Soft and Hard Probe with Hydrodynamic Attractors

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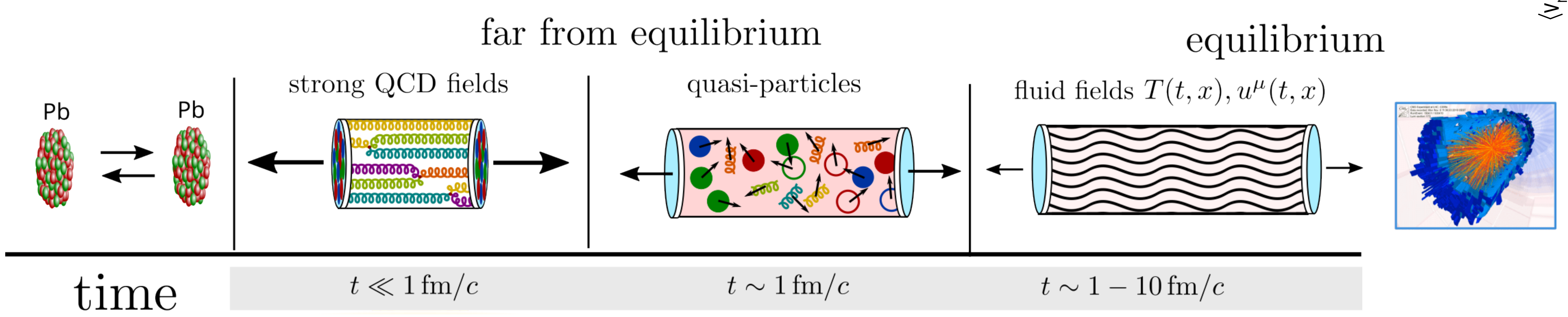
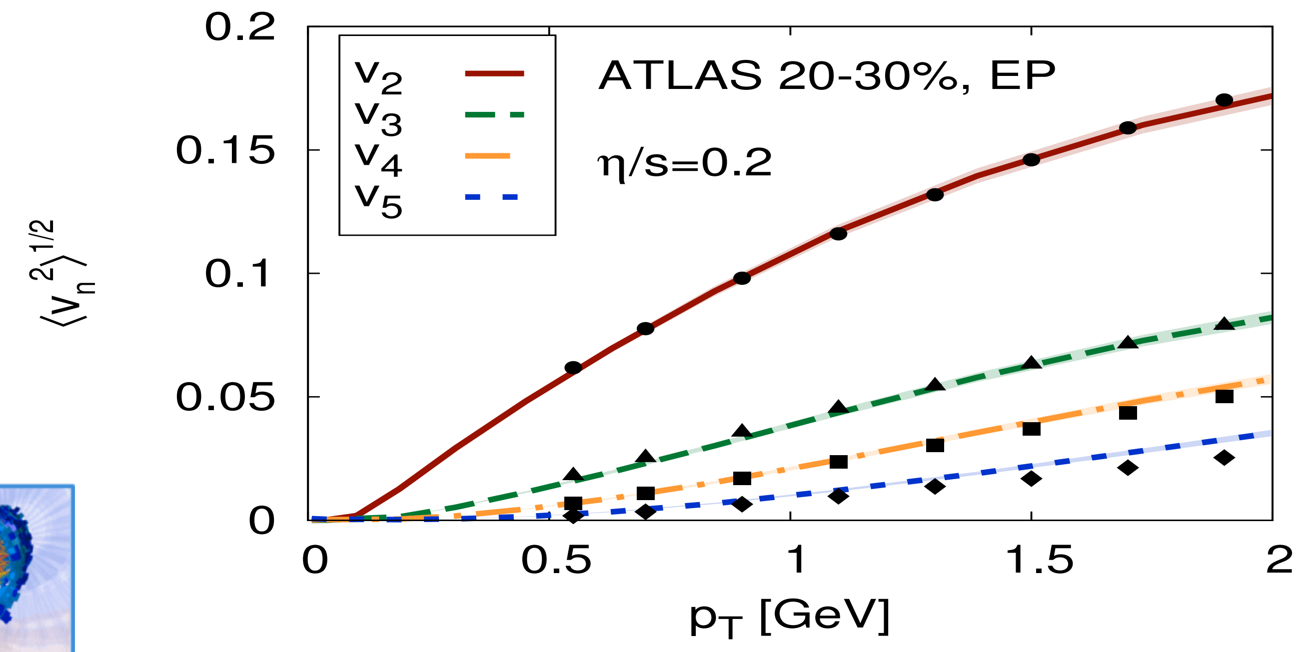
Motivation

Thermalization of QGP far from equilibrium

- Heavy-ion collisions: fast thermalization & hydrodynamization.

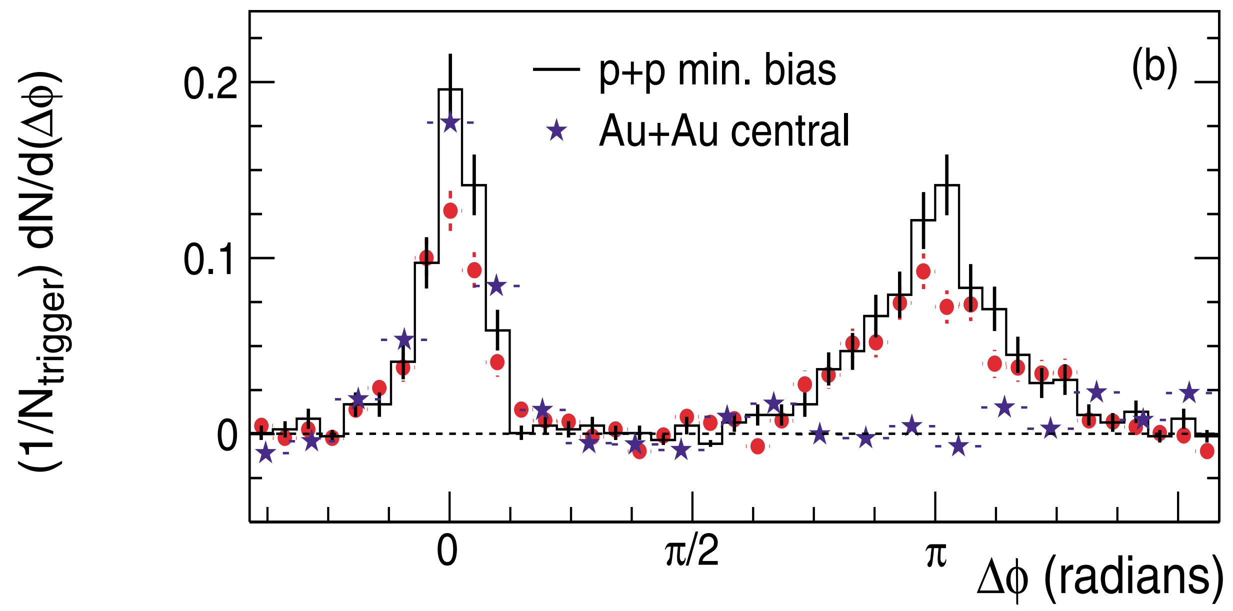


Courtesy of A. Mazeliauskas



Gale et al, 1301.5893

soft probe
flow



PHENIX, 2008

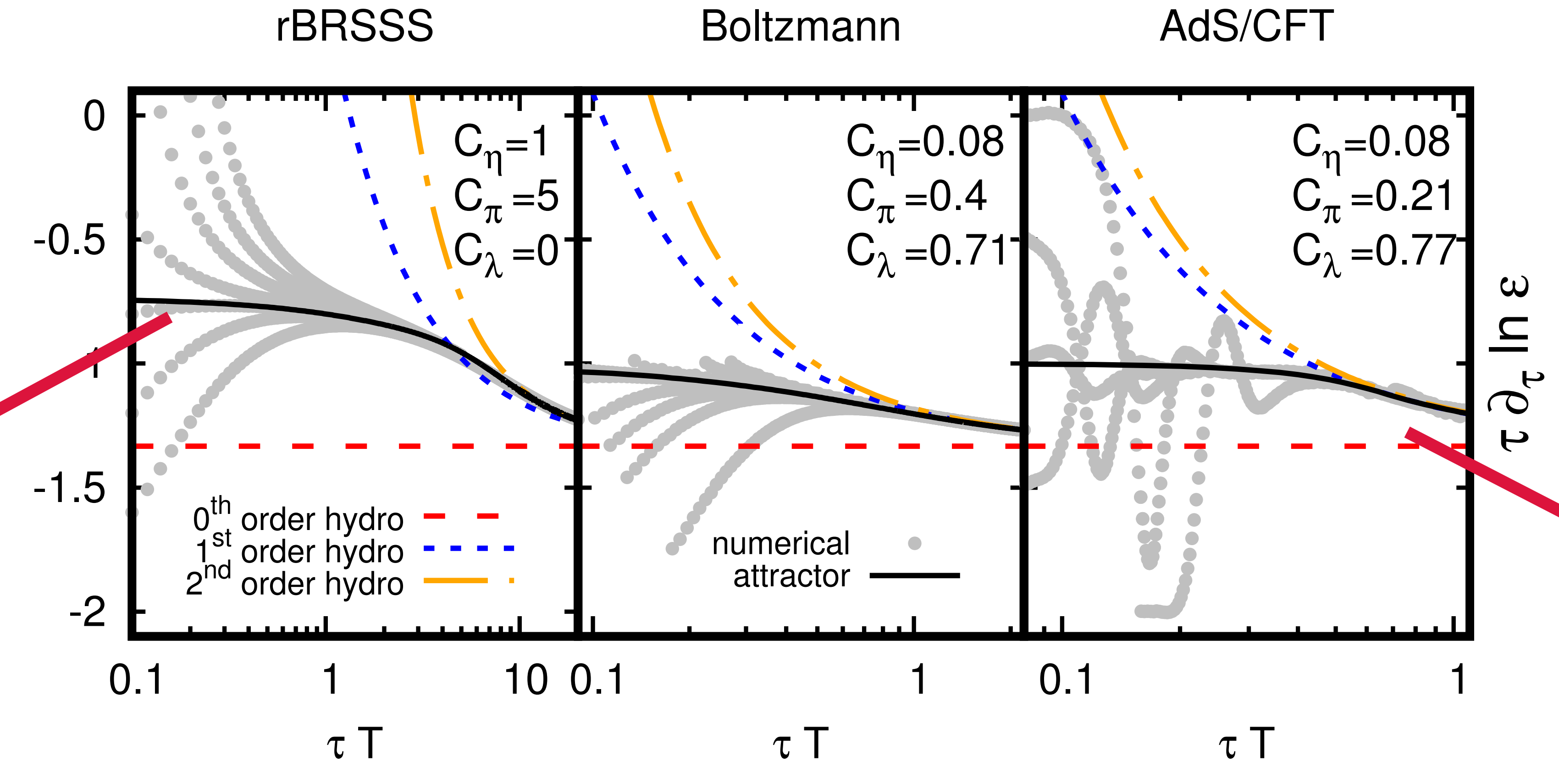
hard probe
jet

- Challenges:
  - **Experiment**: measurement only at freezeout in momentum space;
  - **Theory**: legacy of non-hydrodynamic modes. Yi Yin's talk

Attractors

- **Attractors** serve as a bridge connecting the far-from-equilibrium regime to the near-equilibrium regime.

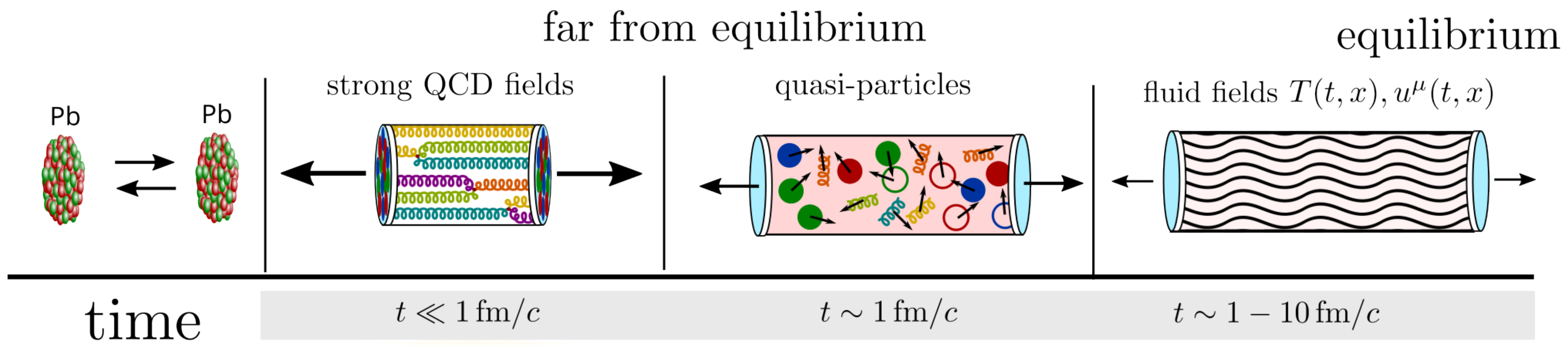
geometry-driven
slow-roll attractor:
onset of hydrodynamization
at early time



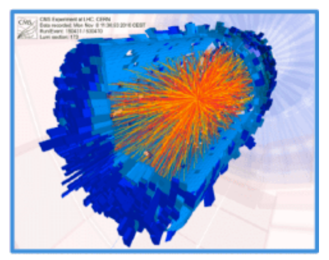
equilibration-driven
dissipative attractor:
lost of initial information
at later time

Florkowski et al, 1707.02282, Romatschke, 1712.05815

Hydrodynamic attractor offers a reliable description of the non-equilibrium fluid dynamics (via resummation) as long as the contribution from all **non-hydrodynamic** modes can be neglected.



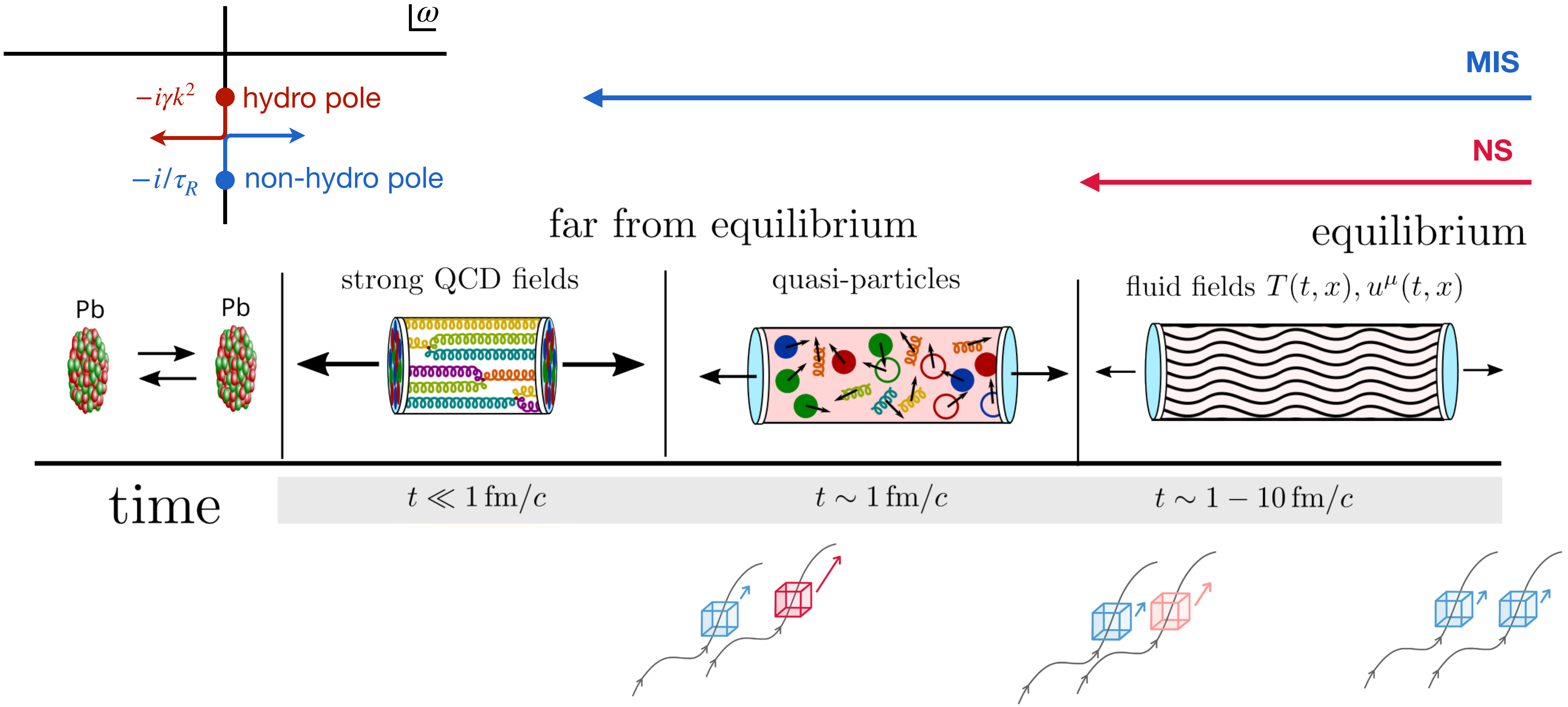
?



- The **motivation** of this talk: understand the role of attractors in interpreting the data from heavy-ion collision experiments.

From NS to MIS equations

- We consider a toy model for QCD, i.e., MIS-like theory, that *extend the applicability* of conventional NS hydrodynamics.



The simplest MIS-like equations

- The simplest scenario: Bjorken fluids that is

Heller et al, 1503.07514

1) 0+1D boost-invariant:

τ (time) dependence only, transversely homogeneous

2) conformal:

T (temperature) measures the energy scale

A (anisotropy) measures the thermalization

- The EOM for the dynamic system $\Psi = (T(\tau), A(\tau))$:

Blaizot et al, 2106.10508

$$\tau \partial_\tau \Psi(\tau) = -M(\tau)\Psi(\tau) + V \qquad \text{where} \qquad M(\tau) = \begin{pmatrix} 1/3 & -T(\tau)/18 \\ \tau A(\tau)/C_\tau & 2A(\tau)/9 \end{pmatrix} \qquad V = \begin{pmatrix} 0 \\ 8C_\eta/C_\tau \end{pmatrix}$$

NB: Quantum gases: $\nabla \cdot u = 1/\tau \sim \dot{a}/a$

Fujii et al, 2404.12921, Heller et al, 2507.02838

Asymptotic solutions

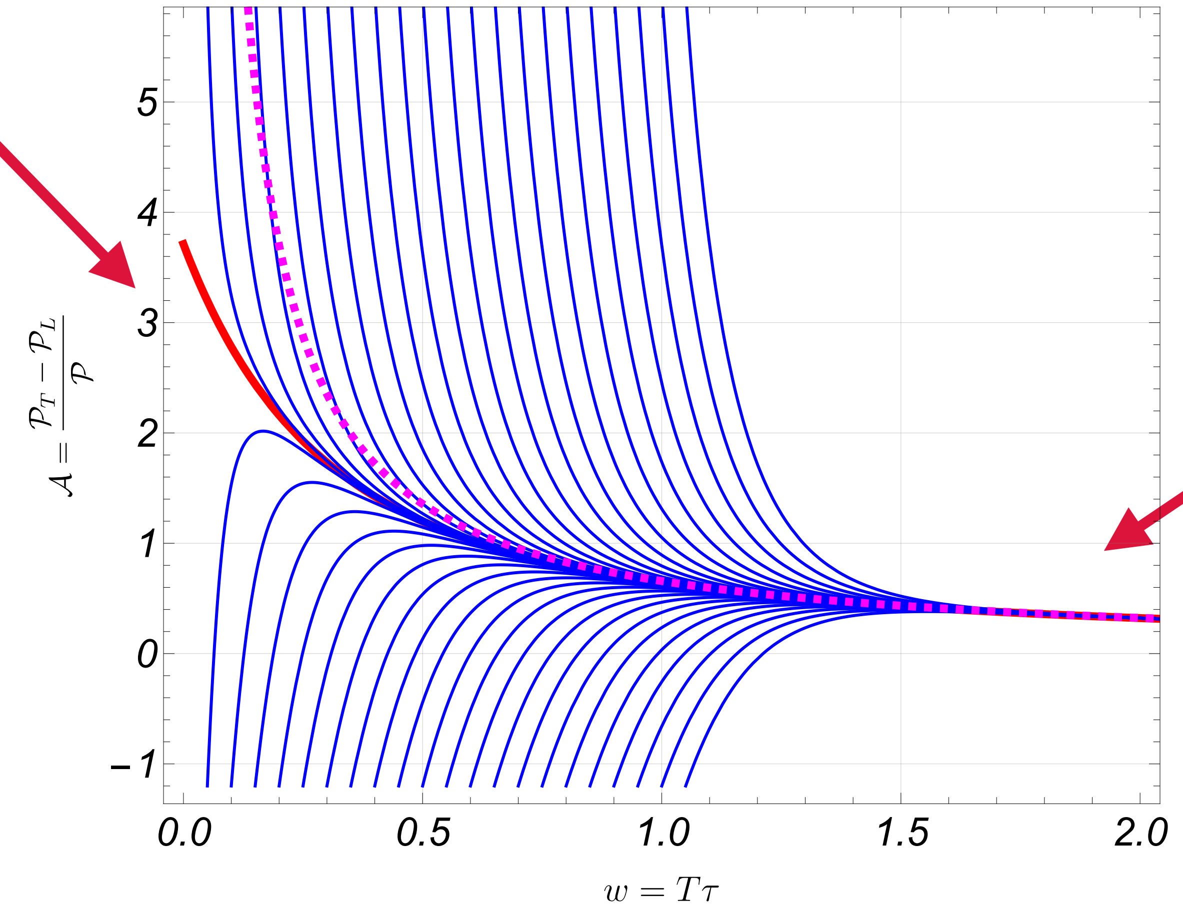
- Early-time *attractor* solutions:

$$T(\tau) \sim \mu(\mu\tau)^{-\frac{1-\alpha}{3}}(1+\ldots)$$

$$A(\tau) \sim 6\alpha(1+\ldots)$$

slow-roll attractor

$$\alpha = \sqrt{C_\eta/C_\tau}$$



Heller et al, 1503.07514

μ: integration constant parametrizing attractor

- Later-time asymptotic solutions

$$T(\tau) \sim \Lambda(\Lambda\tau)^{-\frac{1}{3}}(1+\ldots) + C_\infty e^{-\frac{3}{2C_\tau}(\Lambda\tau)^{2/3}}(\Lambda\tau)^{-\frac{2}{3}(1-\alpha^2)}(1+\ldots)$$

$$A(\tau) \sim 8C_\eta(\Lambda\tau)^{-\frac{2}{3}}(1+\ldots) + f(C_\infty)e^{-\frac{3}{2C_\tau}(\Lambda\tau)^{2/3}}(\Lambda\tau)^{-\frac{1}{3}+\alpha^2}(1+\ldots)$$

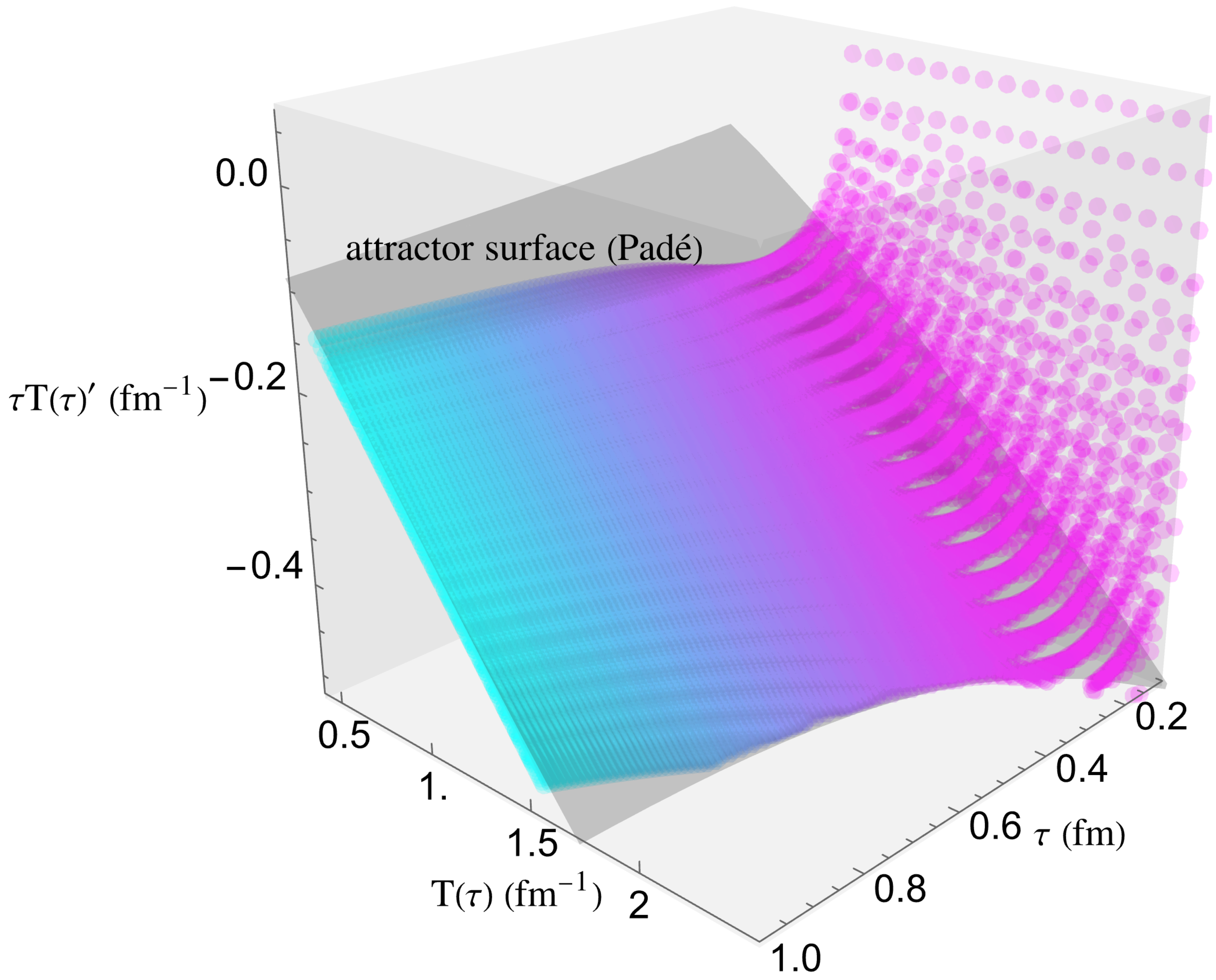
hydrodynamic attractor
+ transseries (non-hydrodynamic) modes

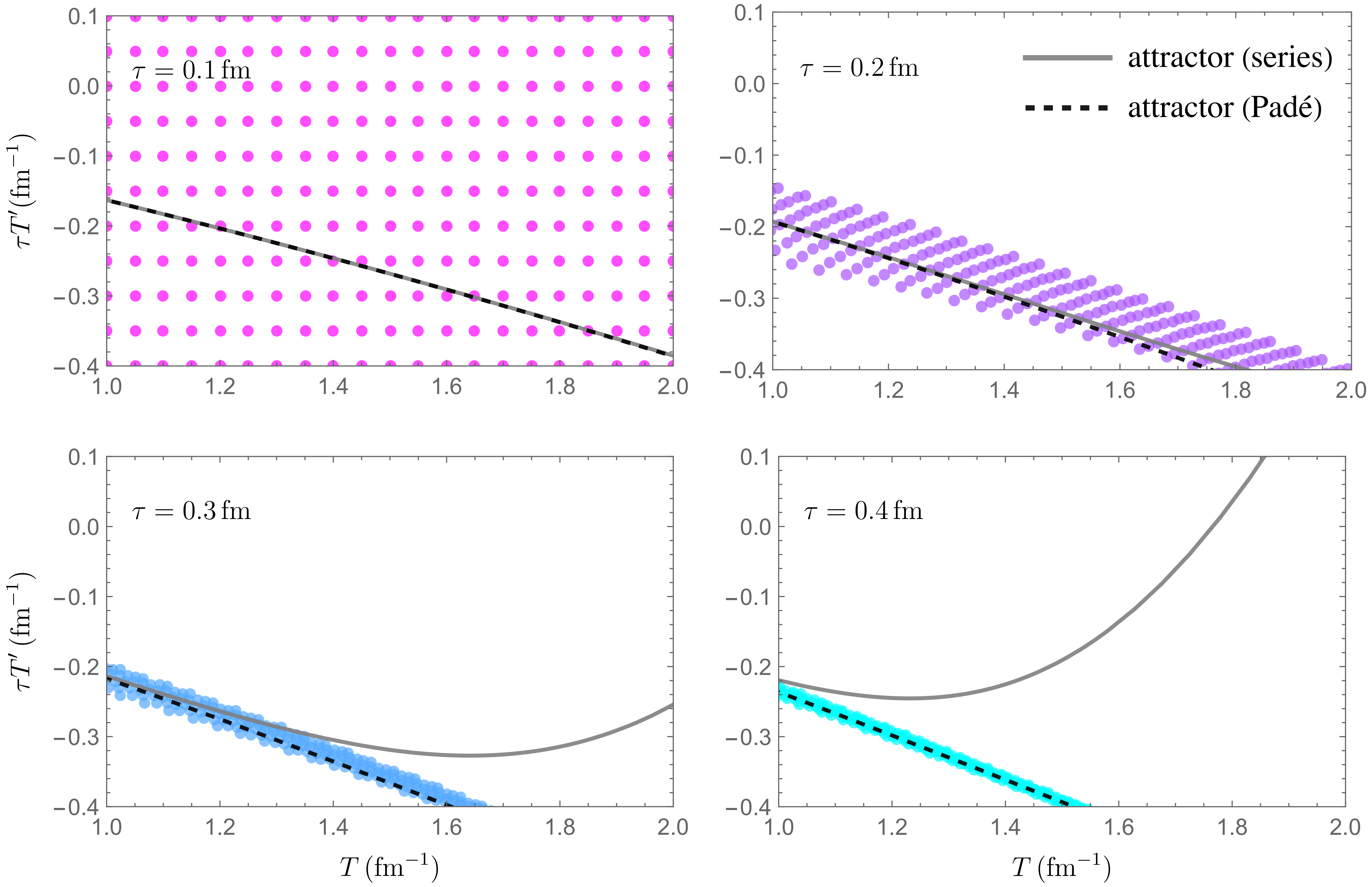
XA and Spalinski, 2312.17237
Aniceto et al, 2401.06750

Λ, C_∞: integration constant

Early-time attractor in phase space

- Trajectories in phase space rapidly approach the early-time *attractor surface*.





snapshot of $(\tau T', T)$ plane at different τ

Beyond attractor

Limitation of the simplest model

- 0+1D Bjorken model: too simple to be true.
  - It assumes infinitely large and homogeneous transverse plane.
  - It has only two initial conditions.
  - It predicts very few observables.

✓ Multiplicity of hadrons
Photon/dilepton spectrum
…

✗ Collective flow
Jet
…

- 2+1D full hydrodynamics: too difficult to handle.

Idea: 0+1D attractor background + 2+1D perturbations

cf Denicol's talk

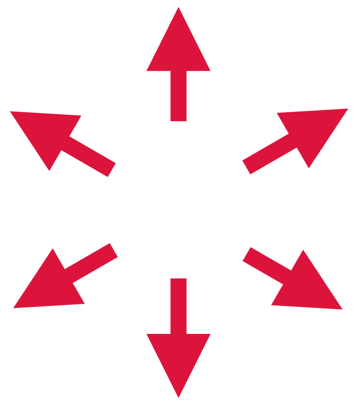
Linear perturbations

- Linearization the full system around attractor background:

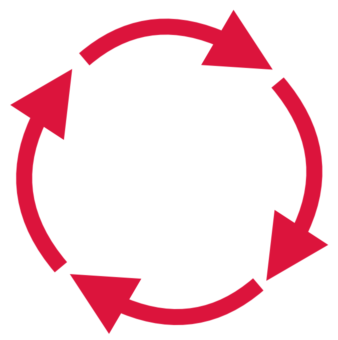
$$\partial_\nu T^{\mu\nu} = \partial_\nu (T^{\mu\nu}_{\text{attractor}} + \delta T^{\mu\nu}) = 0 \quad \longrightarrow \quad \begin{cases} \partial_\nu T^{\mu\nu}_{\text{attractor}} = 0, \\ \partial_\nu \delta T^{\mu\nu} = 0. \end{cases}$$

6 independent fields: $\phi = (\delta T, \delta\theta, \delta\omega, \delta\pi_{11}, \delta\pi_{22}, \delta\pi_{12})(\tau, \mathbf{x})$ $\quad i = 1,2$

fluid divergence $\delta\theta \equiv \partial_i \delta u_i$



vorticity $\delta\omega \equiv \epsilon_{ij}\partial_i \delta u_j$



shear stress tensor $\delta\pi_{ij}$

- The EOM for the dynamic system:

For NS limit, cf Floerchinger et al, 1108.5535

$$\partial_\tau \phi_i(\tau, \mathbf{k}) = M_{ij}(\tau, \mathbf{k})\, \phi_j(\tau, \mathbf{k})$$

Asymptotic solutions at late time

- When $\tau \to \infty, k \neq 0$, solutions perturbed around attractor are transseries:

$$\delta T(\mathbf{k}) \sim C_i(k)\, e^{-S_i \tau^{b_i} + \dots}\, \tau^{\beta_i}\, (1 + \dots) \quad i = 1,2,3,4$$

$$\delta \omega(\mathbf{k}) \sim C_i(k)\, e^{-S_i \tau^{b_i} + \dots}\, \tau^{\beta_i}\, (1 + \dots) \quad i = 5,6$$

$C_1(k), \dots, C_6(k)$: $6N_k$ integration constants

$\delta\hat{\theta}$ and $\delta\hat{\pi}_{ij}$ are determined dependently

- Attractor is asymptotically **stable** ($\mathrm{Re}\, S_i > 0$) against transverse perturbations.
  - Hydrodynamic attractor exists in 2+1D.
  - Non-hydrodynamic (non-perturbative) behavior is important at later time.

Zero wavenumber modes

- When $\tau \to \infty$, $k = 0$ modes need to be considered separately:

$$\delta u_i \sim C_i \tau^{1/3}(1 + \ldots) \qquad i = 1{,}2$$

→ mild growth due to momentum conservation

$$\delta T \sim C_3\,(1 + \ldots) + C_4\, e^{-\frac{3}{2C_\tau}\tau^{2/3}} \tau^{-\frac{2}{3}(1-\alpha^2)}(1 + \ldots)$$

$$\delta \pi_{11} - \delta \pi_{22} \sim C_5\, e^{-\frac{3}{2C_\tau}\tau^{2/3}} \tau^{\frac{2}{3}\alpha^2}(1 + \ldots)$$

$$\delta \pi_{12} \sim C_6\, e^{-\frac{3}{2C_\tau}\tau^{2/3}} \tau^{\frac{2}{3}\alpha^2}(1 + \ldots)$$

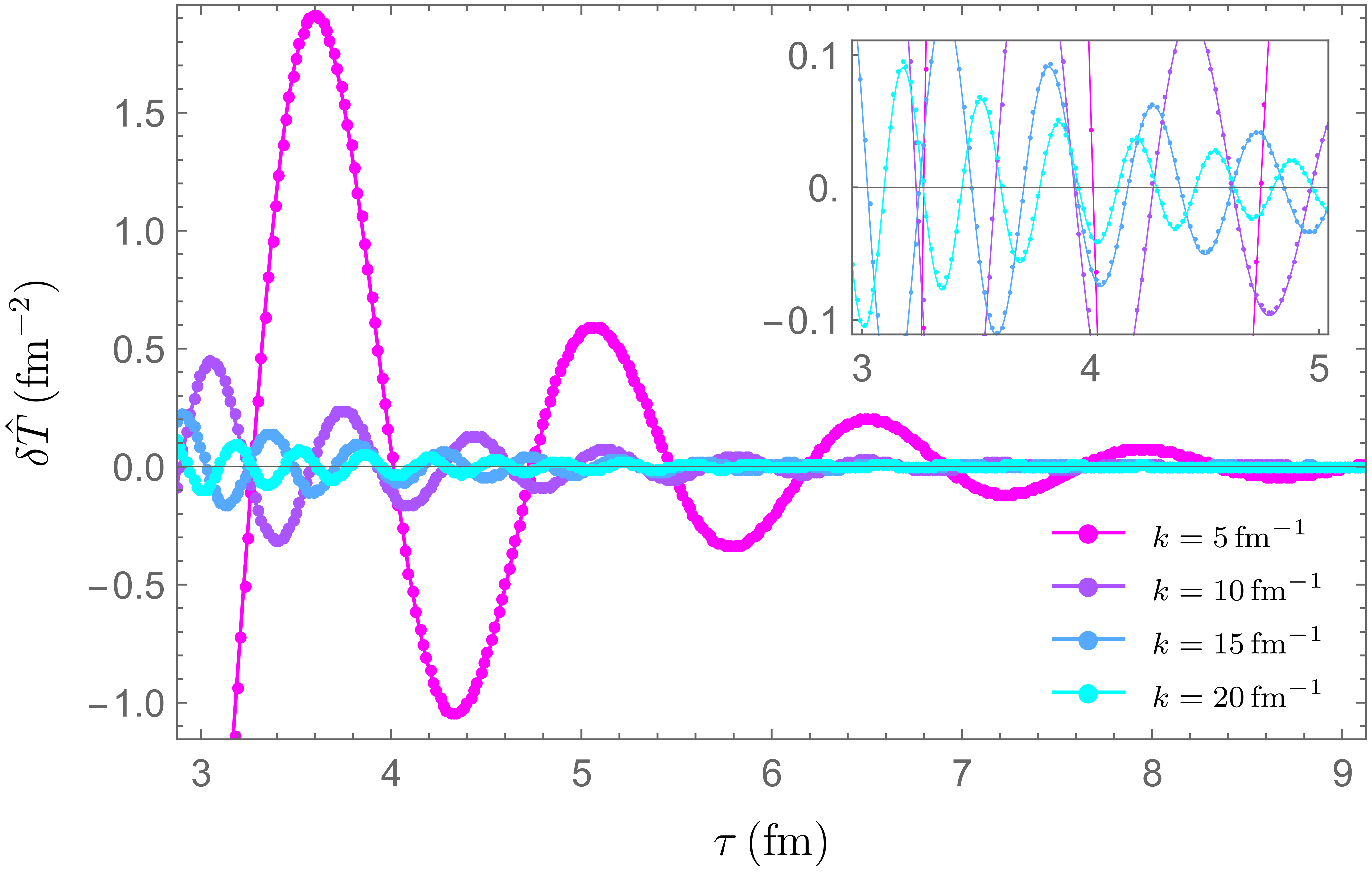
→ reproduces the background transseries

Observables are extracted from a **finite** set of asymptotic data $C_n(\mathbf{k})$.

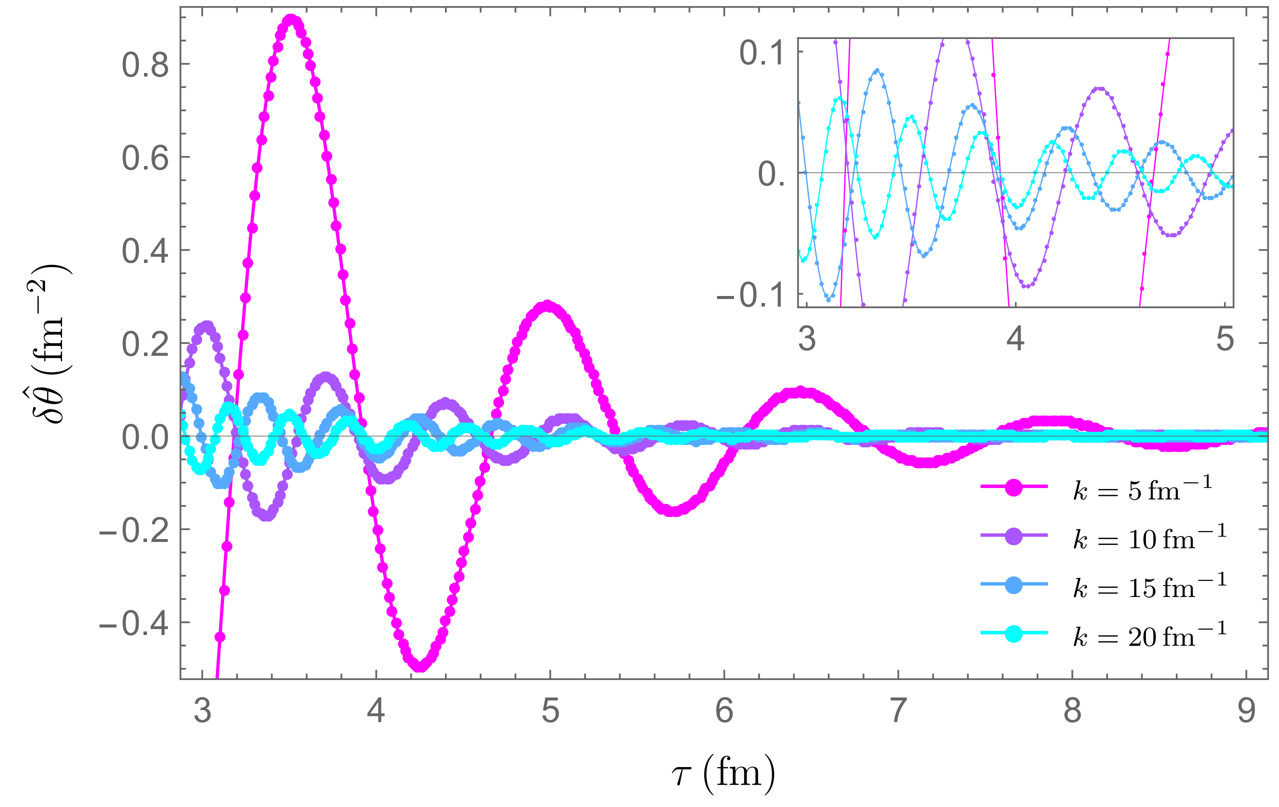
# of data: $6 \times N_k$

Matching to numerics

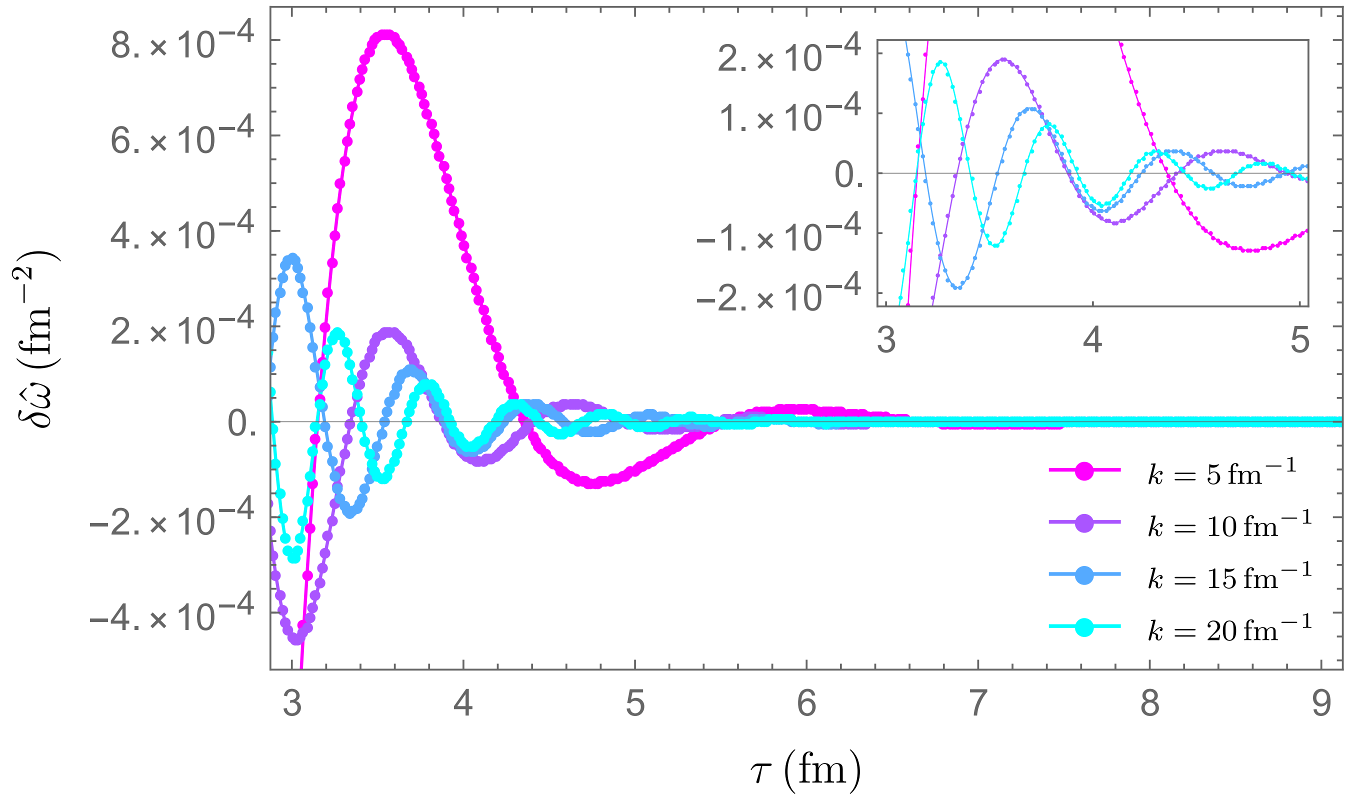
- The analytic solutions (solid curves) fit the numerics (discrete points) even at $\tau = 3$ fm.



temperature



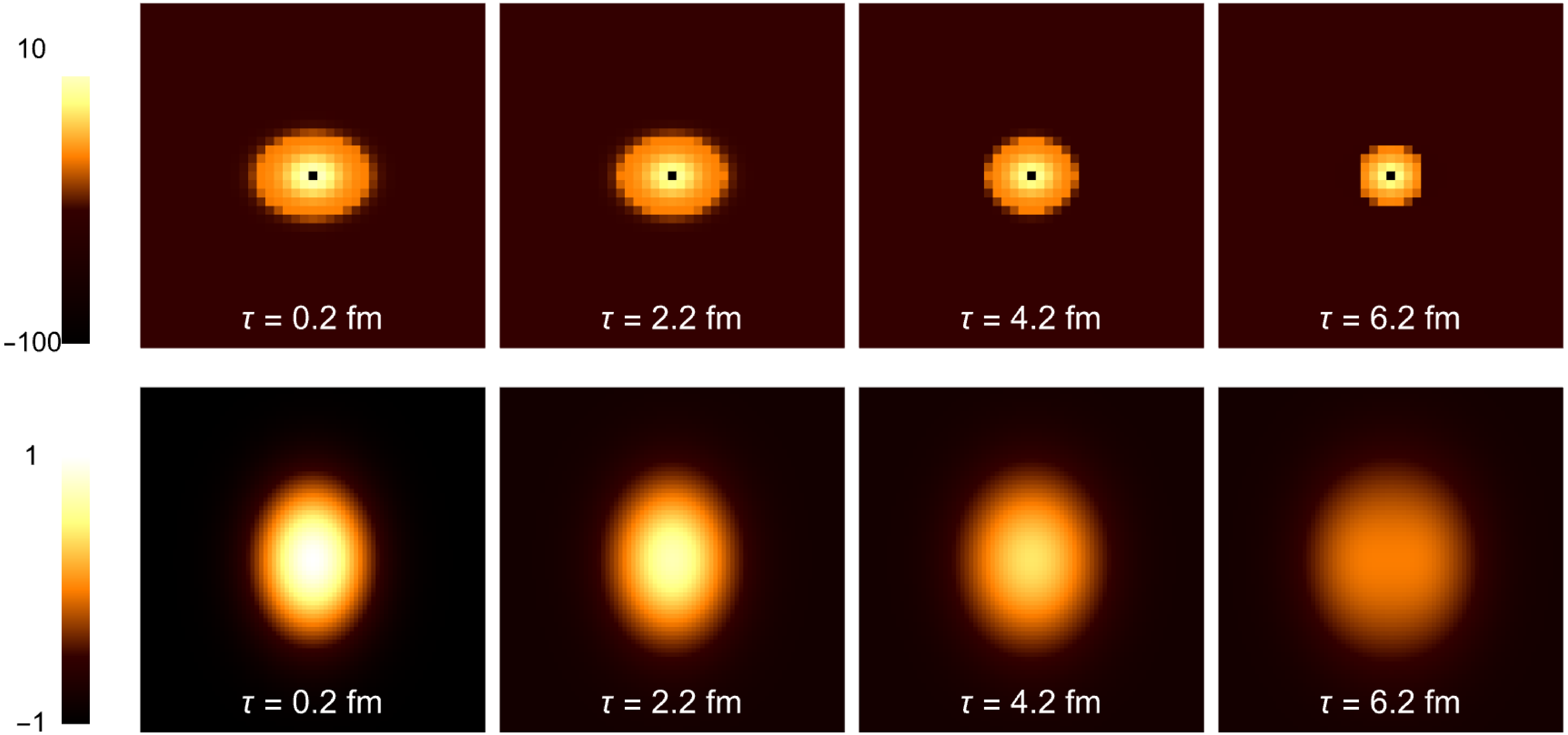
fluid divergence



vorticity

Transverse tomography

- Transverse information is encoded in a finite set of Fourier modes via FFT.



Evolution of temperature (energy density) in transverse spaces

Soft probe: flow

Cooper-Frye freezeout

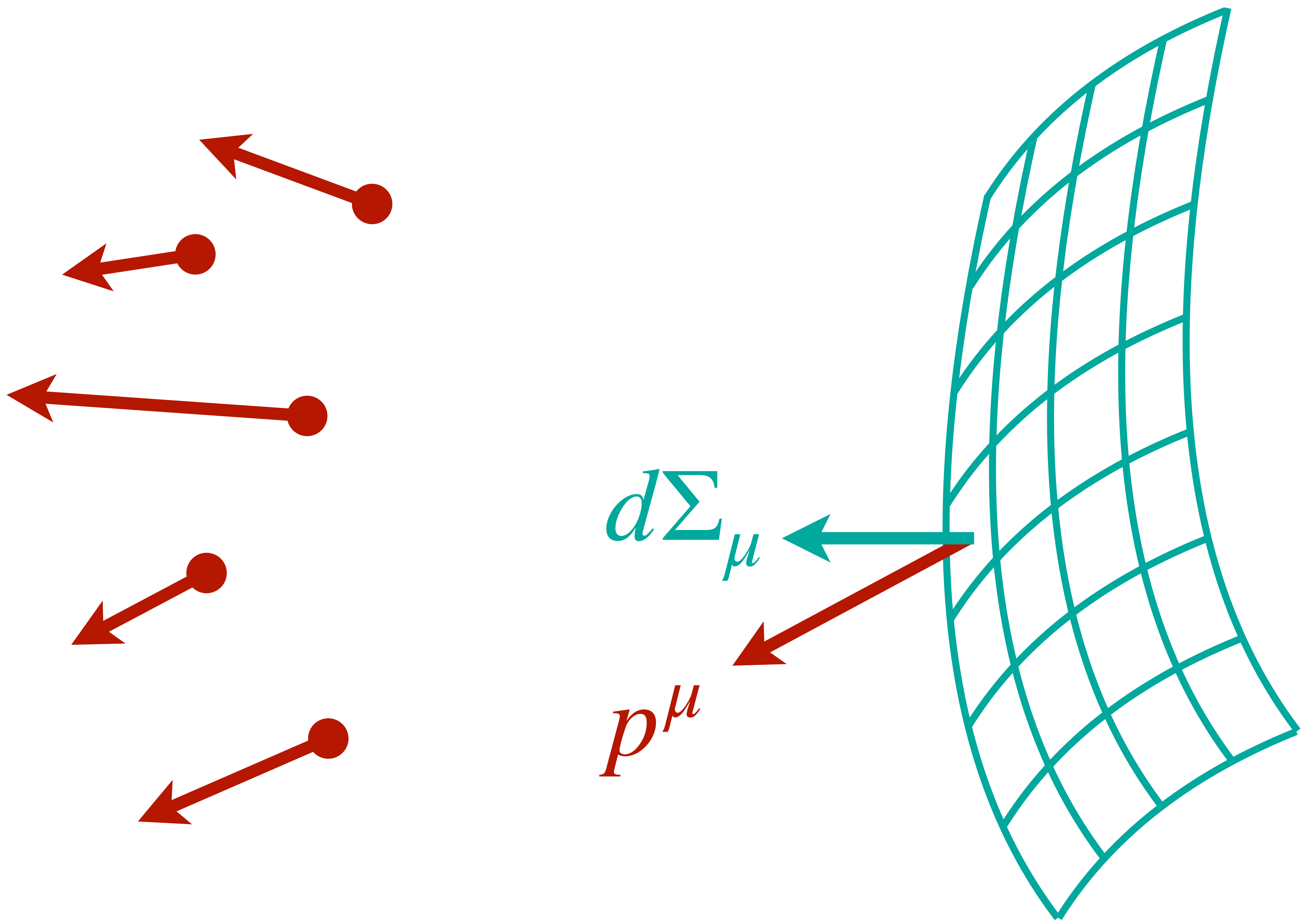
- Cooper-Frye formula   Cooper and Frye, 1974

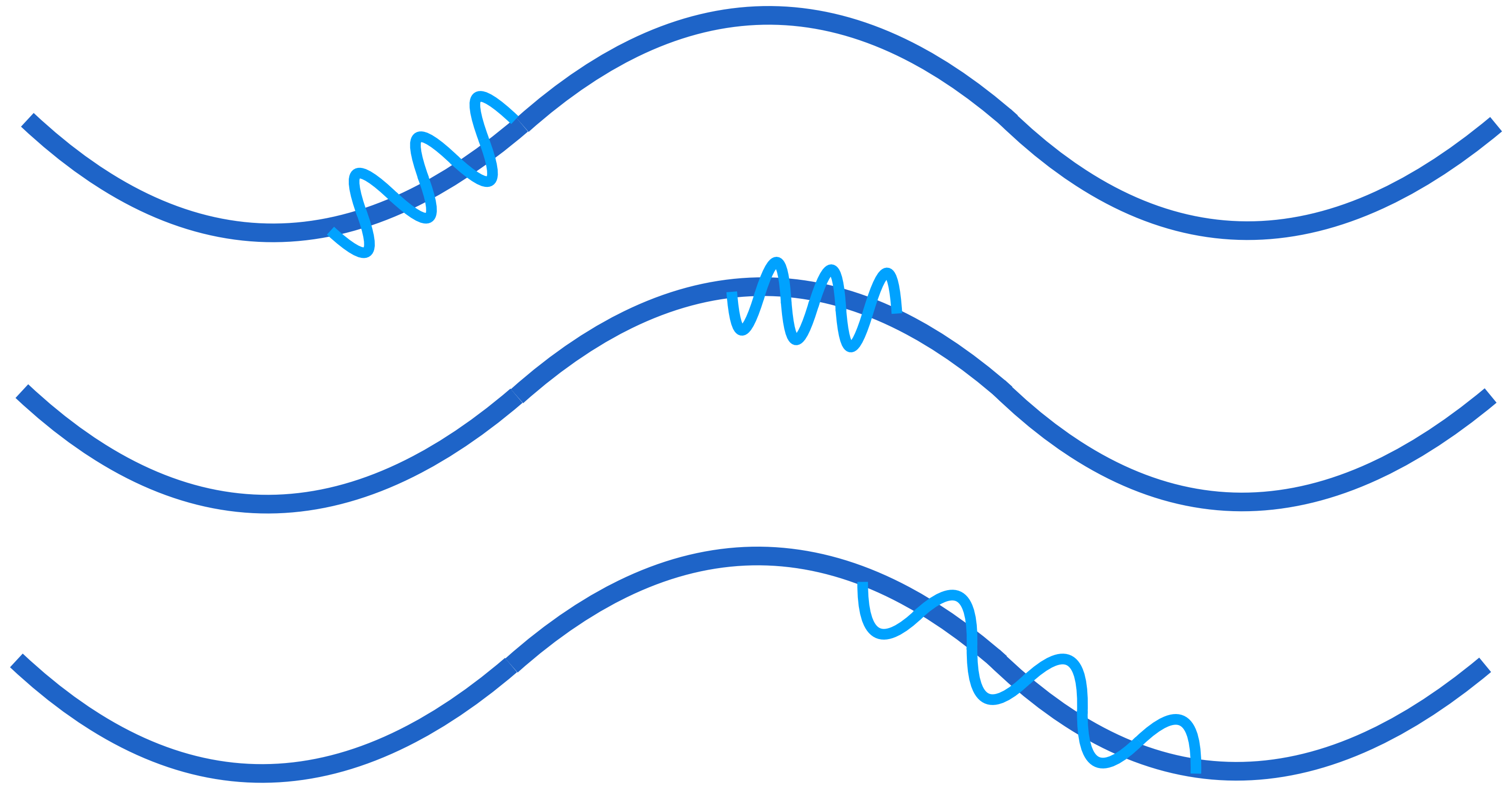
particle
in momentum space

$$E\frac{dN}{dp^3} = \int_\Sigma f(x,p)\, p^\mu d\Sigma_\mu$$

fluids
in position space

$$f(x,p) = f(T^{\mu\nu}(x), p) = e^{u\cdot p/T}(1 + \mathcal{O}(p))$$





Collectivity: analytic results

- Cooper-Frye freezeout

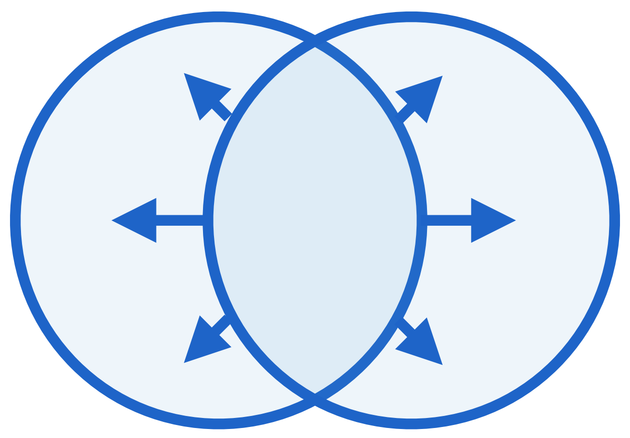
For NS limit w/o perturbations, cf Teaney, 2003

$$\frac{dN(p_\perp,\phi)}{p_\perp dp_\perp d\phi dy} = \frac{m_\perp \tau \Sigma}{(2\pi)^3}\left\{ \underbrace{2K_1(\hat{m}_\perp) + \frac{1}{12}\left[\hat{p}_\perp^2 K_1(\hat{m}_\perp) - 2\hat{m}_\perp K_2(\hat{m}_\perp)\right] A}_{F_0} + \text{perturbations} \right\}$$

pressure anisotropy (A)

- Collective expansion

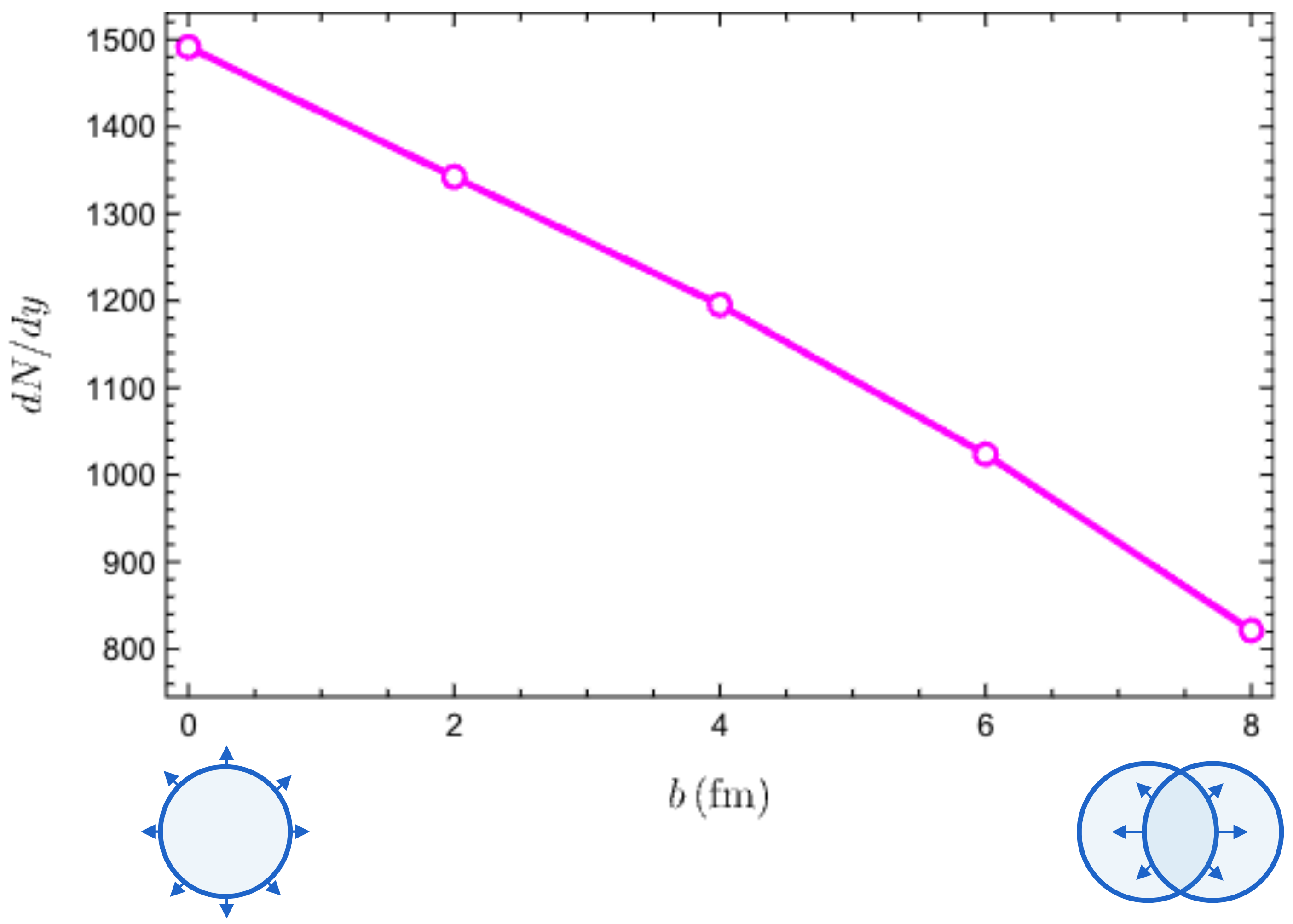
$$\frac{dN(p_\perp,\phi)}{p_\perp dp_\perp d\phi dy} = v_0(p_\perp)\left(1 + \sum_{n=1}^{\infty} 2v_n(p_\perp)\cos(n\phi)\right)$$

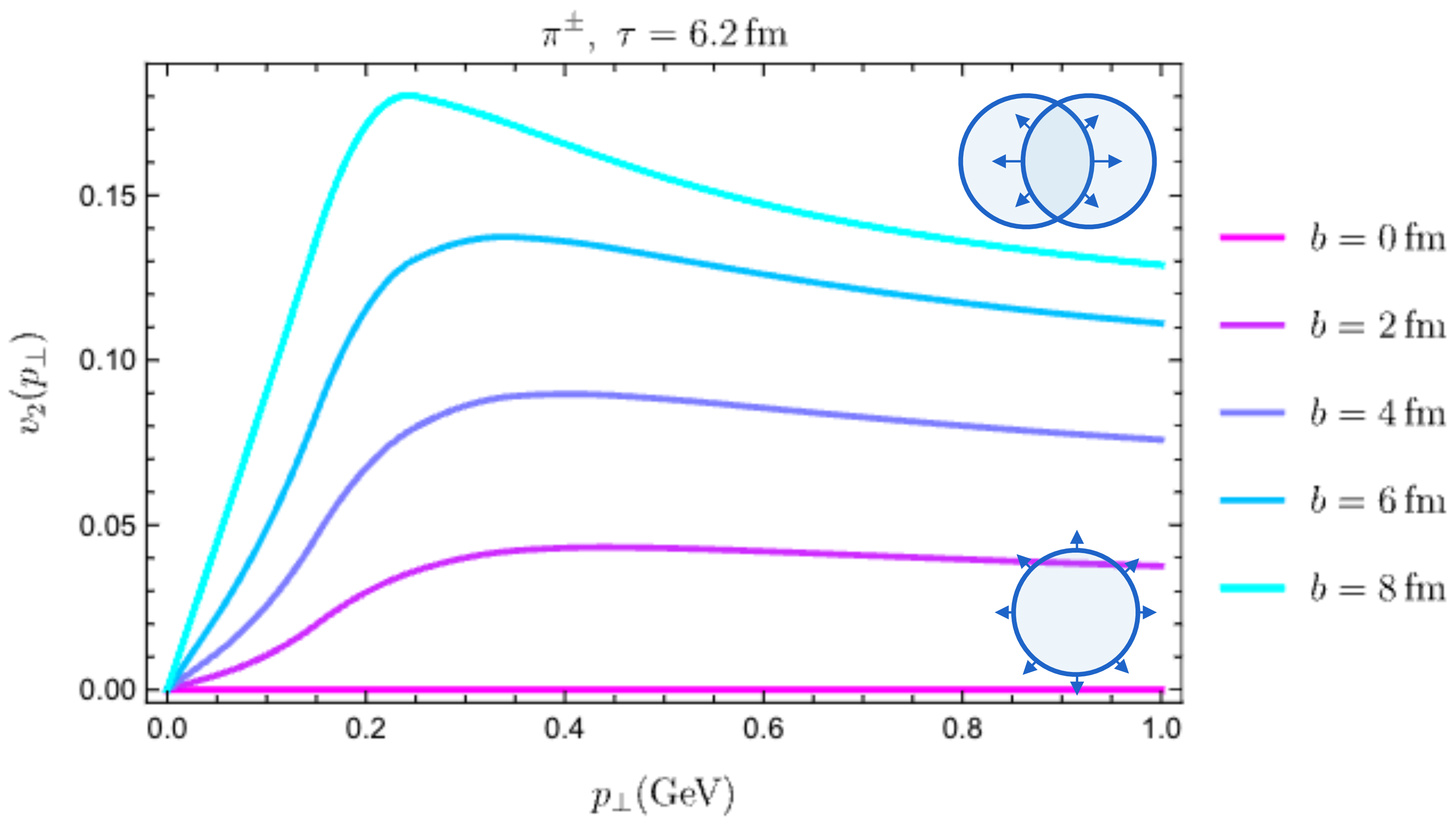


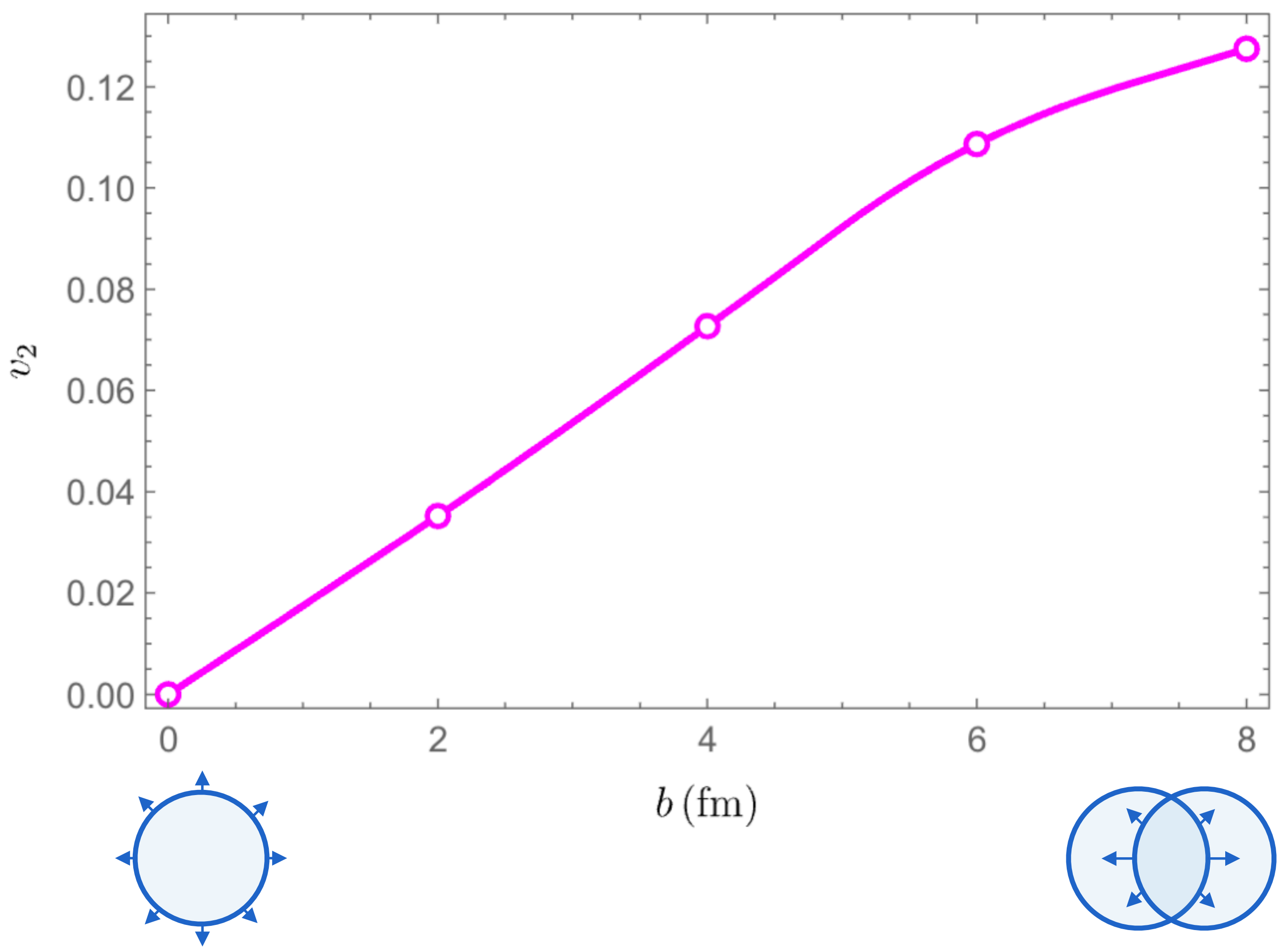
$$v_0(\hat{p}_\perp) \sim \frac{m_\perp \tau_f \Sigma}{(2\pi)^3}\left(F_0 + \text{ perturbations}\right) \qquad v_1(\hat{p}_\perp), v_2(\hat{p}_\perp) \sim \frac{\text{perturbations}}{4F_0 + \text{ perturbations}}$$

Collectivity: numerical results

- Numerical results qualitatively *agree* with experiments.







Hard probe: jet

Jet-medium interaction

- The total energy of jet and fluid system is conserved:

$$\partial_\nu T^{\mu\nu} = \partial_\nu \left(T^{\mu\nu}_{\text{attractor}} + \delta T^{\mu\nu} + T^{\mu\nu}_{\text{jet}} \right) = 0$$

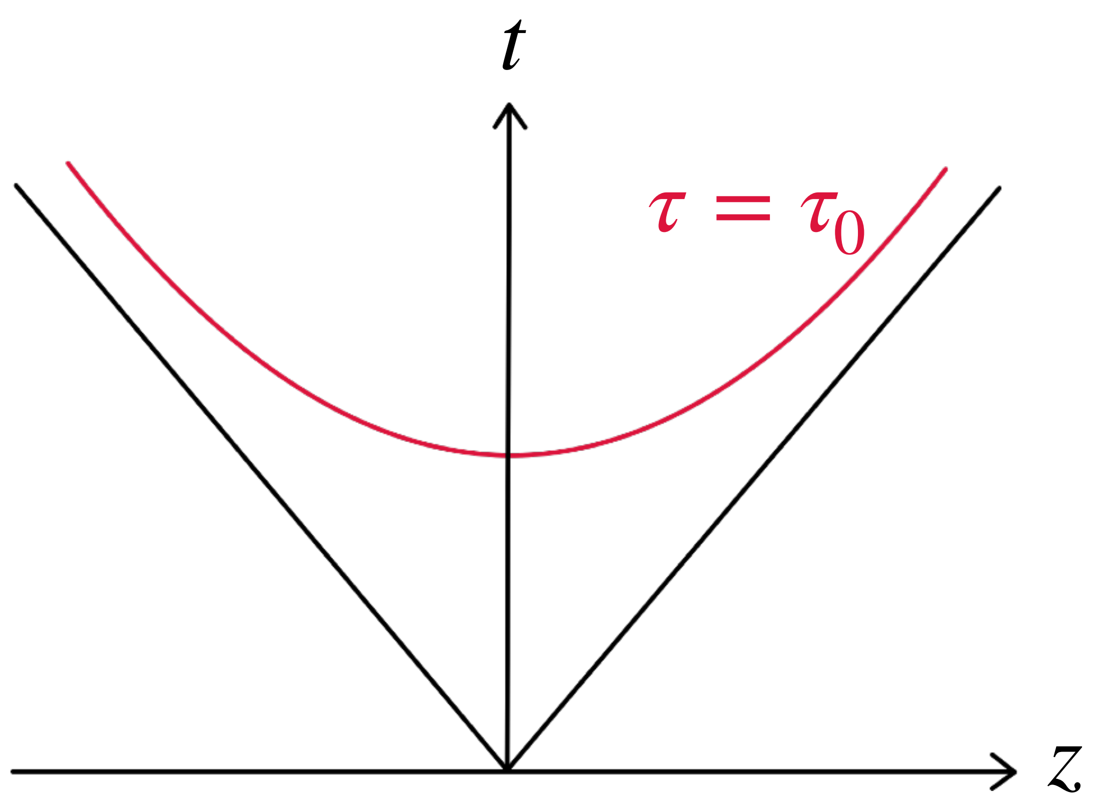
- Attractor provides a background for the jet-medium interactions:

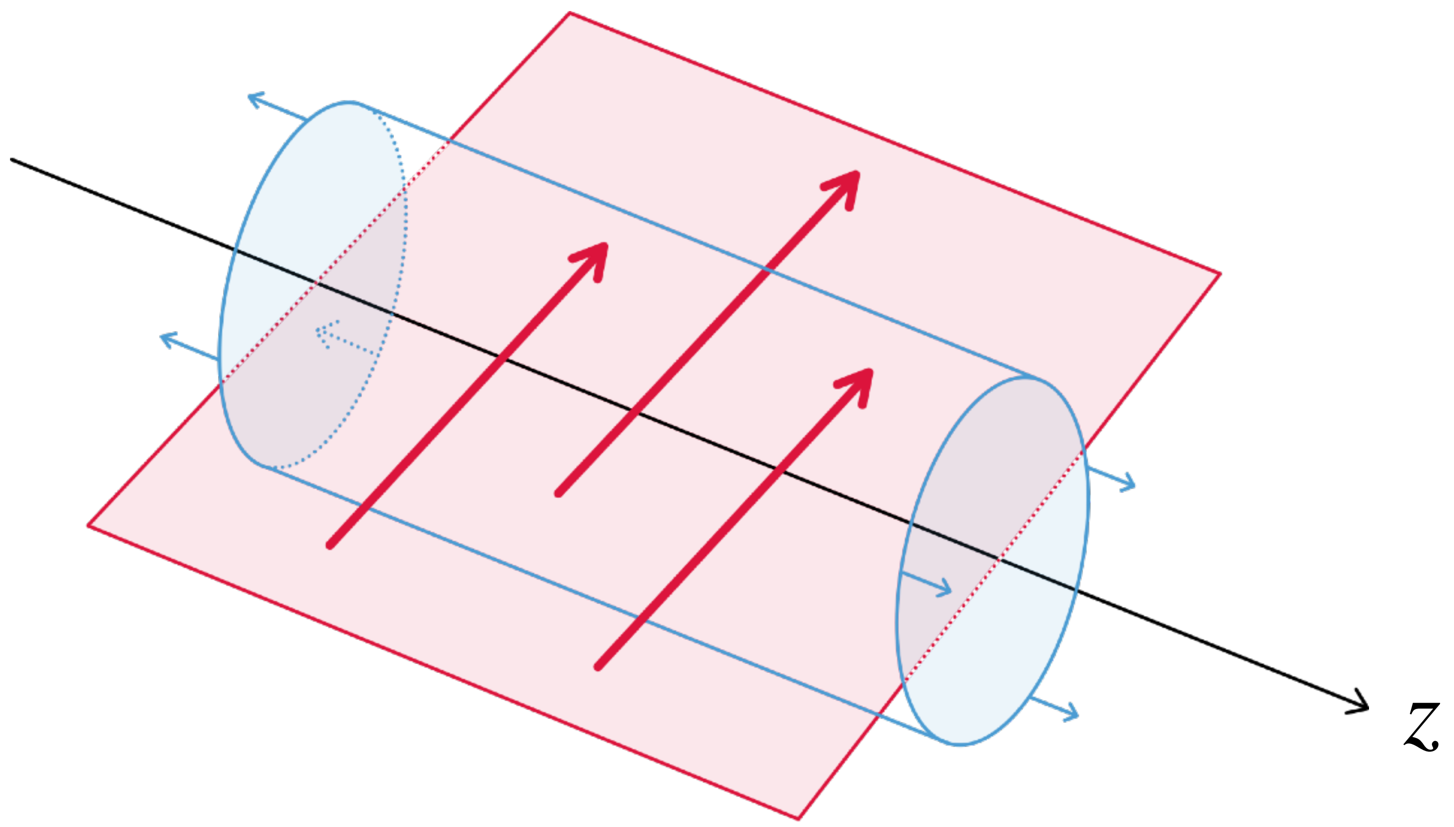
$$\begin{cases} \partial_\nu T^{\mu\nu}_{\text{attractor}} = 0, \\ \partial_\nu \delta T^{\mu\nu} = -\partial_\nu T^{\mu\nu}_{\text{jet}} = J^\mu . \end{cases}$$

Jet in Trento's Adige River

Boost-invariant jet

- A knife-shape jet resulted from boost-invariant assumption, which   Yan et al, 1707.09519
  - captures main effects qualitatively;
  - characterizes the longest wavelength modes along rapidity.





- Jet source current: $J^\mu = f^\mu(t)\, n_{\rm jet}(t, \mathbf{x})$

effective drag force

$$f^\mu(t) = \frac{dE}{dt} u_s^\mu$$

spacetime distribution of source, e.g,

$$n_{\rm jet}(t, \mathbf{x}) \sim \delta^{(2)}(\mathbf{x} - \mathbf{x}_s(\tau))$$

Energy loss

- We assume the BBMG energy loss formalism

Betz, Gyulassy and Torrieri, 1102.5416

$$\frac{dE}{d\tau} = \kappa \left(\frac{E}{T}\right)^a (\tau T)^z T^2$$

κ: jet-medium coupling
a: jet-energy dependence
z: path-length dependence

model	(a, z)	applicable regime
Bethe-Heitler limit	(1, 0)	additive single scattering
N=4 SYM	(0, 0)	pQCD elastic, non-relativistic heavy quark
LPM factorization limit	(0, 1)	pQCD radiative, weakly coupled
AdS/CFT	(0, 2)	light quark, strongly coupled

- Energy loss formula may fall into BBMG classification in certain limit, e.g.,

$$\frac{dE}{d\tau} = \frac{4E_{\text{in}}\tau^2}{\pi \ell_{\text{stop}}^2 \sqrt{\ell_{\text{stop}}^2 - \tau^2}} \sim (\tau T)^2 T^2 \longrightarrow (0,2) \text{ class}$$

Chesler et al, 1402.6756

$\ell_{\text{stop}} \gg \tau, R$
(energetic partons / small systems)

Asymptotic jet solutions at late time

- Inhomogeneous EOM

$$\partial_\tau \phi_i(\tau, \mathbf{k}) = M_{ij}\, \phi_j(\tau, \mathbf{k}) + J_i(\tau, \mathbf{k})$$

- The late-time asymptotic solutions can be found by Wronskian:

$$\delta\phi(\tau, \mathbf{k}) = \sum_i C_i(k)\, \delta\phi_i(\tau, \mathbf{k}) + \delta\phi_p(\tau, \mathbf{k})$$

- The particular solutions have the universal *power-law* behavior, e.g.,

$$\delta T_p(\tau, \mathbf{k}) \sim i\, n_{\rm jet}(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}_s(\tau)} \frac{\Gamma_T(\mathbf{v_s}\cdot\mathbf{k})^2 + (\mathbf{v}_s \times \mathbf{k})^2}{\mathbf{v}_s\cdot\mathbf{k}\left(\Gamma_L(\mathbf{v_s}\cdot\mathbf{k})^2 + (\mathbf{v}_s\times\mathbf{k})^2\right)} (\Lambda\tau)^\beta \left(1 + \mathcal{O}(\tau^{-1/3})\right)$$

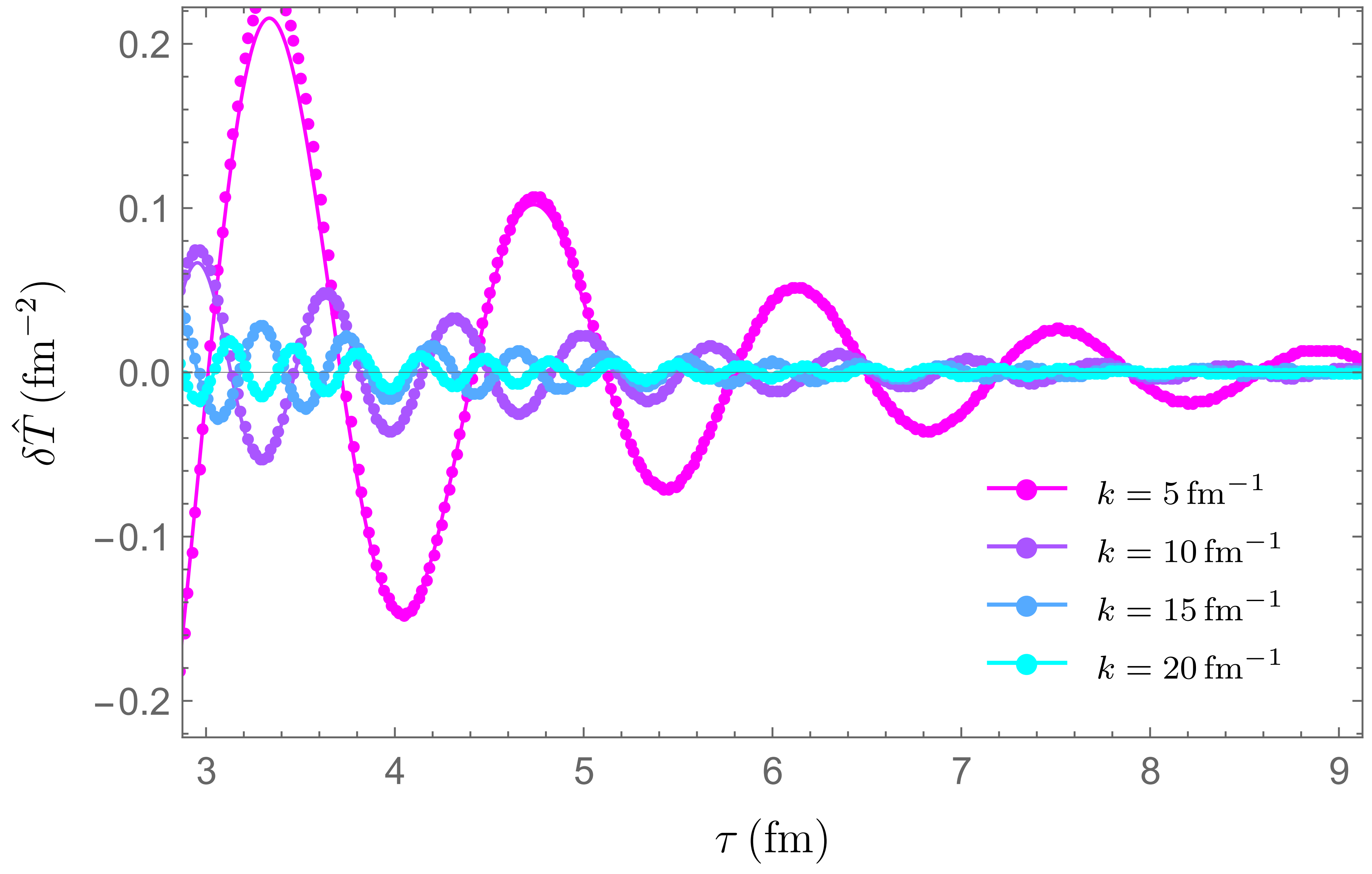
Kevin/Shock wave structure

$$\Gamma_L = 1 - \frac{v_s^2}{c^2}, \quad \Gamma_T = 1 - \frac{3v_s^2}{2\alpha^2}$$

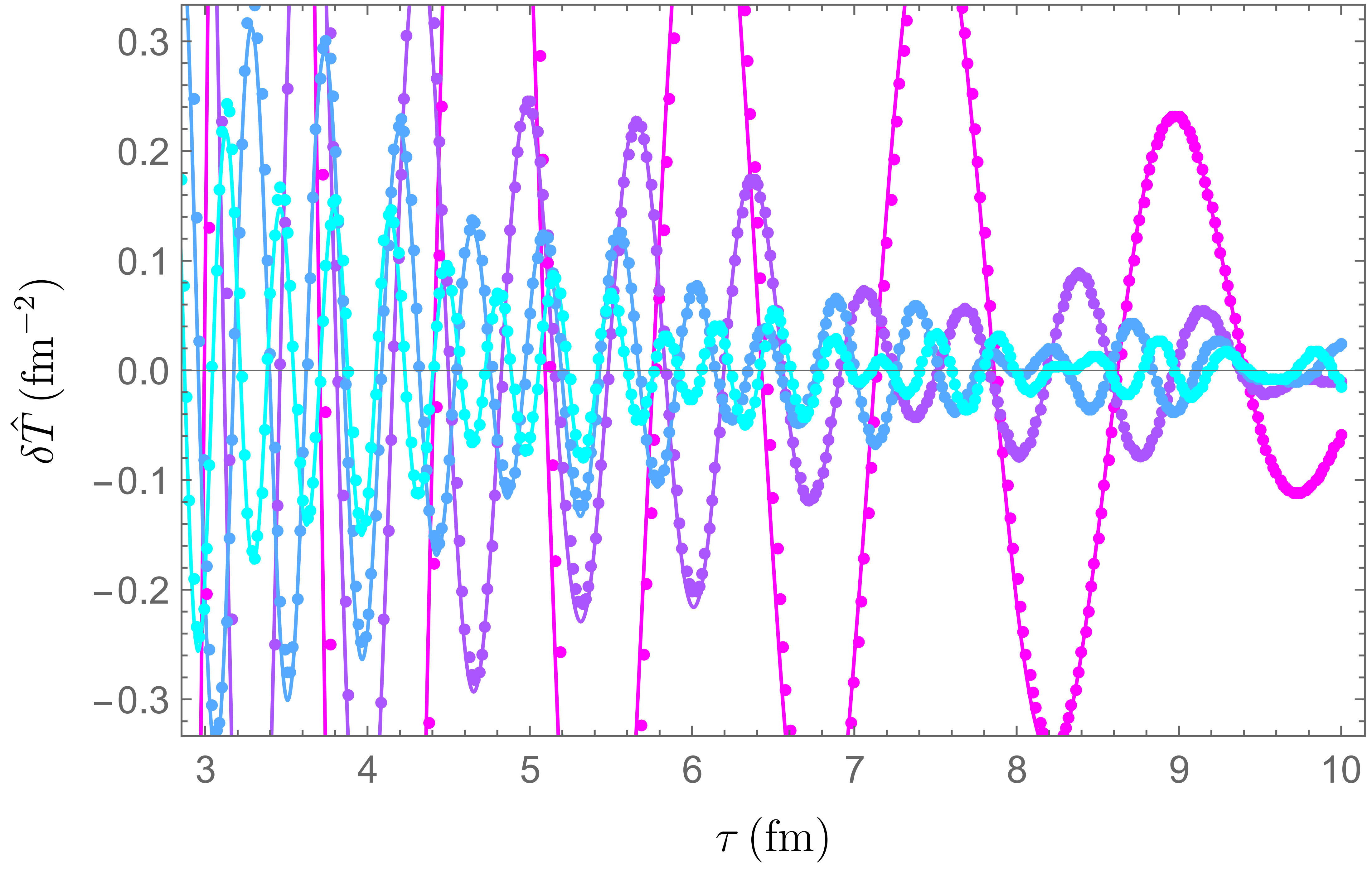
leading power-law exponents $\beta = -\frac{1}{3}\left(1 + \frac{2(a+z)}{a-1}\right)$

Matching to numerics

- The analytic solutions (solid curves) fit the numerics (discrete points) even at $\tau = 3$ fm (with the same initial conditions C_i's).



w/o jet

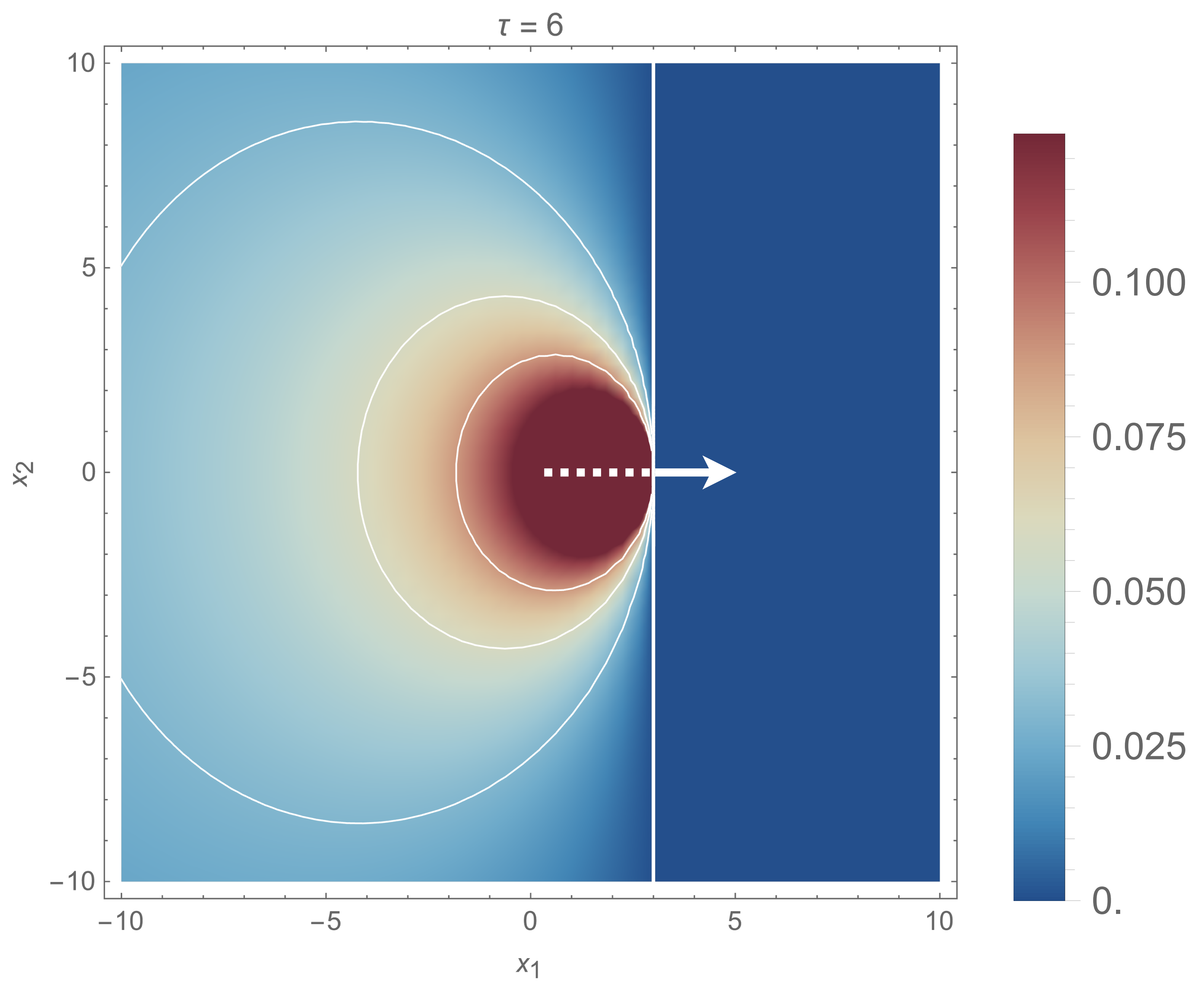


w/ jet

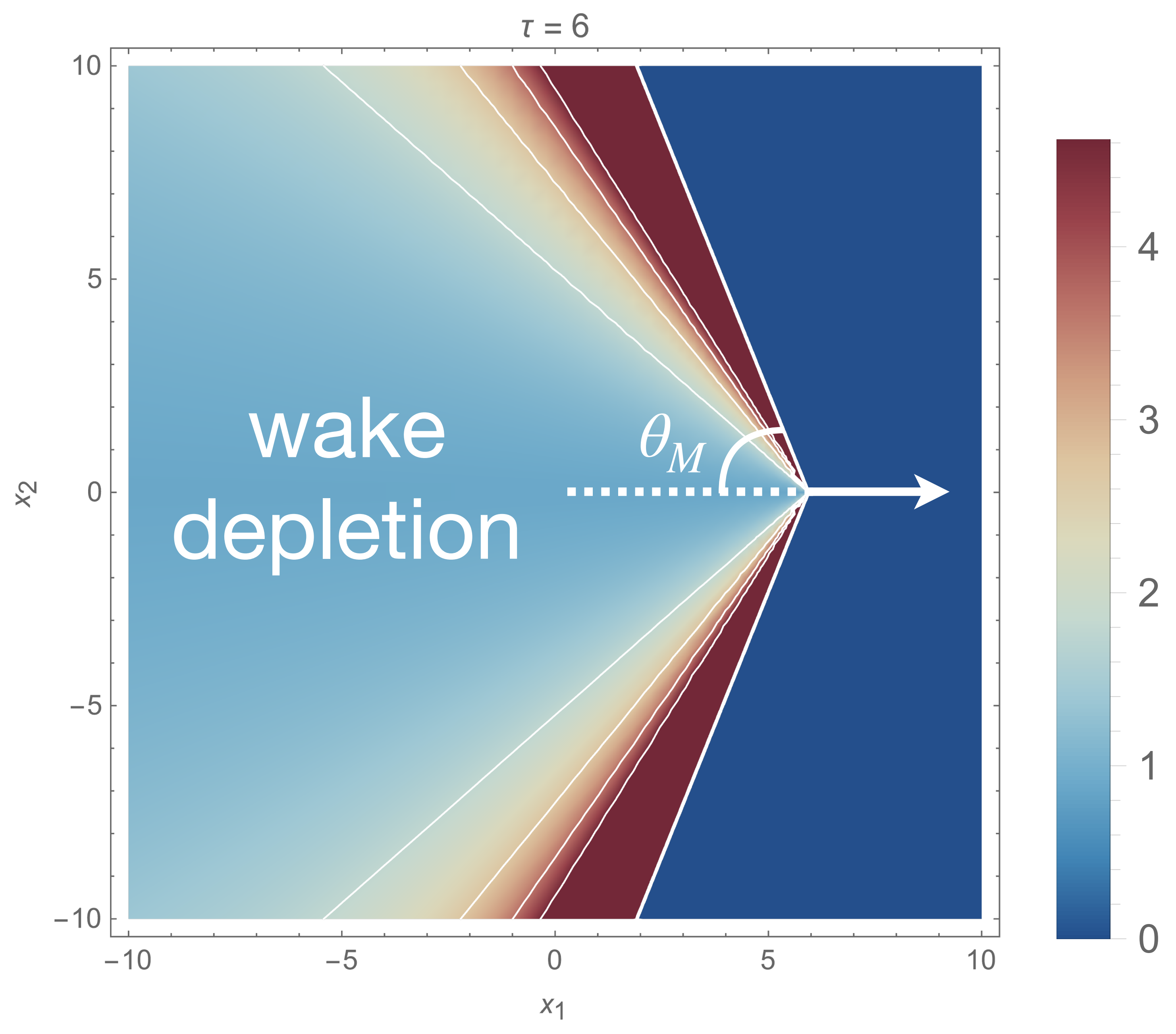
Jet wake: analytic results

- Transition from subsonic to supersonic wave via *analytic* inverse FT of δT_p.

preliminary



Subsonic: $v_s = 0.50$



Supersonic: $v_s = 0.99$

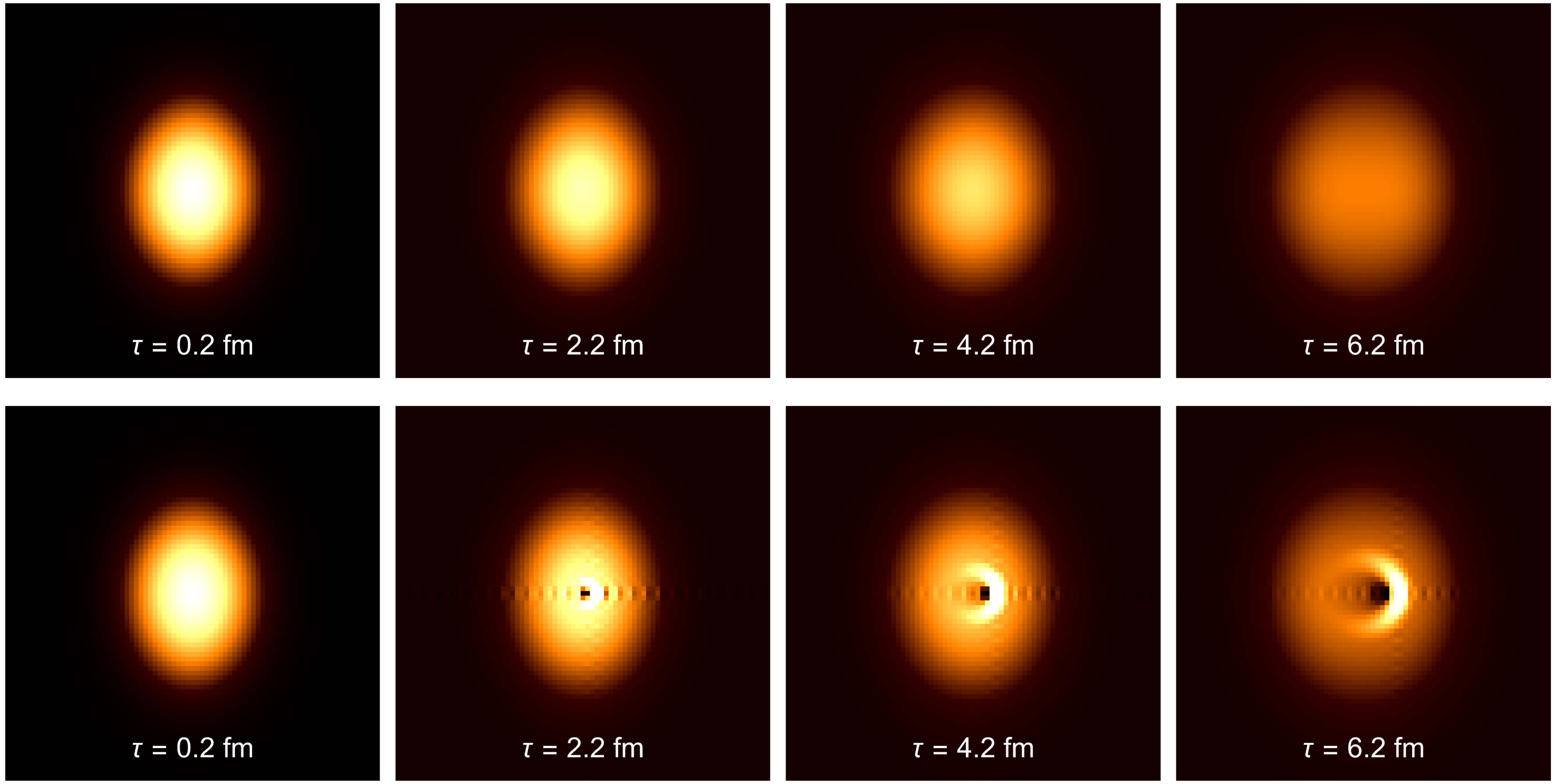
Kevin wave with Doppler effect

Shock wave with Mach cone angle $\arcsin \theta_M = \arcsin c_\infty / v_s \sim 68°$

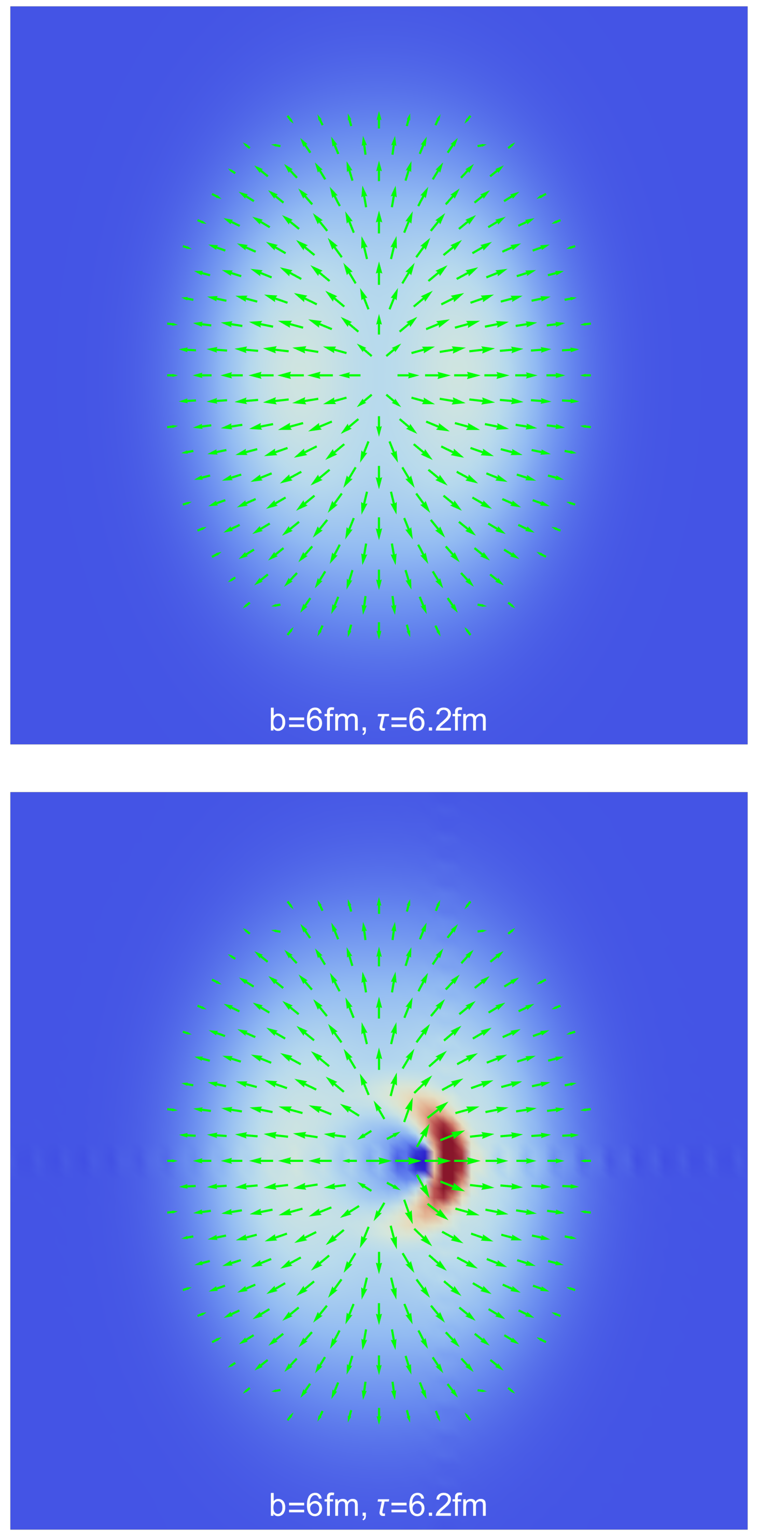
MIS speed of sound $c_\infty = \sqrt{(1+4\alpha^2)/3} \sim 0.92$ (cf conformal speed of sound $\sqrt{1/3} \sim 0.58$)

Jet wake: numerical results

- The transverse tomography *with background*.



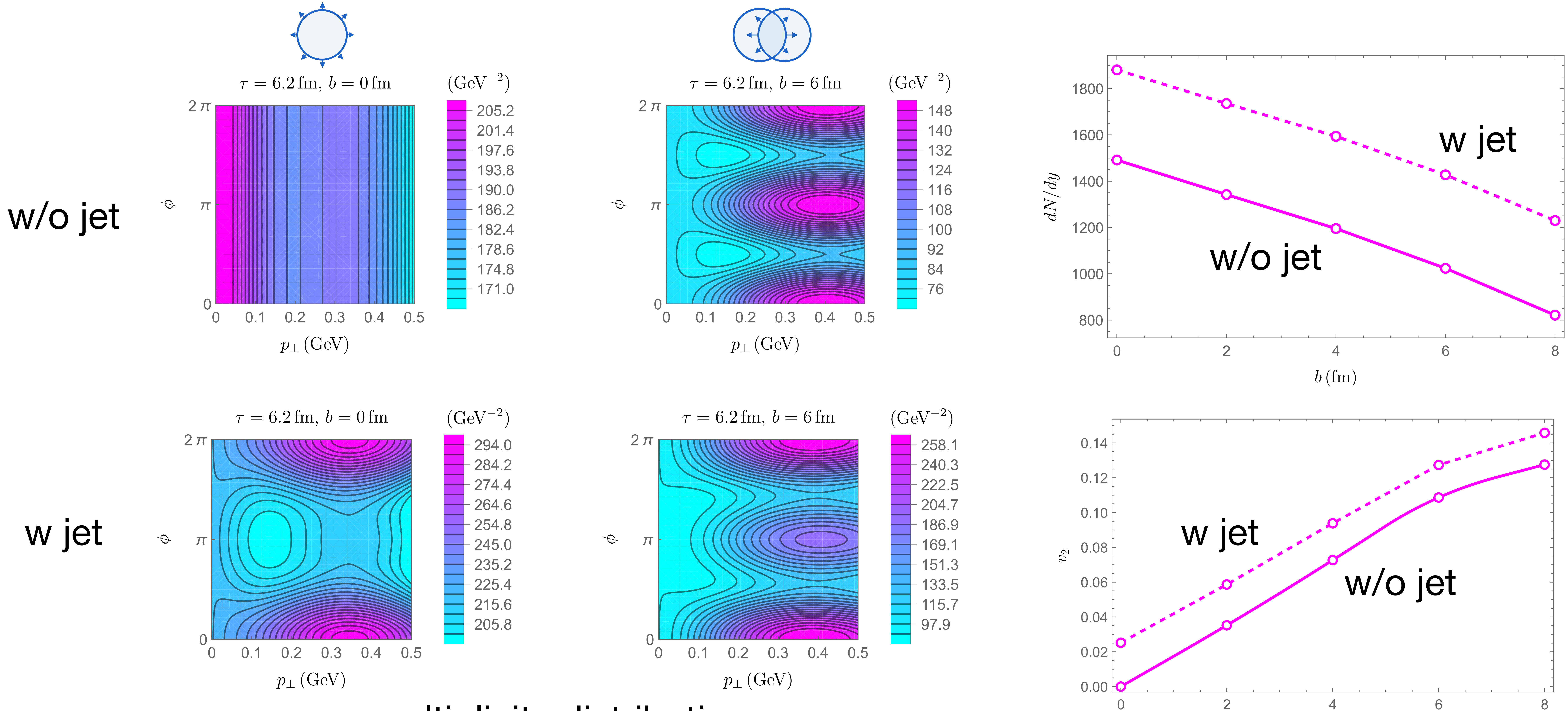
energy density



velocity

Collectivity with jet: multiplicities

- Comparison of flow observables without jet (solid) and with jet (dashed)



multiplicity distribution

Disentangle the jet wake from QGP medium?

- Multiplicity subtracted by background

preliminary

$$\frac{d\Delta N(\phi)}{d\phi dy} \equiv \frac{dN(\phi)}{d\phi dy} - \frac{dN}{2\pi dy} = \frac{dN}{2\pi dy}\Sigma\tau_{\text{fo}}\, f(T_{\text{fo}}, m)\langle\delta\pi_{ii}\rangle\cos(2\phi)$$

Isotropic background

$$\langle\delta\pi_{ii}\rangle \sim \tau^{\beta}$$

$$\beta = -\frac{1}{3}\left(1 + \frac{2(a+z)}{a-1}\right)$$

which suggests at high collision energies, approximately

$$\frac{d\Delta N(\phi)}{d\phi dy} \Big/ \frac{dN}{2\pi dy} \sim \left[\tau_{\text{fo}}(\sqrt{s})\right]^{1+\beta}$$

Can the power-law behavior be measurable in experiments?

Recap

- Hydro attractors provide a background for theoretical analysis for observables.
- Soft modes are dominated by non-perturbative transseries.
- Jet wakes dominate at late time; shock wave and energy loss may be measurable.

Outlook

- More realistic setup: e.g., with rapidity dependence;
- Other contexts: e.g., cold quantum gases, cosmology, stochastic sources;
- Complementary methods: e.g., kinetic theory, holography, top-down approaches.

Thank You