A tale of two sounds

Attractors and thermalization in nuclear collisions and cold quantum gases

ECT* Trento

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Content of the talk

1. Emergence of hydrodynamic behavior in kinetic theory

Relativistic heavy ion collisions, Bjorken flow
Based on work done in collaboration with Li Yan

2. An analogy with collective phenomena in many body systems

"The tale of two sounds"

In both cases we shall be looking at the transition from a (non trivial) collisionless regime to a collision-dominated regime (hydrodynamics).

What is hydrodynamics ?

'Traditional' view

Fluid behavior emerges around local equilibration

Fast relaxation of microscopic degrees of freedom via collisions.
(Collisions have little impact on local conservation laws.)

'Modern' perspective

Effective theory for long wavelength modes

Thermalization

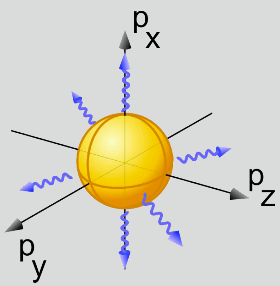
Two main issues

i) relative populations of different momentum modes

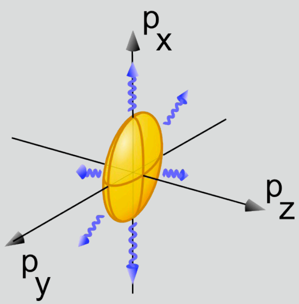
ii) isotropy of momentum distribution ← Main topic for the rest of this talk

In relativist collisions, isotropization involves competition between

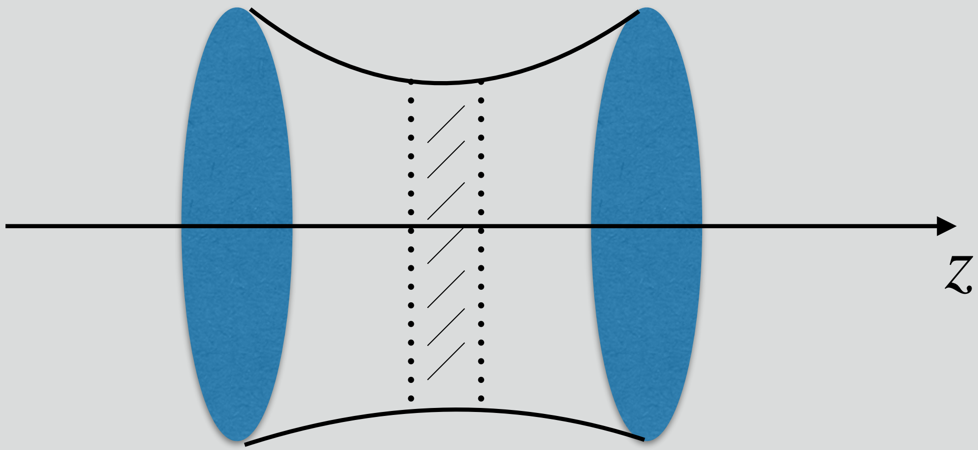
Collisions →



Expansion →



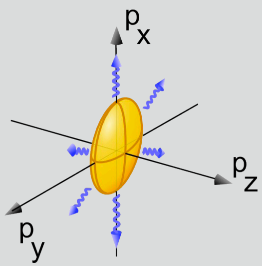
Simple kinetic equation (Bjorken flow)



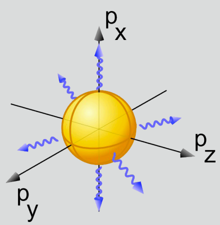
1+1 dimensional expansion, in relaxation time approximation

$$\left[\partial_\tau - \frac{p_z}{\tau}\partial_{p_z}\right] f(\boldsymbol{p}/T) = -\frac{f(\boldsymbol{p},\tau) - f_{\rm eq}(p,\tau)}{\tau_R}$$

collisionless expansion



collisions



Special moments of the momentum distribution

(JPB, Li Yan, 2017, 18, 19, 21)

Special moments $p_z = p\cos\theta$

$$\mathcal{L}_n \equiv \int_p p^2 P_{2n}(\cos\theta) f(\boldsymbol{p})$$

(Legendre polynomial)

- Coarse graining (loss of information)
- Focus on the angular degrees of freedom

The energy momentum tensor is described by first two moments

$$T^{\mu\nu} = \int_p f(\boldsymbol{p}) p^\mu p^\nu \qquad \mathcal{L}_0 = \varepsilon \qquad \mathcal{L}_1 = \mathcal{P}_L - \mathcal{P}_T$$

We are looking for an effective theory for these two moments

Coupled equations for the moments

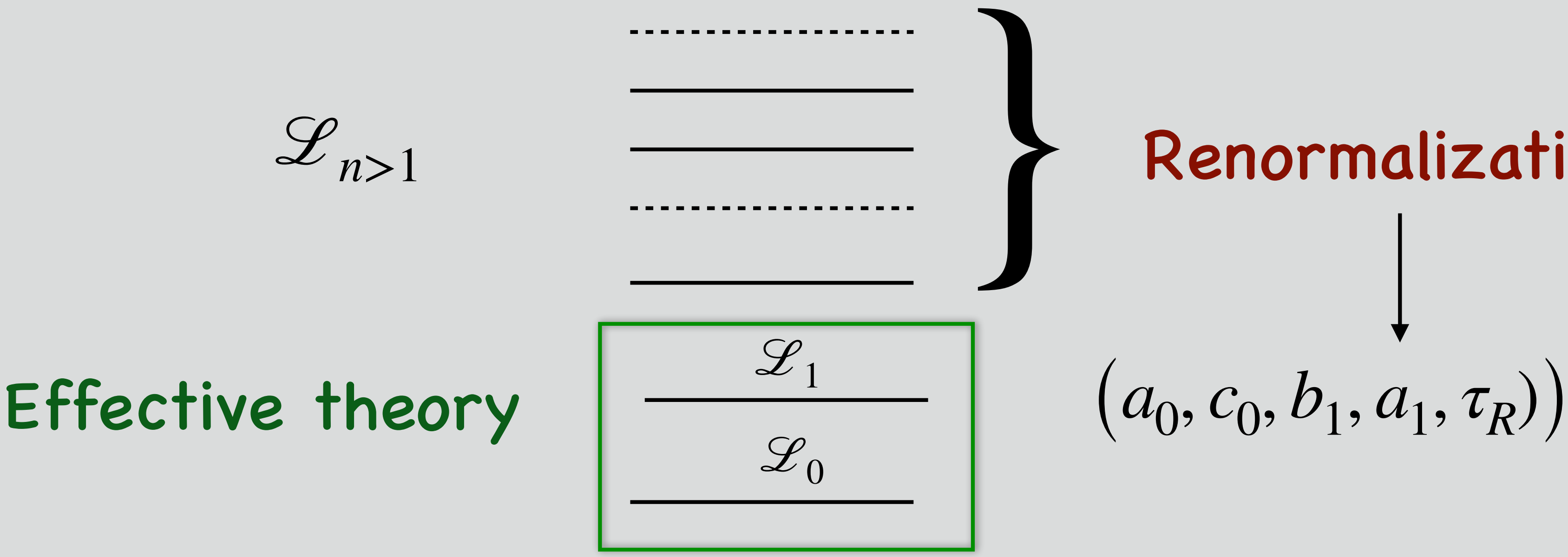
$$\frac{\partial \mathcal{L}_n}{\partial \tau} = -\frac{1}{\tau}\left[a_n \mathcal{L}_n + b_n \mathcal{L}_{n-1} + c_n \mathcal{L}_{n+1}\right] - \frac{\mathcal{L}_n}{\tau_R} \quad (n \geq 1)$$

(collisionless expansion) HARD

(collisions) EASY

$$\frac{\partial \mathcal{L}_0}{\partial \tau} = -\frac{1}{\tau}\left[a_0 \mathcal{L}_0 + c_0 \mathcal{L}_1\right]$$

The coefficients a_n, b_n, c_n **are pure numbers** ($a_0 = 4/3$, $c_0 = 2/3$)



Effective theory = Two-moment truncation

$$\frac{\partial}{\partial\tau}\begin{pmatrix}\mathcal{L}_0\\ \mathcal{L}_1\end{pmatrix} = -\frac{1}{\tau}\begin{pmatrix}a_0 & c_0\\ b_1 & a_1+\frac{\tau}{\tau_R}\end{pmatrix}\begin{pmatrix}\mathcal{L}_0\\ \mathcal{L}_1\end{pmatrix}$$

- **Contains second order viscous hydrodynamics à la "Israel-Stewart"**
- **Amenable to analytic solution, very rich mathematical structure**

The coupled equations can be transformed into a single non linear ODE for:

$$g_0(\tau) = \frac{\tau}{\mathcal{L}_0}\frac{\partial\mathcal{L}_0}{\partial\tau} \qquad \left(\frac{P_L-P_T}{\varepsilon} = -\frac{1}{c_0}(a_0+g_0)\right)$$

Fixed point analysis

$$w\frac{dg_0}{dw} = \beta(g_0, w) \qquad\qquad w \equiv \tau/\tau_R$$

$$\beta(g_0, w) = -g_0^2 - \left(a_0 + a_1 + w\right) g_0 - a_1 a_0 + c_0 b_1 - a_0 w + \left[c_0 c_1 \frac{\mathscr{L}_2}{\mathscr{L}_0}\right]$$

$w \ll 1$ $(\tau \ll \tau_R)$ $\qquad \beta(g_0) = 0$ → **two free streaming fixed points**

$w \gg 1$ $(\tau \gg \tau_R)$ $\qquad g_0 + a_0 = 0, \quad g_0 = -4/3$ **hydrodynamic fixed point**

The 'attractor solution' is the particular solution that starts from the stable collisionless fixed point at time $\tau = 0$ and evolves "slowly" (adiabaticity) to the hydrodynamic fixed point at late time.

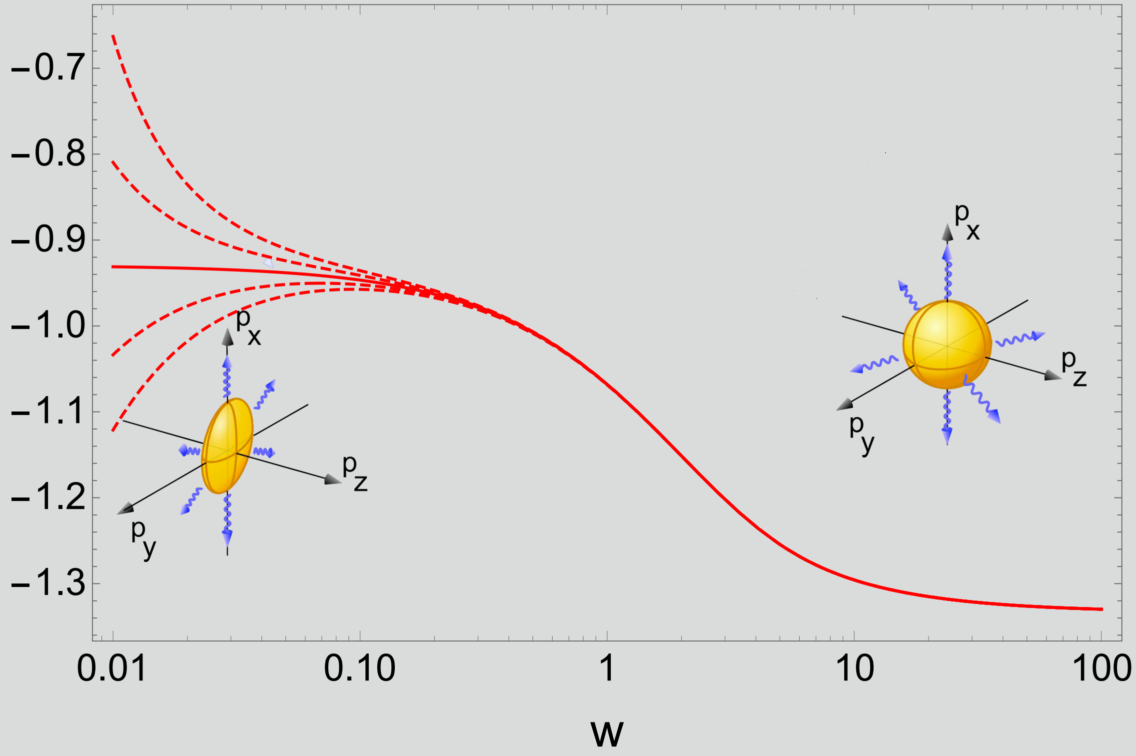
All solutions converge, soon or later depending on the initial conditions, towards the attractor, hence to hydrodynamics at late time (in most cases).

The transition from free streaming to hydrodynamics

Early and late times are controlled by the free streaming and the hydrodynamic fixed points, respectively

$g_0 \approx -0.93$

Collisionless fixed point



$(w \equiv \tau/\tau_R)$

Hydro fixed point

(Universal!)

$g_0 = -\frac{4}{3}$

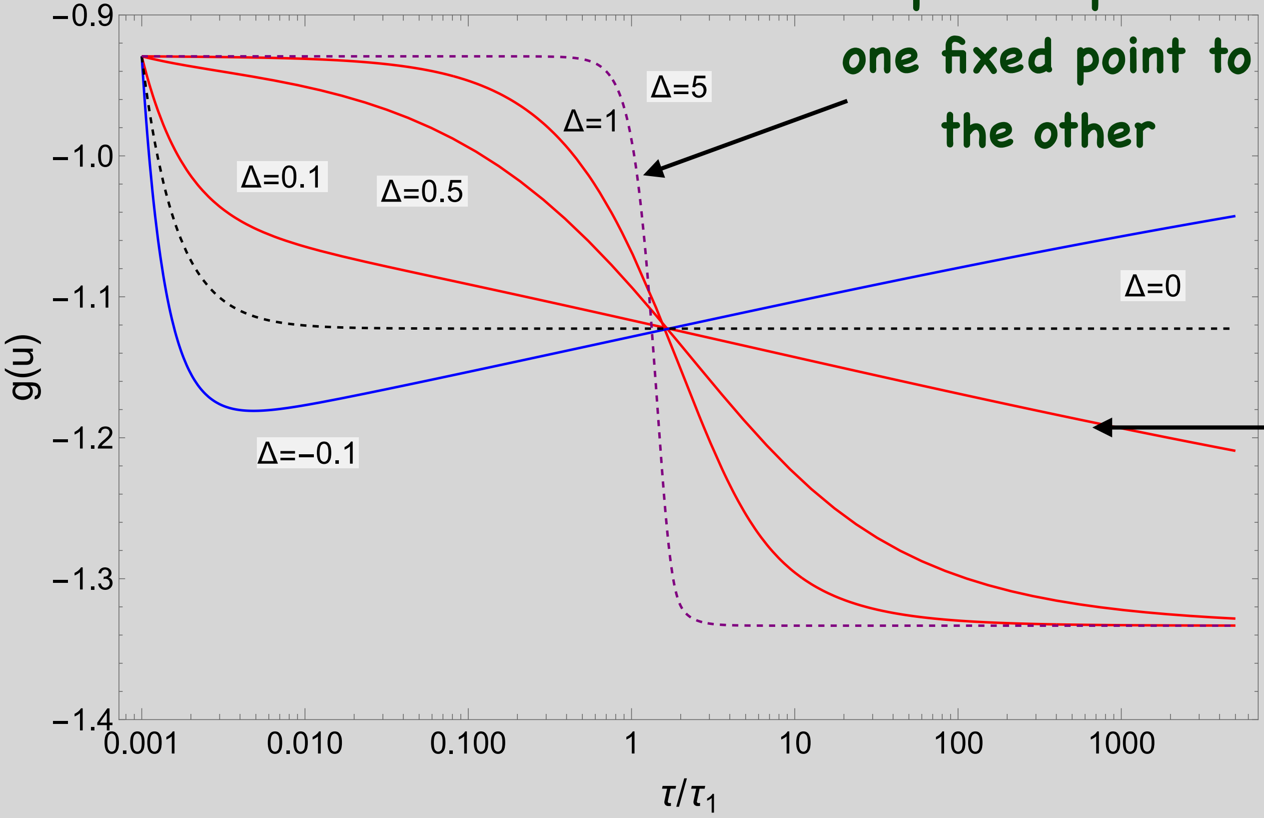
The transition region occurs when the collision rate is comparable to the expansion rate … as expected !

Time dependent relaxation time

$$\tau_R \sim \tau^{1-\Delta}$$

Δ controls the "speed" of the transition

Rapid Jump from one fixed point to the other

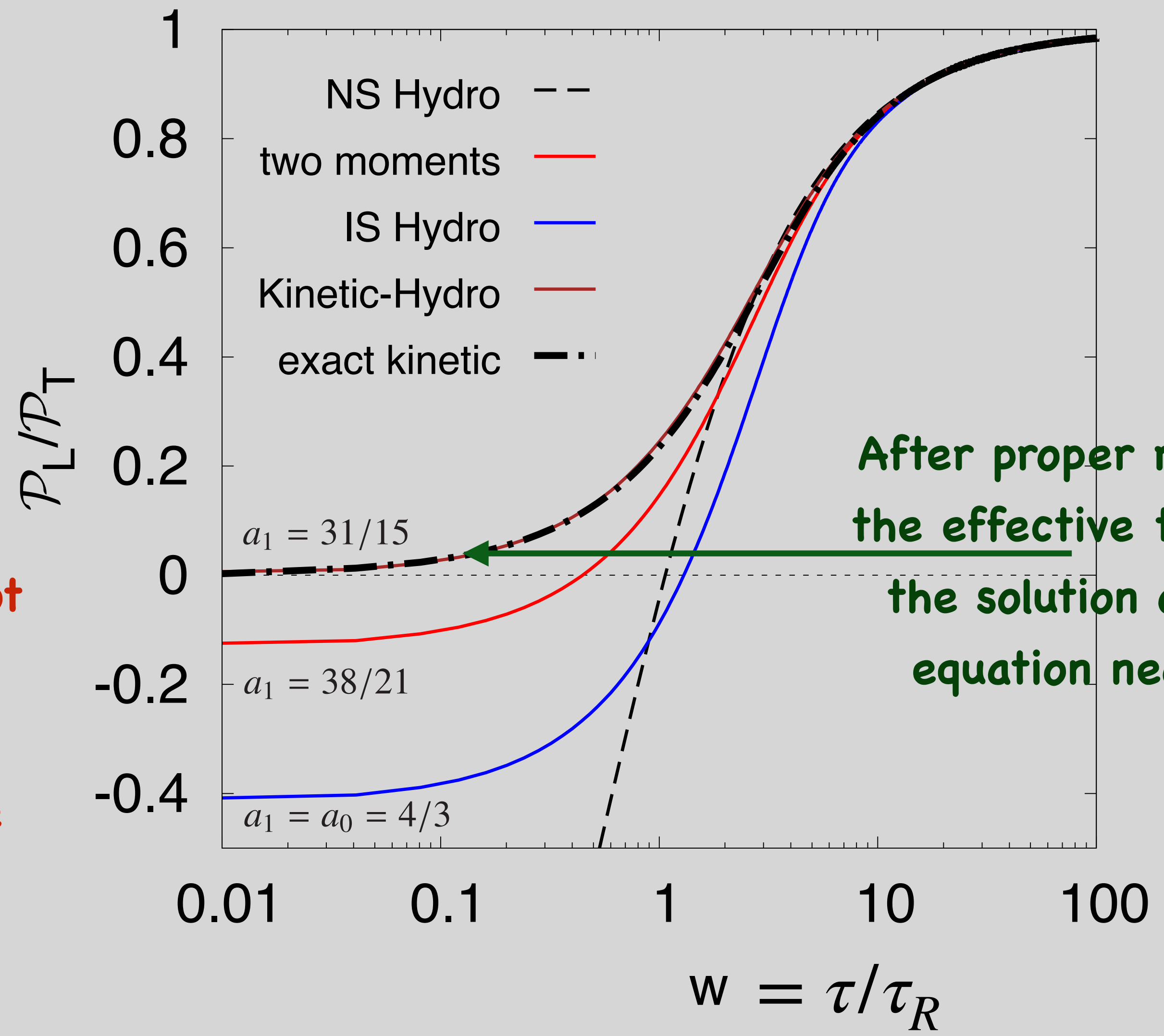


"non thermal fixed point" (?)

(very slow evolution towards the hydro fixed point: the expansion nearly balances the effect of the collisions)

Renormalizing a1 cures unphysical features of two-moment truncation (and other Israel-Stewart calculations)

$$\tau \frac{\mathrm{d}g_0}{\mathrm{d}\tau} + g_0^2 + \left(a_0 + a_1 + \frac{\tau}{\tau_R}\right) g_0 + a_1 a_0 - c_0 b_1 + \frac{a_0 \tau}{\tau_R} = 0$$



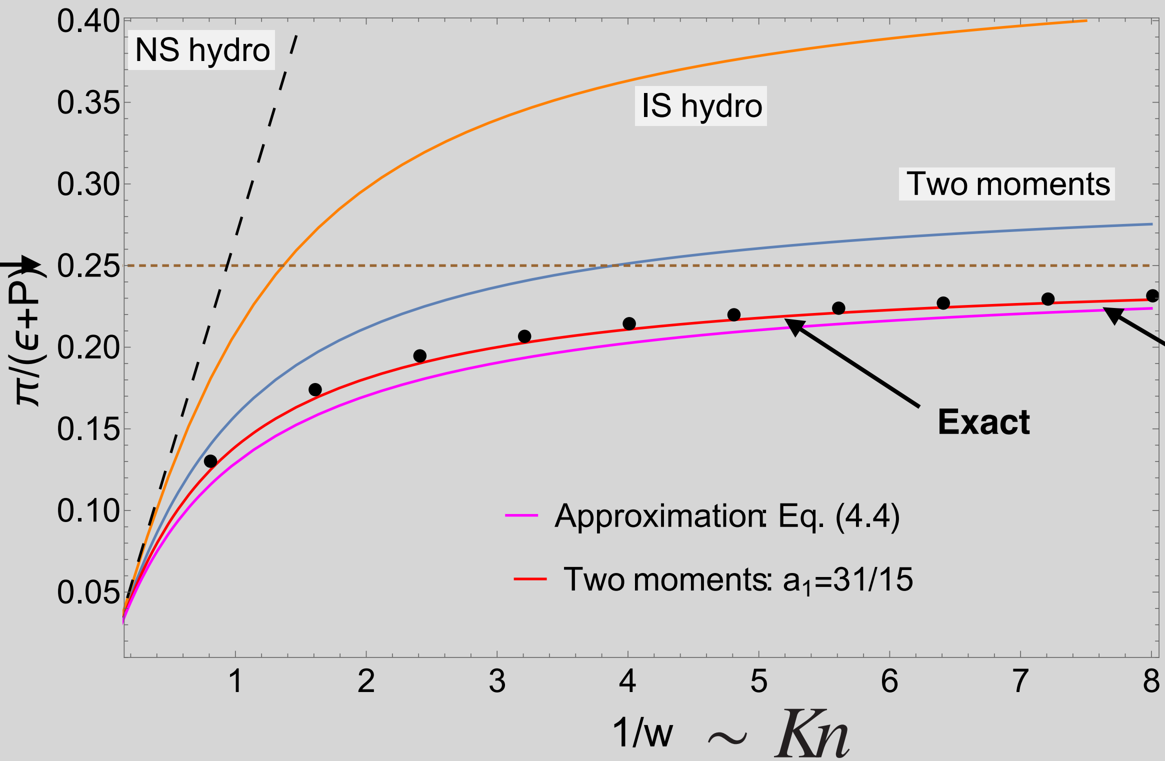
Changing a_1 does not "improve" hydrodynamics, it simply corrects the collisionless regime

After proper renormalisation the effective theory matches the solution of the kinetic equation nearly exactly.

Time dependent relaxation time $(\tau_R \sim \tau)$

Constant cross sections $\tau_R \sim \dfrac{1}{\sigma n} \sim \tau \quad (n \sim 1/\tau)$ **[Denicol, Noronha , 2020]**

free streaming (exact)



$\sim Kn$

$a_1 = a_0 = 4/3$

$a_1 = 38/21$

$a_1 = 31/15$ **(black dots)**

$$\frac{\pi}{\varepsilon + P} \simeq \frac{b_1}{2}\frac{Kn}{1 + 2b_1 Kn}$$

$$\left(\eta = \frac{b_1}{2}\varepsilon\tau_R\right)$$

Changing a1 (a 'second order transport coefficient') does not "improve" hydrodynamics, but rather improves the location of the collisionless fixed point

Renormalization of a1

$$-c_0 c_1 \frac{\mathcal{L}_2}{\mathcal{L}_0} = -c_1 c_0 \frac{A_2}{A_1}\frac{\mathcal{L}_1}{\mathcal{L}_0} = c_1 \frac{A_2}{A_1}(g_0 + a_0) \quad \Rightarrow \quad a_1 \mapsto a_1' = a_1 + c_1 \frac{A_2}{A_1} = \frac{31}{15}$$

'A tale of two sounds'

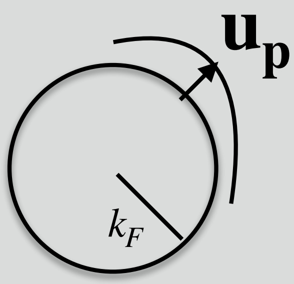
Title from "Collective states in Nuclei, a Tale of Two Sounds, B.K. Jennings and A.D. Jackson, Phys. Rept. 4 (1980)141.

Collective modes in neutral Fermi liquids

Long vavelength excitations are localized at the Fermi surface

$$\delta n_p = \delta(\varepsilon_p - \mu)\, v_F\, u_p$$

Fermi velocity $v_F = \dfrac{\hbar k_F}{m^*}$



Distortion of the Fermi surface

Landau kinetic equation (no collisions but interaction between quasiparticles)

$$(\mathbf{q}\cdot\mathbf{v}_p - \omega)u(\hat{\mathbf{p}}) + \mathbf{q}\cdot\mathbf{v}_p \sum_{\mathbf{p}'} f_{\mathbf{pp}'}\,\delta(\varepsilon_{\mathbf{p}'} - \mu)\, u(\hat{\mathbf{p}}') = 0$$

The dynamics is dominated by the angular degrees of freedom

('Average' over $|\mathbf{p}|$, as done for the $\mathscr{L}_n$, is automatic)

Landau kinetic equation

$$\cos\theta = \hat{\mathbf{q}} \cdot \hat{\mathbf{p}}$$

$$(\cos\theta - \lambda)u(\hat{p}) + \frac{\cos\theta}{8\pi}\int \mathrm{d}\Omega'\, F(\hat{p}\cdot\hat{p}')\, u(\hat{p}') = 0 \qquad \lambda \equiv \frac{\omega}{q v_F}$$

Expand in Legendre polynomials: all values of ℓ are coupled.

(Note the analogy with the $\mathscr{L}_n$ moments)

Simple solution for constant interaction $F(\hat{p}\cdot\hat{p}') \mapsto F_0$

$$u(\hat{p}) = C\frac{\cos\theta}{\lambda - \cos\theta} \qquad \frac{\lambda}{2}\ln\frac{\lambda+1}{\lambda-1} - 1 = \frac{1}{F_0}$$

When $F_0 = 0, \lambda = 1$, and $\omega = q v_F$: single particle excitation.

When $F_0 > 0$, there exists an undamped collective mode with $\lambda > 1$: the zero sound.

When $F_0 \gg 1, \lambda \sim \sqrt{\frac{F_0}{3}}$

Transition from the collisionless zero sound to the (collision-dominated) first sound

Collisions suppress all moments except the lowest ones (associated with conservation laws). Same as for the $\mathscr{L}_n$ moments.

The distorsion of the Fermi surface is then simply $u(\hat{p}) = A + B\cos\theta$

The collective mode is the first sound $\omega = qv_F\sqrt{\frac{1+F_0}{3}}$

When $F_0 \gg 1$ the two sounds have analogous dispersion relations...
but zero sound is NOT "hydrodynamics out of equilibrium"

Conclusions

Kinetic Bjorken flow can be analysed in terms of a restricted set of moments of the distribution function, leading to an effective theory that contains in particular second order hydrodynamics à la Israel-Steward.

The generic flow exhibits two regimes: a collisionless regime (driven by the expansion), and a collision-dominated regime (hydrodynamics).

The transition between the two regimes occurs where it is supposed to occur, namely when the collision rate is comparable to the expansion rate.

The collisionless regime is non trivial. Its underlying simple fixed point structure allows us to justifies the two-moment truncation. A simple renormalisation brings the effective theory in nearly perfect agreement with the exact solution of the kinetic equation.