FACULTY OF SCIENCES

EFFECTIVE THERMALIZATION RATES

Robbe Brants – 23/09/2025

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THERMALIZATION

- How (quickly) are attractors approached?
- For single time scale: $\sim e^{-t/\tau_R} = e^{-\Gamma t}$

  → often far too simplistic!

- Sum of exponentials, e.g. hydrodynamic attractor, AdS/CFT

  Talks by Ines, Bruno, Yi, Navid
- Subexponential, e.g. high temperature QFT

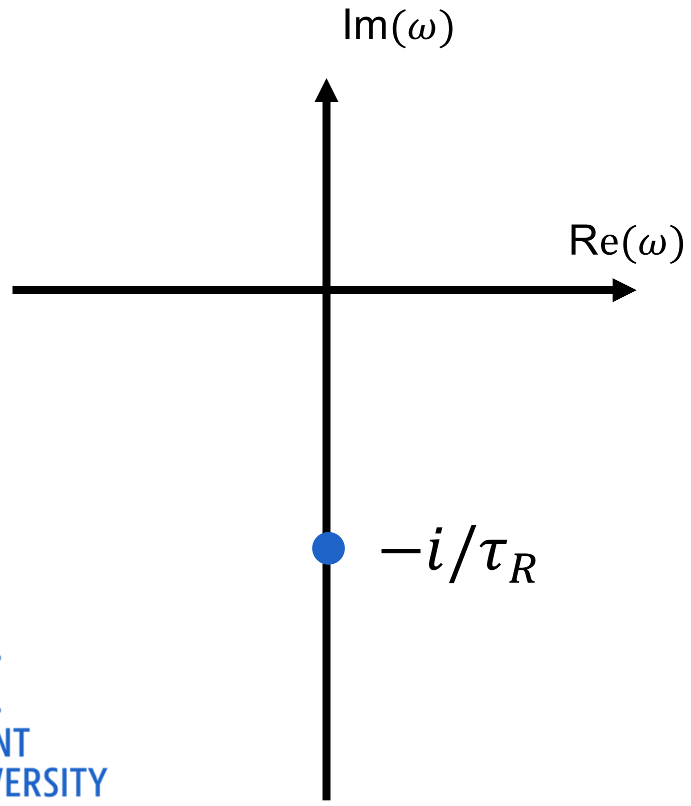
  Talks by Xiaojian, Florian
- Power law, e.g. non-thermal fixed points

  Talks by Matisse, Thimo

HOW TO DEFINE?

– Mode decays over single timescale

$$G(t) \propto e^{-i\omega t}$$

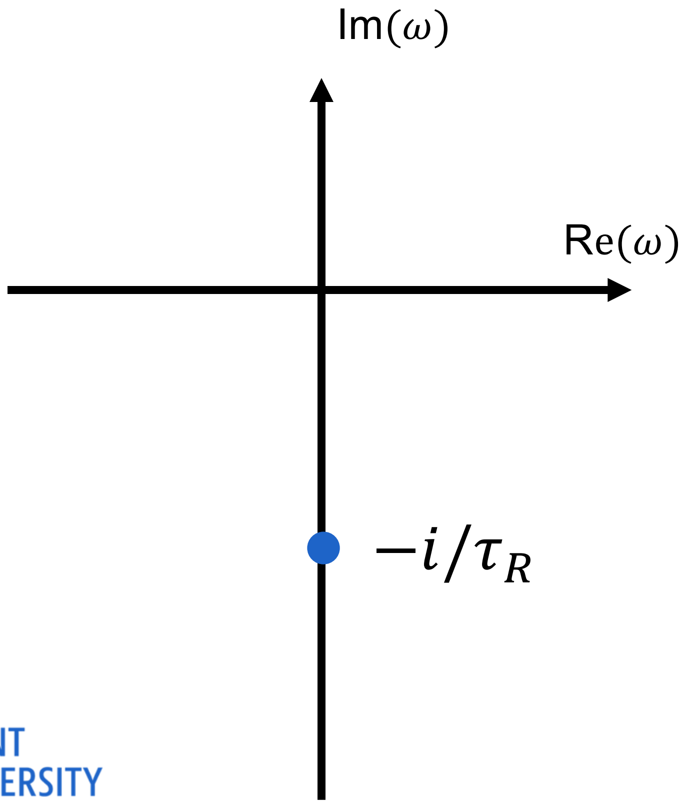


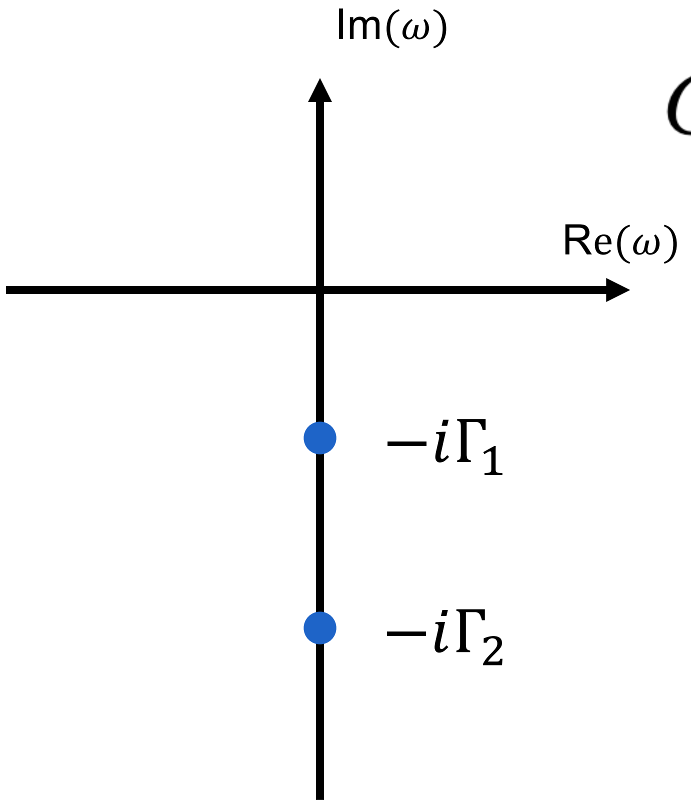
$$G(t) \propto e^{-t/\tau_R} = e^{-\Gamma_R t}$$

HOW TO DEFINE?

– Multiple poles?

$$G(t) \propto e^{-i\omega t}$$





$$G(t) \propto c_1 e^{-\Gamma_1 t} + c_2 e^{-\Gamma_2 t}$$

$$\approx (c_1 + c_2) e^{-\Gamma t}$$

$$\Downarrow \quad \Gamma_2 t \ll 1$$

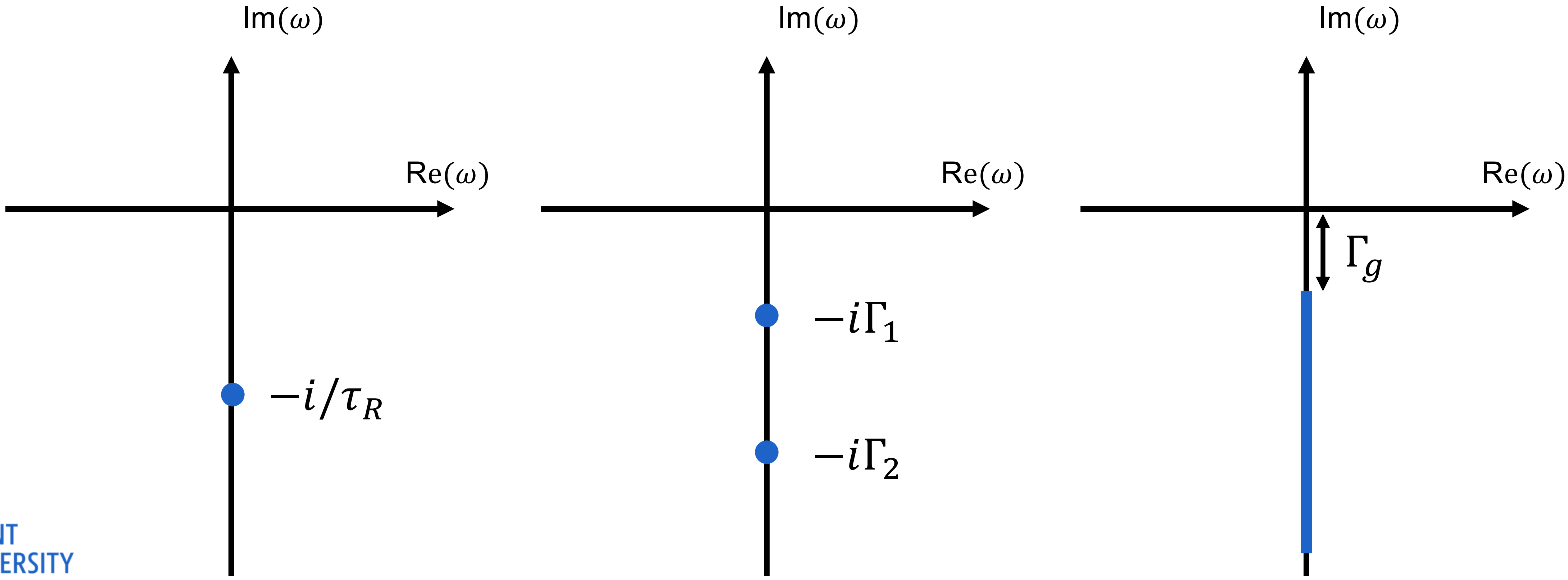
$$\Gamma = \frac{c_1 \Gamma_1 + c_2 \Gamma_2}{c_1 + c_2}$$

HOW TO DEFINE?

- Continuum of poles?
- Gapped spectrum Γ_g

$$G(t) \propto e^{-i\omega t}$$

$$G(\Gamma_g t \gg 1) \approx e^{-\Gamma_g t}$$

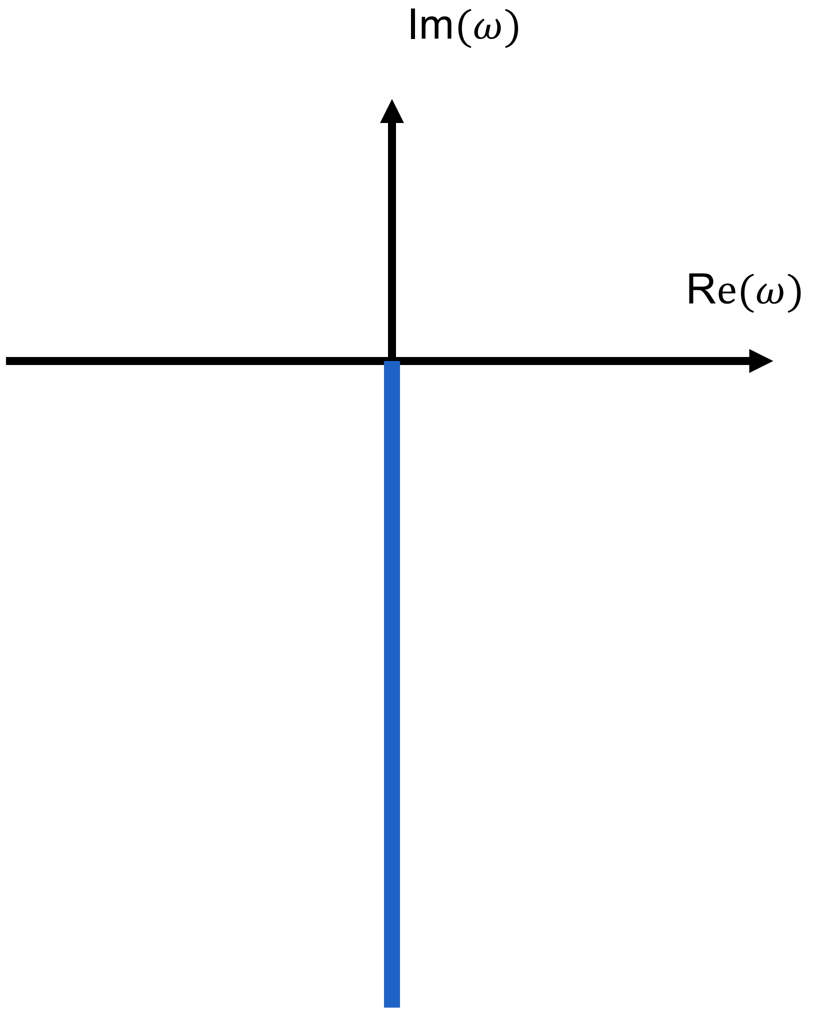


GAPLESS EXCITATIONS

- Poles reach up to real axis
- No lower limit on cross section[1]
- $\lambda\phi^4$ theory[2,3]
- QCD kinetic theory[4] :

$$2 \leftrightarrow 2: \quad \tau_R(p) \propto p/\log(p)$$

$$1 \leftrightarrow 2: \quad \tau_R(p) \propto \sqrt{p}$$



[1] Gavassino (2404.12327)

[2] Denicol, Noronha et al. (2404.04679)

[3] Schlichting et al. (2308.04491)

[4] Teaney et al. (0909.0754)

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POSSIBLE DEFINITIONS

- First order hydro (single mode approximation): τ_π

$$(1 + \tau_\pi \partial_t)\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$$

Possible issue: transport coefficient not related to decay

Fun example: shear channel in 2+1d RTA[5]

$$\omega_H(k) = -i\frac{\tau_R}{4}k^2 \to \tau_\pi = 0$$

- Recent proposal based on Greens function[6]

$$\frac{1}{\tau_\pi} \equiv \frac{1}{\eta}\int \frac{d\omega}{2\pi}\frac{\mathcal{G}(\omega)}{\omega}$$

[5] Soloviev et al. (2403.17769)

[6] Denicol, Noronha (2406.15569)

NON-ANALYTIC DENSITY

- Distribution of modes in complex plane

$$\sum_n c_n f(\omega_n) \to \int f(\omega)\rho(\omega)d\omega$$

- Single poles (residues): $\rho(\omega) = 2\pi i \sum_n \mathrm{Res}[G(\omega_n)]\delta(\omega - \omega_n)$
- Branch cut (discontinuity): $\rho(\omega) = \mathrm{Disc}[G(\omega)]$
- Generally: area density for entire region[5]

[7] RB (2409.09022)

GENERAL FORMULATION

Preliminary

EFFECTIVE DECAY RATE

– Single mode decay rate $\Gamma(\omega_n) = i\omega_n$

$\Downarrow$

– Collective decay rate $\Gamma = \frac{\int i\omega\rho(\omega)d\omega}{\int \rho(\omega)d\omega} = \langle i\omega \rangle_\rho$

– Previous proposal[6]:

$$\frac{1}{\tau_\pi} \equiv \frac{1}{\eta}\int \frac{d\omega}{2\pi}\frac{\mathcal{G}(\omega)}{\omega} = \frac{\int \frac{1}{i\omega}\rho(\omega)d\omega}{\int \left(\frac{1}{i\omega}\right)^2 \rho(\omega)d\omega}$$

[6] Denicol, Noronha (2406.15569)

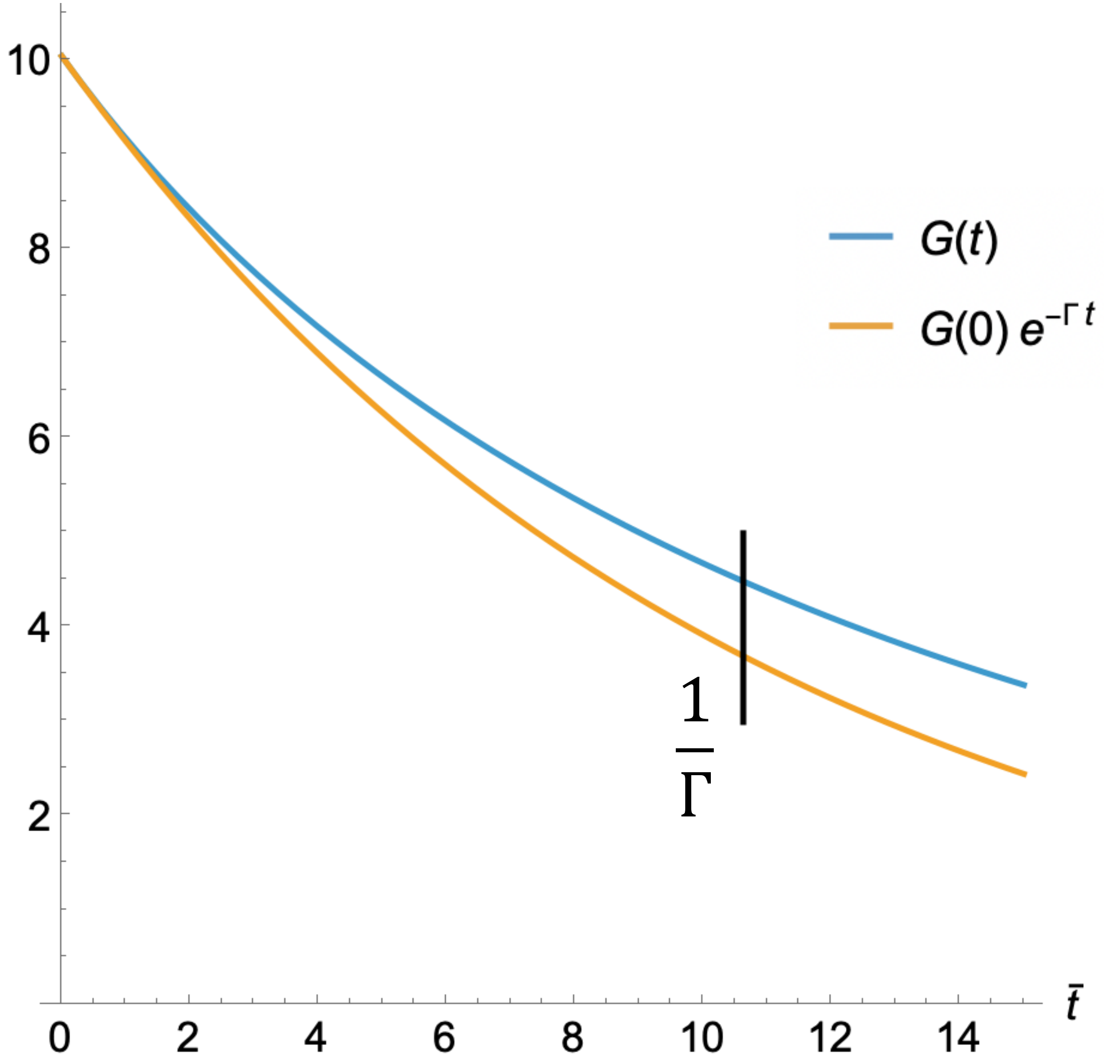
TIME DEPENDENCE

- Γ is only early time approximation
- Modes start decaying:

$$\rho(\omega) \to \rho(\omega)e^{-i\omega t} \equiv \rho_t(\omega)$$

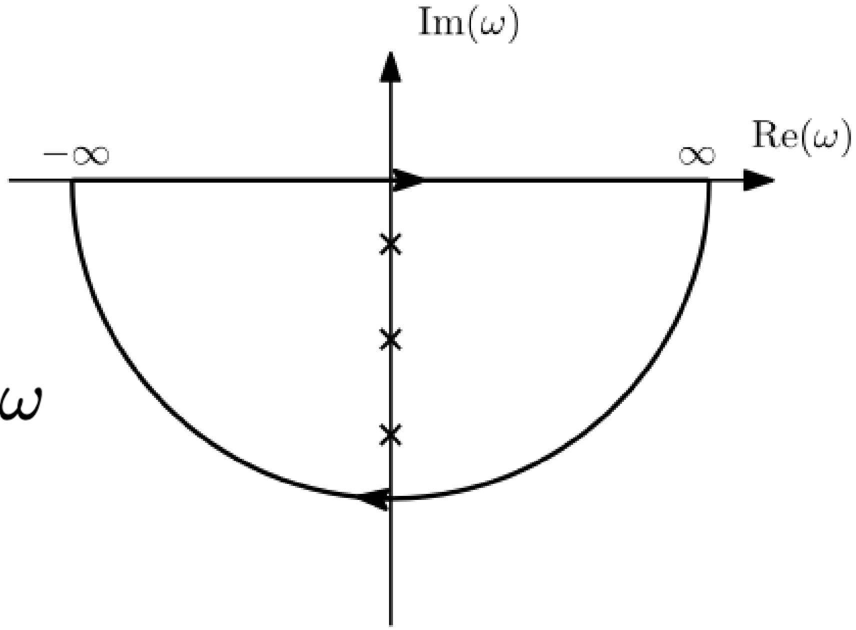
$$\Downarrow$$

$$\Gamma(t) = \frac{\int i\omega\rho(\omega)e^{-i\omega t}d\omega}{\int \rho(\omega)e^{-i\omega t}d\omega} = \langle i\omega \rangle_{\rho_t}$$



INTERPRETATION

– Without UV divergences:



$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{-i\omega t}d\omega = -\frac{1}{2\pi}\int \rho(\omega)e^{-i\omega t}d\omega$$

$$\Rightarrow \Gamma(t) = \frac{\int i\omega\rho(\omega)e^{-i\omega t}d\omega}{\int \rho(\omega)e^{-i\omega t}d\omega} = -\frac{G'(t)}{G(t)}$$

– $\Gamma(t)$ corresponds to momentaneous decay rate:

$$G(t_0 + t) \approx G(t_0)e^{-\Gamma(t_0)t}$$

$$G(t) = G(0)e^{-\bar{\Gamma}(t)t} \qquad \bar{\Gamma}(t) = \frac{1}{t}\int_0^t \Gamma(t')dt'$$

KINETIC THEORY

– Boltzmann equation

$$p^\mu \partial_\mu f = C[f] \quad \Longrightarrow \quad -i\omega p f(\omega, p) - p f_0(p) = C[f]$$

– Macroscopic variable

$$\mathcal{O}(t) \propto \int p^n f(t,p) dp \quad \Longrightarrow \quad \Gamma = -\frac{\int p^{n-1} C[f_0(p)] dp}{\int p^n f_0(p) dp}$$

– Time dependence:

$$\Gamma(t) = -\frac{\int p^{n-1} C[f(t,p)] dp}{\int p^n f(t,p) dp}$$

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APPLICATIONS

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RELAXATION TIME APPROXIMATION

- Momentum dependent relaxation time $\tau_R(p)$[8]

$$C[f] = -\frac{p}{\tau_R(p)} f(p)$$

$$\Gamma = \frac{\int \frac{1}{\tau_R(p)} f_0(p) p^n dp}{\int f_0(p) p^n dp} = \left\langle \frac{1}{\tau_R(p)} \right\rangle_{f_0}$$

- Explicit time dependence:

$$\Gamma(t) = \frac{\int \frac{1}{\tau_R(p)} f_0(p) e^{-t/\tau_R(p)} p^n dp}{\int f_0(p) e^{-t/\tau_R(p)} p^n dp}$$

[8] Kurkela, Wiedemann (1712.04376)

POWER LAW RTA

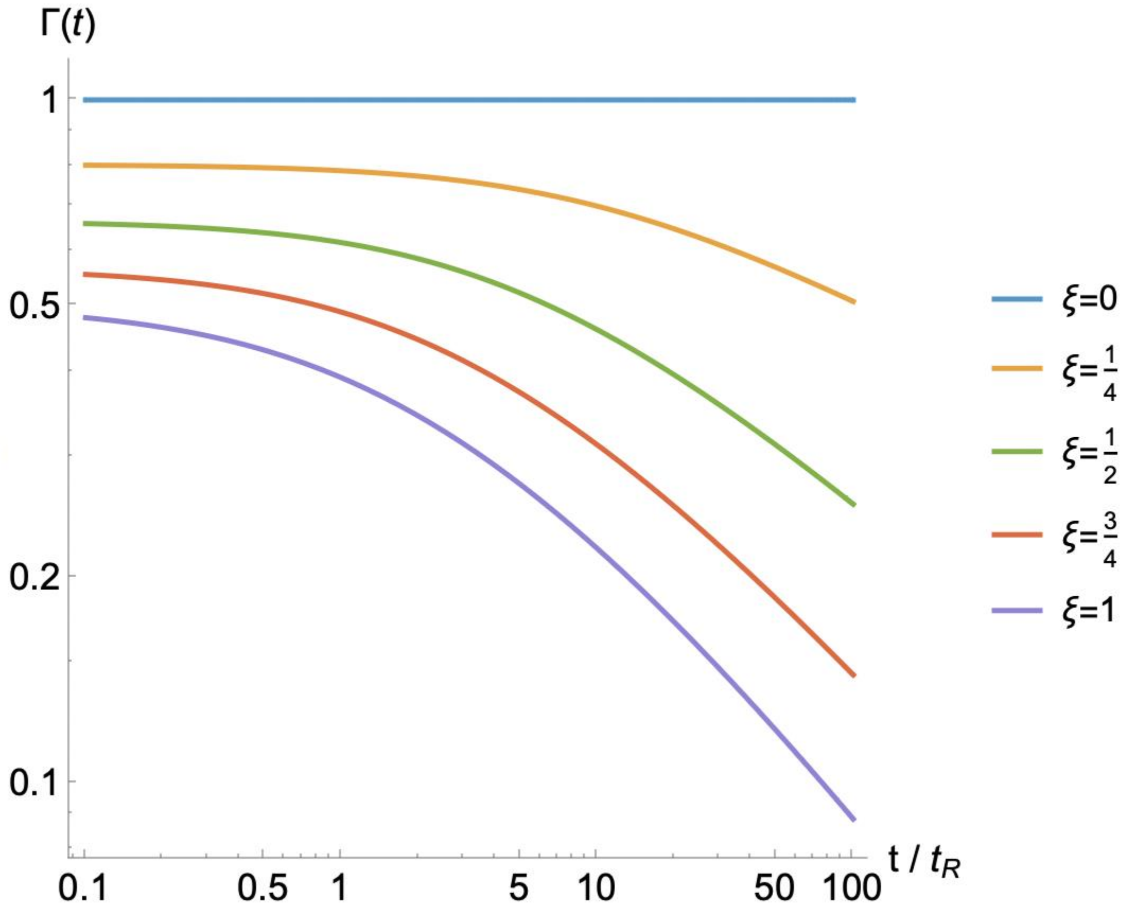
$$\tau_R(p) = (p/T)^{\xi} t_R$$

$$f_0(p) = e^{-p/T}$$

– Early times $\Gamma = \frac{(n-\xi)!}{n!}\frac{1}{t_R}$

– Late times

$$\Gamma(t) \sim \frac{1}{t_R}\left(\frac{\xi t}{t_R}\right)^{-\frac{\xi}{1+\xi}} - \frac{2n+1}{2\xi+2}\frac{1}{t}$$



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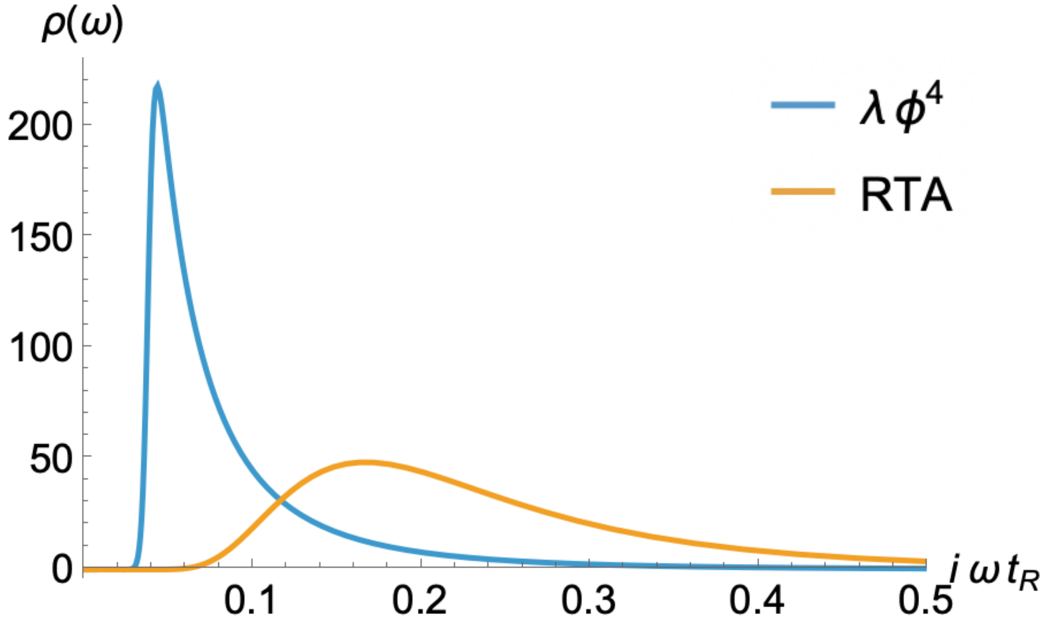
$\lambda\phi^4$ SCALAR THEORY

– Approximate $\rho(\omega)$ using eigenmoments[2]

– Effective RTA: $\tau_R(p) = \frac{2p}{g\mathcal{M}} \equiv \frac{p}{T} t_R$

$$\Gamma_{\mathrm{RTA}} = \frac{1}{4t_R}$$

$$\Gamma_{\lambda\phi^4} = \left(\frac{119}{6} - 2\pi^2\right)\frac{1}{t_R} \approx \frac{0.094}{t_R}$$



[2] Denicol, Noronha et al. (2404.04679)

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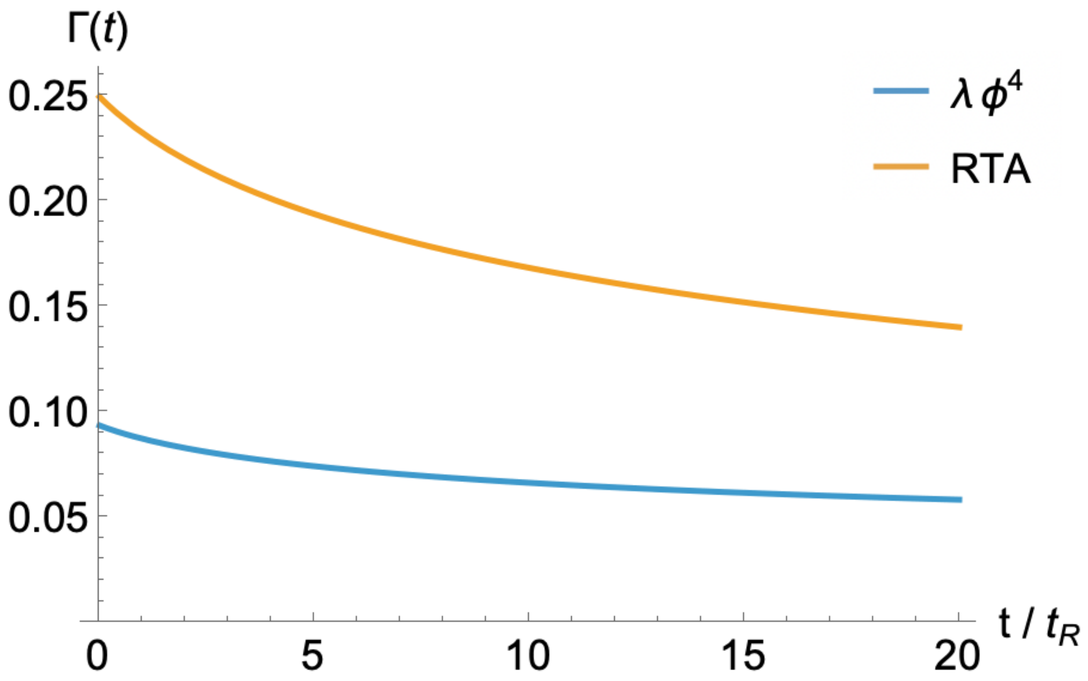
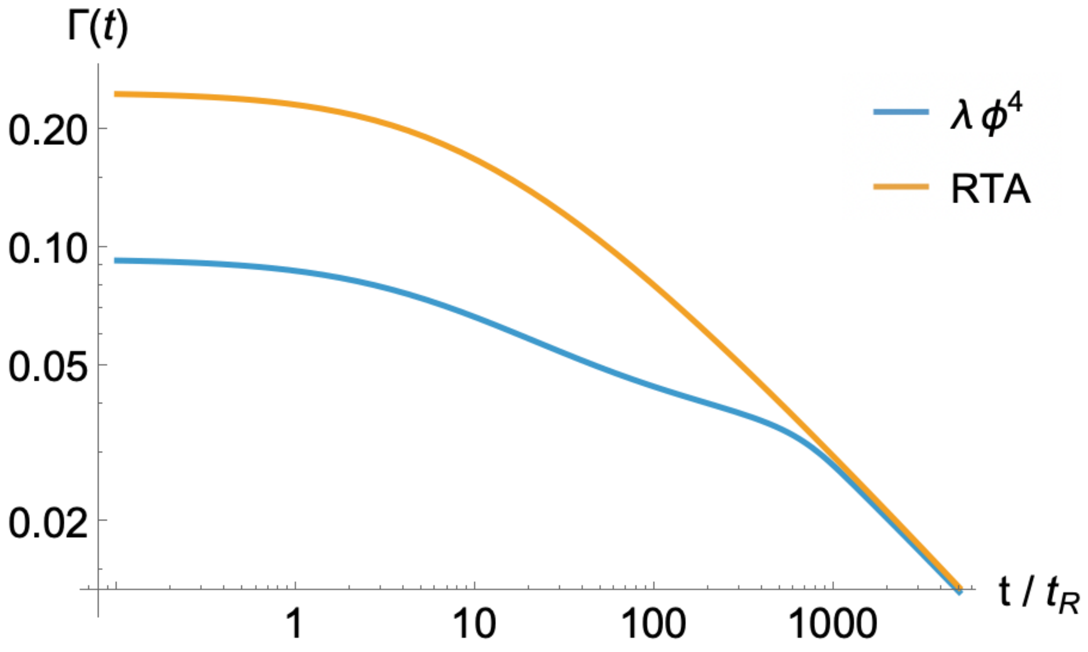
$\lambda\phi^4$ SCALAR THEORY

– Late time behavior converges (almost)

$$\Gamma_{\mathrm{RTA}} \sim \frac{1}{\sqrt{t \cdot t_R}} - \frac{9}{4}\frac{1}{t}$$

$$\Gamma_{\lambda\phi^4} \sim \frac{1}{\sqrt{t \cdot t_R}} - \frac{17}{4}\frac{1}{t}$$





DISTRIBUTION MOMENTS

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EARLY TIME DEPENDENCE

– First order correction:

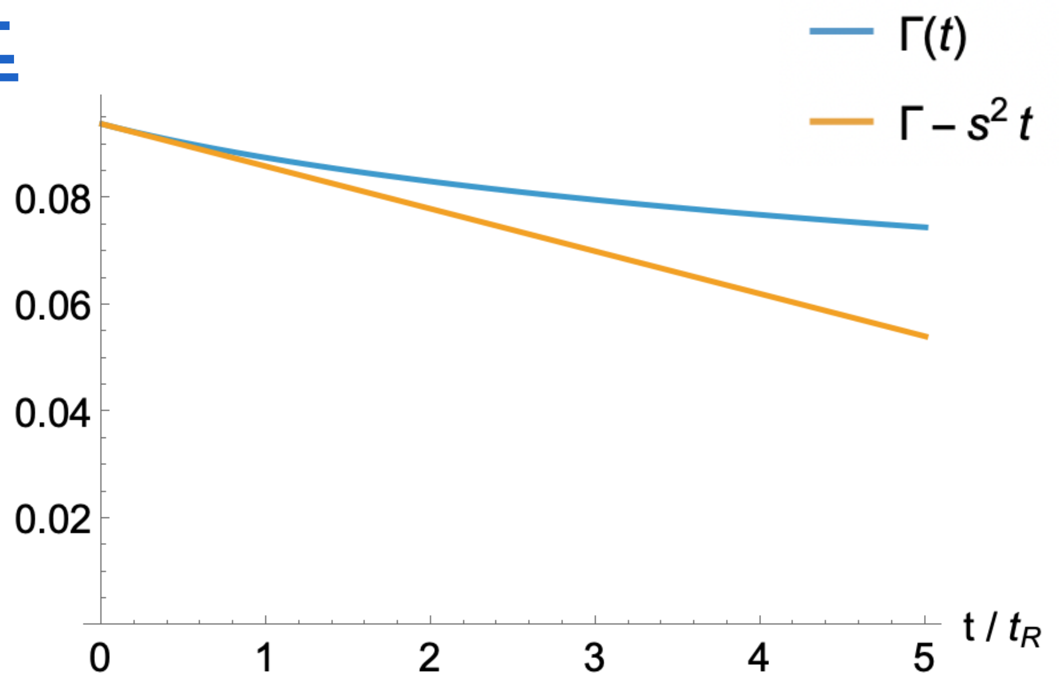
$$\Gamma(t) \approx \Gamma - s^2 t$$

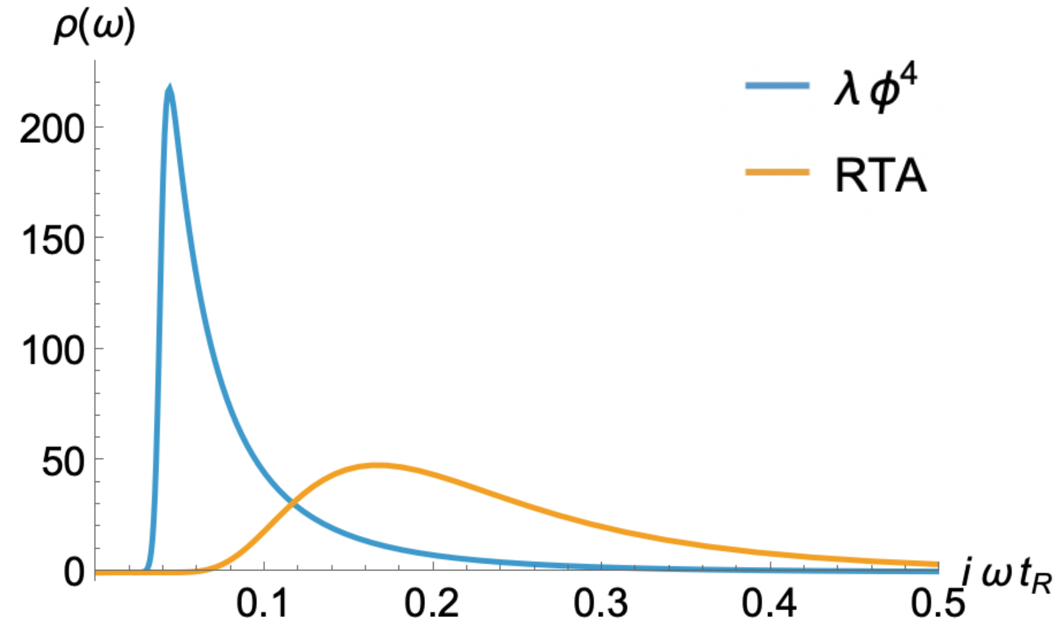
$$\Gamma = \langle i\omega \rangle_\rho$$

$$s^2 = \langle (i\omega)^2 \rangle_\rho - \langle i\omega \rangle_\rho^2$$

– Kinetic theory:

$$\langle (i\omega)^2 \rangle_\rho = \frac{\int p^n (-\frac{1}{p} C)^2 [f_0(p)] dp}{\int p^n f_0(p) dp}$$





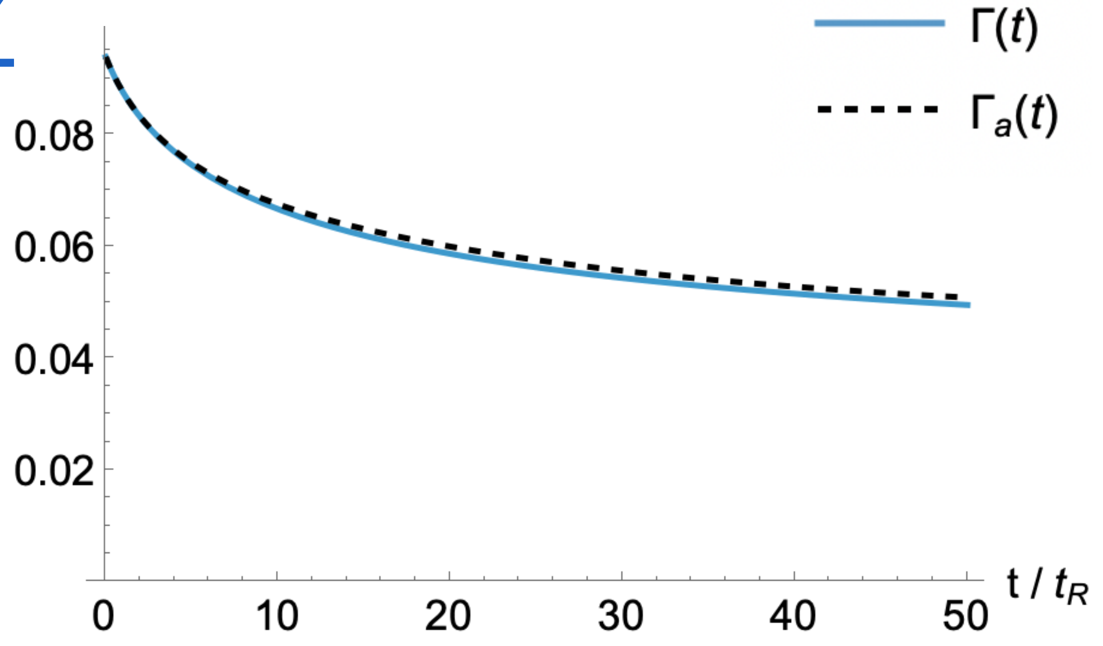
APPROXIMATING DENSITY

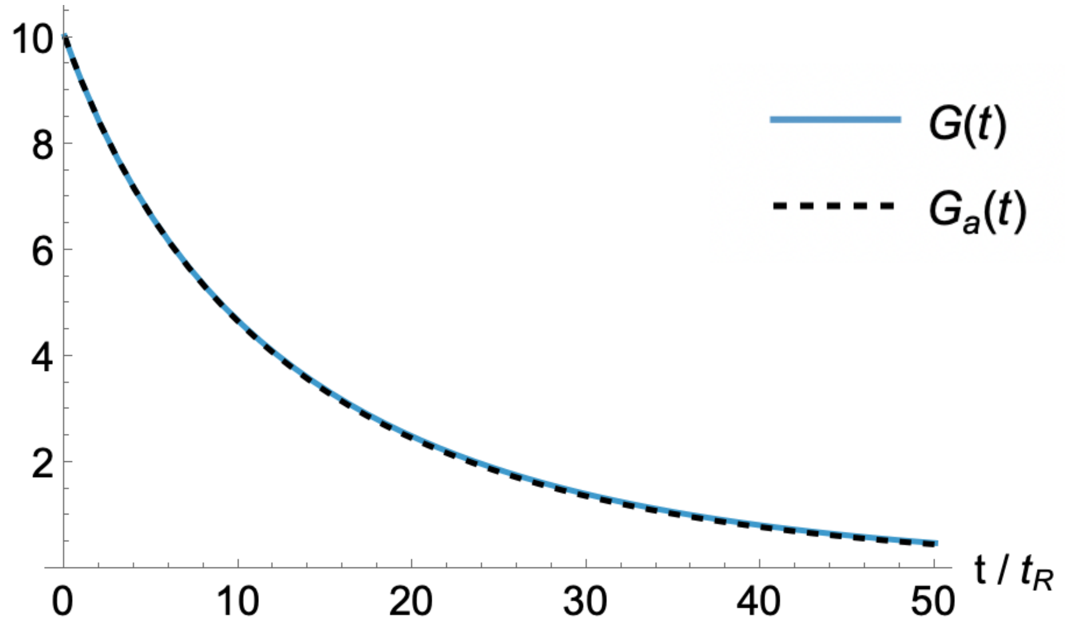
– Use first 5 moments

$$\mu'_n = \langle (i\omega)^n \rangle_\rho = \frac{\int p^n (-\frac{1}{p}C)^n [f_0(p)] dp}{\int p^n f_0(p) dp}$$

– Approx with same moments

$$\rho_a(\omega) = e^{-1/i\omega} \left(\sum_{m=3}^{7} c_m \omega^{-m} \right)^2$$





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CONCLUSIONS & OUTLOOK

- From mode structures, thermalization rates are easily accessible
- Decay rates often change significantly as system thermalizes
  - Limiting behavior still predictable!

- Relevant decay rate in QCD kinetic theory? Florian’s talk
- Corrections from nonlinear effects or time dependent background
  → Adiabatic framework? Bruno’s talk
- Decay towards non-thermal fixed points Matisse’s talk
- Hydrodynamics: hydrodynamization timescale, causality