_C - \sum_k \frac{g_k^2}{\hbar\omega_k}\right)}_{4\widetilde{E}_C}\hat{N}^2 - \underbrace{E_J\cos\left(\hat{\varphi} + \sum_k \frac{g_k}{\hbar\omega_k}(\hat{a}_k + \hat{a}_k^\dagger)\right)}_{\text{with zero ``new'' photons (``integrated out'')}} + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k,$$

$$e^{-\frac{1}{2}\sum_k \frac{g_k^2}{\hbar^2\omega_k^2}} E_J \sum_N (|N+1\rangle\langle N| + |N\rangle\langle N+1|)$$

Reduced capacitive energy

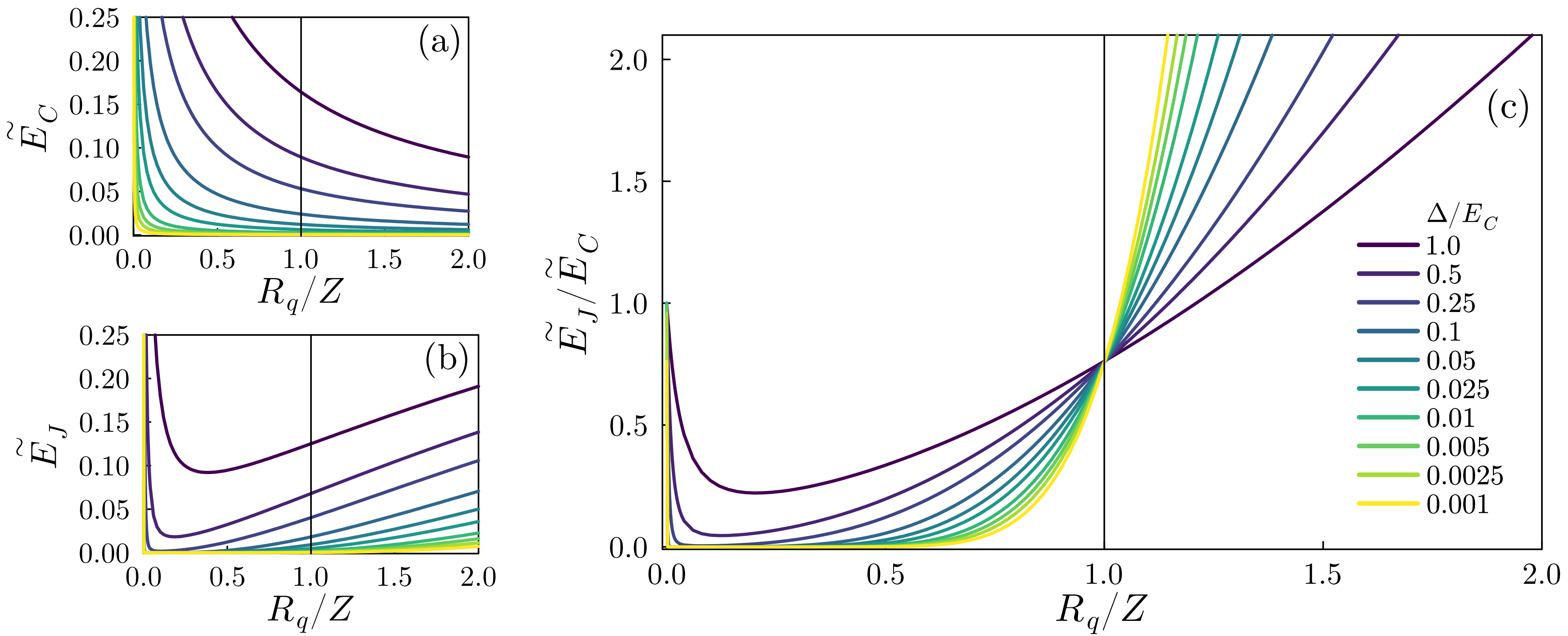
Reduced Josephson energy

"Adiabatic" renormalization of the junction



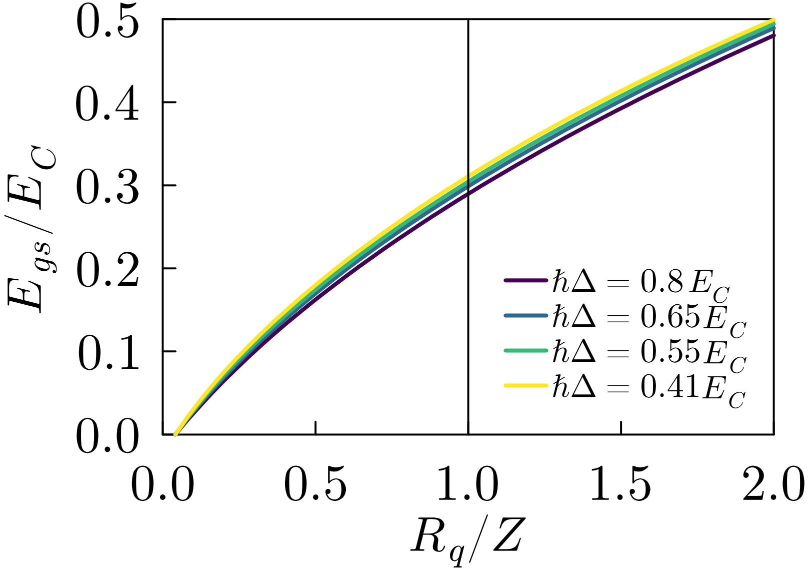
$$\widetilde{E}_C = E_C - \frac{1}{4}\sum_k \frac{g_k^2}{\omega_k}$$

$$\widetilde{E}_J = \exp\left(-\frac{1}{2}\sum_k \frac{g_k^2}{\hbar^2\omega_k^2}\right) E_J$$



Ground state and excited states

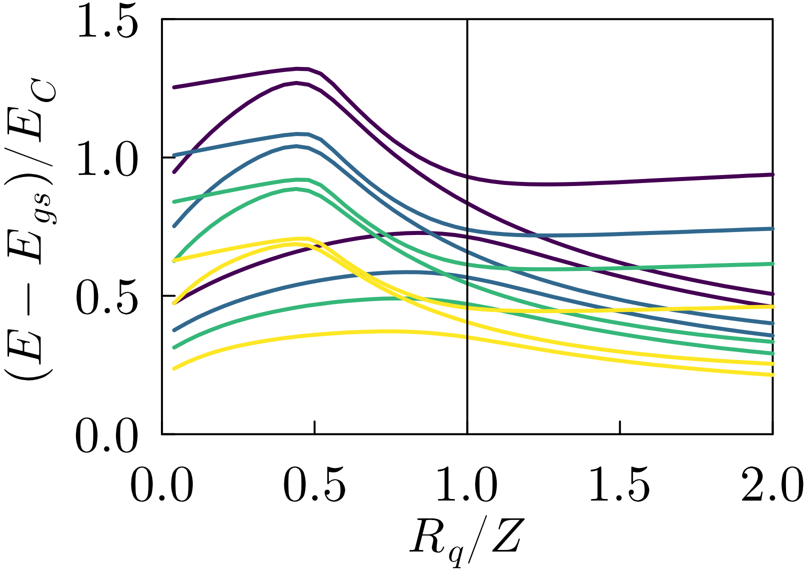
GS energy



Ground state does not show exciting features

A closing gap is not the first thing that emerges

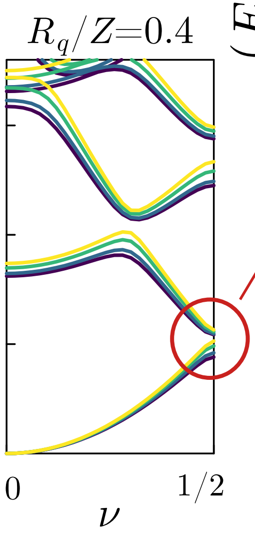
First excited states



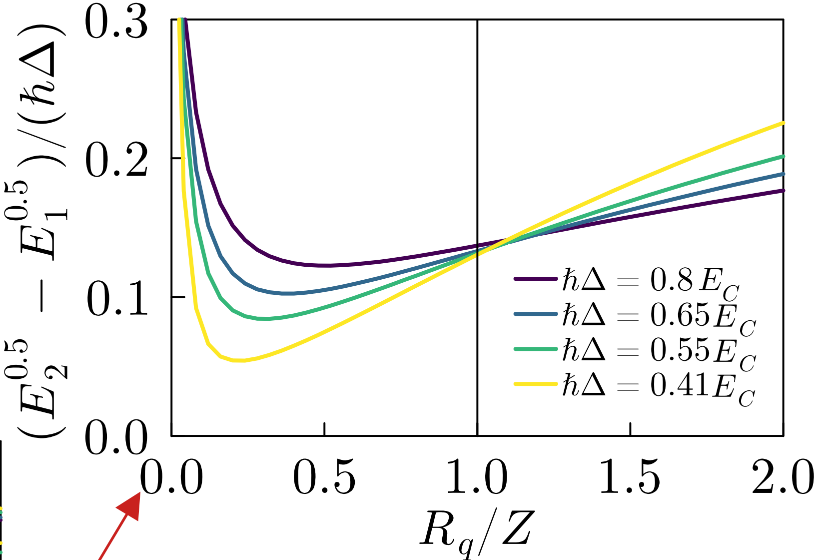
Peculiar anticrossing in the first excited state

Collapses on GS in TD limit

Ground state and excited states

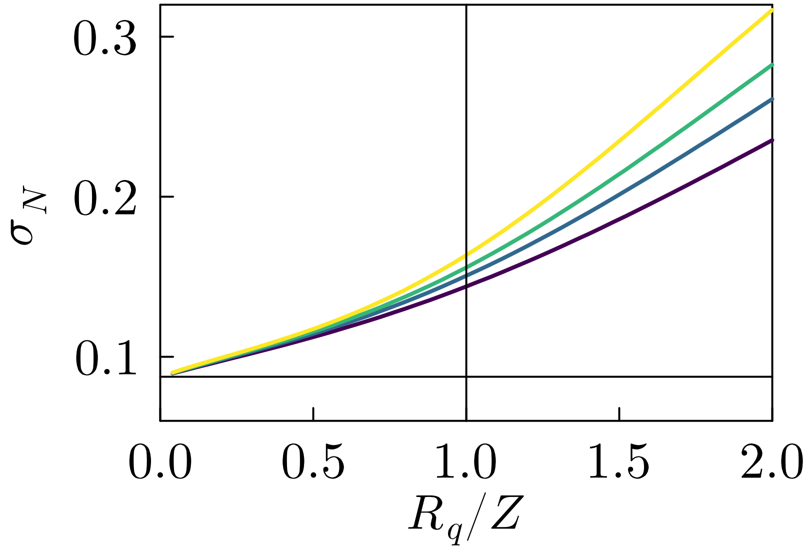


Bandgap



Bands approach capacitive behaviour for $R_q/Z < 1$

Charge fluctuations

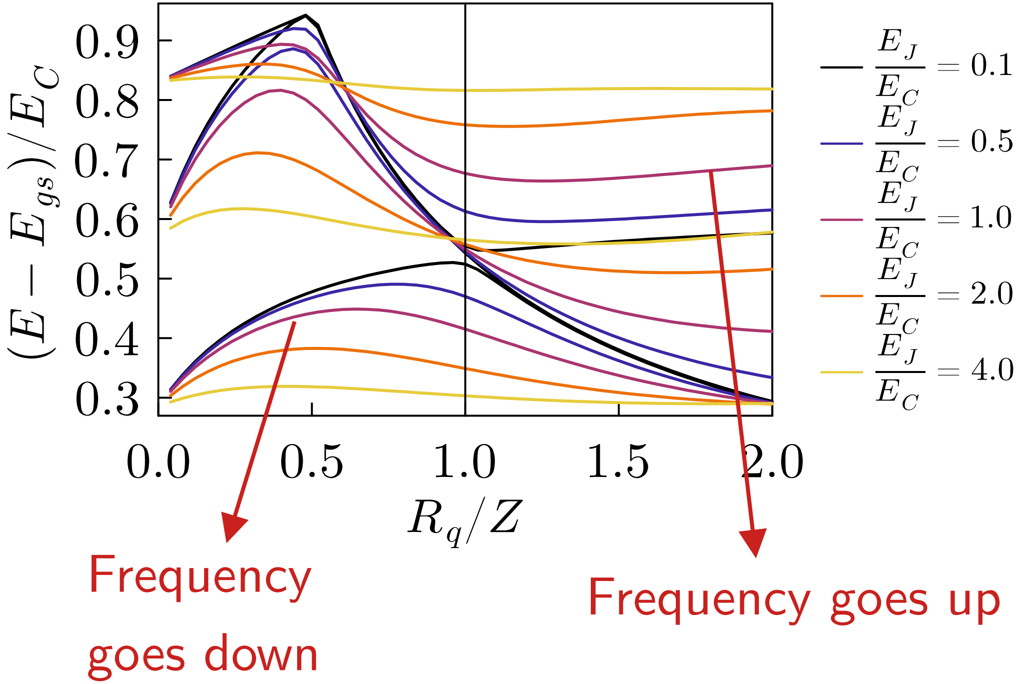


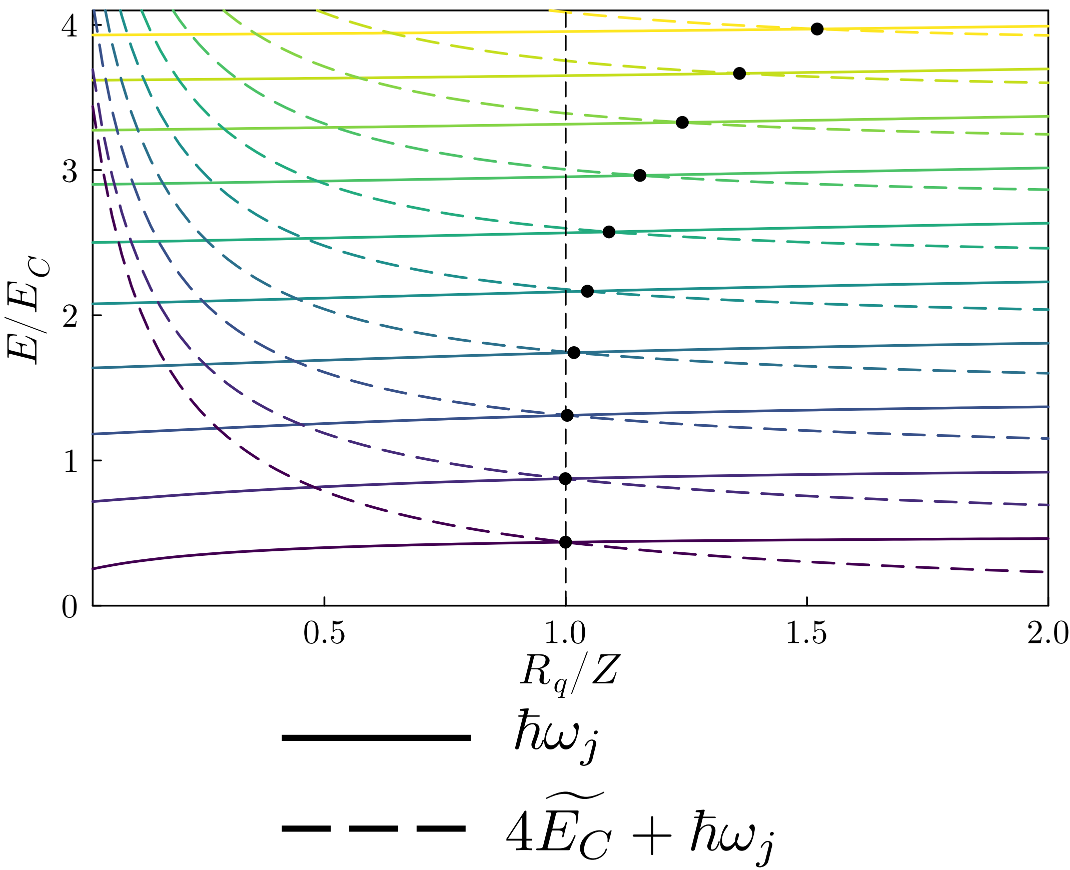
$$\sigma_N = \langle \hat{N}^2 \rangle_{GS} - \langle \hat{N} \rangle_{GS}^2$$

But charge fluctuations do not go to zero in GS

Effect on photons

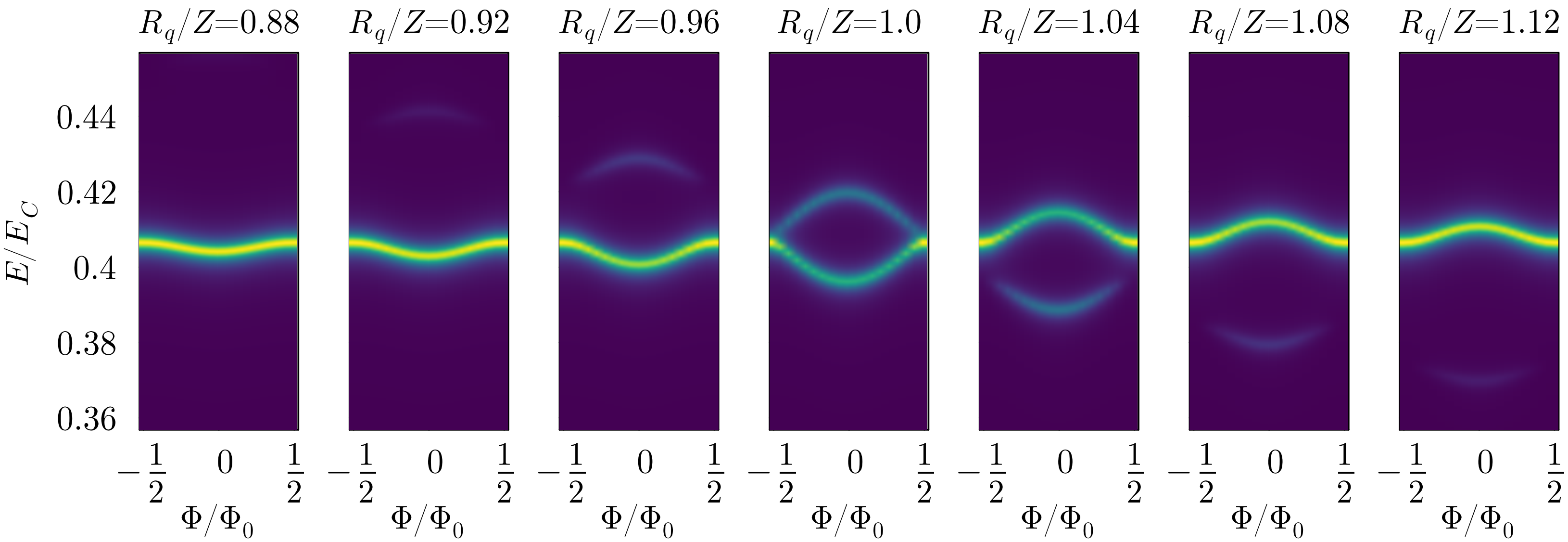
- The width of anti-crossings is fixed by E_J ⟶ for $E_J \to 0$ "no interaction"
- First excited level: $\varepsilon^1_{\nu=0}(E_J = 0) = 4\widetilde{E_C}$
- Photon frequency depends on impedance





Spectroscopy of photons

- Photonic spectral function $D(\omega) = \sum_n \frac{\gamma^2}{\gamma^2 + (\omega - E_n + E_G)^2} \left| \langle G | \sum_k (a_k + a_k^\dagger) | E_n \rangle \right|^2$
- Tune E_J of a SQUID: $E_J(\Phi) \longrightarrow \begin{cases} E_J(0) = E_J^{max} \\ E_J(\Phi_0) = 0 \end{cases} \quad \left(\Phi_0 = \frac{h}{2e} \right)$

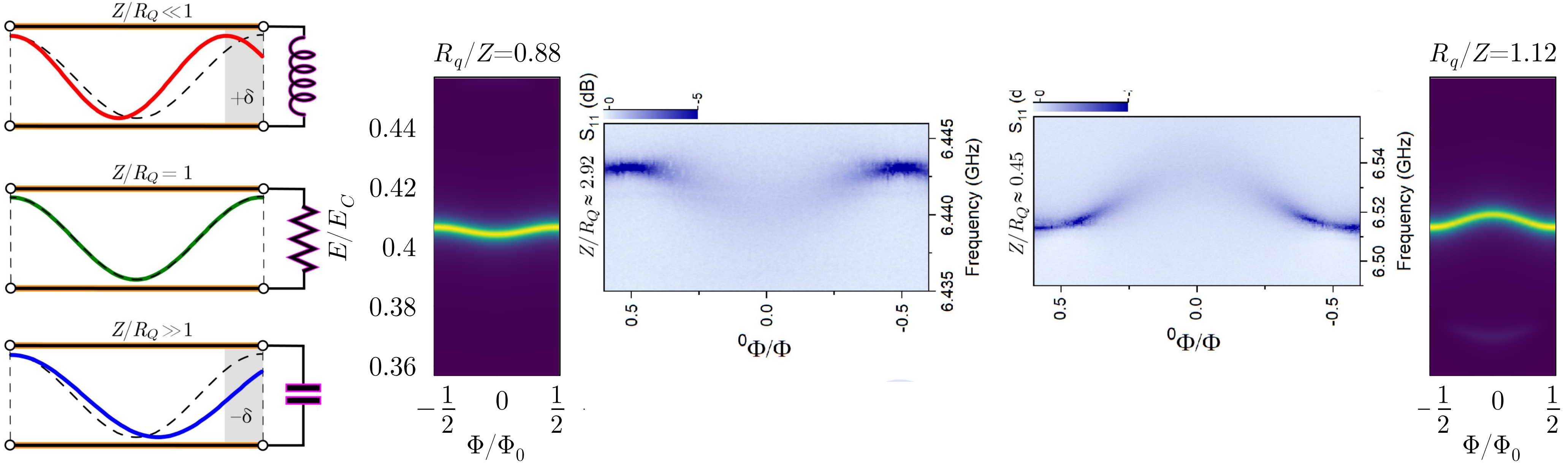


Spectroscopy of photons

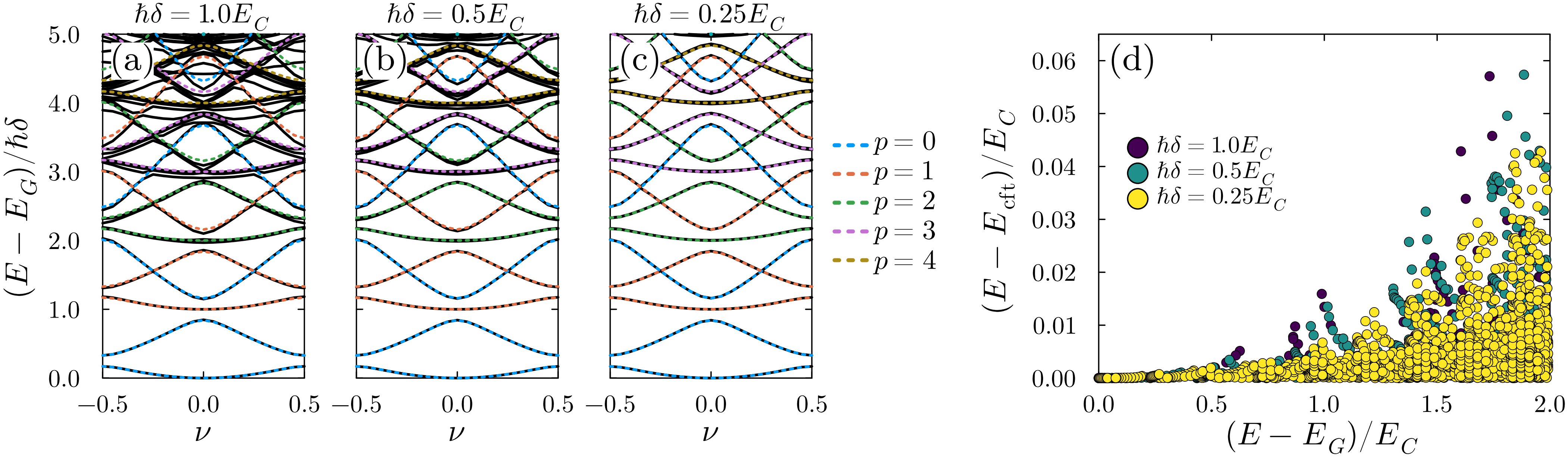
- Photonic spectral function $D(\omega) = \sum_n \frac{\gamma^2}{\gamma^2+(\omega-E_n+E_G)^2} |\langle G| \sum_k (a_k + a_k^\dagger) |E_n\rangle|^2$

- Tune E_J of a SQUID: $E_J(\Phi) \longrightarrow \begin{cases} E_J(0) = E_J^{max} \\ E_J(\Phi_0) = 0 \end{cases} \quad \left(\Phi_0 = \frac{h}{2e}\right)$

- Shift observed in recent experiment [Kuzmin et al. Nature Physics 21.1 (2025)]

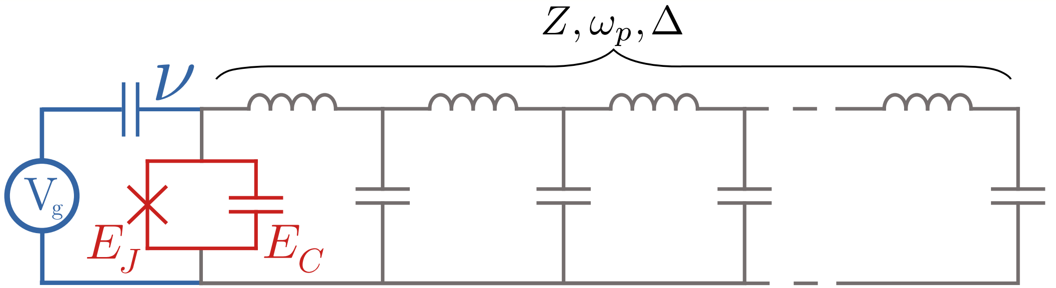


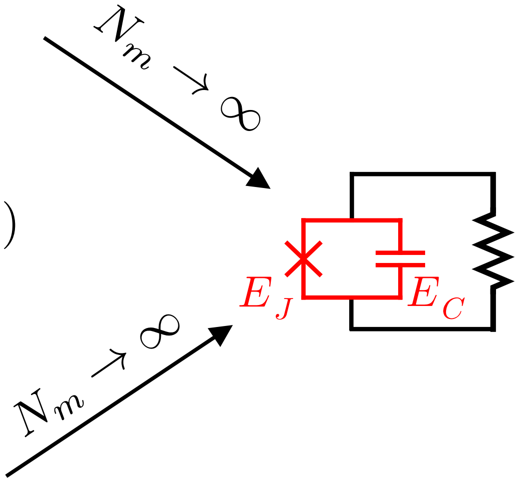
Size dependence of the CFT fit



Dual (microscopic) circuits

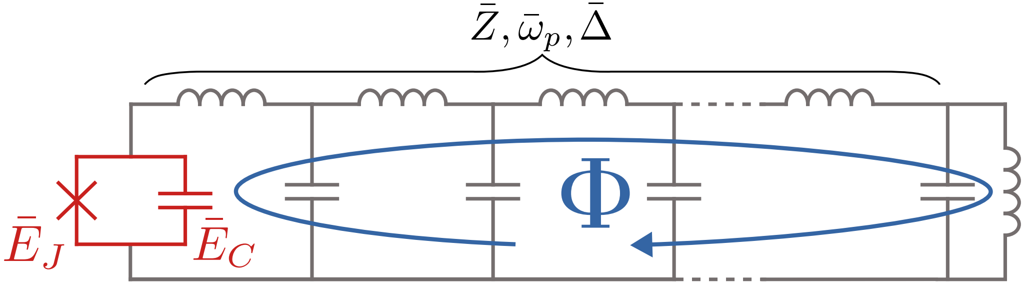
Charge is conserved





$$\hat{\mathcal{H}}_1^{ch} = 4E_C(\hat{N} - \nu)^2 - E_J \cos\hat{\varphi} + \sum_{k=1}^{N_m} \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + i(\hat{N} - \nu) \sum_{k=1}^{N_m} g_k(\hat{a}_k^\dagger - \hat{a}_k)$$

Flux is conserved



$$\hat{\mathcal{H}}_2^{fl} = \frac{\Phi_0^2}{4}\frac{\bar{\omega}_p}{\bar{Z}}(\hat{\varphi} - 2\pi\Phi)^2 + \underbrace{4\bar{E}_C\hat{N}^2 - \bar{E}_J\cos\hat{\varphi}}_{\sim\, -U_0 \cos(2\pi\hat{N}) \text{ for } E_J \gg E_C} + \sum_{k=1}^{N_m} \hbar\Omega_k \hat{a}_k^\dagger \hat{a}_k + (\hat{\varphi} - 2\pi\Phi) \sum_{k=1}^{N_m} f_k(\hat{a}_k + \hat{a}_k^\dagger)$$

$\sim -U_0 \cos(2\pi\hat{N})$ for $E_J \gg E_C$ Single band approx.