Relaxation to Nonthermal Attractors: Prescaling and Hydrodynamics

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ECT* workshop, 23.09.2025,
based on Heller, Mazeliauskas, TP, Phys. Rev. Lett. 132, 071602 (2024),
Berges, Denicol, Heller, TP, 2504.18754

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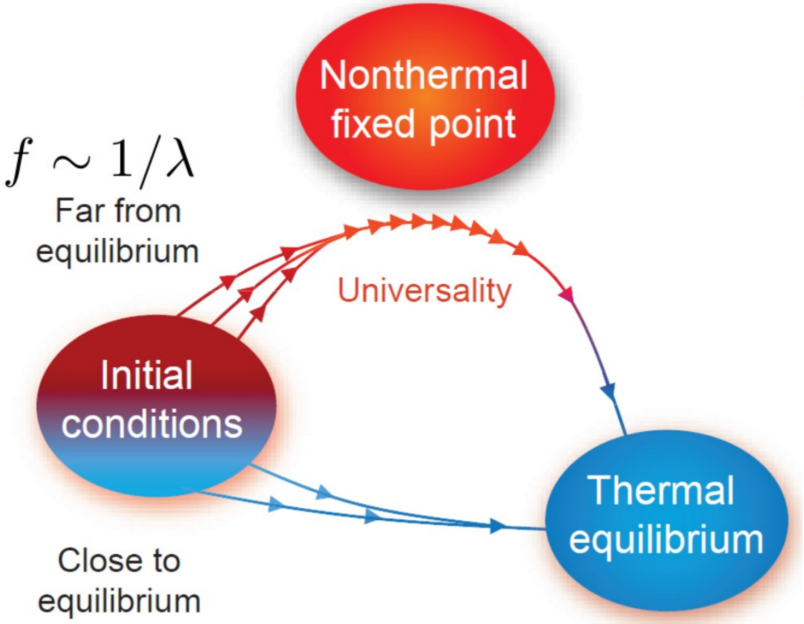
STRUCTURES CLUSTER OF EXCELLENCE

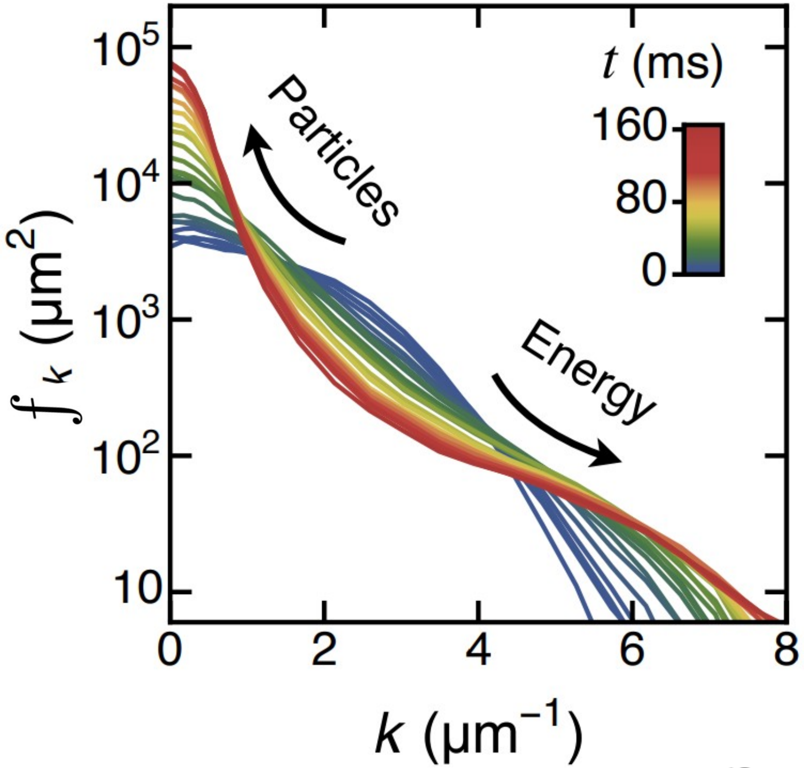
Nonthermal attractors

$$f(t,\mathbf{p}) = t^{\alpha} f_S(t^{\beta}\mathbf{p})$$

$f \sim 1/\lambda$



$$f(t,\mathbf{p}) - f_{\beta}(\mathbf{p}) \sim e^{-\gamma_{\mathbf{p}} t}$$



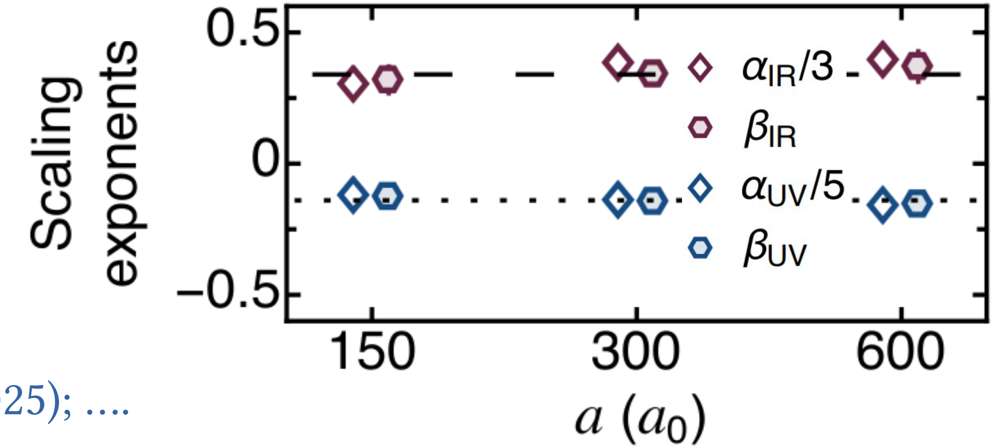
$$\omega_{\mathbf{k}} \sim |\mathbf{k}|^{z} \qquad (\alpha,\beta)_{\infty} = \left(\frac{d}{2}, \frac{1}{2}\right)$$

$$n(t) \sim \int_{\mathbf{k}} f(t,\mathbf{k}) \equiv t^{\alpha - d\beta}\,\bar{n}$$

$$\varepsilon(t) \sim \int_{\mathbf{k}} \omega_{\mathbf{k}} f(t,\mathbf{k}) \equiv t^{\alpha-(d+z)\beta}\,\bar{\varepsilon}$$

$$(\alpha,\beta)_{\infty,\mathrm{rel}} = \left(-\frac{d+1}{7}, -\frac{1}{7}\right)$$

$$(\alpha,\beta)_{\infty,\mathrm{non-rel}} = \left(-\frac{d+2}{6}, -\frac{1}{6}\right)$$



Prüfer et al., Nature 563, 217 (2018); Erne et al., Nature 563 (2018);
Glidden et al., Nature Phys. 17, 457 (2021); Gazo et al Science 389, 802 (2025); ....

The prescaling solution

$$\alpha(t) \to \alpha_\infty$$

Make prescaling ansatz

with $A(t) = \exp\left[\int_{t_0}^{t} dt' \frac{\alpha(t')}{t'}\right] \to t^{\alpha_\infty}$

$$f(t,\mathbf{p}) = A(t) f_S(B(t)\mathbf{p})$$

Determine prescaling exponents from Boltzmann equation via separation of variables

$$1/\beta_\infty = (1-\mu_\alpha)\sigma - \mu_\beta$$

$$\partial_t f(t,\mathbf{p}) = A(t)\left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\bar{\mathbf{p}}\cdot\partial_{\bar{\mathbf{p}}}\right] f_S(\bar{\mathbf{p}})$$
$$= A(t)^{\mu_\alpha} B(t)^{\mu_\beta} \mathcal{C}[f_S](\bar{\mathbf{p}})$$

$$\longrightarrow \quad \frac{B(t)^{1-1/\beta_\infty}}{\partial_t B(t)} = \frac{1}{D_1} = \frac{[\sigma + \bar{\mathbf{p}}\cdot\partial_{\bar{\mathbf{p}}}] f_S(\bar{\mathbf{p}})}{\mathcal{C}[f_S](\bar{\mathbf{p}})}$$

and relevant conservation laws

$$A(t) = B(t)^\sigma$$
$$\alpha(t) = \sigma\beta(t)$$

$$\sigma = d, d+z$$

Full prescaling dynamics from same ingredients as scaling !

$$B(t) = \left(\frac{t-t_*}{t_{\mathrm{ref}}}\right)^{\beta_\infty} \approx \left(\frac{t}{t_{\mathrm{ref}}}\right)^{\beta_\infty} \left(1 - \beta_\infty \frac{t_*}{t} + \ldots\right)$$
$$\beta(t) = \beta_\infty \frac{t}{t-t_*} \approx \beta_\infty + \beta_\infty \frac{t_*}{t} + \ldots \quad .$$

$$t_* = t_0(1 - \beta_\infty/\beta_0)$$

Prescaling coined: Berges, Mazeliauskas, PRL 122, 122301 (2019)

Approaching the IR attractor

$$(\alpha, \beta)_\infty = \left(\frac{d}{2}, \frac{1}{2}\right)$$

$$\alpha(t) \to \alpha_\infty$$

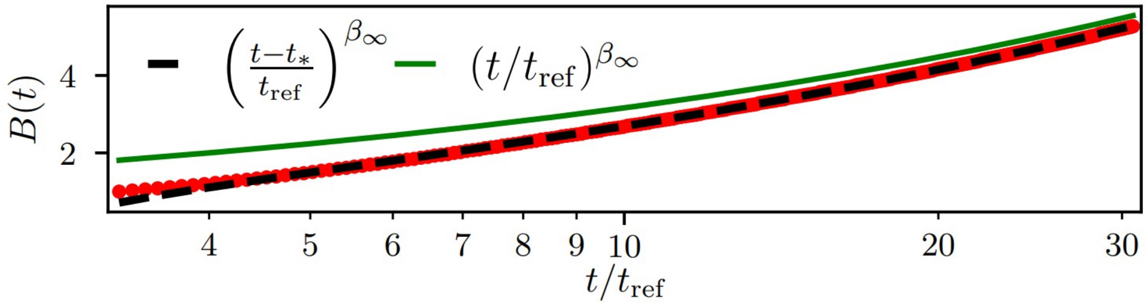
Prescaling ansatz $$f(t, \mathbf{p}) = A(t) f_S(B(t)\mathbf{p})$$

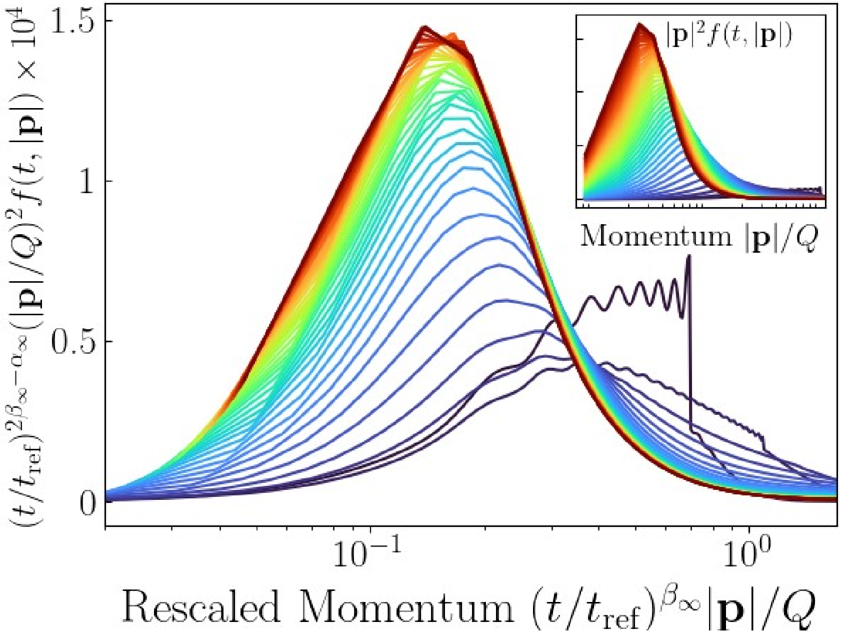
$$A(t) = \exp\left[\int_{t_0}^{t} dt' \frac{\alpha(t')}{t'}\right] \to t^{\alpha_\infty}$$

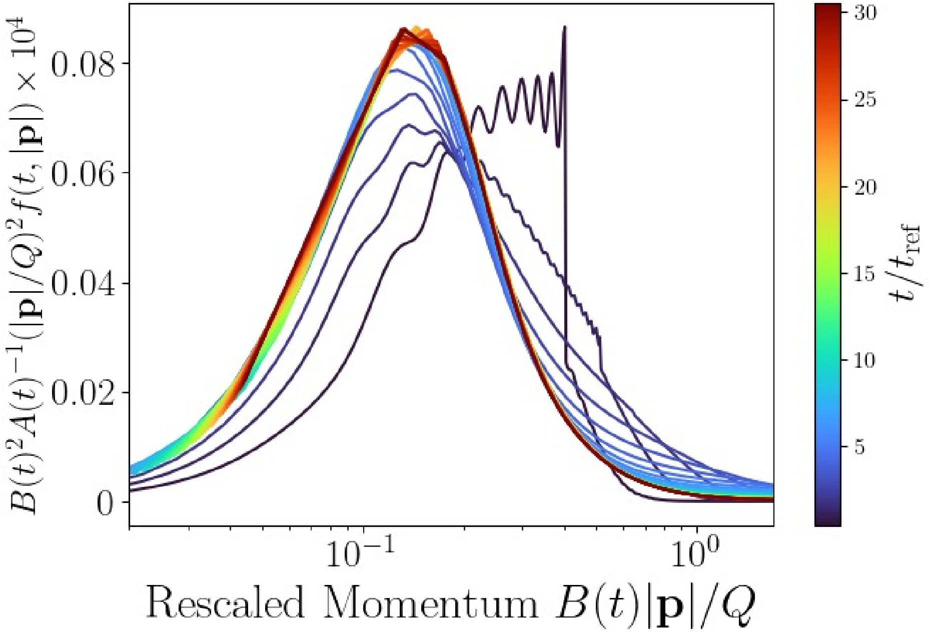
$$B(t) = \left(\frac{t-t_*}{t_{\text{ref}}}\right)^{\beta_\infty} \approx \left(\frac{t}{t_{\text{ref}}}\right)^{\beta_\infty} \left(1 - \beta_\infty \frac{t_*}{t} + \dots\right)$$

$$\beta(t) = \beta_\infty \frac{t}{t-t_*} \approx \beta_\infty + \beta_\infty \frac{t_*}{t} + \dots \quad .$$

$$t_* = t_0(1 - \beta_\infty/\beta_0)$$



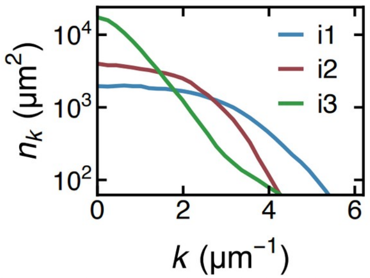




$$A(t) = B(t)^d$$

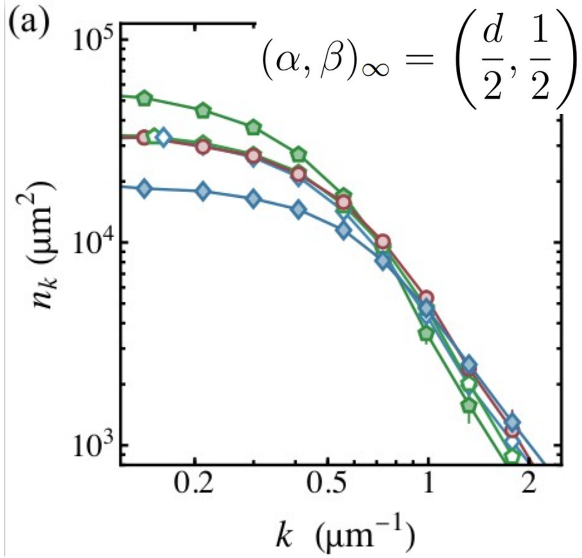
$$B(t) = n(t)\mathcal{E}(t_0)/(\mathcal{E}(t)n(t_0))$$

Prescaling in a Bose gas



Gazo et al., Science 389, 802 (2025)

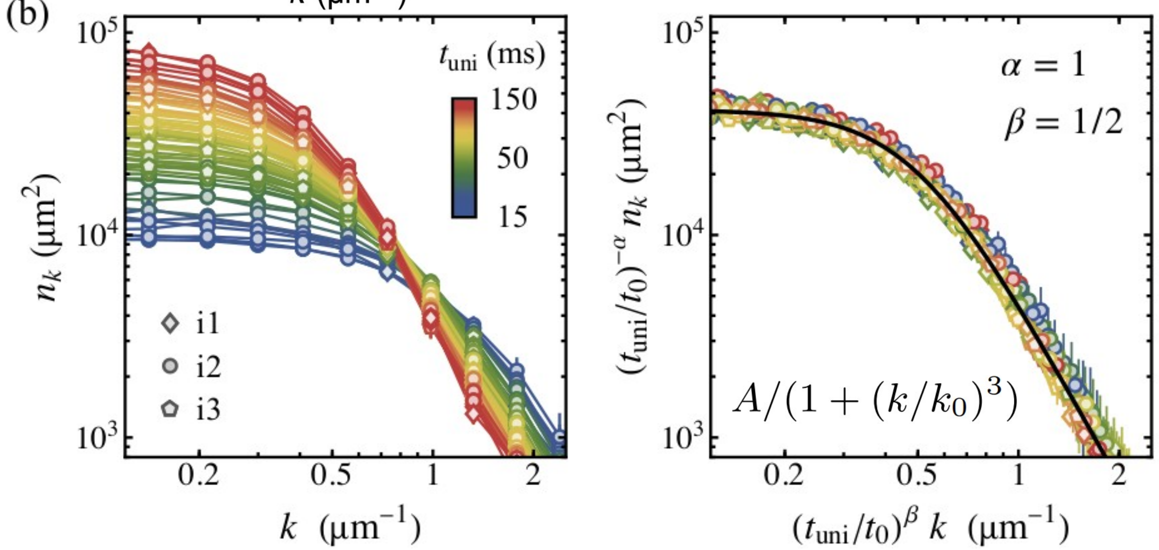
IR



$$(\alpha, \beta)_\infty = \left(\frac{d}{2}, \frac{1}{2}\right)$$

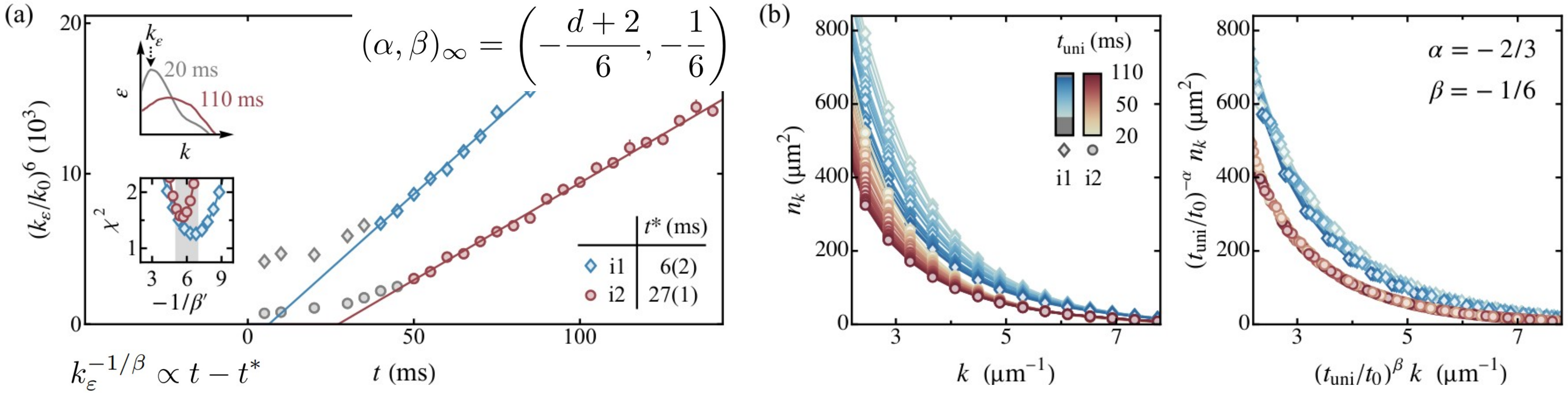
$$t_{\mathrm{uni}} = t - t_*$$

		t (ms)	t_{uni} (ms)
◆	i1	65	33(3)
◇	i1	95	63(3)
○	i2	65	58(2)
⬠	i3	30	62(3)
⬟	i3	65	97(3)



UV

$$(\alpha, \beta)_\infty = \left(-\frac{d+2}{6}, -\frac{1}{6}\right)$$



$k_\varepsilon^{-1/\beta} \propto t - t^*$

Breaking of scaling in QCD KT

$$\alpha(t) = (d+z)\beta(t)$$
$$(\alpha,\beta)_\infty = (-4/7, -1/7)$$
$$1/\lambda \gg f_{\mathbf{p}} \gg 1$$

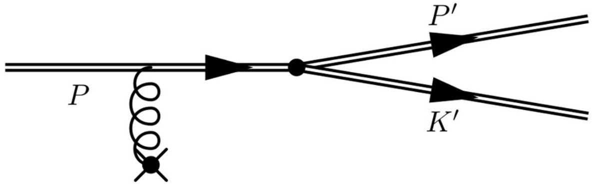
Debye mass in QCD kinetic theory breaks overall scaling

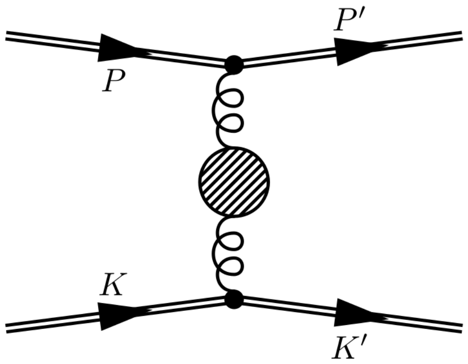
$$\partial_t f(t,\mathbf{p}) = \mathcal{C}^{2\leftrightarrow 2}_{\rm QCD}[f](t,\mathbf{p}) + \mathcal{C}^{1\leftrightarrow 2}_{\rm QCD}[f](t,\mathbf{p})$$

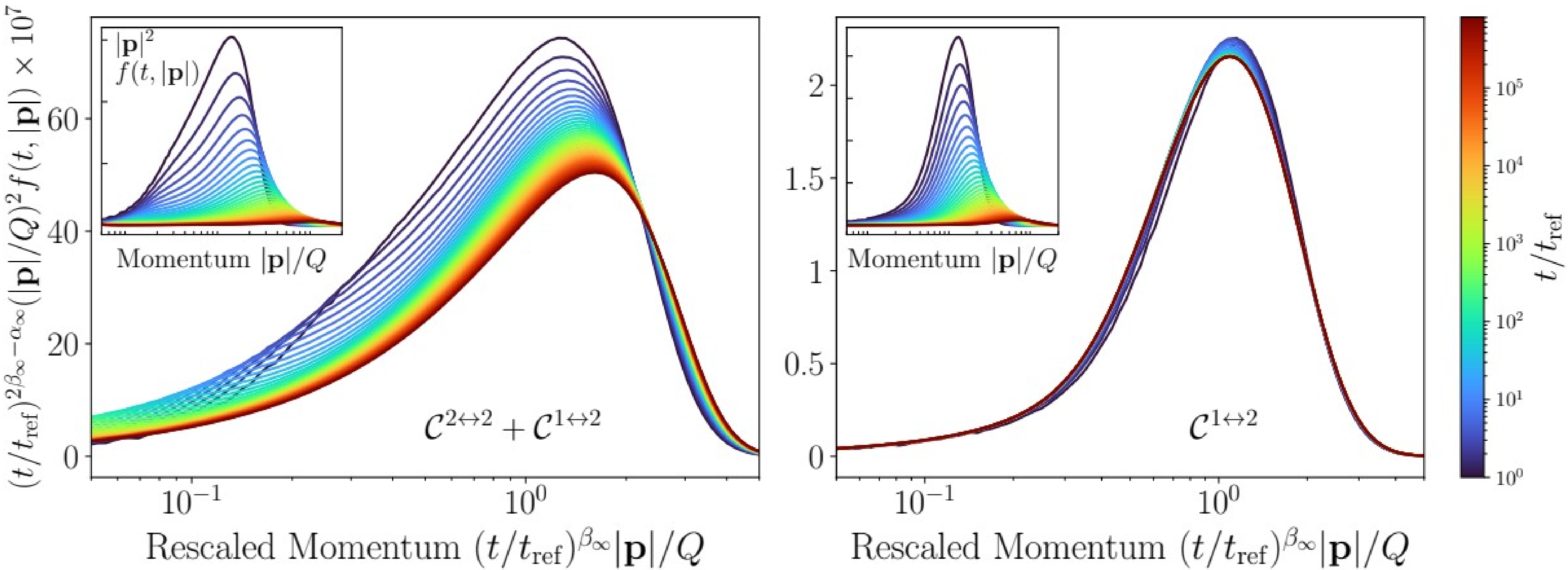
$$\mathcal{C}^{2\leftrightarrow 2}_{\rm QCD}[f](t,\mathbf{p}) \leftarrow |\mathcal{M}^{gg}_{gg}|^2 = |\mathcal{M}^{gg}_{gg}|^2[m_D(t)]$$

$$m_D^2(t) \sim \int d^3\mathbf{p} f(t,\mathbf{p})/p \qquad m_D(t) \sim t^{-1/7}$$

Arnold, Moore, Yaffe, JHEP 0301 (2003)







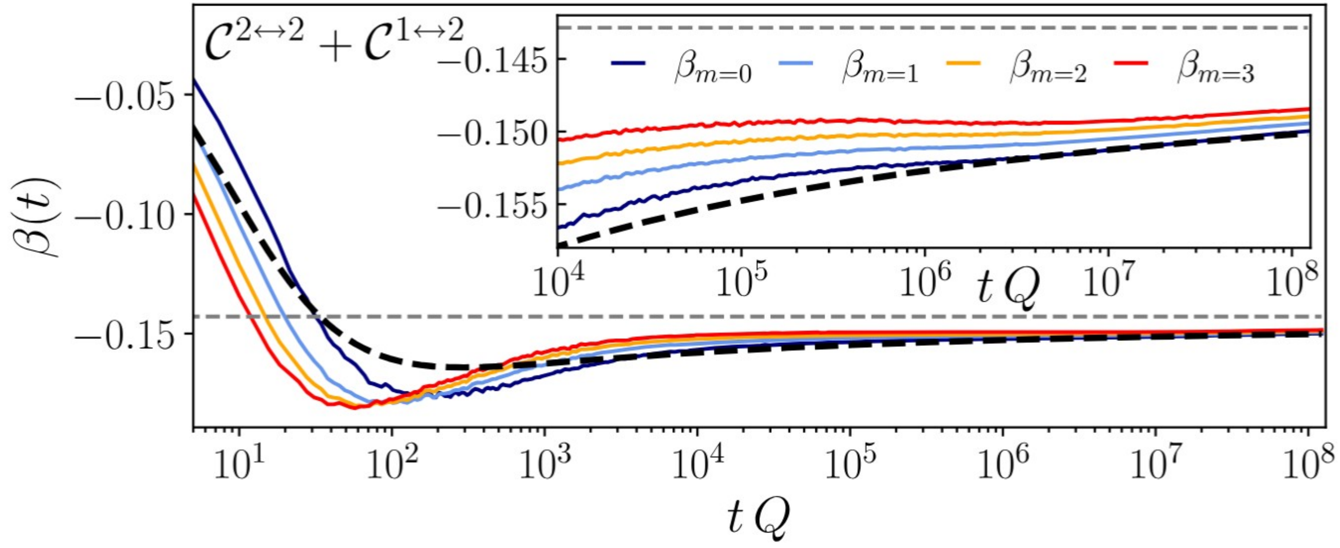
$$f(t_i,\mathbf{p}) = n_0/g^2 \exp\left[-\mathbf{p}^2/Q^2\right]$$

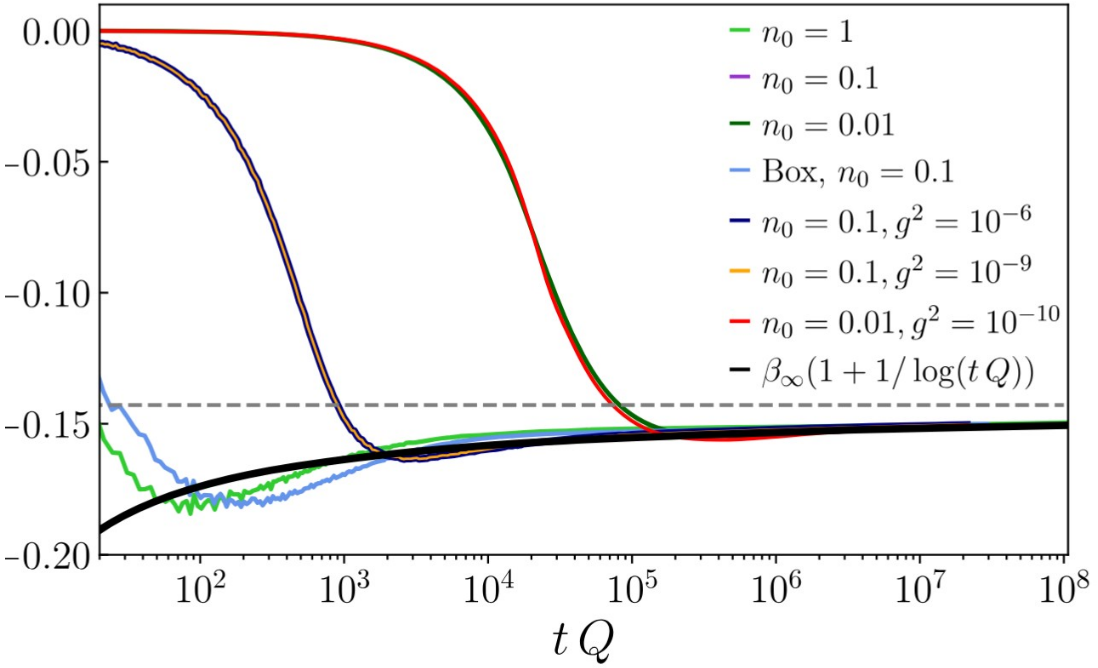
Effect of scaling breaking terms in FP

$$\alpha(t) = (d+z)\beta(t)$$
$$(\alpha,\beta)_\infty = (-4/7, -1/7)$$
$$1/\lambda \gg f_{\mathbf{p}} \gg 1$$

$$\mathcal{C}^{\mathrm{FP}}[f](t,\mathbf{p}) = \frac{A(t)^3}{B(t)} \log\left[\frac{\langle \bar{p} \rangle}{A(t)^{\frac{1}{2}}\bar{m}_D}\right] \tilde{C}^{\mathrm{FP}}[f_S](\bar{\mathbf{p}})$$

$n_m(t) = \int_{\mathbf{p}} |\mathbf{p}|^m f(t,\mathbf{p}) = A(t)B^{-(3+m)}(t)n_m(t_{\mathrm{ref}})$ Modifies Scaling !





$$\beta(t) \approx \beta_\infty + \sum_{m=1}^{\infty}\sum_{n=0}^{m-1} \beta_{m,n} \frac{\log(\log(t\,Q))^n}{\log(t\,Q)^m}$$

$$\approx \beta_\infty\left[1 + \frac{1}{\log(t\,Q)}\right],$$

Transseries structure with universal late-time behavior

$\beta_{m+1,0}/\beta_{m,0} \sim m$ vanishing radius of convergence

Also in Bjorken expansion: Brewer et. al JHEP 145 (2022)

Scaling in FP: Tanji et al, PRD 95 (2017), Brewer et. al JHEP 145 (2022), Mikheev et al, PRD 105 (2022)

Closing hydrodynamics with kinetic theory

Energy-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = (\mathcal{E} + \mathcal{P})u^\mu u^\nu - \mathcal{P}g^{\mu\nu} + \pi^{\mu\nu}$$

Closure →

Israel-Stewart: Causality & Stability

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} = -\pi^{\mu\nu} + 2\eta\sigma^{\mu\nu} + \ldots$$

$$\mathcal{E} = 3\mathcal{P}$$

$$\sigma^{\mu\nu} \equiv \partial^{\langle\mu}u^{\nu\rangle}$$

Construct equations through kinetic theory:

$$T^{\mu\nu} = \int dK\, k^\mu k^\nu f_{\mathbf{k}}$$

$$\mathcal{E} = \int dK\, (u_\mu k^\mu)^2 f_{\mathbf{k}},$$

$$\pi^{\mu\nu} = \int dK\, k^{\langle\mu}k^{\nu\rangle} f_{\mathbf{k}}.$$

$$k^\mu \partial_\mu f_{\mathbf{k}} = \mathcal{C}[f]$$

$$k^\mu = (\omega_{\mathbf{k}}, \mathbf{k})$$

$f_{\mathbf{k}} \gg 1$

$$C[f] \approx \frac{1}{2}\int dK' dP dP' W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} \times \left[\left(f_{\mathbf{p}}f_{\mathbf{p}'}\left(f_{\mathbf{k}} + f_{\mathbf{k}'}\right) - f_{\mathbf{k}}f_{\mathbf{k}'}\left(f_{\mathbf{p}} + f_{\mathbf{p}'}\right)\right)\right]$$

Expand around any isotropic background:

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}$$

f_β $\quad$ $A(t) f_S(B(t)|\mathbf{p}|)$

Initial idea: Denicol & Noronha, NPA 10005, 121996 (2021)

Far from equilibrium hydrodynamics

14-moment: $\delta f_{\mathbf{k}} = f_{0\mathbf{k}} \frac{2p^\mu p^\nu}{15 N_4(t)} \pi_{\mu\nu}(t)$

$$N_4(t) = \int dK |\mathbf{k}|^4 f_{0\mathbf{k}}$$

$$\tau_\pi(t)\dot{\pi}^{\langle\mu\nu\rangle} = -\pi^{\mu\nu} + 2\eta(t)\sigma^{\mu\nu} + \tau_\pi(t)\left[-\frac{4}{3}\pi^{\mu\nu}\theta - \frac{10}{7}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} - 2\pi^{\lambda\langle\mu}\omega_\lambda^{\nu\rangle}\right]$$

$\theta = \nabla_\mu u^\mu$ $\quad\omega^{\mu\nu} = \left(\nabla^\mu u^\nu - \nabla^\nu u^\mu\right)/2.$

With time-dependent transport coefficients

$$\eta(t) = \frac{8}{15^2}\frac{\mathcal{E} N_4(t)}{\delta\mathcal{C}(t)},$$

$$\tau_\pi(t) = \frac{2}{15}\frac{N_4(t)}{\delta\mathcal{C}(t)}.$$

$$\frac{\eta(t)}{\tau_\pi(t)\left(\mathcal{E}+\mathcal{P}\right)} = \frac{1}{5}.$$

Thermal comparison

KT $\quad \frac{\eta}{\tau_\pi(\mathcal{E}+\mathcal{P})} = \frac{1}{5}$

Pu et al., PRD 81, 114039 (2010),
Ghiglieri et al., PRL 121,052302 (2018).

Holography

$$\frac{\eta}{\tau_\pi\left(\mathcal{E}+\mathcal{P}\right)} \approx 0.65$$

$\eta = \frac{1}{4\pi}\frac{\mathcal{E}+\mathcal{P}}{T} \qquad \tau_\pi T \approx 1.3 \times 2\pi$

Kovtun et al., PRL 94, 111601 (2005),
Kovtun et al., PRD 72, 086009 (2005).

Direct energy cascade:

$$(\alpha, \beta)_{\infty,\mathrm{rel}} = \left(-\frac{d+1}{7}, -\frac{1}{7}\right)$$

$$(\alpha, \beta)_{\infty,\mathrm{non-rel}} = \left(-\frac{d+2}{6}, -\frac{1}{6}\right)$$

Dynamics of transport coefficients:

$$\eta(t) = \left(\frac{t - t_*}{t_{\mathrm{ref}}}\right) \bar{\eta}, \qquad \tau_\pi(t) = \left(\frac{t - t_*}{t_{\mathrm{ref}}}\right) \bar{\tau}_\pi$$

Far from equilibrium:

$$\Delta\mathcal{P}(t) = \Delta\mathcal{P}(t_0) \exp\left[-\int_{t_0}^{t} \frac{dt'}{\tau_\pi(t')}\right]$$

$$= \Delta\mathcal{P}(t_0) \left(\frac{t_0 - t_*}{t - t_*}\right)^{\frac{t_{\mathrm{ref}}}{\bar{\tau}_\pi}},$$

Homogeneous Isotropization dynamics:

$$\tau_\pi(t)\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 0$$

$$\pi^{\mu\nu} = \mathrm{diag}\left(0, \frac{1}{3}\Delta\mathcal{P}(t), \frac{1}{3}\Delta\mathcal{P}(t), -\frac{2}{3}\Delta\mathcal{P}(t)\right)$$

Thermal:

$$\Delta\mathcal{P}(t)\Big|_{\mathrm{thermal}} = \Delta\mathcal{P}(t_0) \exp\left(-\frac{t - t_0}{\tau_\pi}\right)$$

Florkowski et al., RPR 81, 046001 (2018)

Direct energy cascade: Hydrodynamic excitations

$x^3: \quad e^{i\,q\,x^3}$

$$\delta u,\, \delta\mathcal{E},\, \delta\pi \sim \; e^{-i\int_{t_0}^{t} dt'\Omega(t',q)}$$

$$\bar\Omega(\bar q) = \Omega(t,q)\times\left(\frac{t-t_*}{t_{\rm ref}}\right), \quad \bar q \quad = q\times\left(\frac{t-t_*}{t_{\rm ref}}\right)$$

Shear channel: $\delta u^1,\, \delta\pi^{13}$

$$\bar\Omega(\bar q) + i\frac{3\bar\eta}{4\mathcal{E}}\bar q^2 - i\bar\tau_\pi\bar\Omega(\bar q)^2 = 0.$$

$$\bar\Omega_{\rm shear}(\bar q) \approx -i\frac{3\bar\eta}{4\mathcal{E}}\bar q^2 \;\longrightarrow\; e^{-\frac{3\bar\eta}{4\mathcal{E}}q^2\frac{(t-t_*)^2}{2t_{\rm ref}}}$$

Thermal: $e^{-\frac{3\eta}{4\mathcal{E}}q^2\,t}$

Shear coefficient

$+$ Transient $\sim\left(\frac{t_{\rm ref}}{t-t_*}\right)^{\frac{\bar\tau_\pi}{t_{\rm ref}}}$

Zero mode and tower of transient power-law decaying excitations: → Talk De Lescluze

Sound channel:

$$\delta\mathcal{E},\, \delta u^3,\, \delta\pi^{33},\, \delta\pi^{11} = \delta\pi^{22} = -\frac{1}{2}\delta\pi^{33}$$

$$+i\frac{\bar\eta}{\mathcal{E}}\bar q^2\bar\Omega(\bar q) + \frac{1}{3}i\,\bar\tau_\pi\bar q^2\bar\Omega(\bar q) - i\,\bar\tau_\pi\bar\Omega(\bar q)^3$$
$$+\bar\Omega(\bar q)^2 - \frac{1}{3}\bar q^2 = 0.$$

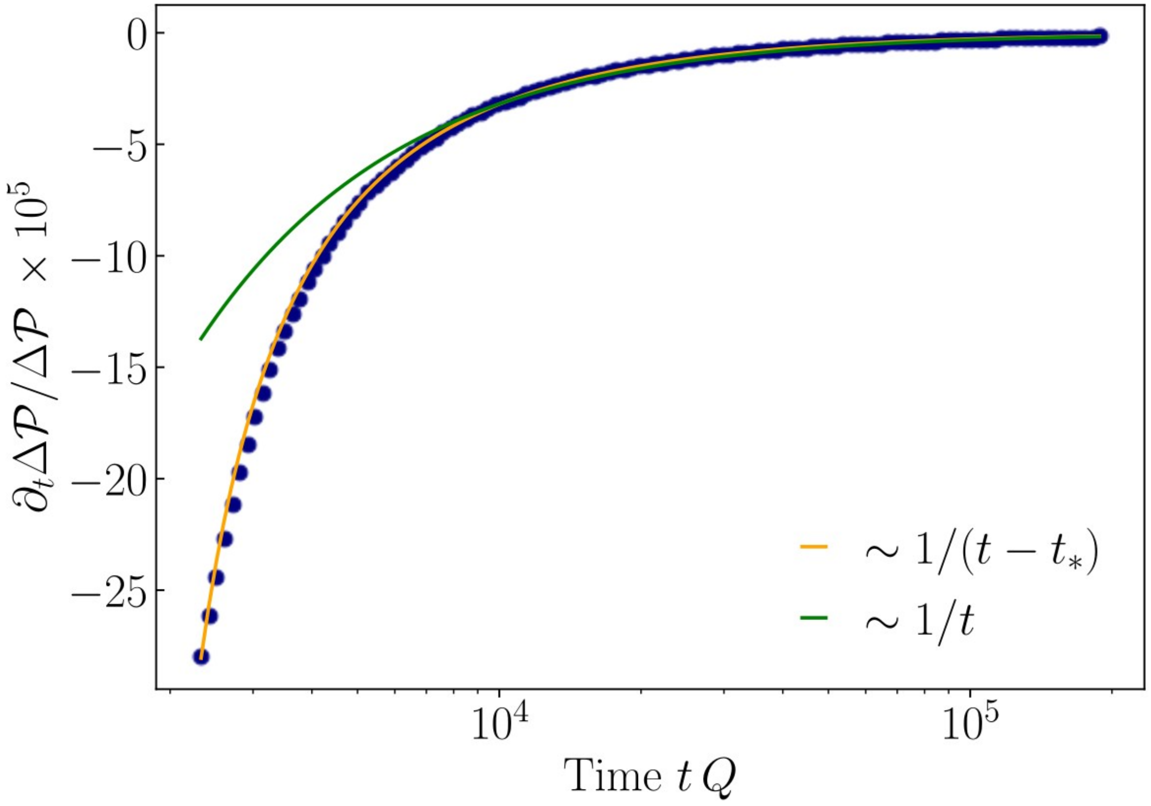
$$\bar\Omega_{\rm sound}(\bar q) \approx \pm\frac{1}{\sqrt{3}}\bar q \;\longrightarrow\; e^{\pm i\frac{1}{\sqrt{3}}q\,t}$$

Speed of sound

$+$ Transient $\sim \Delta\mathcal{P}(t)$

In QCD Kinetic Theory

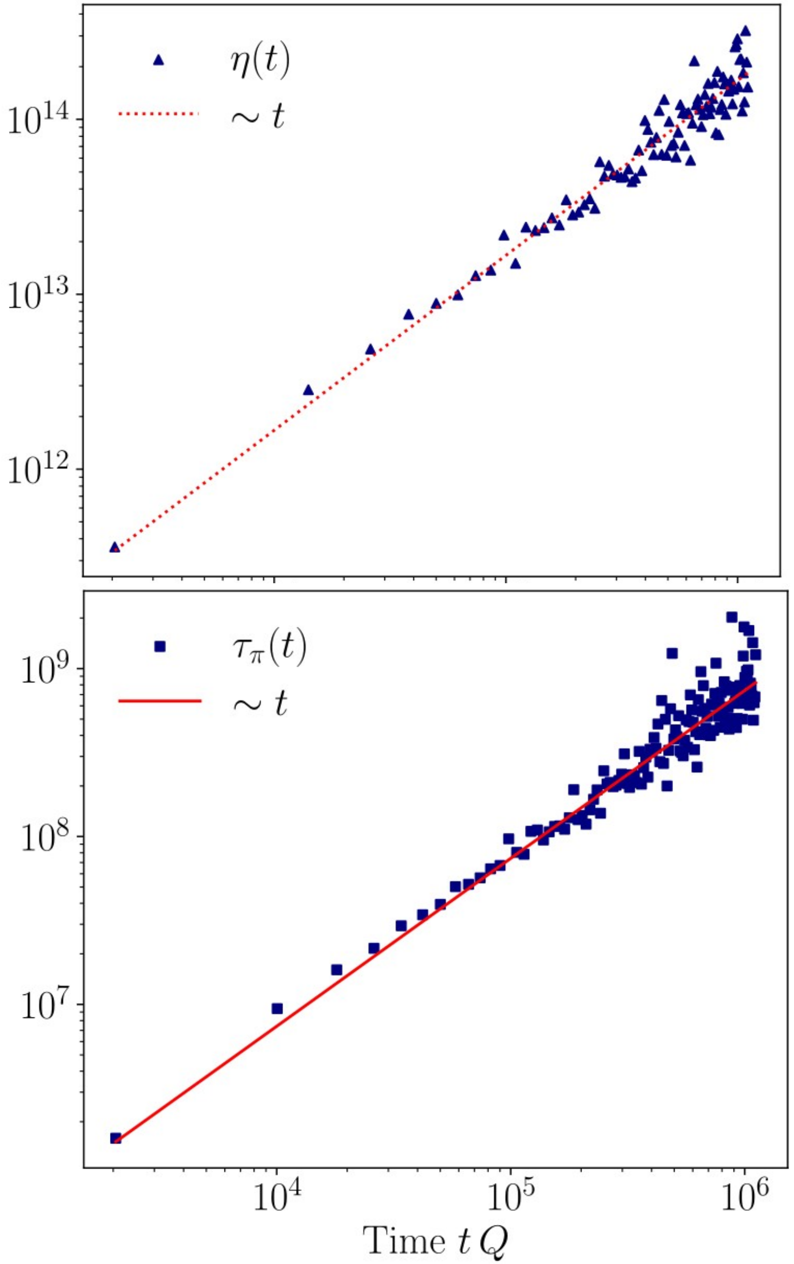
Applicability beyond 2-2 processes in QCD KT:



Anisotropic perturbation near isotropic attractor:

$$\delta f_{\mathbf{p}} = f_0(t_0, \mathbf{p})\, \frac{A}{3} \left[-(p^1)^2 - (p^2)^2 + 2(p^3)^2\right]$$

$$f(t_i, \mathbf{p}) \;=\; n_0/g^2 \exp\left[-\mathbf{p}^2/Q^2\right]$$



Summary

based on Heller, Mazeliauskas, TP, Phys. Rev. Lett. 132, 071602 (2024), Berges, Denicol, Heller, TP, 2504.18754

Prescaling: Find universal scaling in the experimentally accessible finite-time dynamics by accounting for the initial-state dependent effects.

Hydrodynamics: Hydrodynamic excitations near nonthermal attractors characterized by time-dependent transport coefficients.

Outlook

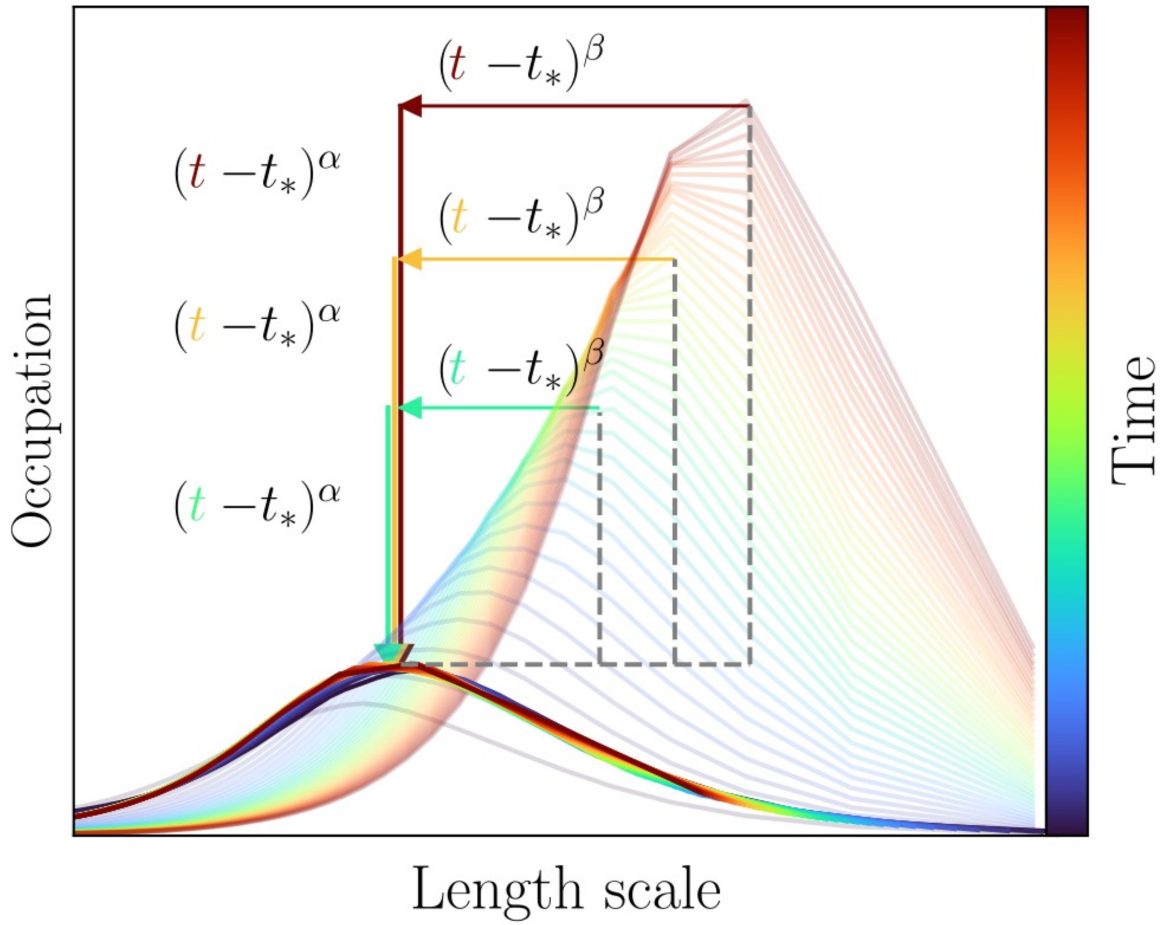
Applicability to ab initio simulations with broken homogeneity, e.g. via KøMPøST

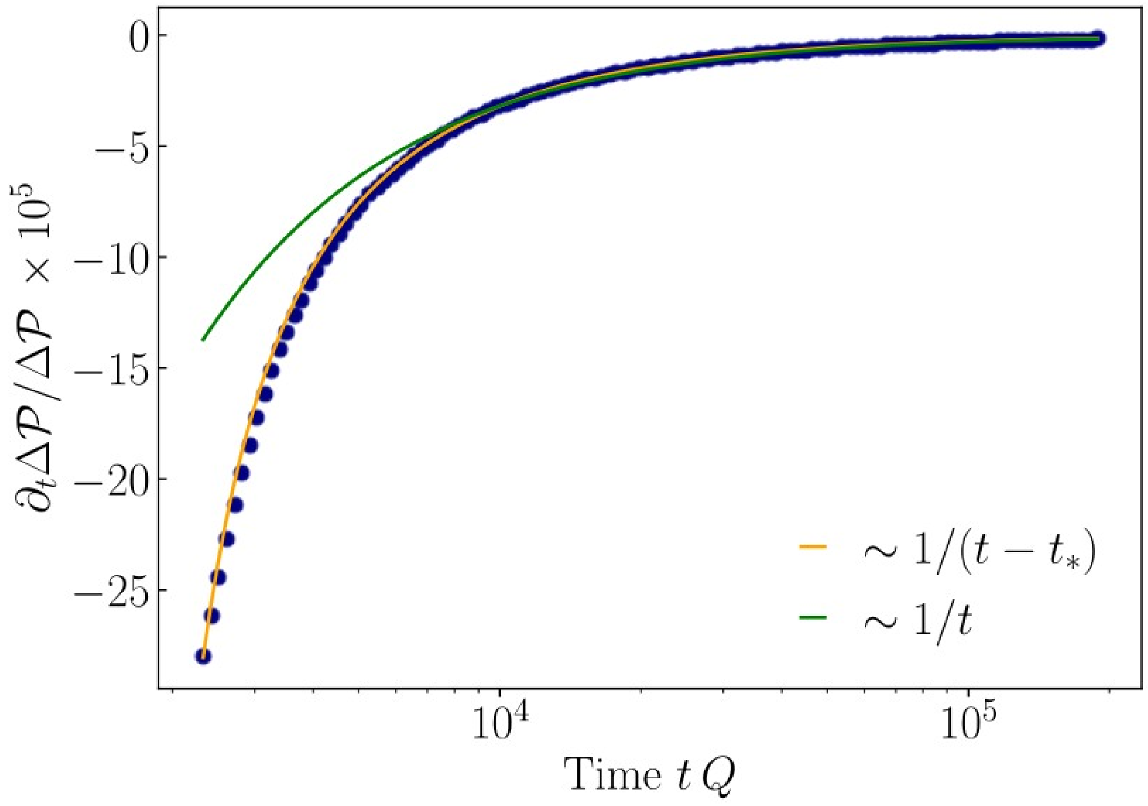
Study universal scaling in cold quantum gases far from equilibrium:

→ Scaling breaking perturbations through time-dependent tuning of interactions

→ Perturbations of isotropy or spatial homogeneity to study power-law isotropization of pressure or evolution of far from equilibrium transport coefficients, e.g. $\eta(t) \sim t$

Thank you for your attention !





Backup

The derivation of scaling

See also: Micha, Tkachev, PRL 90, 121301 (2003), Berges, Rothkopf, Schmidt, PRL 101, 041603 (2008), Berges et al., PRL 114 (2015).

$$\partial_t f(t,\mathbf{p}) = C[f](t,\mathbf{p})$$

Make scaling ansatz:

$$f(t,\mathbf{p}) = (t/t_{\mathrm{ref}})^{\alpha_\infty} f_S((t/t_{\mathrm{ref}})^{\beta_\infty}|\mathbf{p}|)$$

Determine scaling exponents via scaling relations from Boltzmann equation:

$$(t/t_{\mathrm{ref}})^{\alpha_\infty - 1}\left[\alpha_\infty + \beta_\infty \bar{\mathbf{p}}\cdot\partial_{\bar{\mathbf{p}}}\right] f_S(\bar{\mathbf{p}}) = (t/t_{\mathrm{ref}})^{\mu} C[f_S](\bar{\mathbf{p}} = (t/t_{\mathrm{ref}})^{\beta_\infty}\mathbf{p})$$

and relevant conservation laws such as energy or particle number conservation:

$$\omega_{\mathbf{p}} \sim |\mathbf{p}|^z$$

$$\mathcal{E}(t) = \int \frac{d^d\mathbf{p}}{(2\pi)^d}\omega_{\mathbf{p}} f(t,\mathbf{p}) = (t/t_{\mathrm{ref}})^{\alpha_\infty - (d+z)\beta_\infty}\bar{\mathcal{E}}$$

$$n(t) = \int \frac{d^d\mathbf{p}}{(2\pi)^d} f(t,\mathbf{p}) = (t/t_{\mathrm{ref}})^{\alpha_\infty - d\beta_\infty}\bar{n}$$

$$\mu = \alpha_\infty - 1$$

$$\alpha_\infty = (d+z)\beta_\infty$$

$$\alpha_\infty = d\beta_\infty$$

$$\alpha_\infty = \sigma\beta_\infty$$

Direct energy cascade

$$f(t,\mathbf{p}) = A(t) f_S(B(t)\mathbf{p})$$

$$C^{2\leftrightarrow2}[f](t,\mathbf{p}) = \frac{\lambda}{2}\int_{\mathbf{qlr}} \frac{(2\pi)^4\delta^{(3)}(\mathbf{p}+\mathbf{l}-\mathbf{q}-\mathbf{r})}{2\omega_{\mathbf{p}}2\omega_{\mathbf{l}}2\omega_{\mathbf{q}}2\omega_{\mathbf{r}}}\delta(\omega_{\mathbf{p}}+\omega_{\mathbf{l}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}})$$
$$\times\left[(f_{\mathbf{p}}+1)(f_{\mathbf{l}}+1)f_{\mathbf{q}}f_{\mathbf{r}} - f_{\mathbf{p}}f_{\mathbf{l}}(f_{\mathbf{q}}+1)(f_{\mathbf{r}}+1)\right]$$
$$= A(t)^3 B(t)^{-1} C[f_S](\bar{\mathbf{p}})$$

$$\omega_{\mathbf{p}} \sim |\mathbf{p}|$$

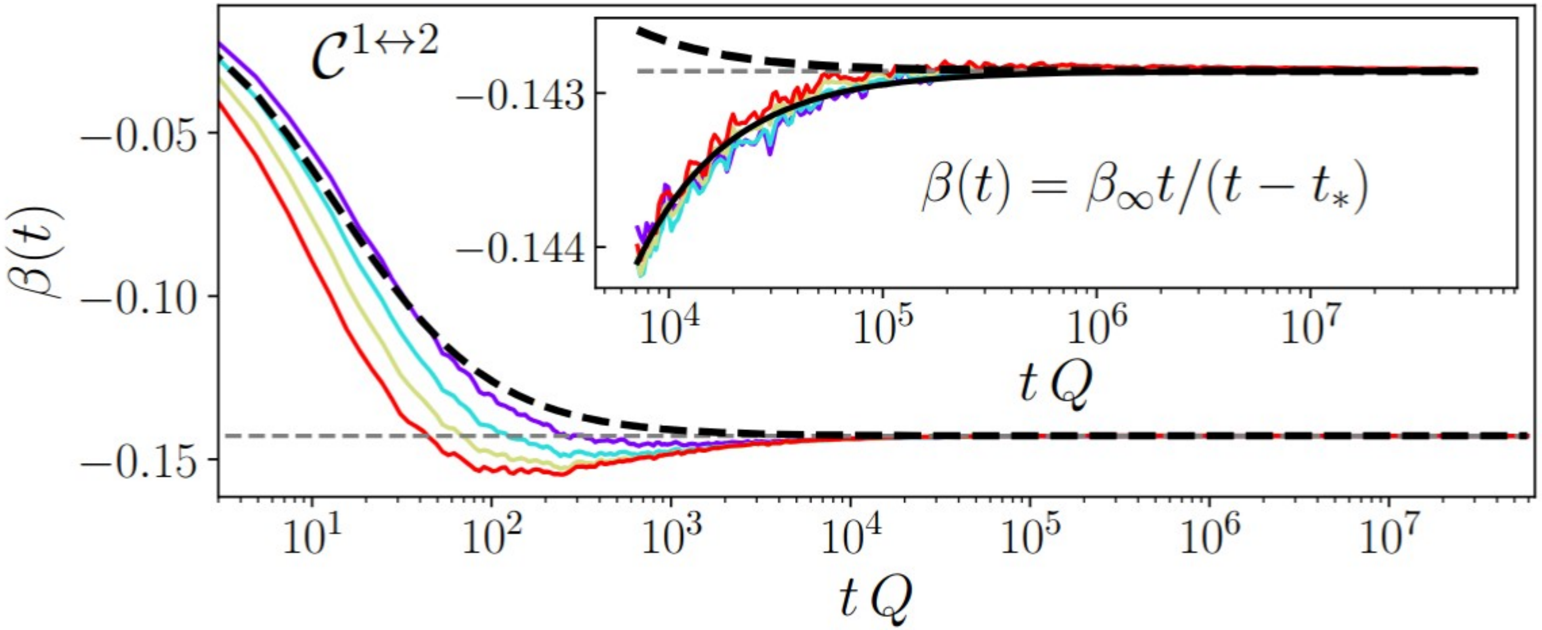
$$\mathcal{C}^{1\leftrightarrow2}_{\mathrm{QCD}}[f](t,\mathbf{p}) = A(t)^3 B(t)^{-1}\mathcal{C}^{1\leftrightarrow2}_{\mathrm{QCD}}[f_S](\bar{\mathbf{p}})$$

$$\alpha(t) = (d+z)\beta(t)$$
$$1/\lambda \gg f_{\mathbf{p}} \gg 1$$

$$(\alpha,\beta)_\infty = (-4/7, -1/7)$$

$$f(t_i,\mathbf{p}) = n_0/g^2 \exp\left[-\mathbf{p}^2/Q^2\right]$$

$$\partial_t f(t,\mathbf{p}) = \mathcal{C}^{2\leftrightarrow2}_{\mathrm{QCD}}[f](t,\mathbf{p}) + \mathcal{C}^{1\leftrightarrow2}_{\mathrm{QCD}}[f](t,\mathbf{p})$$



Also, Inflationary Cosmology:
Micha, Tkachev, PRL 90, 121301 (2003), ...

$$n_m(t) = \int_{\mathbf{p}} |\mathbf{p}|^m f(t,\mathbf{p}) = A(t)B^{-(3+m)}(t)\, n_m(t_{\mathrm{ref}})$$

Elastic collision kernel

$$C\,[f] = \frac{1}{2}\int dK' dP dP' W_{\mathbf{kk'}\to\mathbf{pp'}} I[f](\mathbf{k},\mathbf{k}',\mathbf{p},\mathbf{p}')$$

$$I[f](\mathbf{k},\mathbf{k}',\mathbf{p},\mathbf{p}') = f_{\mathbf{p}} f_{\mathbf{p}'}[1+f_{\mathbf{k}}][1+f_{\mathbf{k}'}] - f_{\mathbf{k}} f_{\mathbf{k}'}[1+f_{\mathbf{p}}][1+f_{\mathbf{p}'}]$$
$$\approx f_{\mathbf{p}} f_{\mathbf{p}'}\left(f_{\mathbf{k}} + f_{\mathbf{k}'}\right) - f_{\mathbf{k}} f_{\mathbf{k}'}\left(f_{\mathbf{p}} + f_{\mathbf{p}'}\right).$$

$$W_{\mathbf{kk'}\to\mathbf{pp'}} = |\mathcal{M}|^2_{\mathbf{kk'}\to\mathbf{pp'}}(2\pi)^{d+1}\delta^{(1)}(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'} - \omega_{\mathbf{p}} - \omega_{\mathbf{p}'})\delta^{(d)}(\mathbf{k}+\mathbf{k}'-\mathbf{p}-\mathbf{p}')$$

$$|\mathcal{M}_{\mathrm{O}(N)}|^2 = \lambda^2\frac{N+2}{6N^2}$$

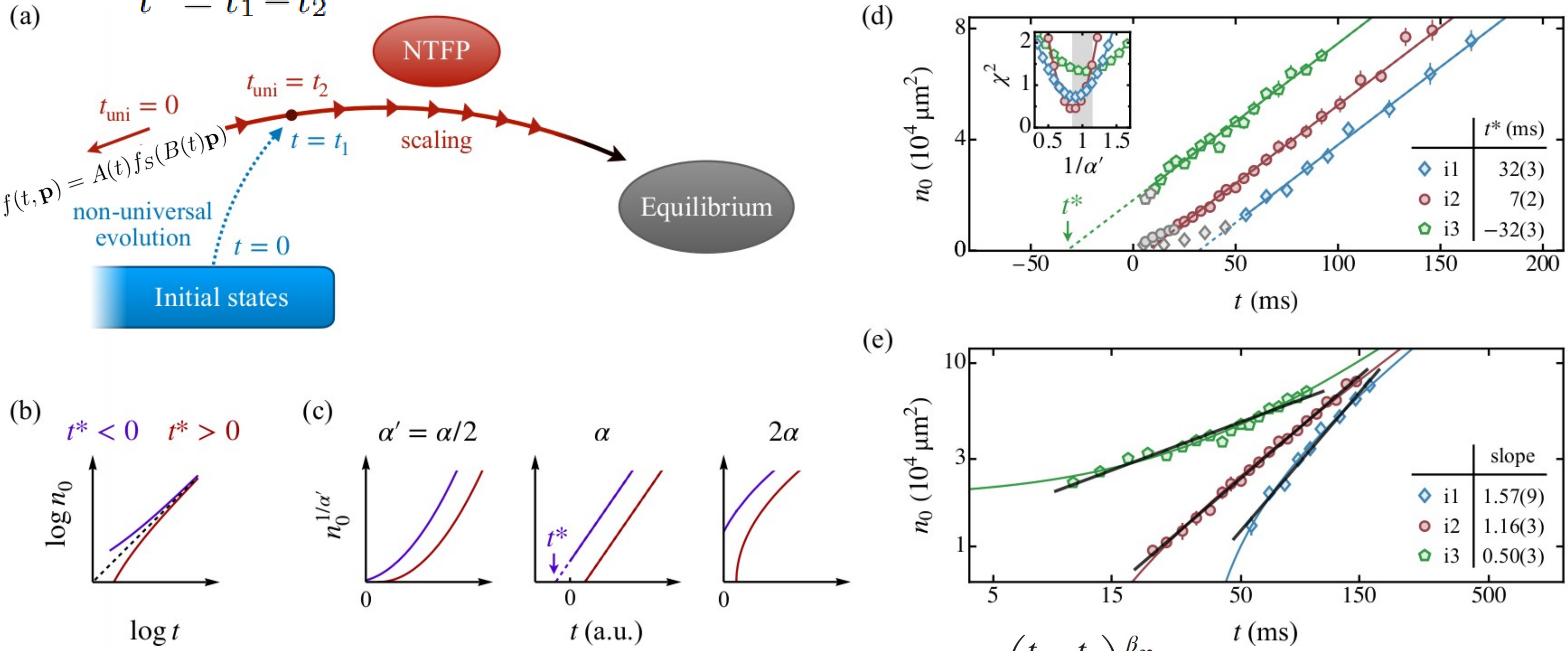
$$|\mathcal{M}_{O(N),\mathrm{non-rel}}|^2 = 4g^2$$

$$|\mathcal{M}^{gg}_{gg}|^2(k,k',p,p') = 16(N_c^2-1)N_c^2 g^4\left(3 - \frac{s_M t_M}{u_M^2} - \frac{s_M u_M}{t_M^2} - \frac{t_M u_M}{s_M^2}\right)$$

Dependence on an experimental clock variable

$$t_{\text{uni}} = t - t^*$$
$$t^* = t_1 - t_2$$

Gazo et al., Science 389, 802 (2025)



$$n_0 \propto (t - t^*)^{\alpha}$$

$$f(t,\mathbf{p}) = A(t) f_S(B(t)\mathbf{p})$$

$$B(t) = \left(\frac{t - t_*}{t_{\text{ref}}}\right)^{\beta_\infty}$$

$$A(t) = B(t)^{\sigma}$$

Weak wave turbulence

$1/\lambda \gg f_{\mathbf{p}} \gg 1$

Micha, Tkachev, PRD 70 (2004)

$\int_{\mathbf{p}} \equiv d^d p/(2\pi)^d$

Make stationary turbulence ansatz $\quad f_{\mathbf{p}} = |\mathbf{p}|^{-\kappa} f_1 \qquad \omega_{\mathbf{p}} = |\mathbf{p}|\omega_1$

$$C^{2\leftrightarrow 2}[f](\mathbf{p}) = \frac{\lambda^2}{2}\int_{\mathbf{lqr}} \frac{(2\pi)^{d+1}\delta^{(3)}(\mathbf{p}+\mathbf{l}-\mathbf{q}-\mathbf{r})}{2\omega_{\mathbf{p}}2\omega_{\mathbf{l}}2\omega_{\mathbf{q}}2\omega_{\mathbf{r}}}$$
$$\times\, \delta^{(1)}(\omega_{\mathbf{p}}+\omega_{\mathbf{l}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}})\left[(f_{\mathbf{p}}+1)(f_{\mathbf{l}}+1)f_{\mathbf{q}}f_{\mathbf{r}} - f_{\mathbf{p}}f_{\mathbf{l}}(f_{\mathbf{q}}+1)(f_{\mathbf{r}}+1)\right]$$
$$= |\mathbf{p}|^{\mu} C(\mathbf{1}) \qquad \mu = 2d-5-3\kappa$$

and obtain solution from e.g. energy conservation by demanding a scale-independent flux A(k)

$$\partial_t(\omega_{\mathbf{p}} f_{\mathbf{p}}) + \nabla_{\mathbf{p}}\cdot \mathbf{j}_{\mathbf{p}} = 0 \qquad \int_{\mathbf{p}}^{k} \nabla_{\mathbf{p}}\cdot\mathbf{j}_{\mathbf{p}} = \int_{\partial k} \mathbf{p}\cdot d\mathbf{A}_{\mathbf{p}} \equiv (2\pi)^d A(k)$$

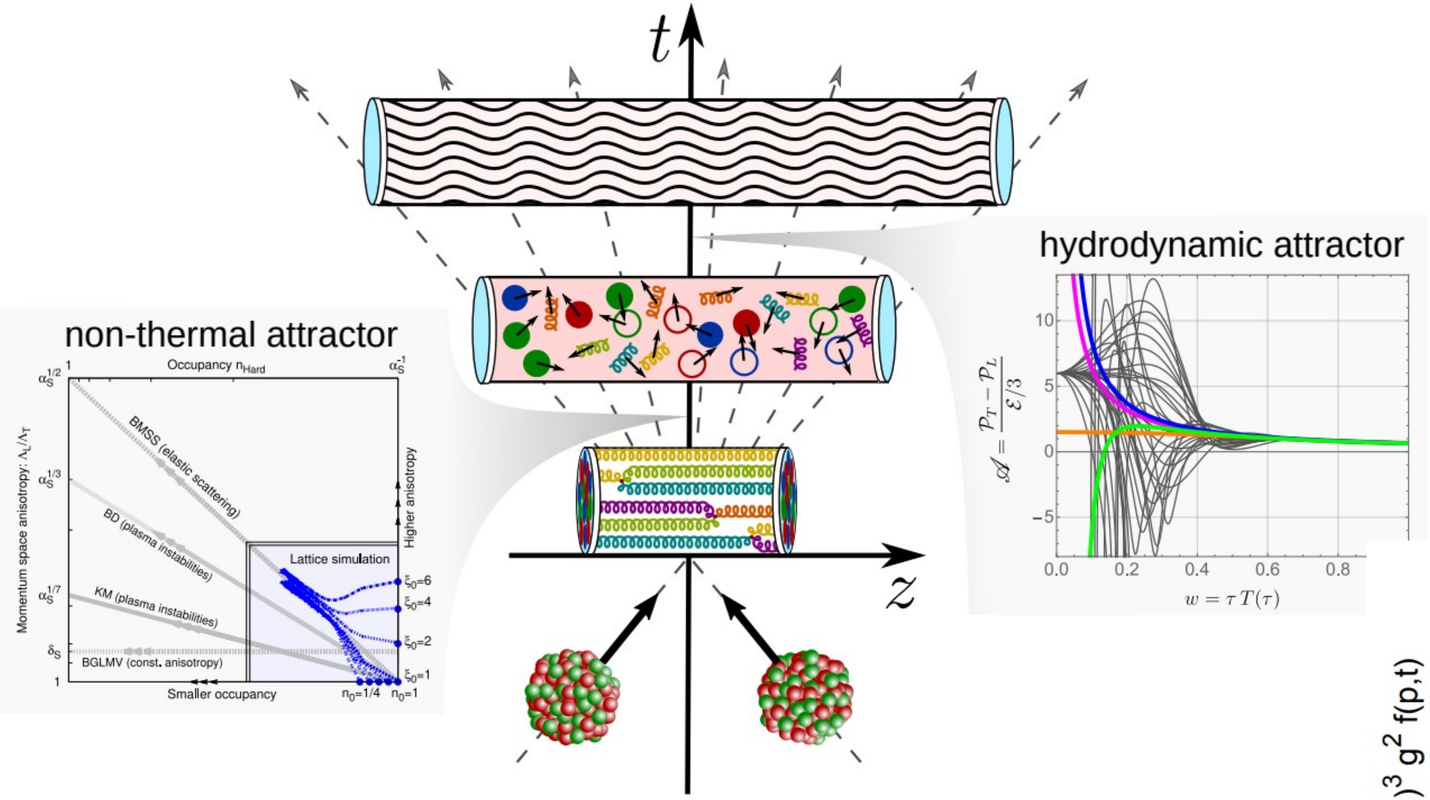
$$A(k) = -\frac{1}{2^d \pi^{d/2}\Gamma(d/2+1)}\int^{k} dp|\mathbf{p}|^{d+\mu}\omega_1 C(\mathbf{1}) \sim \frac{k^{d+1+\mu}}{d+1+\mu}\omega_1 C(\mathbf{1})$$

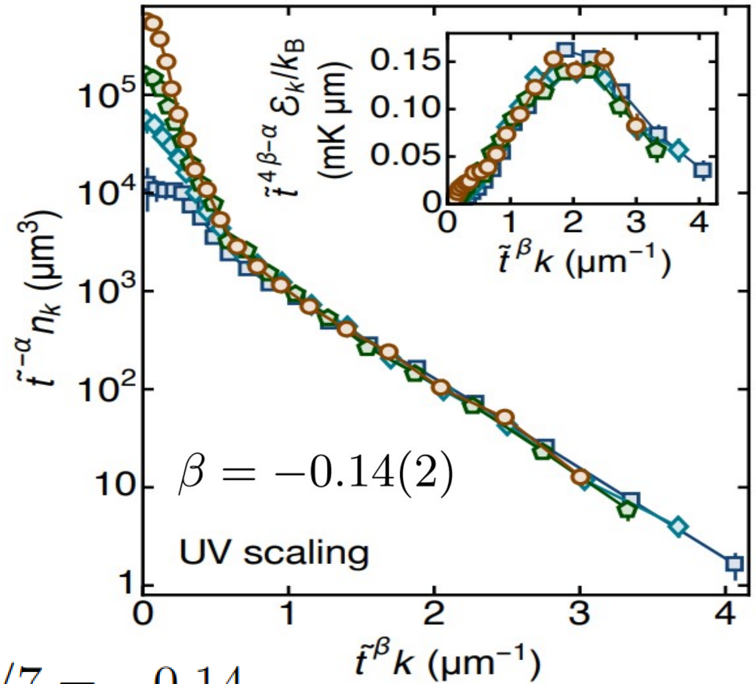
$$d+1+\mu = 0$$
$$\kappa = d - \frac{4}{3}$$

Similarly, for relativistic particle cascade $\quad \kappa = d - \frac{5}{3}$

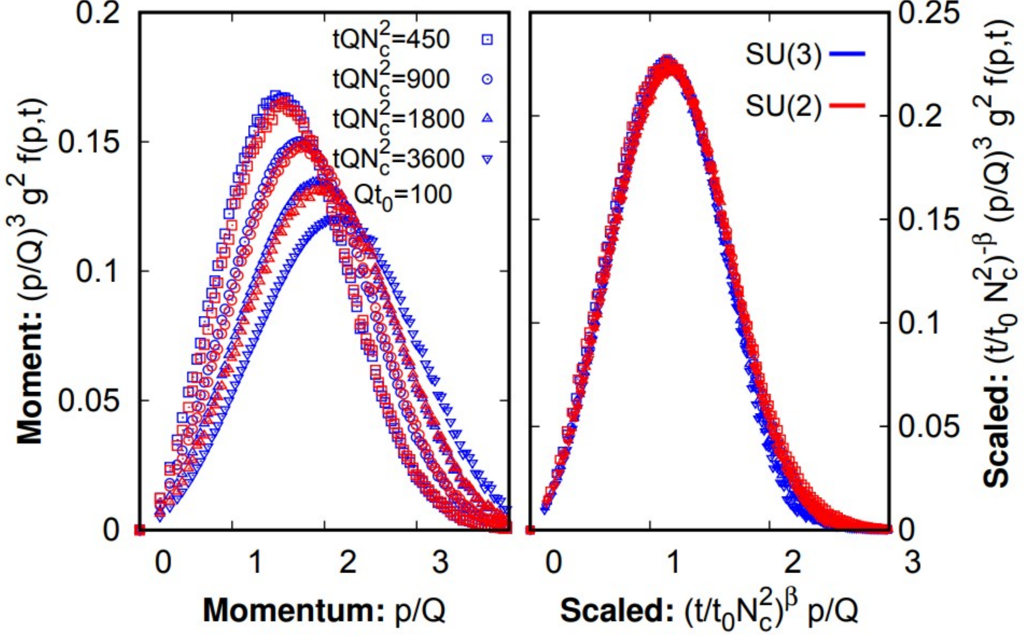
Relativistic energy cascade

The non-thermal attractor in QCD





$\beta = -1/7 = -0.14$



Berges, Mace, Schlichting, PRL 118, 192005 (2017)

From $\tau Q_S \geq \log^2 \alpha_S^{-1}$ to $\tau Q_S \geq \alpha_S^{-3/2}$

Berges, Heller, Mazeliauskas, Venugopalan, 93 RMP (2020).
See also: Baier et al., PLB 502 (2001); Gelis et al., Ann. Rev. Nucl. Part. Sci. 60 (2010); Heller, Janik, Witaszczyk, PRL 108 (2012); Kurkela, Zhu, PRL 115 (2015); Romatschke, PRL 120 (2018); ...

Kinetic theory calculation

$$u^\mu \partial_\mu f_{\mathbf{k}} = -\frac{1}{u_\alpha k^\alpha} k^\mu \nabla_\mu f_{\mathbf{k}} + \frac{1}{u_\alpha k^\alpha} \mathcal{C}[f]$$

$$\nabla_\mu \equiv \Delta_\mu^\nu \partial_\nu$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} \frac{d}{d\tau} \int dK k^{\langle\alpha} k^{\beta\rangle} f_{\mathbf{k}}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = \mathcal{C}^{\mu\nu}_{-1} + \frac{8}{15}\mathcal{E}\sigma^{\mu\nu} - 2\pi^{\lambda\langle\mu}\omega^{\nu\rangle}_{\lambda} - \frac{10}{7}\sigma^{\langle\mu}_{\lambda}\pi^{\nu\rangle\lambda} - \frac{4}{3}\pi^{\mu\nu}\theta - \rho^{\mu\nu\alpha\beta}_{-2}\sigma_{\alpha\beta} - \Delta^{\mu\nu}_{\mu_1\mu_2}\nabla_{\mu_3}\rho^{\mu_1\mu_2\mu_3}_{-1}$$

With non-hydro fields

$$\mathcal{C}^{\mu\nu}_{-1} = \int dK k^{\langle\mu} k^{\nu\rangle} \mathcal{C}[f]/(u_\mu k^\mu)$$

$$\rho^{\mu\nu\alpha}_{-1} = \int dK k^{\langle\mu} k^{\nu} k^{\alpha\rangle} f_{\mathbf{k}}/(u_\mu k^\mu)$$

$$\rho^{\mu\nu\alpha\beta}_{-2} = \int dK k^{\langle\mu} k^{\nu} k^{\alpha} k^{\beta\rangle} f_{\mathbf{k}}/(u_\mu k^\mu)^2$$

14-Moment expansion

Expand around isotropic background in moments up to second order:

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}, \qquad \delta f_{\mathbf{k}} \simeq f_{0\mathbf{k}} \left(\epsilon + k^{\mu} \epsilon_{\mu} + k^{\mu} k^{\nu} \epsilon_{\mu\nu} \right)$$

With linearized collision kernel:

$$\mathcal{C}_{-1}^{\mu\nu} = -\delta\mathcal{C} \frac{15}{2N_4(t)} \pi^{\mu\nu}$$

$$\delta\mathcal{C} = \int dK dK' dP dP' \frac{1}{u_{\mu} k^{\mu}} k_{\langle\alpha} k_{\beta\rangle} W_{\mathbf{k}\mathbf{k}'\to\mathbf{p}\mathbf{p}'} \left[-\frac{1}{10} f_{0\mathbf{p}} f_{0\mathbf{p}'} f_{0\mathbf{k}} \left(k^{\langle\alpha} k^{\beta\rangle} + 2p^{\langle\alpha} p^{\beta\rangle} \right) \right.$$
$$\left. - \frac{1}{10} f_{0\mathbf{p}} f_{0\mathbf{p}'} f_{0\mathbf{k}'} \left(k'^{\langle\alpha} k'^{\beta\rangle} + 2p^{\langle\alpha} p^{\beta\rangle} \right) + \frac{1}{5} f_{0\mathbf{k}} f_{0\mathbf{k}'} f_{0\mathbf{p}} \left(k^{\langle\alpha} k^{\beta\rangle} + k'^{\langle\alpha} k'^{\beta\rangle} + p^{\langle\alpha} p^{\beta\rangle} \right) \right]$$