# Adiabatic Hydrodynamization

## the adiabatic picture of attractors

**Attractors and Thermalization in Nuclear and Cold Quantum Gases**
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**based on 2203.02427 (with Jasmine Brewer and Yi Yin), 2405.17545 and 2507.21232 (with Krishna Rajagopal and Rachel Steinhorst)**

# Outline

## and the main messages

1. Adiabatic Hydrodynamization
   - A framework to *explain* memory loss and out-of-equilibrium universality.

     $\Longleftrightarrow$ explain what attracts to attractors
2. Non-thermal fixed points
   - Scaling (and even time-dependent scaling) attractors are often *exactly* adiabatic.
3. Bottom-up thermalization in weakly coupled QCD
   - Memory loss of the initial condition is a sequential, multi-stage process
4. Take-home message: AH is a powerful, versatile framework to study attractor phenomena

# 1. Adiabatic Hydrodynamization (AH)

# Adiabatic Hydrodynamization

## as proposed by Brewer, Yan, and Yin

- Idea: the essential feature of an attractor is a reduction in the number of quantities needed to describe the system.
- Brewer, Yan and Yin conjectured that this is due to an emergent timescale $\tau_{\mathrm{Redu}} \ll \tau_{\mathrm{Hydro}}$ after which a set of “pre-hydrodynamic” slow modes (that gradually evolve into hydrodynamic modes) govern the system.
- Their proposal: try to understand the emergence of $\tau_{\mathrm{Redu}}$ (at which only slow modes remain) using the machinery of the adiabatic approximation in quantum mechanics.

J. Brewer, L. Yan, Y. Yin “Adiabatic hydrodynamization in rapidly-expanding quark-gluon plasma” Phys. Lett. B 816, 136189 (2021)

# Adiabatic Hydrodynamization

## The analogy between kinetic theory and quantum mechanics

- A kinetic equation $\partial_t f = - C[f]$ is first-order in time derivatives, just like a Schrödinger equation:

$$\partial_t \psi = - i\mathcal{H}\psi$$

- The parallel becomes clear if we are able to write the kinetic equation as

$$\partial_t f = - H[f]\, f\,,$$

because then we can study $H[f]$ as a generator of time evolution.

- To use QM techniques, let us write $H[f] \longrightarrow H(\tau)$.

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- To use QM techniques, let us write $H[f] \longrightarrow H(\tau)$.

Note: I have not told you (yet) *how* this rewriting takes place.

# Adiabatic Hydrodynamization

## adiabatic theorem and the notion of adiabaticity

- Consider a system whose evolution is given by

$$\partial_\tau |\psi\rangle = -H(\tau)|\psi\rangle,$$

where $H(\tau)$ has eigenstates/eigenvalues $\{|n(\tau)\rangle, E_n(\tau)\}_{n=0}^{\infty}$:

$$H(\tau)|n(\tau)\rangle = E_n(\tau)|n(\tau)\rangle.$$

- Then, one may write the system's evolution as

$$|\psi\rangle = \sum_{n=0}^{\infty} a_n(\tau) e^{-\int^\tau E_n(\tau')d\tau'} |n(\tau)\rangle.$$

- Adiabaticity is the degree to which transitions between different instantaneous eigenstates are suppressed:

$$\text{Adiabaticity for the } n\text{-th eigenstate} \iff \frac{\dot{a}_n}{a_n} \ll |E_n - E_m|, \text{ for } n \neq m.$$

- When this is the case, provided there is an "energy" gap between the ground state and the excited states, one has

$$\begin{aligned} |\psi\rangle &= \sum_{n=0}^{\infty} a_n(\tau) e^{-\int^{\tau} E_n(\tau')d\tau'} |n(\tau)\rangle \\ &\approx a_0\, e^{-\int^{\tau} E_0(\tau')d\tau'} |0(\tau)\rangle\,, \end{aligned}$$

that is to say, the dynamics of the system collapses onto a single form.

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$$|\psi\rangle = \sum_{n=0}^{\infty} a_n(\tau) e^{-\int^{\tau} E_n(\tau')d\tau'} |n(\tau)\rangle \quad \rightarrow \text{sequential memory loss!}$$

$$\approx a_0\, e^{-\int^{\tau} E_0(\tau')d\tau'} |0(\tau)\rangle\,, \qquad \rightarrow \text{attractor}$$

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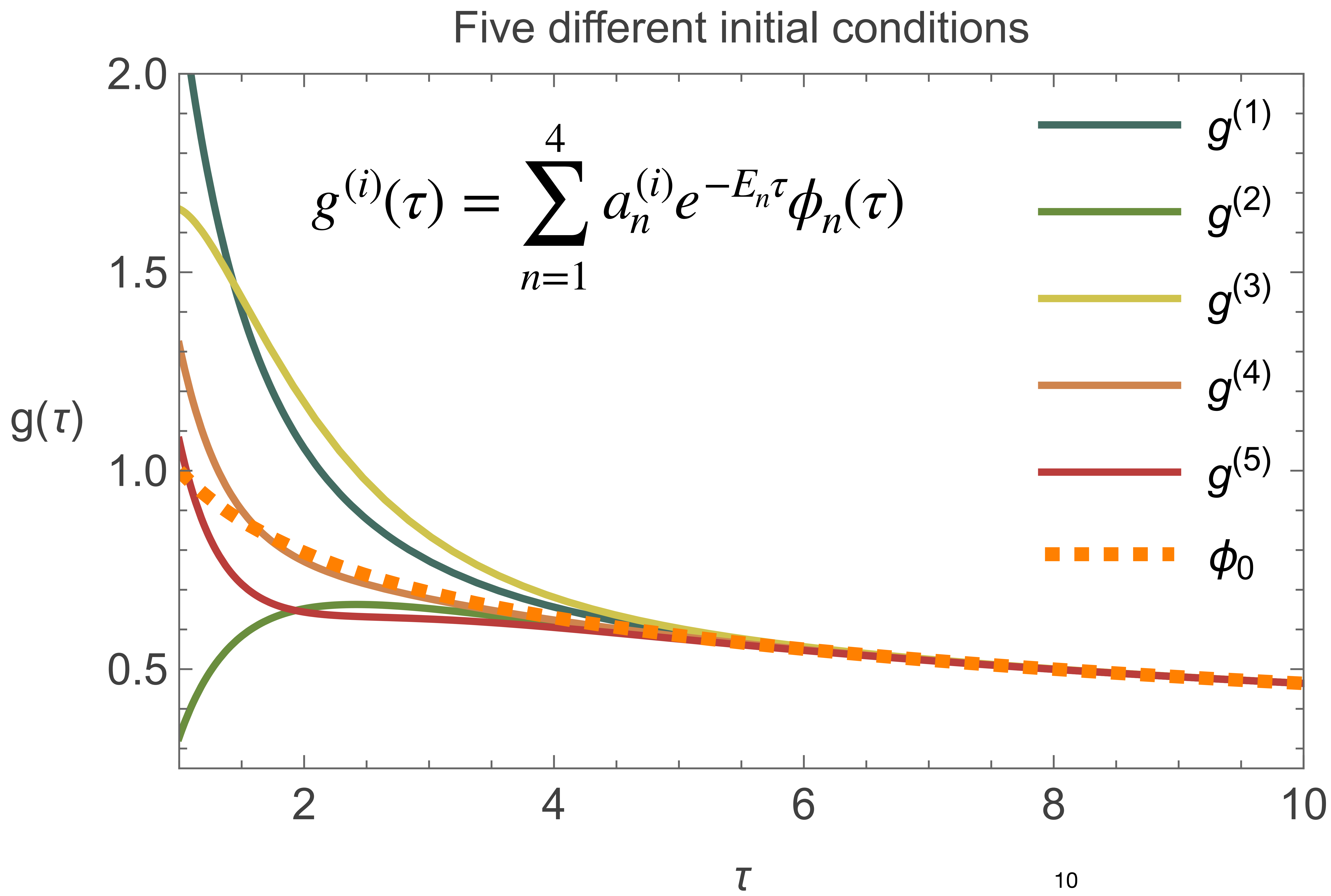
# A schematic picture

Let's say we had eigenvalues $E_n = 0{,}1{,}2{,}3$
and eigenfunctions $\phi_n(\tau) = \tau^{-1/3}, \tau^{-1/4}, \tau^{-1/2}, 1$




- The "attractor" is described by the slowest mode, $\phi_0(\tau)$
- The timescale in which the attractor is reached is set by the energy gap (in this example, $\Delta E = 1$).
- The AH framework allows us to do this for the whole distribution function

$$f(\mathbf{x}, \mathbf{p}, \tau).$$

(more information than $g(\tau)$)

# 2. Scaling and Adiabaticity

[1] J. Brewer, B. Scheihing-Hitschfeld, and Y. Yin, *Scaling and adiabaticity in a rapidly expanding gluon plasma*, *JHEP* **05** (2022) 145, [arXiv:2203.02427].

# A case study

## with applications to QCD EKT (later)

- Consider the following kinetic equation

$$\frac{\partial f}{\partial t} = x\frac{\partial f}{\partial x} + D[f;t]\frac{\partial^2 f}{\partial x^2}\ .$$

(e.g., $D = \int_{-\infty}^{\infty} f(x,t)^2\, dx$)

where $D[f;t]$ is an *arbitrary* diffusion coefficient independent of $x$.

- I will show you that:

1. This equation can be reduced analytically to a single ODE. (for *any* $D[f;t]$)

2. It features a scaling fixed point whose “attractiveness” is explained in terms of the AH framework.

# Scaling

$$\frac{\partial f}{\partial t} = x\frac{\partial f}{\partial x} + D[f;t]\frac{\partial^2 f}{\partial x^2}$$

## and new variables

- Let's introduce two time-dependent functions $A(t), B(t)$ and a rescaled distribution function $w$ as

$$f(x,t) = A(t)\, w\Big(x/B(t), t\Big).$$

  - Note: I haven't done anything. (I just introduced dummy variables)
  - Motivation to do this: if scaling behavior appears in the system, then we will be able to *choose* $A(t), B(t)$ such that $w(\xi, t)$ is stationary.
    Scaling $\iff$ $\exists A(t), B(t) \quad \mathrm{s.t.} \quad w(\xi,t) = w(\xi)$.
- I will call a given choice of $A(t), B(t)$ a "frame."
- Next step: rewrite the equation for $f$ in terms of $w$.

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# Finding the adiabatic frame

## what we want to get from choosing $A, B$

- Goal: choose $A(t), B(t)$ such that if we write

$$\frac{\partial w}{\partial t} = -\mathcal{H}[A, B, \ldots](t) w \ ,$$

then the eigenstates $\{|n\rangle\}_n$ of $\mathcal{H}$ become time-independent (or as much as possible). This is quantified by the adiabaticity criterion

$$\delta_A = \delta_A^{(n,m)} \equiv \left| \frac{\langle n|_L \partial_t |m\rangle_R}{E_n - E_m} \right| \ .$$

- In practice, for attractor behavior to emerge, all we need is that the ground state of $\mathcal{H}$ evolves adiabatically $\delta_A^{(n,0)} \ll 1$.

# Finding the adiabatic frame

$$\frac{\partial w}{\partial t} = -\mathscr{H}[A, B, \ldots](t)w$$

## what we want to get from choosing $A, B$

- Rationale: if $\{\phi_n\}_n \iff \{|n\rangle\}_n$ is the eigenbasis of $\mathscr{H}$,

$$w(\xi, t) = \sum_n a_n(t)\phi_n(\xi, t) \ ,$$

the coefficients $a_n$ evolve as

$$\partial_t a_n = -E_n(t)\, a_n - \sum_m a_m \langle n|_L \partial_t |m\rangle_R \ .$$

- Both terms on the r.h.s. are frame-dependent. We want to find $A, B$ such that

$$\partial_t a_n = -E_n(t)\, a_n \ .$$

$$\delta_A = \delta_A^{(n,m)} \equiv \left| \frac{\langle n|_L \partial_t |m\rangle_R}{E_n - E_m} \right| \ll 1$$

$$\frac{\partial f}{\partial t} = x\frac{\partial f}{\partial x} + D[f;t]\frac{\partial^2 f}{\partial x^2} \qquad f(x,t) = A(t)\, w\big(x/B(t), t\big) \qquad \alpha \equiv \dot{A}/A \qquad \beta \equiv -\dot{B}/B$$

Let me *choose* $B$ such that $\dfrac{D}{B^2(1-\beta)} = 1$, which is to say $\dfrac{\dot{B}}{B} = -1 + \dfrac{D}{B^2}$. Then

$$\frac{\partial w}{\partial t} = -\left(\alpha - (1-\beta)\left[\xi\frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \xi^2}\right]\right) w \equiv -\mathscr{H} w$$

- Now diagonalize the operator $\mathscr{H}$. The answer is

$$\mathscr{H}\phi_n^R(\xi) = E_n\phi_n^R(\xi)\,, \qquad \phi_n^L(\xi)\mathscr{H} = \phi_n^L(\xi)E_n\,,$$

$$E_n = \alpha + (1-\beta)(n+1)\,,$$

$$\phi_n^L(\xi) = \mathrm{He}_n\big(\xi\big)\,, \qquad \phi_n^R(\xi) = \mathrm{He}_n\big(\xi\big)\exp\left(-\frac{\xi^2}{2}\right).$$

- It is convenient to *choose* $A(t)$ such that $\alpha = \beta - 1$ and then $E_n = (1-\beta)\,n$.

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# The solution

## in the frame we have specified

- We have then obtained that the general solution for $w$ is

$$w(\xi, t) = \sum_{n=0}^{\infty} a_n e^{-\int_0^t E_n(t')dt'} \phi_n^R(\xi)$$

$$= \sum_{n=0}^{\infty} a_n e^{-nt} \left( \frac{B(0)}{B(t)} \right)^n \mathrm{He}_n\left(\xi\right) \exp(-\xi^2/2)$$

where $\{a_n\}_{n=0}^{\infty}$ and $B(0)$ specify the initial condition. All that remains to close the system is to solve

$$\frac{\dot{B}}{B} = -1 + \frac{D}{B^2} \quad \text{and} \quad \frac{\dot{A}}{A} = -1 - \frac{\dot{B}}{B} .$$

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$$w(\xi, t) = \sum_{n=0}^{\infty} a_n \, e^{-\int_0^t E_n(t')dt'} \phi_n^R(\xi) \quad \text{→ sequential memory loss!}$$

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where $\{a_n\}_{n=}^{\infty}$ … remains to close the system is

$$\xrightarrow{t \gg 1} a_0 \exp(-\xi^2/2) \quad \Longrightarrow \quad \textbf{scaling!}$$

$$\frac{\quad}{B} = -1 + \frac{\quad}{B^2} \quad \text{and} \quad \frac{\quad}{A} = -1 - \frac{\quad}{B}.$$

$$\frac{\dot{B}}{B} = -1 + \frac{D}{B^2} \quad \text{and} \quad \frac{\dot{A}}{A} = -1 - \frac{\dot{B}}{B}$$

- The second equation can be integrated directly: $A(t) = \dfrac{A(0)B(0)}{B(t)} e^{-t}$ .

- To solve the first equation, we have to specify $D$ as an explicit function of time and $B(t)$. At this point, this is in fact straightforward:

$$\begin{aligned} D &= D[f; t] \\ &= D\left[A(0) \sum_{n=0}^{\infty} a_n \, e^{-(n+1)t} \left(\frac{B(0)}{B(t)}\right)^{n+1} \mathrm{He}_n\left(\frac{x}{B(t)}\right) \exp\left(-\frac{x^2}{2B(t)^2}\right); t\right] \end{aligned}$$

- Then, given initial conditions for $f(x, t = 0)$, which are specified by $\{a_n\}_{n=0}^{\infty}$ , $A(0)$ and $B(0)$, the problem has been reduced to solving *one* ordinary differential equation for $B(t)$.

Solutions of $\dfrac{\partial f}{\partial t} = x\dfrac{\partial f}{\partial x} + D[f;t]\dfrac{\partial^2 f}{\partial x^2}$

# Example:

Solutions for

$$D[f;t] = e^t \int_x f^2$$

with

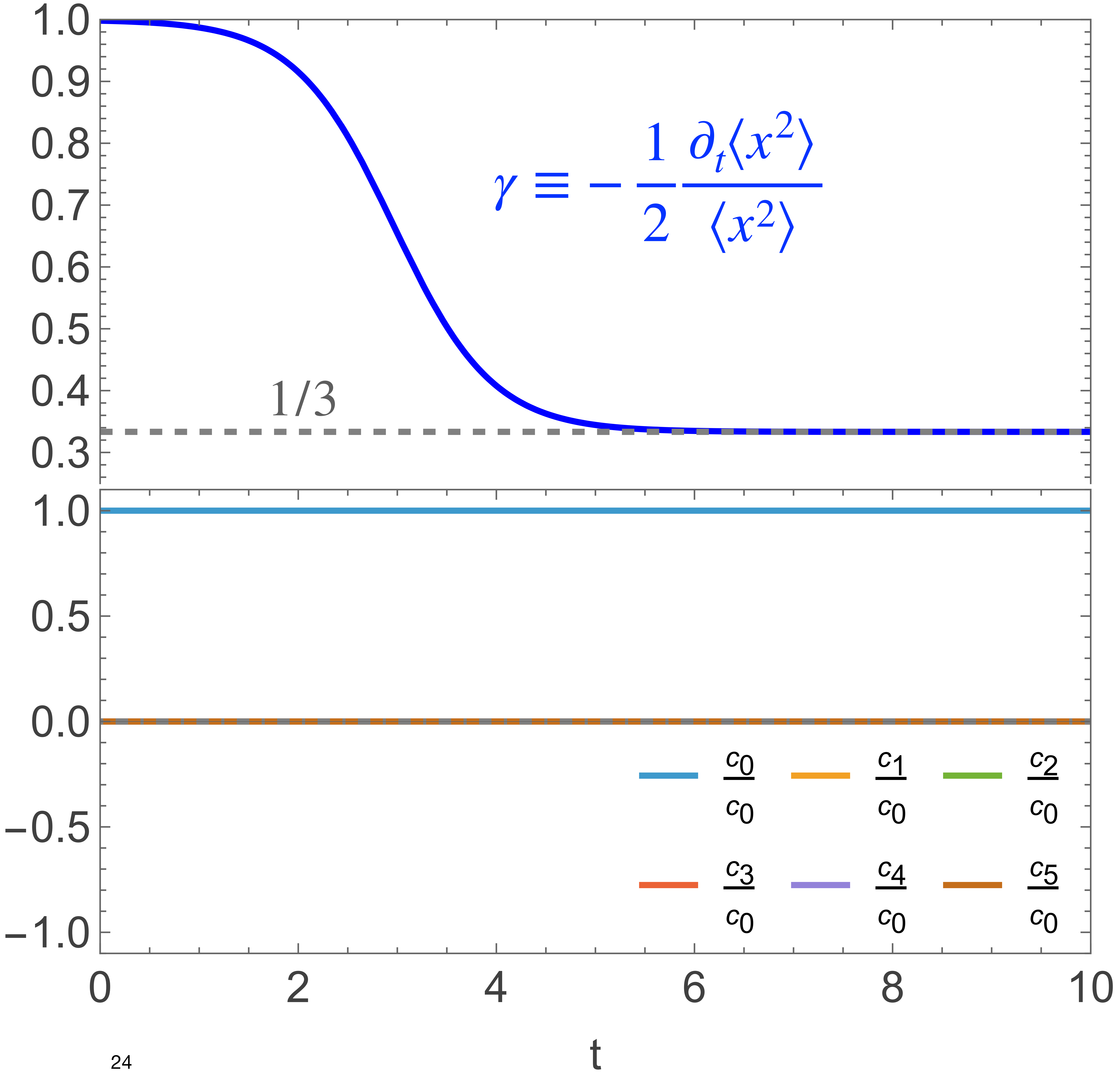
$$\xi = x/B(t)\,, \quad f(x,t) = A(t)\, w\big(x/B(t), t\big)$$

$$w(\xi, t) = \sum_{n=0}^{5} c_n(t)\, \mathrm{He}_n\big(\xi\big)\, \exp(-\xi^2/2)$$

$$c_n(t) = a_n\, e^{-\int_0^t E_n(t')dt'}$$

Scaling $\Longleftrightarrow$ unique $n$ s.t. $a_n \neq 0$

Note: if the system starts in the ground state, prescaling!



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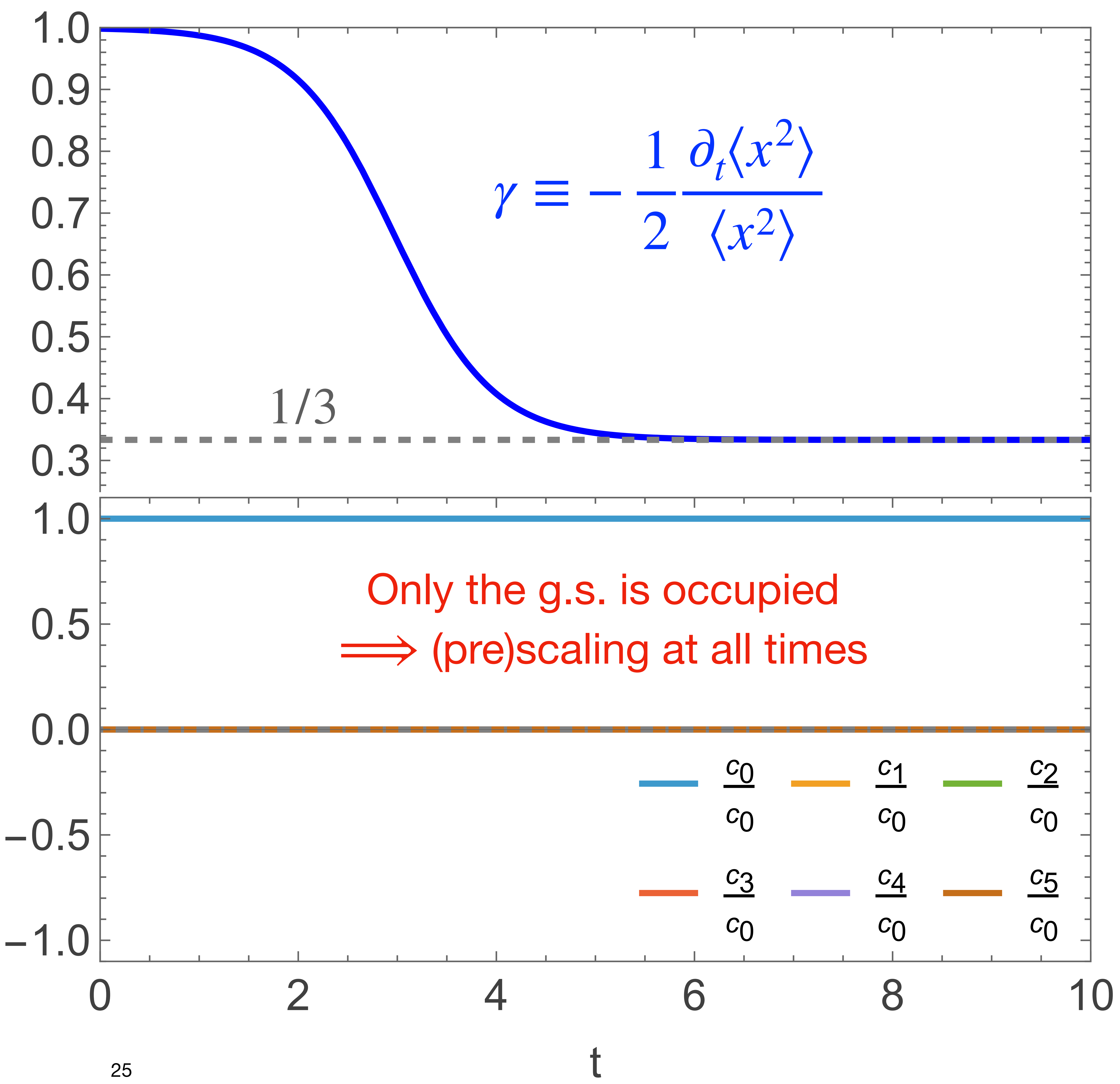

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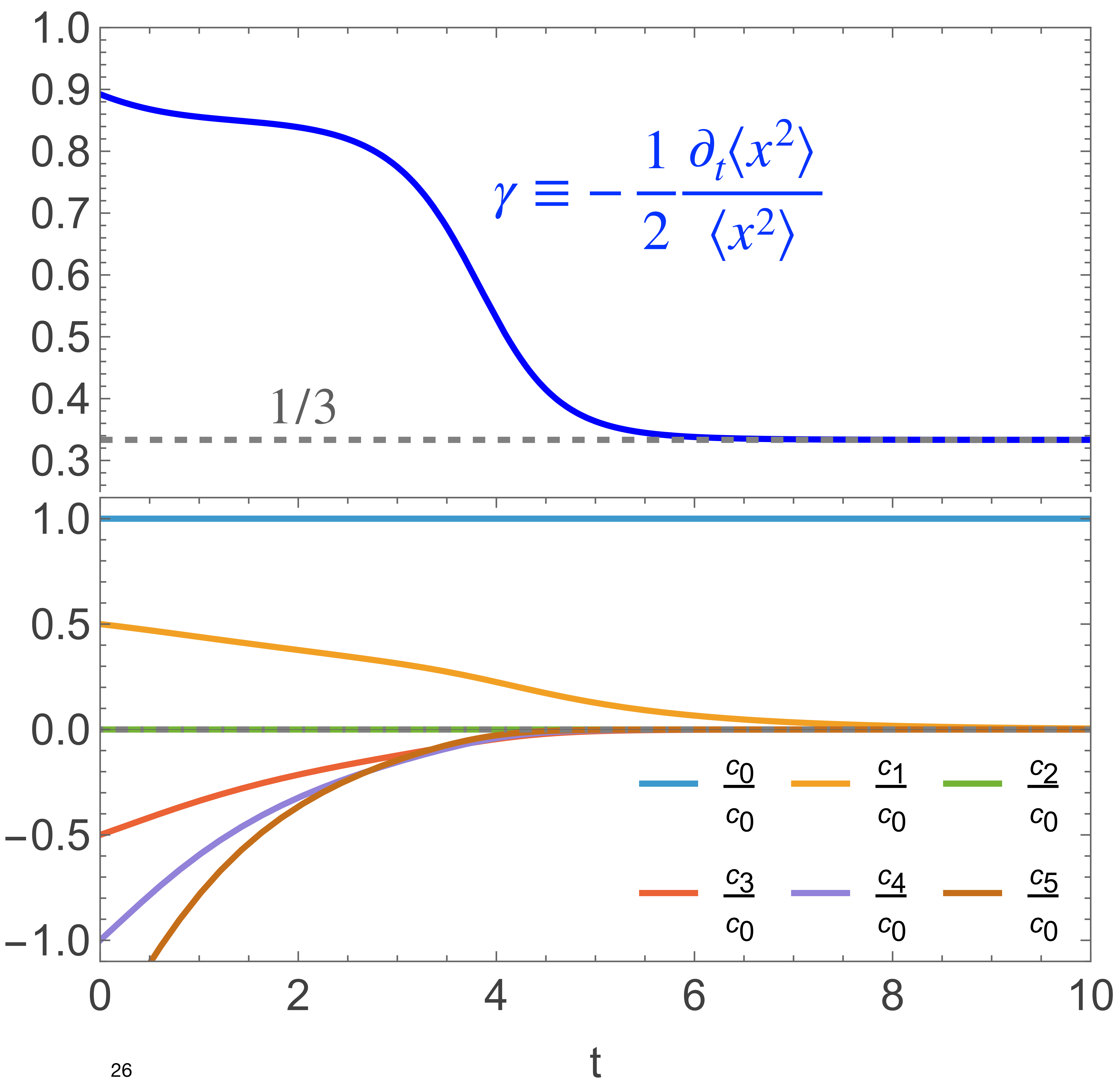

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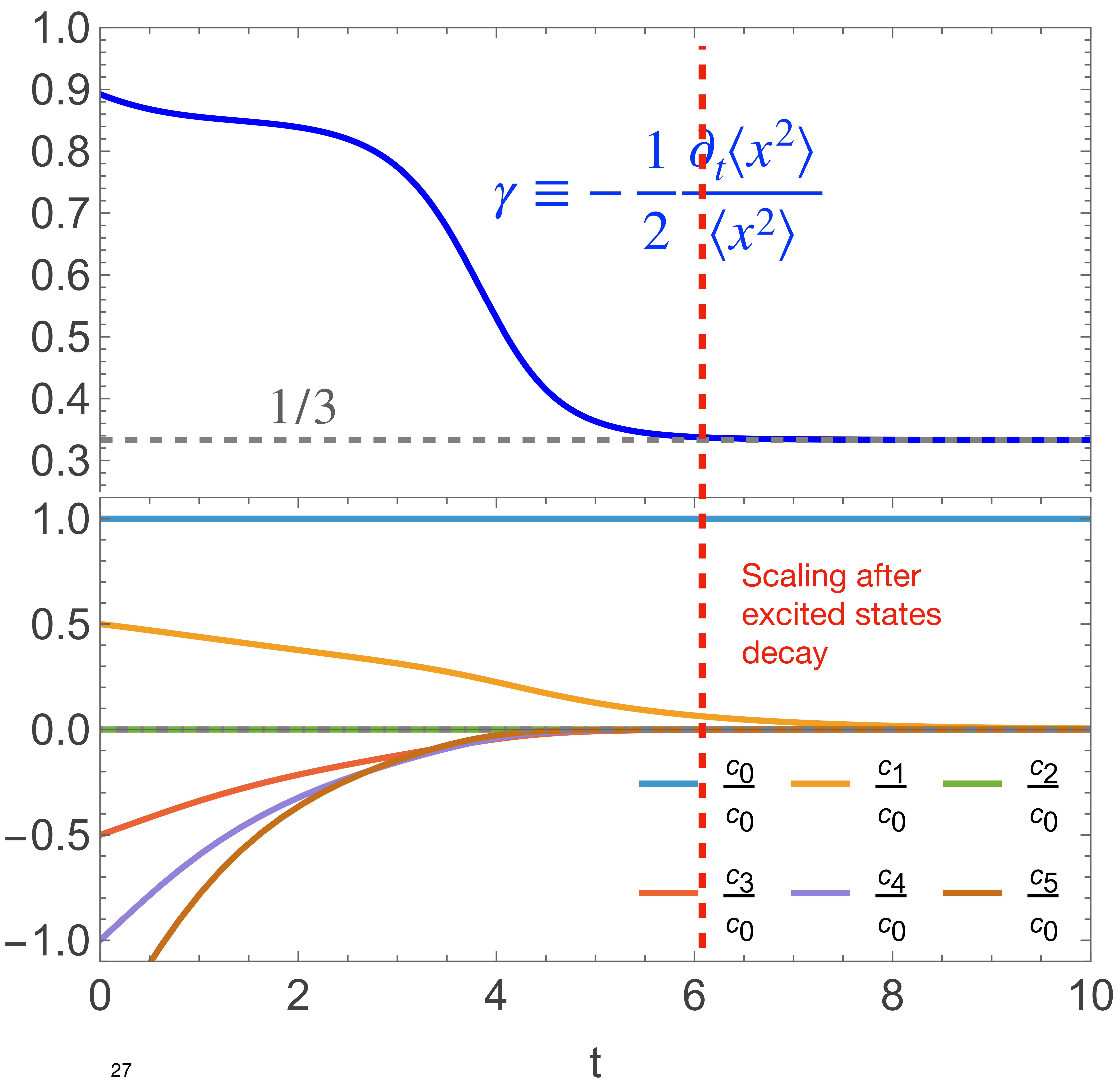

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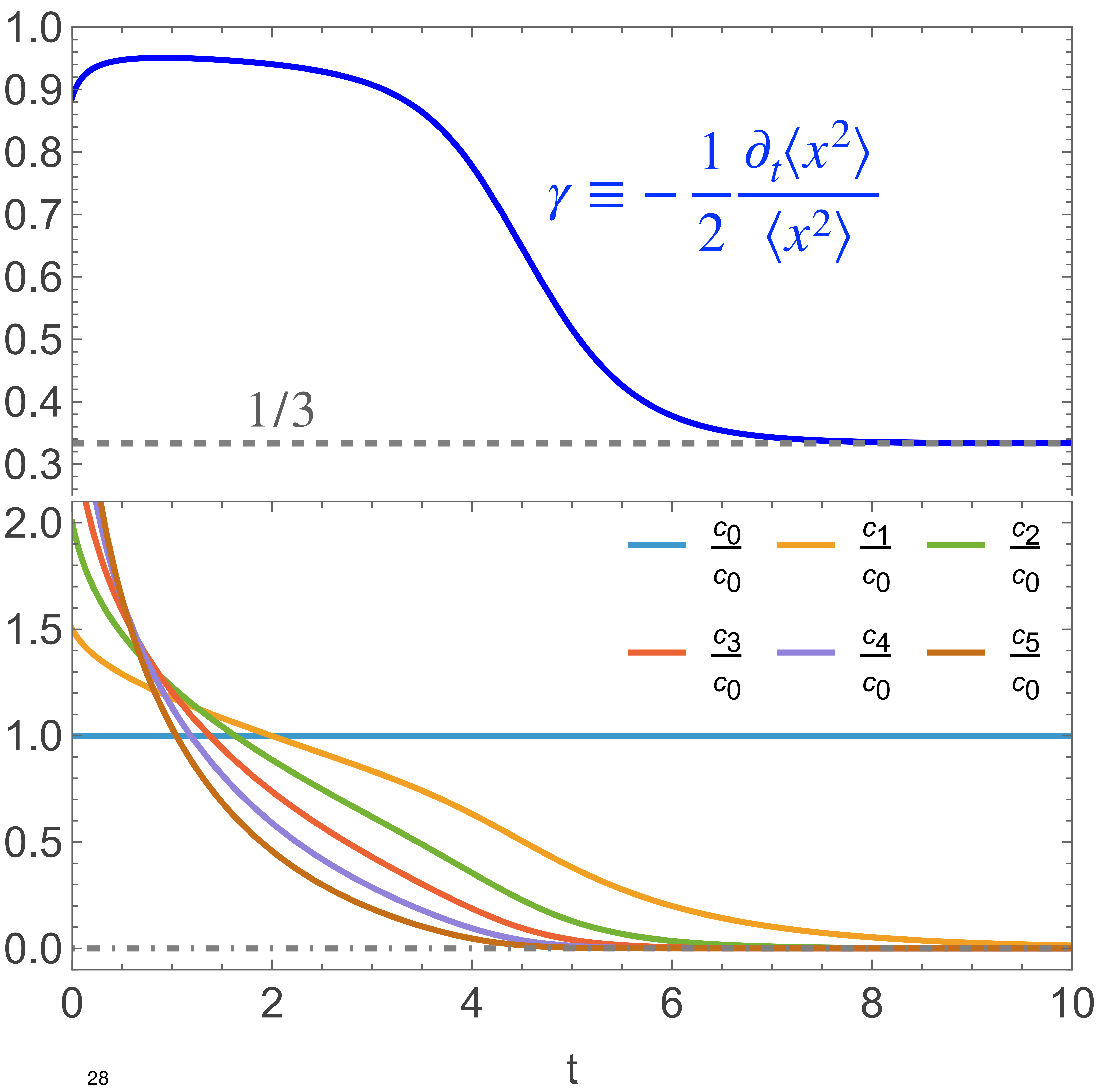

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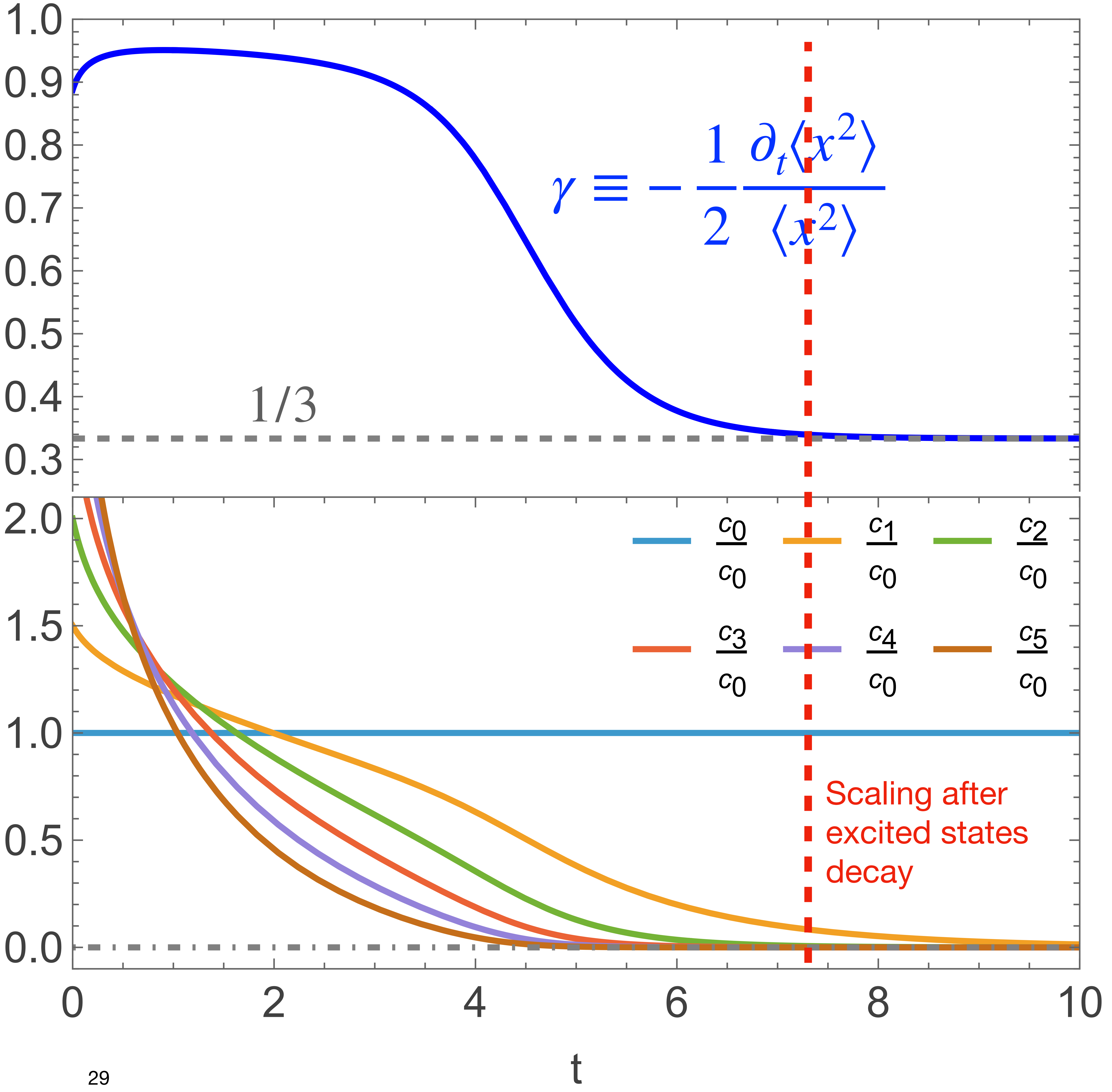

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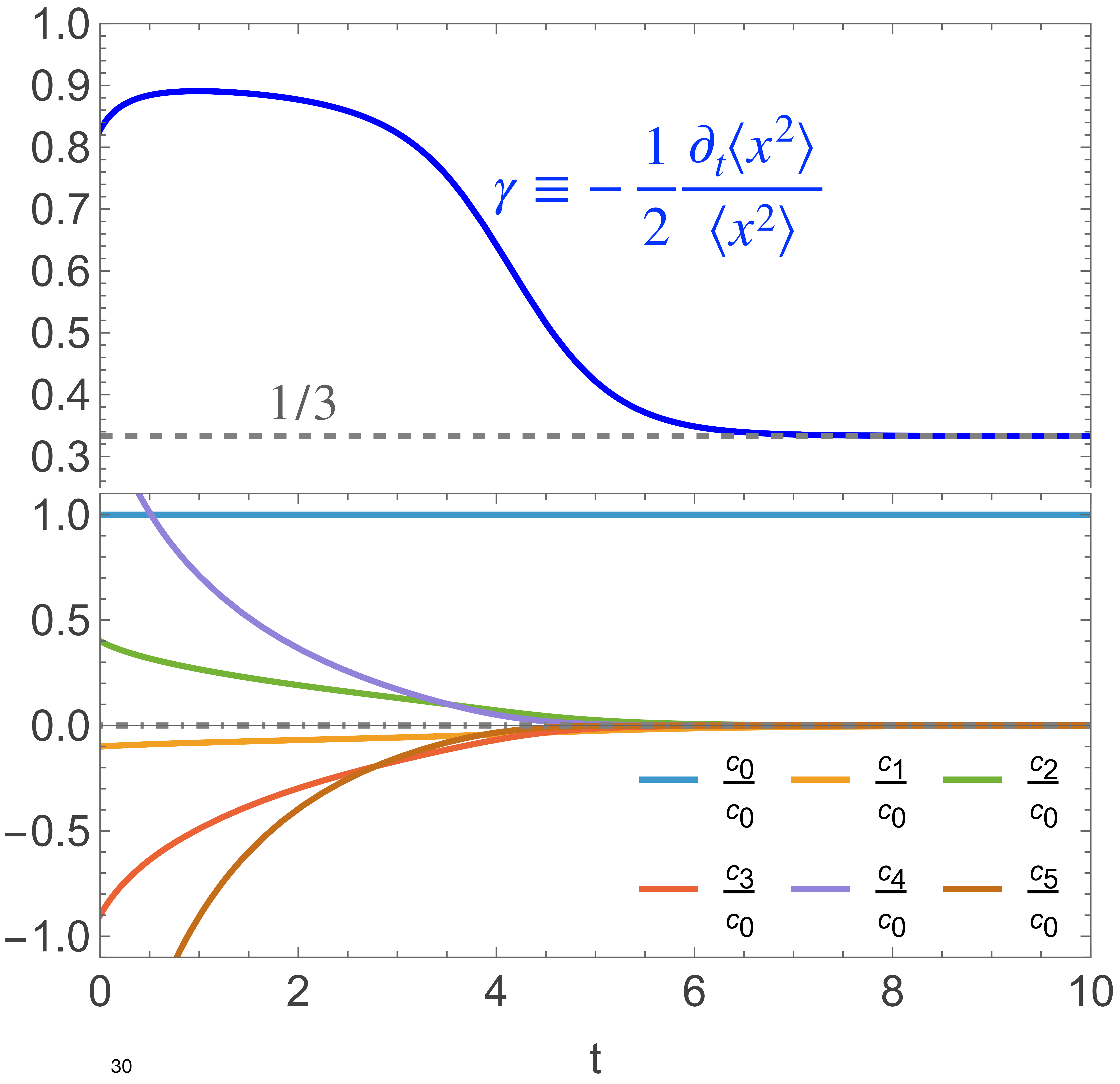

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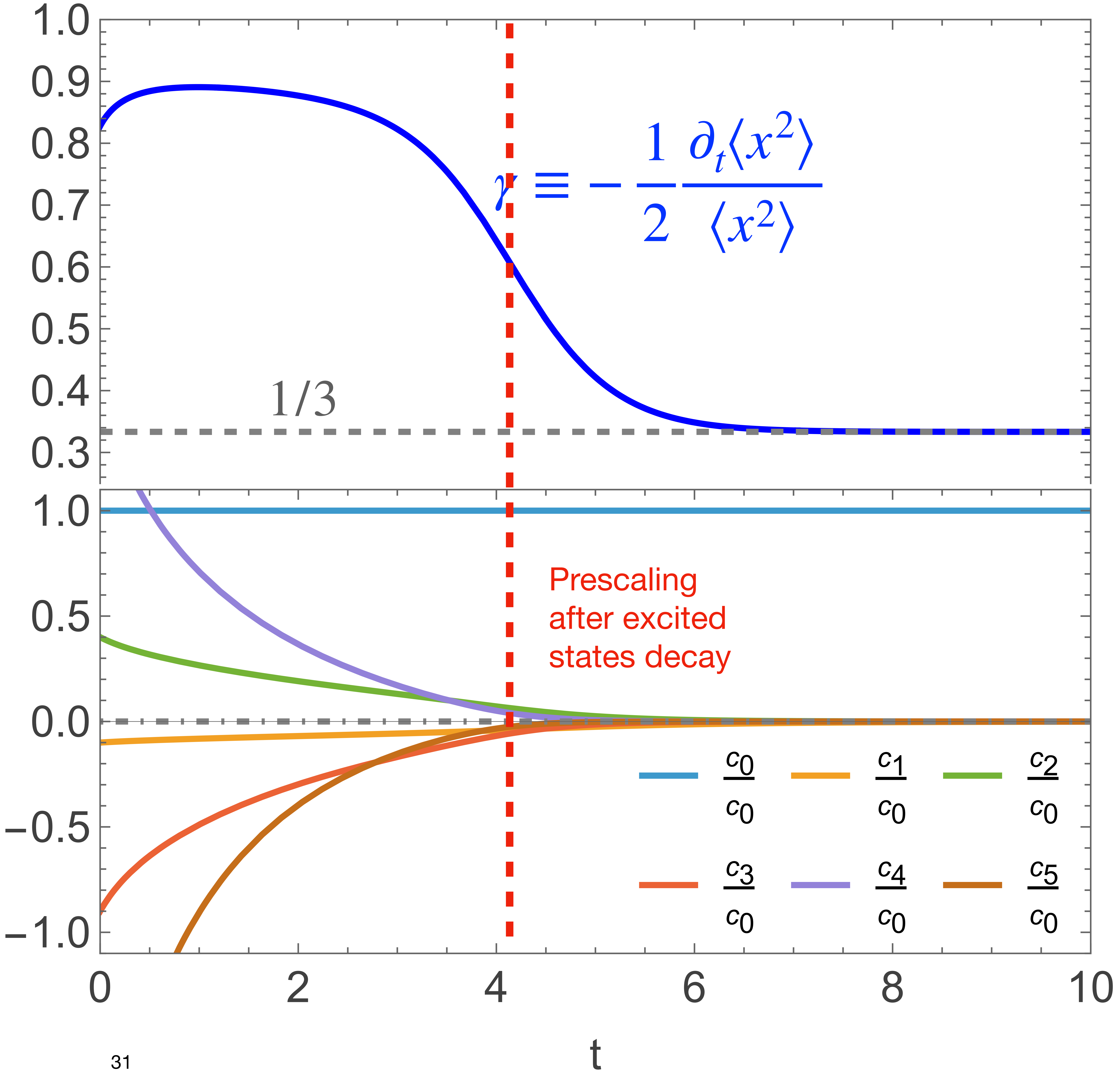

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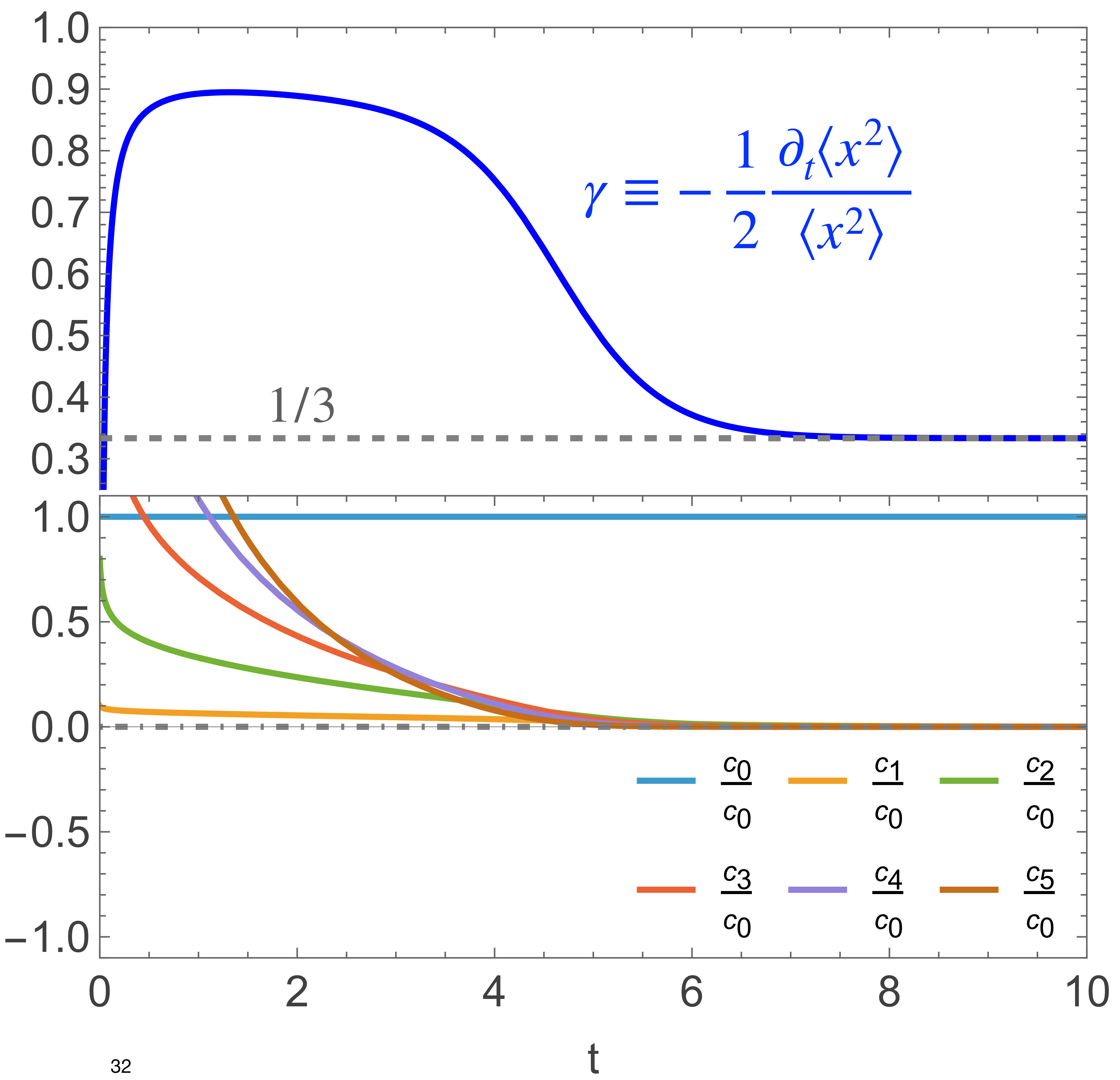
$$\xi = x/B(t)\,, \quad f(x,t) = A(t)\, w\big(x/B(t), t\big)$$

$$w(\xi,t) = \sum_{n=0}^{5} c_n(t)\,\mathrm{He}_n\big(\xi\big)\,\exp(-\xi^2/2)$$

$$c_n(t) = a_n\, e^{-\int_0^t E_n(t')dt'}$$

Scaling $\Longleftrightarrow$ unique $n$ s.t. $a_n \neq 0$

Note: if the system starts in the ground state, prescaling!



Solutions of $\frac{\partial f}{\partial t} = x\frac{\partial f}{\partial x} + D[f;t]\frac{\partial^2 f}{\partial x^2}$

# Example:

Solutions for

$$D[f;t] = e^t \int_x f^2$$

with

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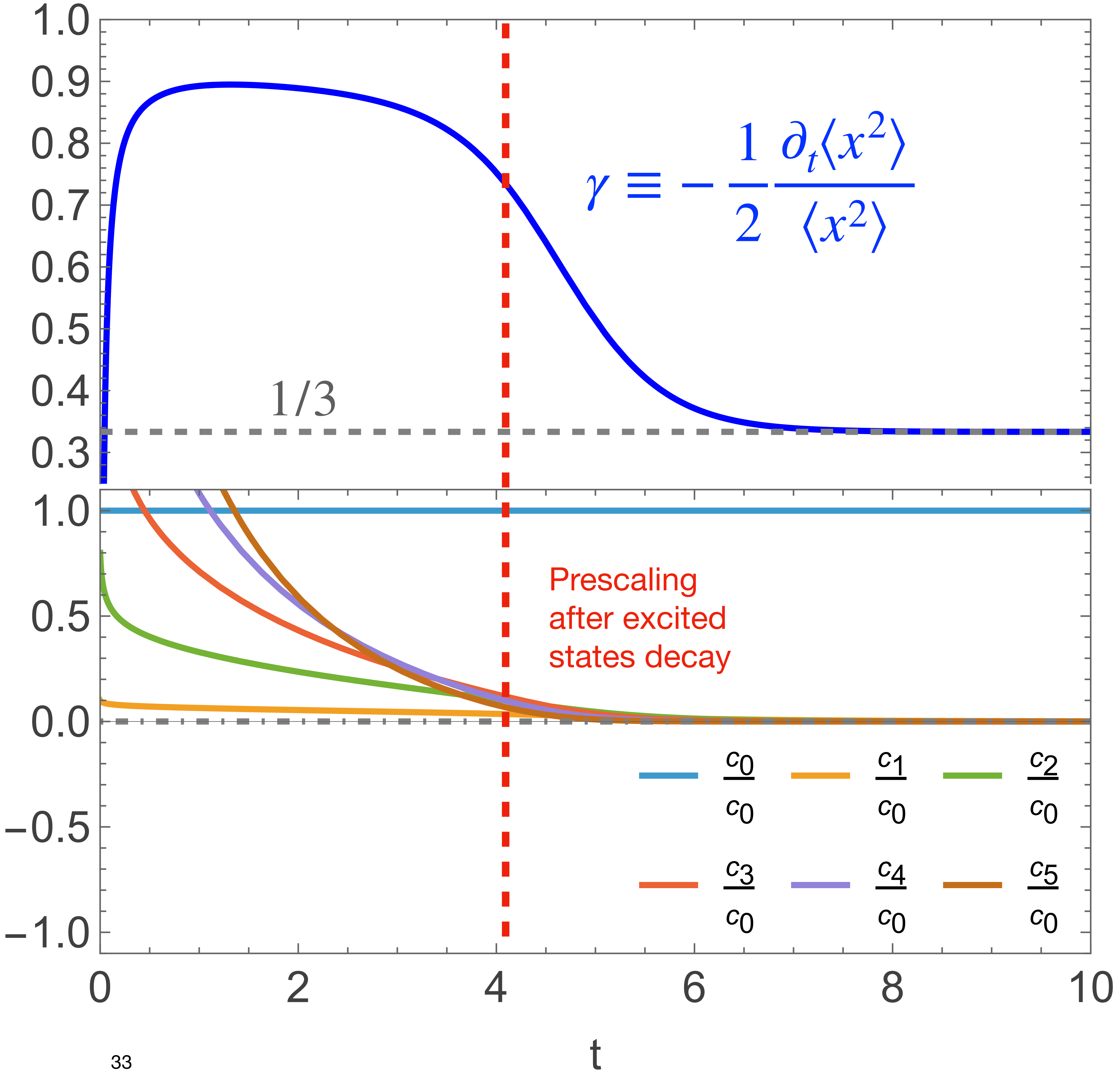

# What just happened?

- We started with a kinetic equation

$$\frac{\partial f}{\partial t} = -\, C[f]\,,$$

and we introduced $A(t)$, $B(t)$ and $w(\xi, t)$ such that

$$f(x,t) = A(t)\, w\Big(x/B(t), t\Big).$$

- We then wrote the kinetic equation as $\partial_t w = -\, \mathscr{H} w$ and found the spectrum of $\mathscr{H}$ by making a convenient choice for $A(t), B(t)$.

- What is special about this choice?

*It makes the eigenstates of $\mathscr{H}$ evolve adiabatically.*

# What just happened?

Conclusion:
If a scaling attractor is present, it can be identified as the ground state in the adiabatic ($\delta_A$-minimizing) frame.

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*It makes the eigenstates of $\mathscr{H}$ evolve adiabatically.*

# How general is this?

## what happens for more general collision kernels?

- For more realistic setups, we will not always be able to choose $A(t), B(t)$ such that the eigenstates of $\mathscr{H}$ are time-independent.
- The point of the AH framework is to provide a prescription to choose the frame in an “optimal” way. The desired features are:
  - a gapped and slowly varying spectrum, so that
  - the lowest energy state(s) can be identified as an “attractor” (surface).
- The proposal in 1910.00021, 2203.02427, 2405.17545, 2507.21232: define a measure of “adiabaticity” and derive equations for the “frame” variables by extremizing this quantity. I’ll come back to this later.

# When does prescaling happen?

## in this model

- If the system starts on the ground state, then prescaling takes place automatically.

- If the shape of the distribution function at the initial time is *not* the scaling form (i.e., if the “excited” states have nonzero occupancy), then two possibilities emerge:

    1. The excited states decay before $\beta$ approaches its fixed point value.

    2. The excited states decay as $\beta$ approaches its fixed point value.

- In practice, prescaling will always be approximate if excited states are present. How close it gets to being exact is determined by the size of the excited states’ initial conditions.

# 2.B. Application: A model of QCD EKT

at very early times in a weakly coupled, boost-invariant setup

$$\partial_\tau f - \frac{p_z}{\tau}\frac{\partial f}{\partial p_z} = 4\pi\,\alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f] I_a[f] \frac{\partial^2 f}{\partial p_z^2}$$

$$\Longleftrightarrow \quad \frac{\partial f}{\partial y} = x\frac{\partial f}{\partial x} + D[f;t]\frac{\partial^2 f}{\partial x^2} \qquad \text{with} \qquad y \equiv \ln(\tau/\tau_I)\,, \quad x \equiv p_z\,, \quad D[f] \equiv 4\pi\,\alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f] I_a[f]\,.$$

[6] A. Mazeliauskas, J. Berges, "Prescaling and far-from-equilibrium hydrodynamics in the quark-gluon plasma" Phys. Rev. Lett. 122, 122301 (2019)
see also [7] A. N. Mikheev, A. Mazeliauskas, J. Berges, "Stability analysis of nonthermal fixed points in longitudinally expanding kinetic theory" Phys. Rev. D **105**, 116025 (2022)

# Scaling exponents

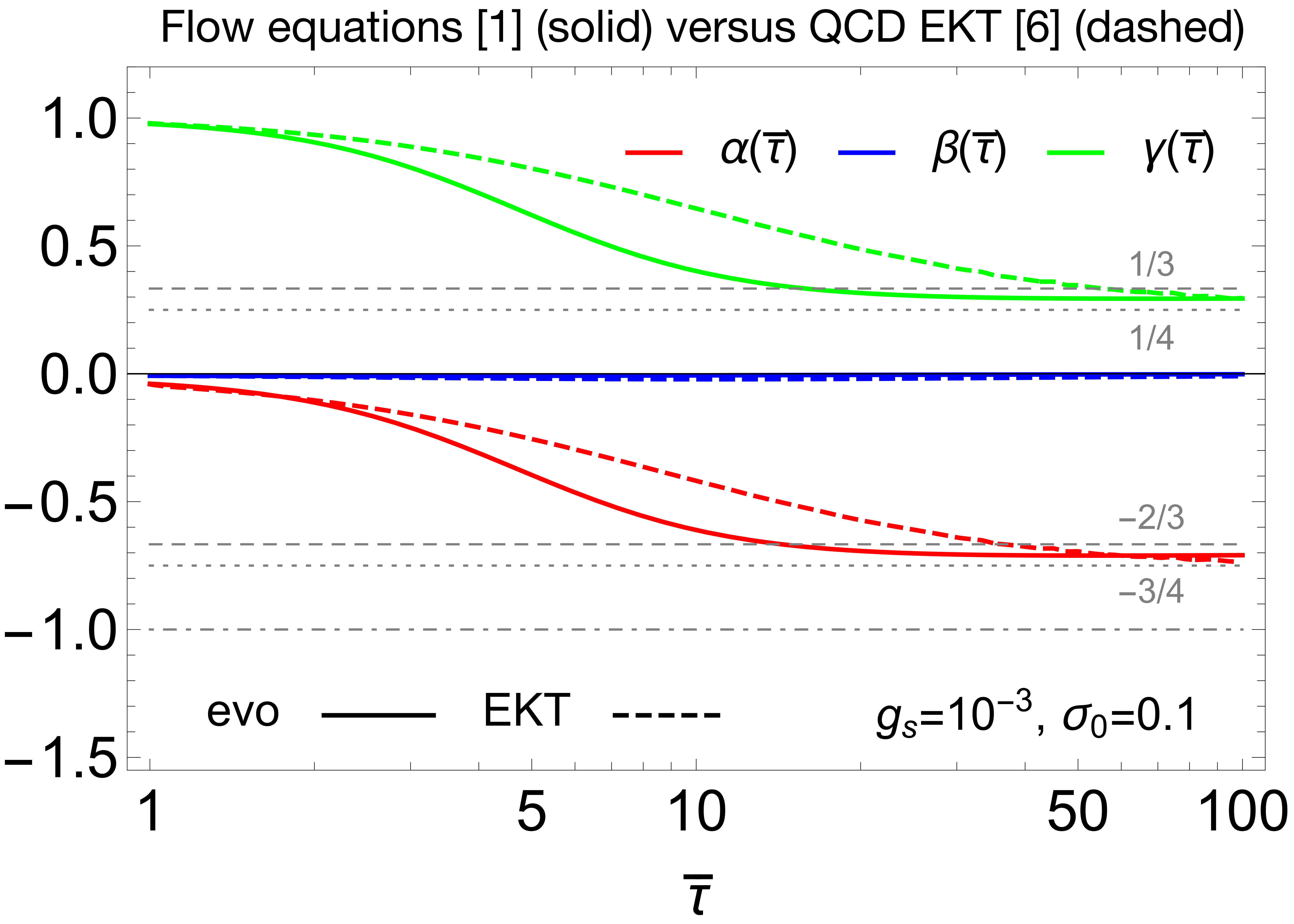
## comparison with QCD EKT

- We compare our results with those of [6], using the same initial condition:

$$f(\tau_I) = \frac{\sigma_0}{g_s^2} \exp\left( -\frac{p_\perp^2 + \xi^2 p_z^2}{Q_s^2} \right).$$

- In our description, for this initial condition we predict a deviation from the BMSS scaling exponents given by:

$$\delta\gamma \equiv \gamma - \frac{1}{3} = -\frac{1}{3 \ln\left(\frac{4\pi\tau}{N_c \tau_I \sigma_0}\right)}$$



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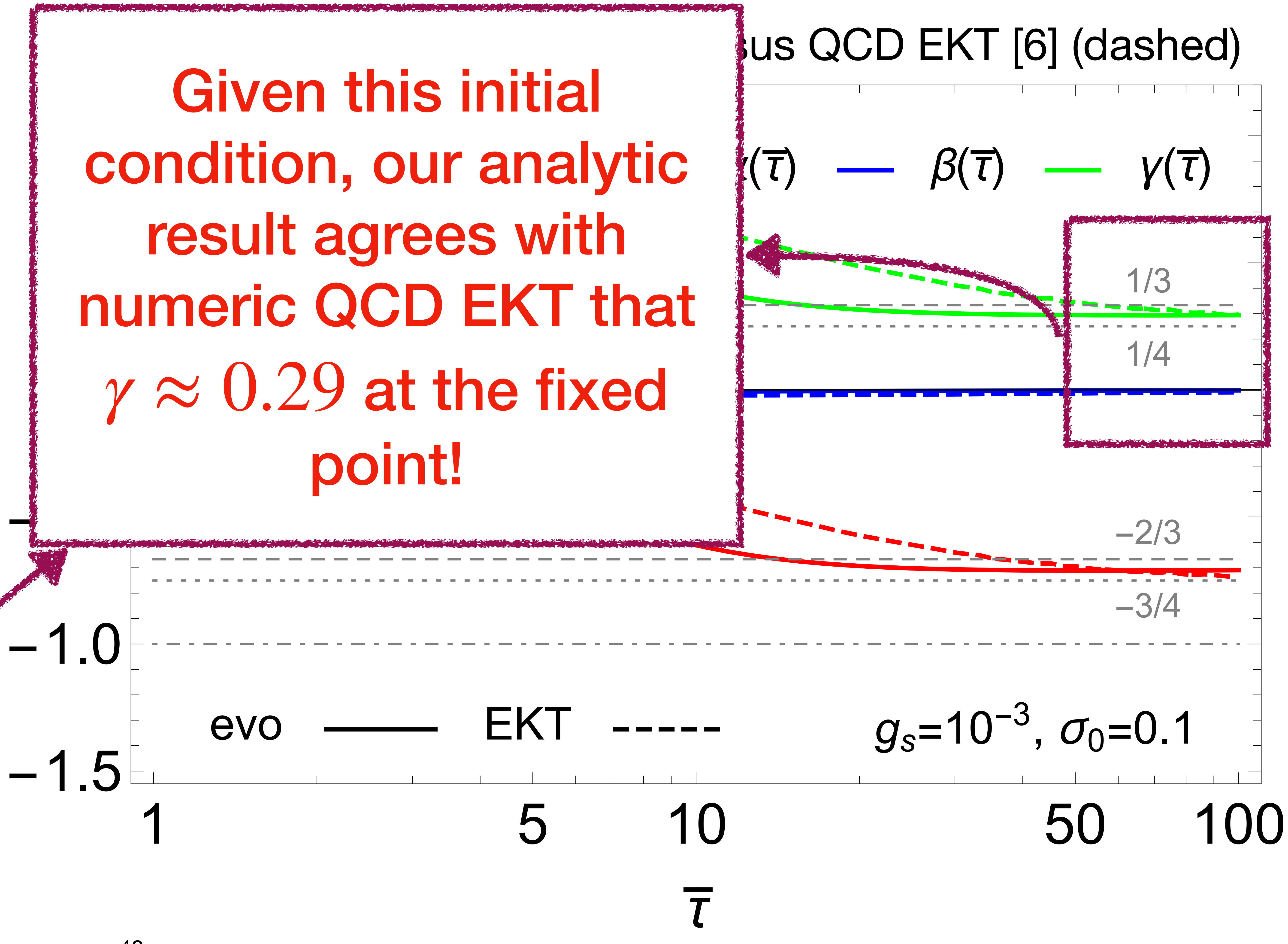


Given this initial condition, our analytic result agrees with numeric QCD EKT that $\gamma \approx 0.29$ at the fixed point!

Typical time evolution of the gluon occupation number in a weakly-coupled Bjorken-expanding plasma

# Takeaway message from this example

## what adiabaticity can do for you

- Essentially, what we have done is to rewrite $f(x,t)$ as

$$f(x,t) = \sum_{n=0}^{\infty} a_n(t)\,\phi_n(x,t)\ ,$$

*choosing* $\phi_n$ in such a way that the dynamics of $\{a_n\}_n$ is as simple as possible.

- The Adiabatic Hydrodynamization framework provides a way to identify the "optimal" choice for $\phi_n(x,t)$: they are the instantaneous eigenstates of the time evolution operator in the frame that gives the most adiabatic description.

# 3. Bottom-up thermalization

Adiabaticity beyond scaling

[2] K. Rajagopal, B. Scheihing-Hitschfeld, and R. Steinhorst, *Adiabatic Hydrodynamization and the Emergence of Attractors: a Unified Description of Hydrodynamization in Kinetic Theory*, arXiv:2405.17545.
[3] K. Rajagopal, B. Scheihing-Hitschfeld, and R. Steinhorst, *Attractors Without Scaling: Adiabatic Hydrodynamization With and Without Inelastic Scattering*, arXiv:2507.21232.

# Breakdown of the scaling regime

## a necessary stage in the hydrodynamization process

- In the previous discussion, a distribution function $f$ of the form

$$f = A(y)\, w\left(\frac{p_\perp}{B(y)}, \frac{p_z}{C(y)}\right), \text{with } w(\zeta, \xi) = \exp[-(\zeta^2 + \xi^2)/2]$$

is the instantaneous ground state that explains an initial stage of memory loss.

- However, at late times in a locally equilibrated expanding system

$$f = w\left(\frac{p}{T(y)}\right), \text{with } w(\chi) = [\exp(\chi) - s]^{-1},\ s \in \{-1,0,1\},$$

where the different values of $s$ correspond to fermions, classical particles, and bosons, respectively.

# A more complete model of QCD EKT

## including number-changing processes

… and I will drop Bose enhancement

see also Xiaojian Du's talk Tue 16:00

- In the previous discussion, we omitted the 1 <—> 2 terms in

$$\frac{\partial f}{\partial \tau} - \frac{p_z}{\tau}\frac{\partial f}{\partial p_z} = -\mathscr{C}_{1\leftrightarrow 2}[f] - \mathscr{C}_{2\leftrightarrow 2}[f]\,.$$

- For the 2 <—> 2 part, we will use the diffusion approximation:

$$\mathscr{C}_{2\leftrightarrow 2}[f] = -4\pi\,\alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f]\left[I_a[f]\,\nabla_{\mathbf{p}}^2 f + I_b[f]\,\nabla_{\mathbf{p}}\cdot\left(\hat{p}f\right)\right]$$

- For the 1 <—> 2 part,

$$\mathscr{C}_{1\leftrightarrow 2}[f] = -8\pi\alpha_s^2 N_c^2\sqrt{\frac{I_a[f]\ell_{\mathrm{Cb}}[f]}{2\pi^3 p}}\,[1-f(p=0)]\left(\frac{7}{2}+\mathbf{p}\cdot\nabla_{\mathbf{p}}\right)f$$

# Adiabaticity beyond scaling

## how to choose a frame with adiabatic ground state evolution

- The description in the previous discussion may be cast as an expansion

$$f(p_\perp = \zeta B(\tau), p_z = \xi C(\tau), \tau) = \sum_{i,j} c_{ij}(\tau)\, P_{ij}(\zeta, \xi)\ \exp\{-\left(\xi^2 + \zeta^2\right)/2\},$$

where $P_{ij}$ is a polynomial of degree $i$ in $\zeta$ and $j$ in $\xi$. This, by construction, is well-adapted to describe the ground state at early times. It is, in fact, the eigenbasis

- To accommodate the transition to a hydrodynamic state, we write a new basis

$$f(p = \chi D(\tau), u, \tau) = \sum_{n,l} c_{nl}(\tau)\, P_{nl}(\chi, u; r(\tau))\ \exp\{-\left(u^2 r^2(\tau)/2 + \chi\right)\},$$

*Not* the eigenbasis (but hopefully close) — also, *not* scaling

where we introduced a new time-dependent variable $r(\tau)$ and $u \equiv p_z/p = \cos\theta$. We define $\mathscr{H}$ as the operator that evolves the coefficients $c_{nl}$ .

# Results

**2507.21232**

What comes next is the result of:

1. Solving the dynamics numerically
2. Calculating the eigenvalues and eigenstate occupations
3. Diagnose memory loss and adiabaticity

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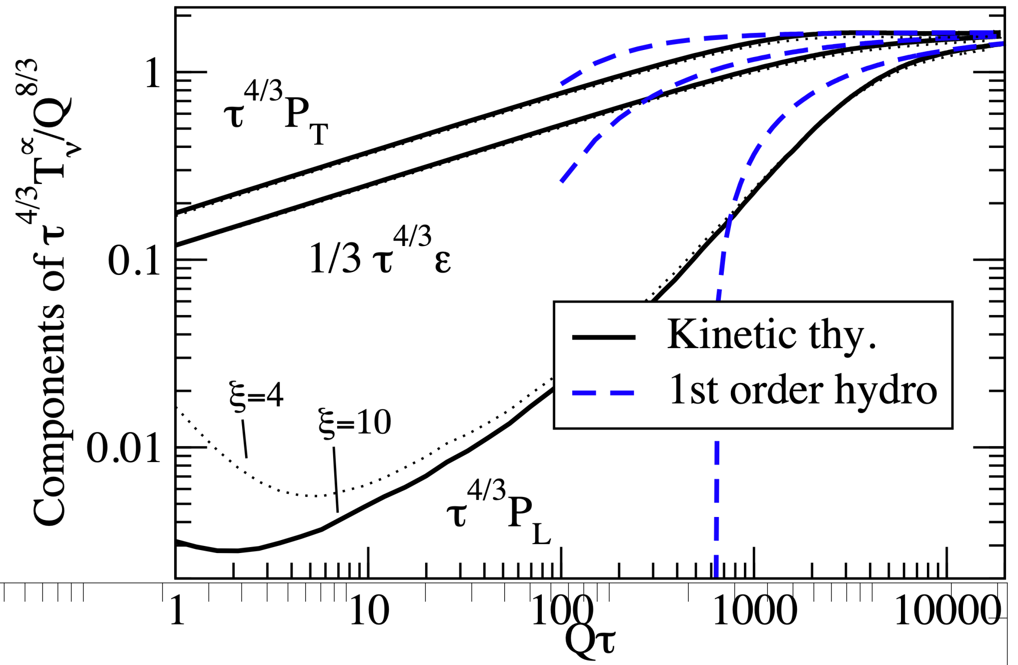
# Adiabatic Hydrodynamization

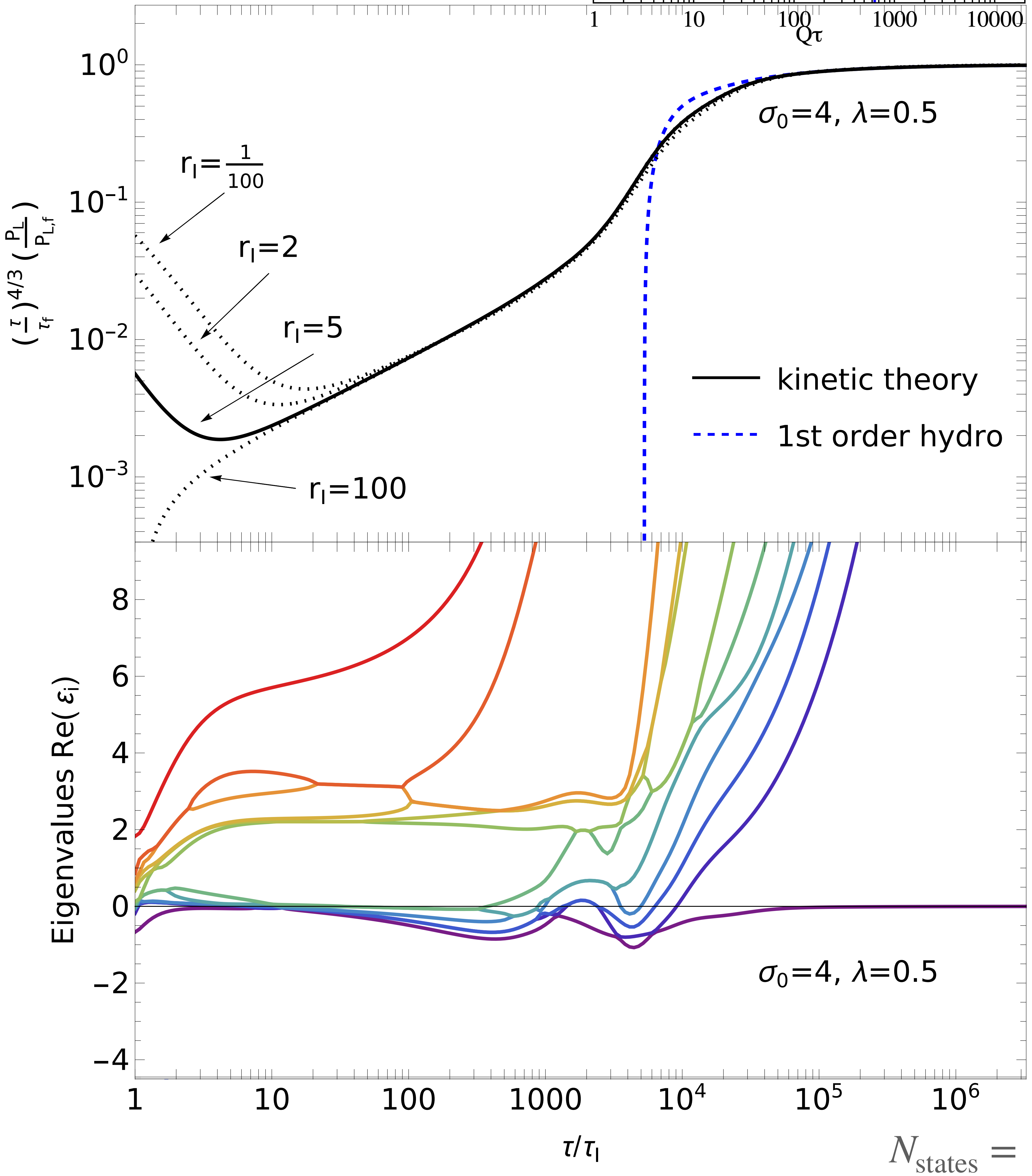
## the stages of the bottom-up scenario

At the initial time,

$$|\psi\rangle = \sum_n a_n(\tau) e^{-\int^\tau E_n(\tau')d\tau'} |n(\tau)\rangle$$

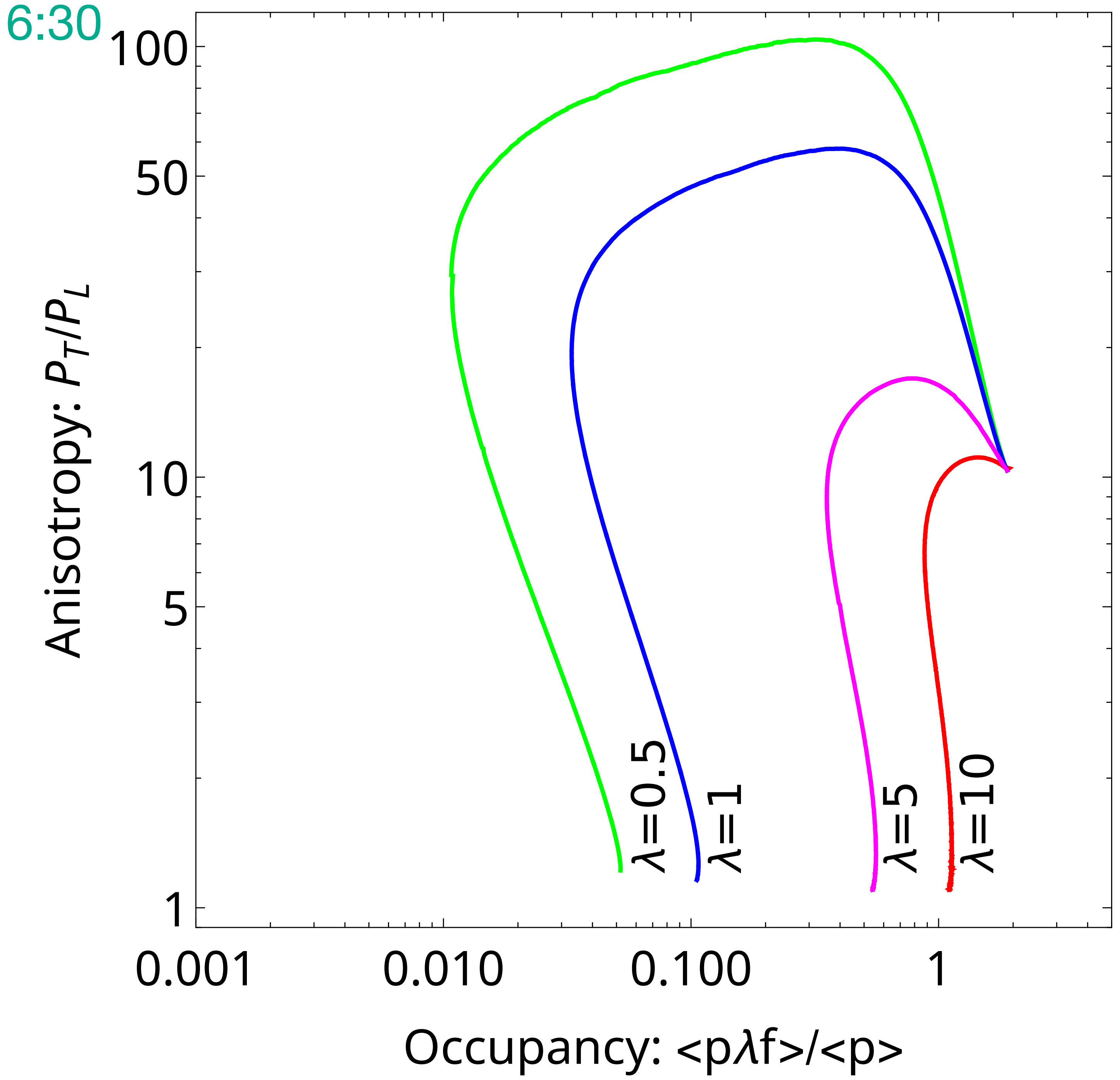
arXiv:2507.2132 AH framework with BEDA + (1 <—> 2)* - (Bose enhancement)
c.f. Kurkela & Zhu (2015) on the right $(\lambda = 1)$




see Florian's talk
Tue 16:30

QCD EKT (Figure by F. Lindenbauer)
*PLB* 852 (2024) 138623

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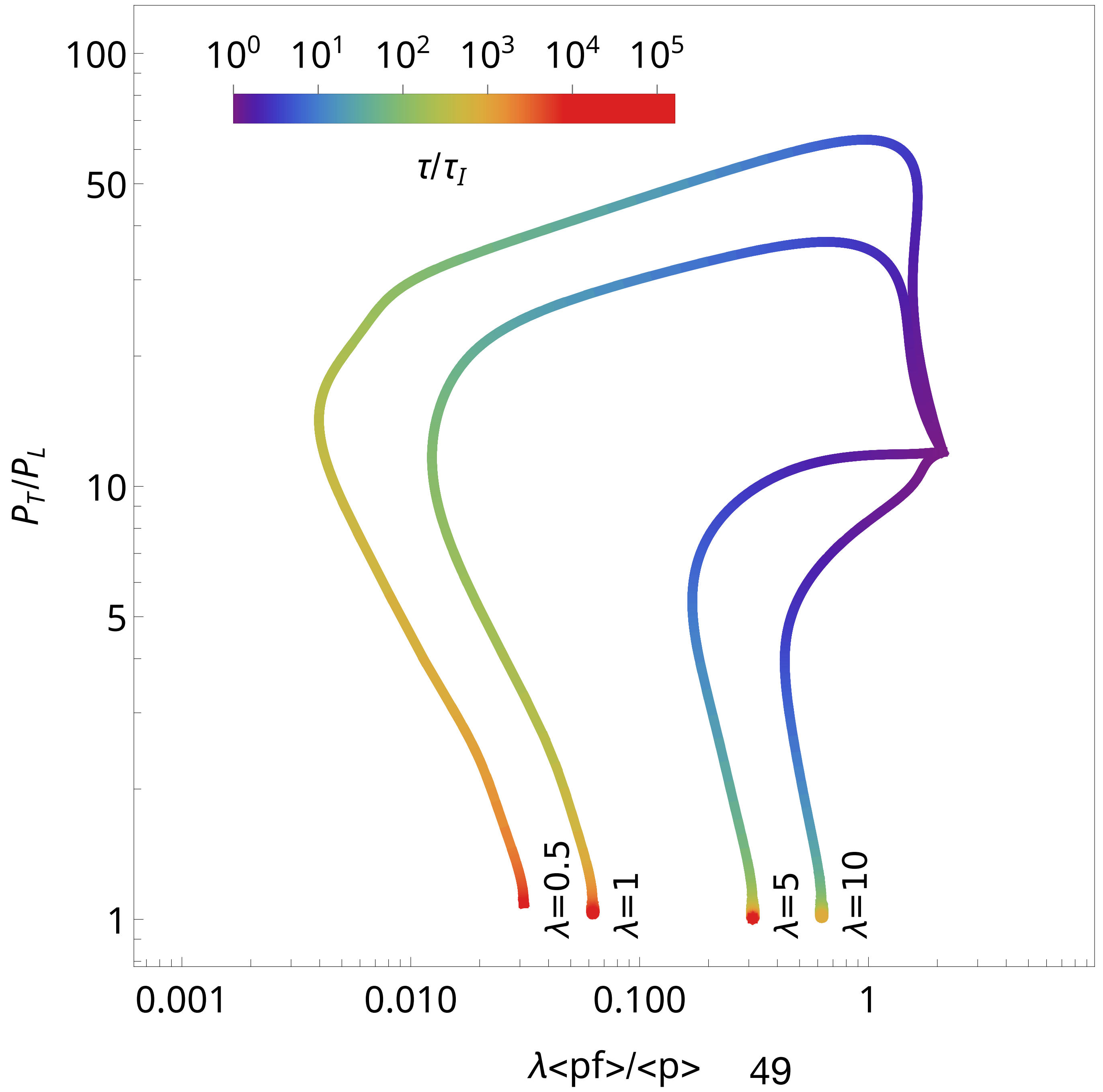




$N_{\text{states}} = 12$

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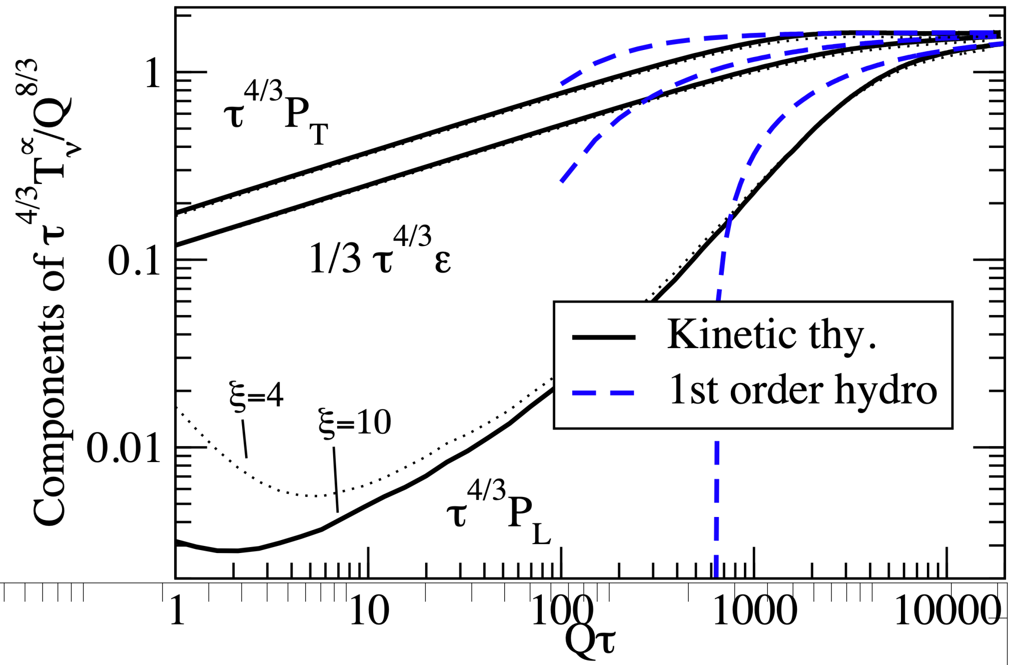


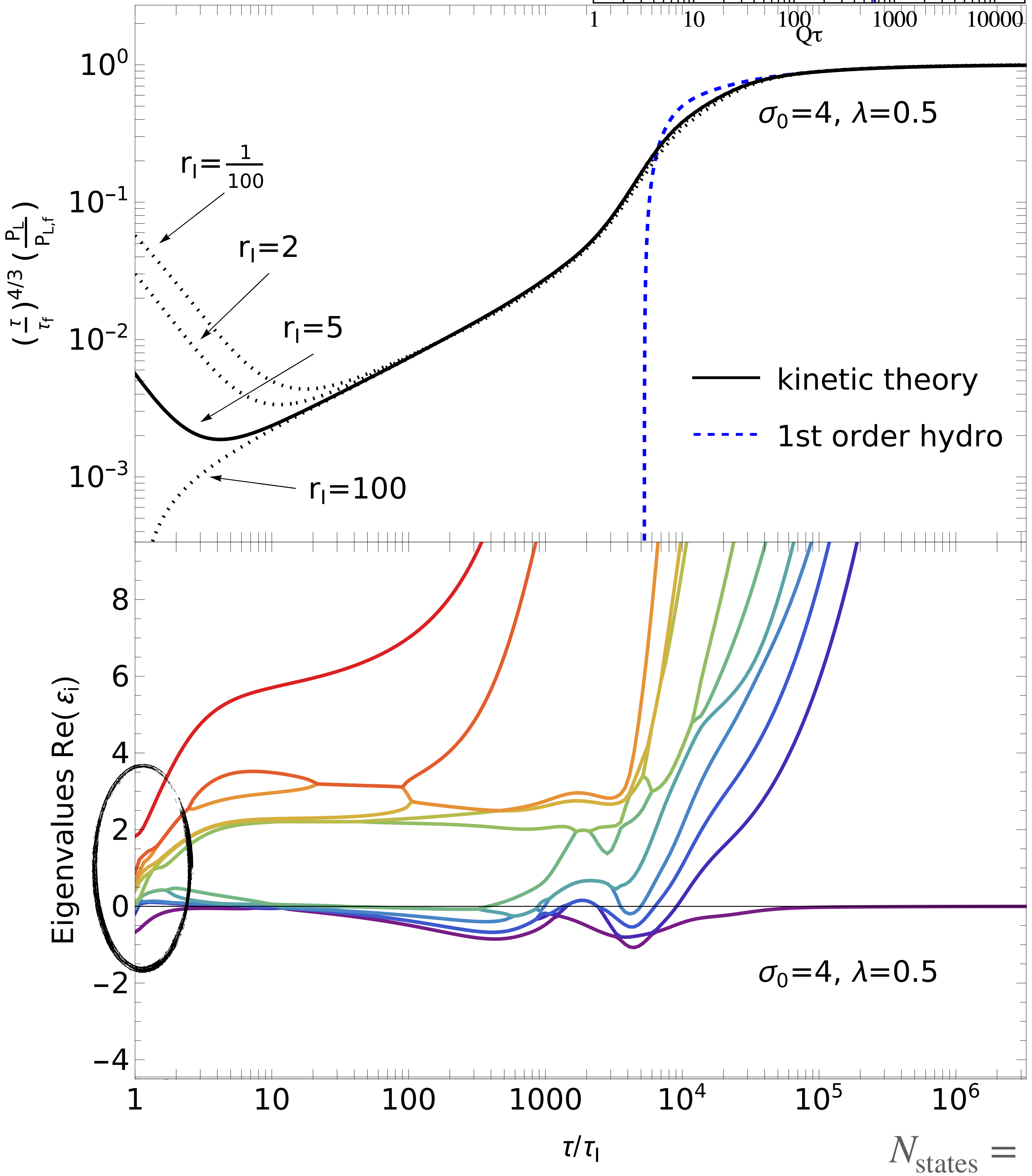


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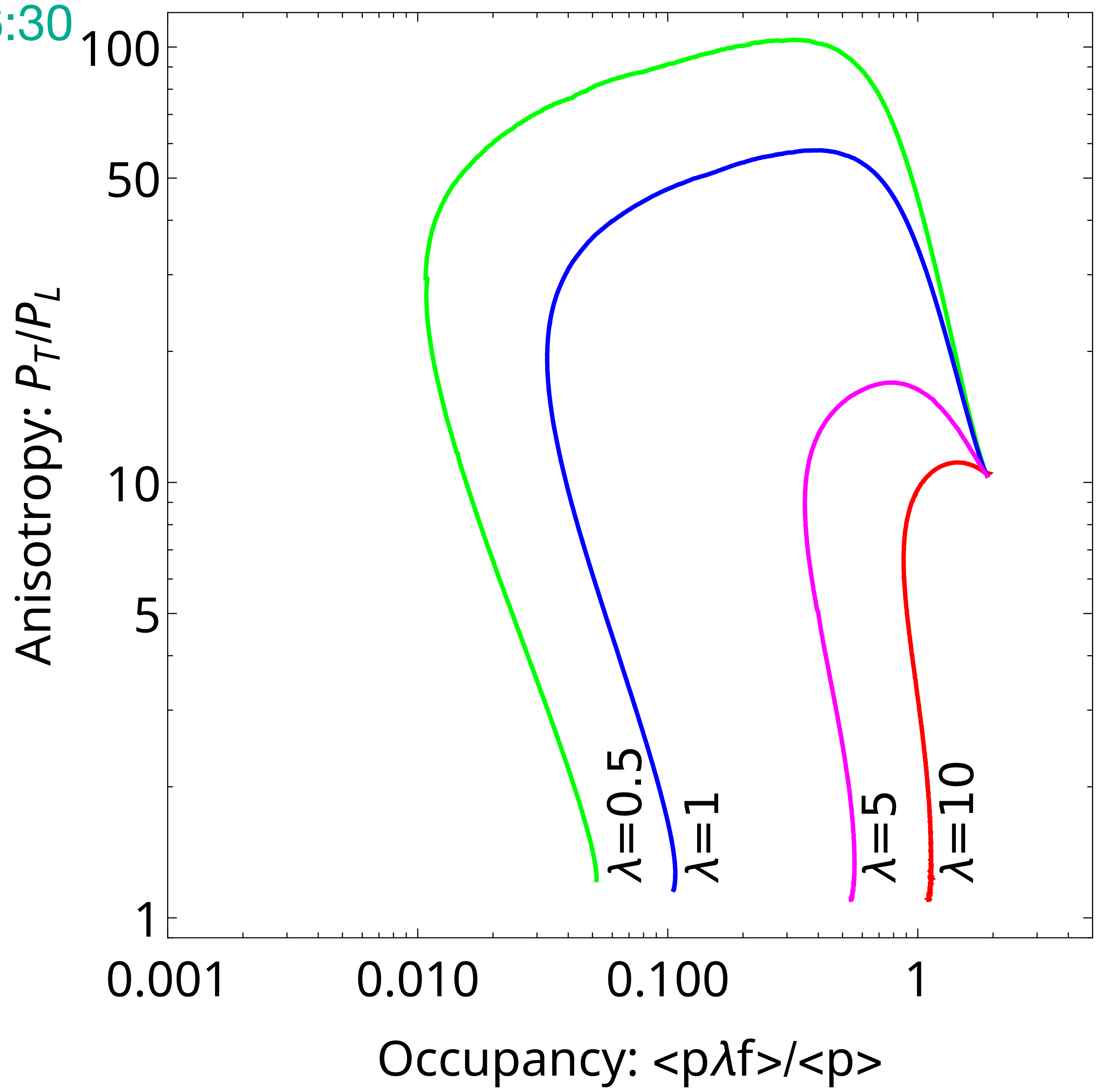


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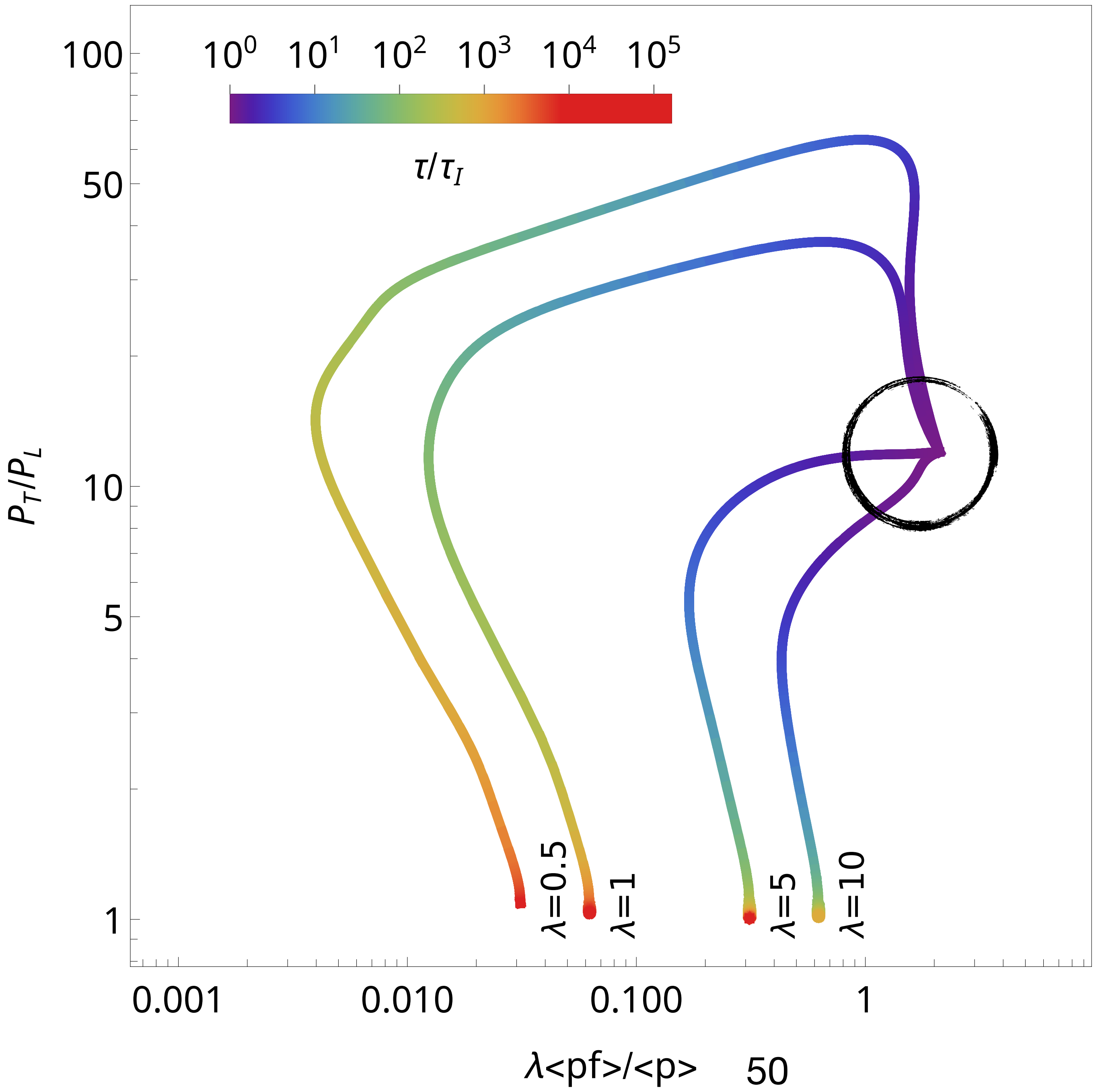




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# Adiabatic Hydrodynamization

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In the dilute regime,

$$|\psi\rangle = \sum_{n\in \text{g.s.b.}} a_n(\tau)e^{-\int^\tau E_n(\tau')d\tau'}|n(\tau)\rangle + \sum_{n\in \text{e.s.}} a_n(\tau)e^{-\int^\tau E_n(\tau')d\tau'}|n(\tau)\rangle$$

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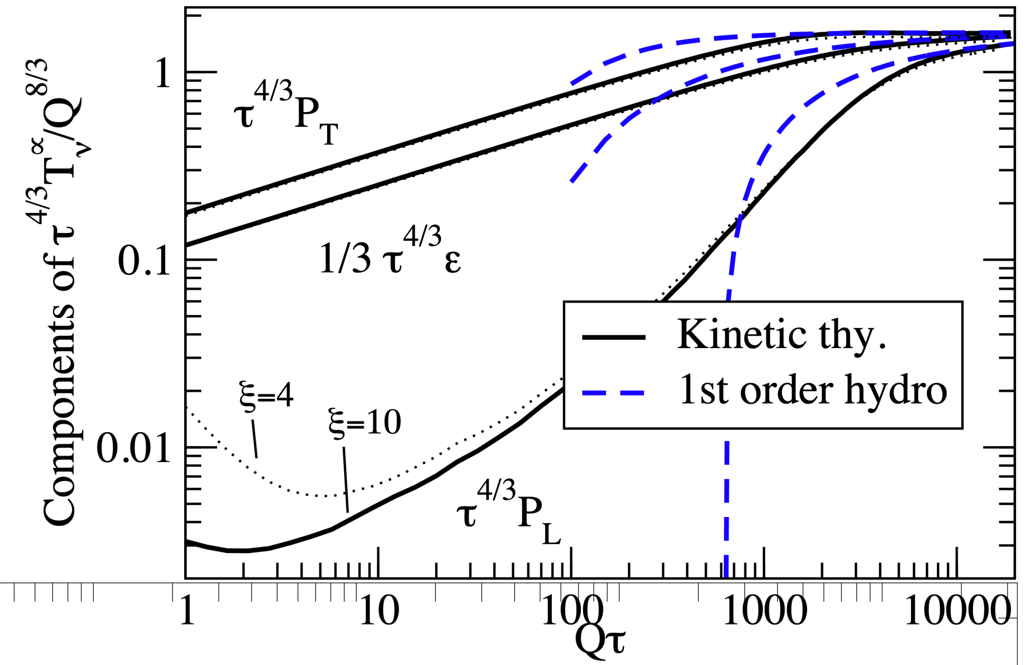


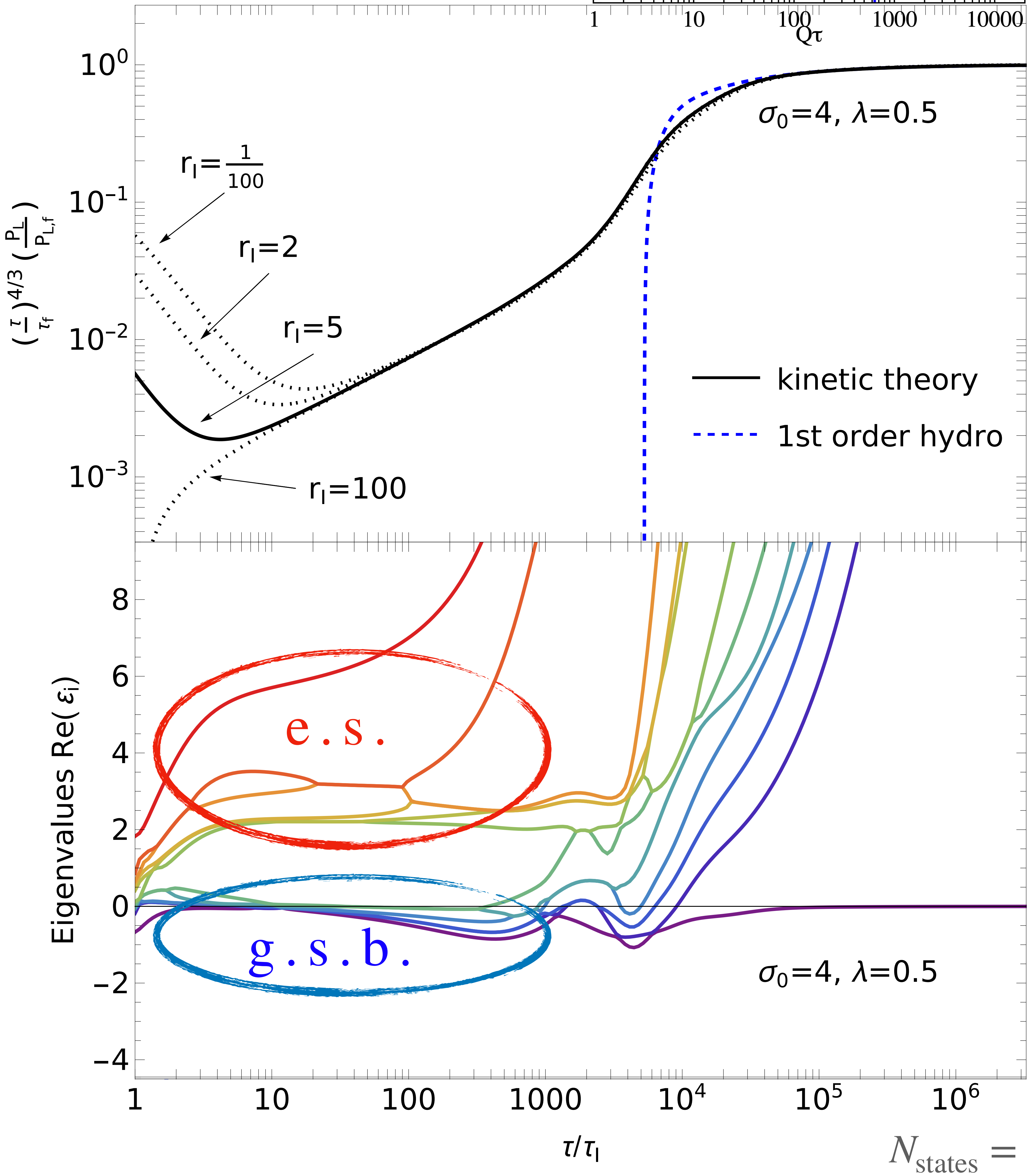


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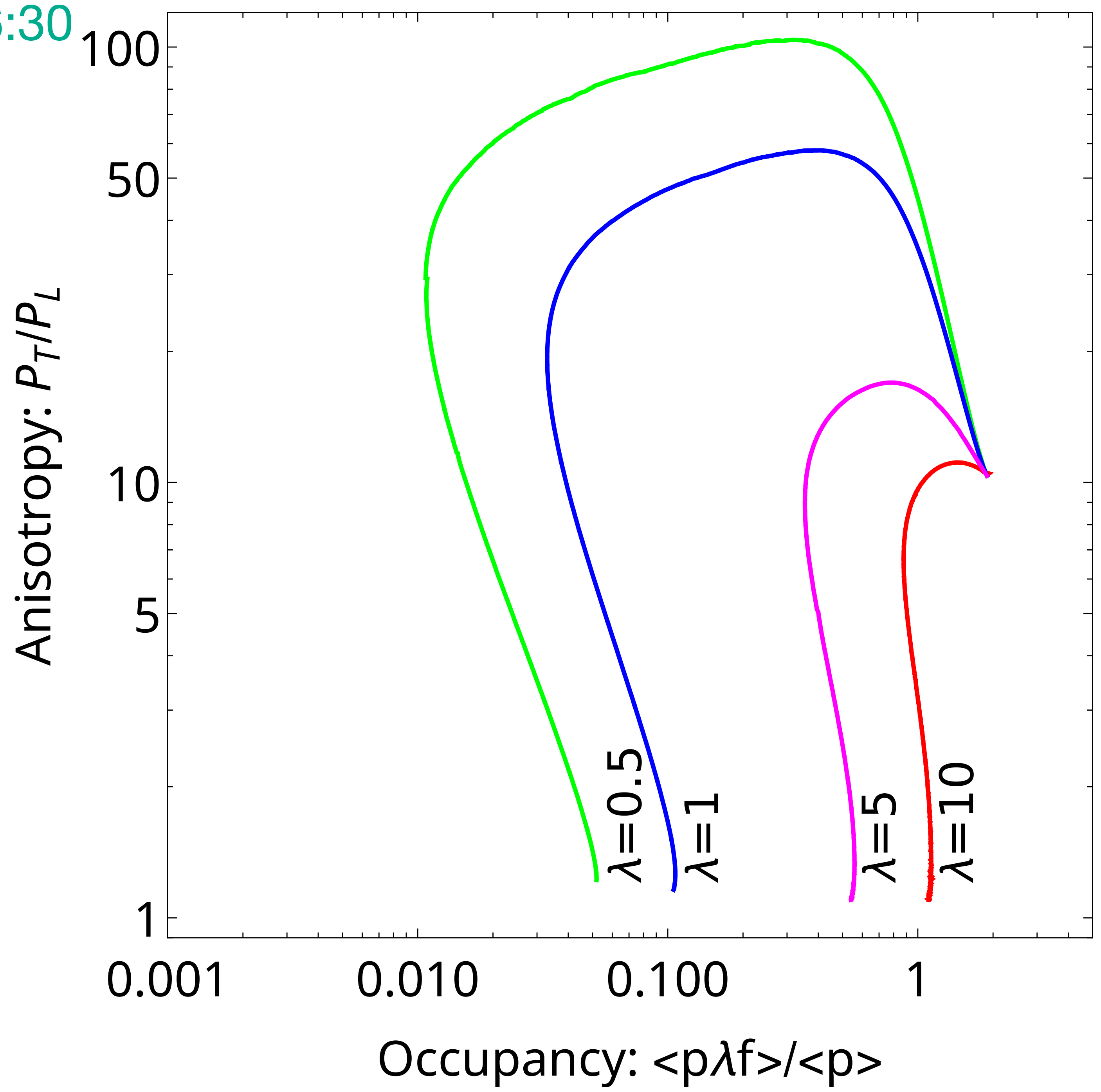


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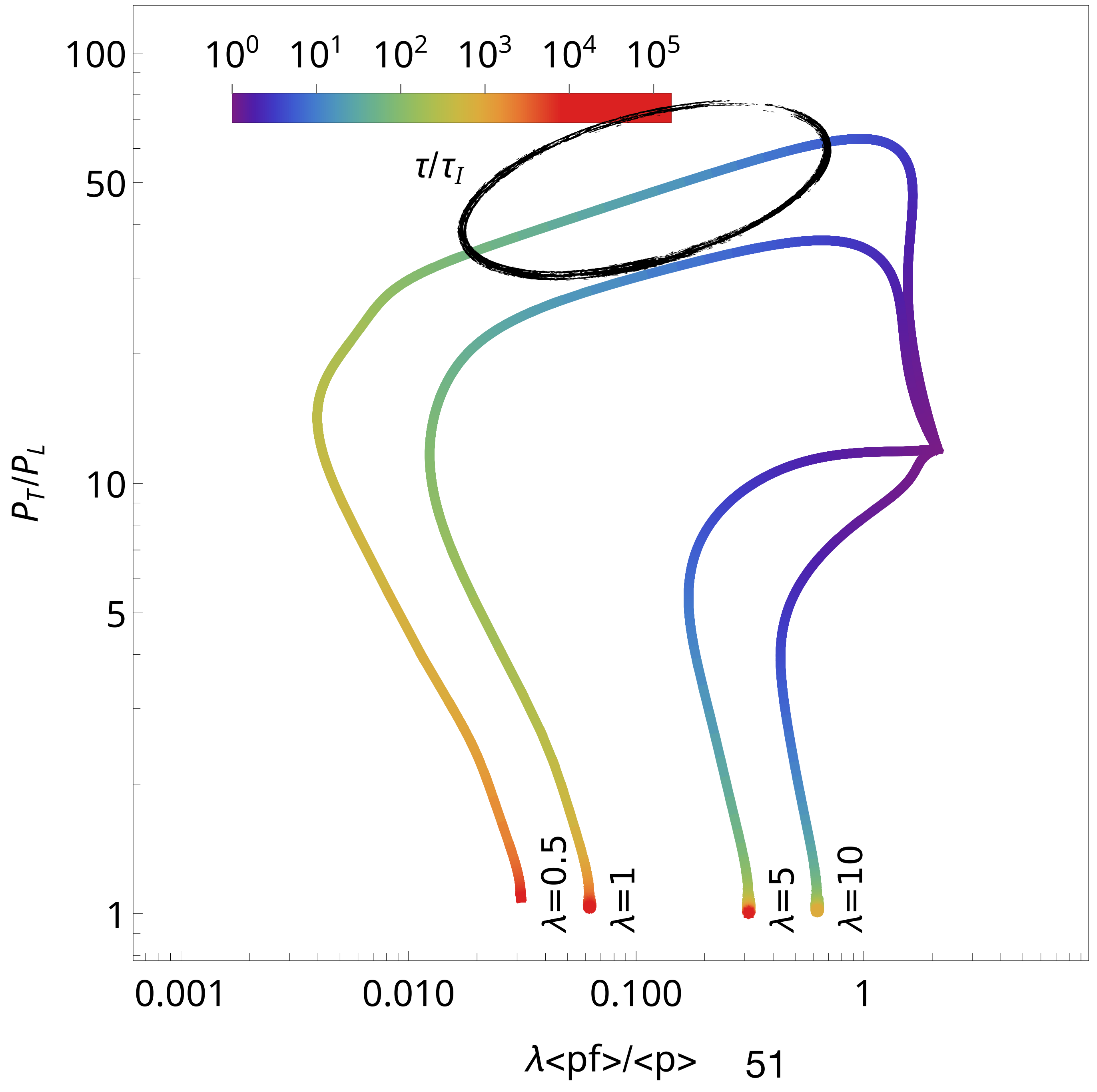




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see Florian's talk Tue 16:30

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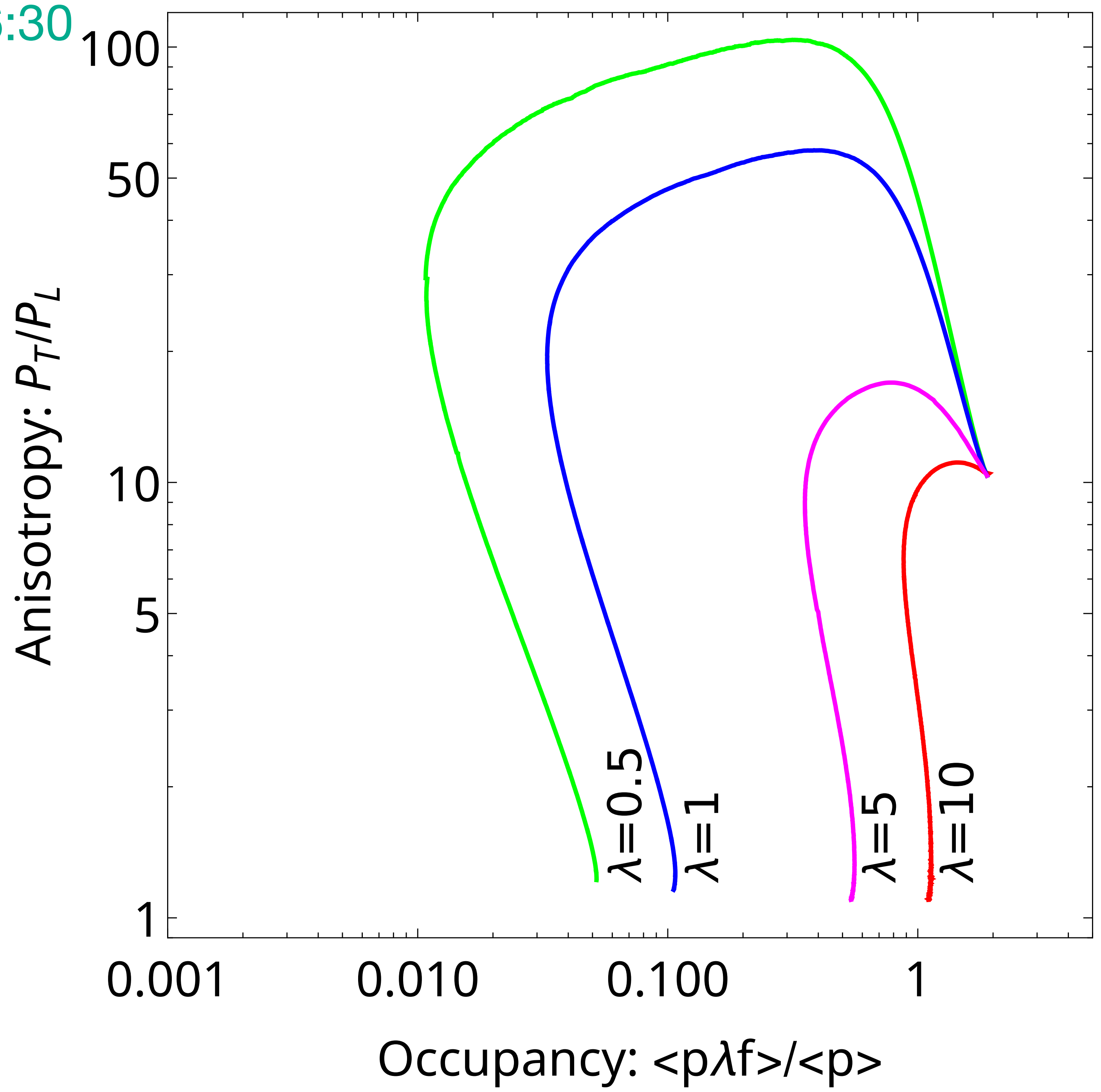


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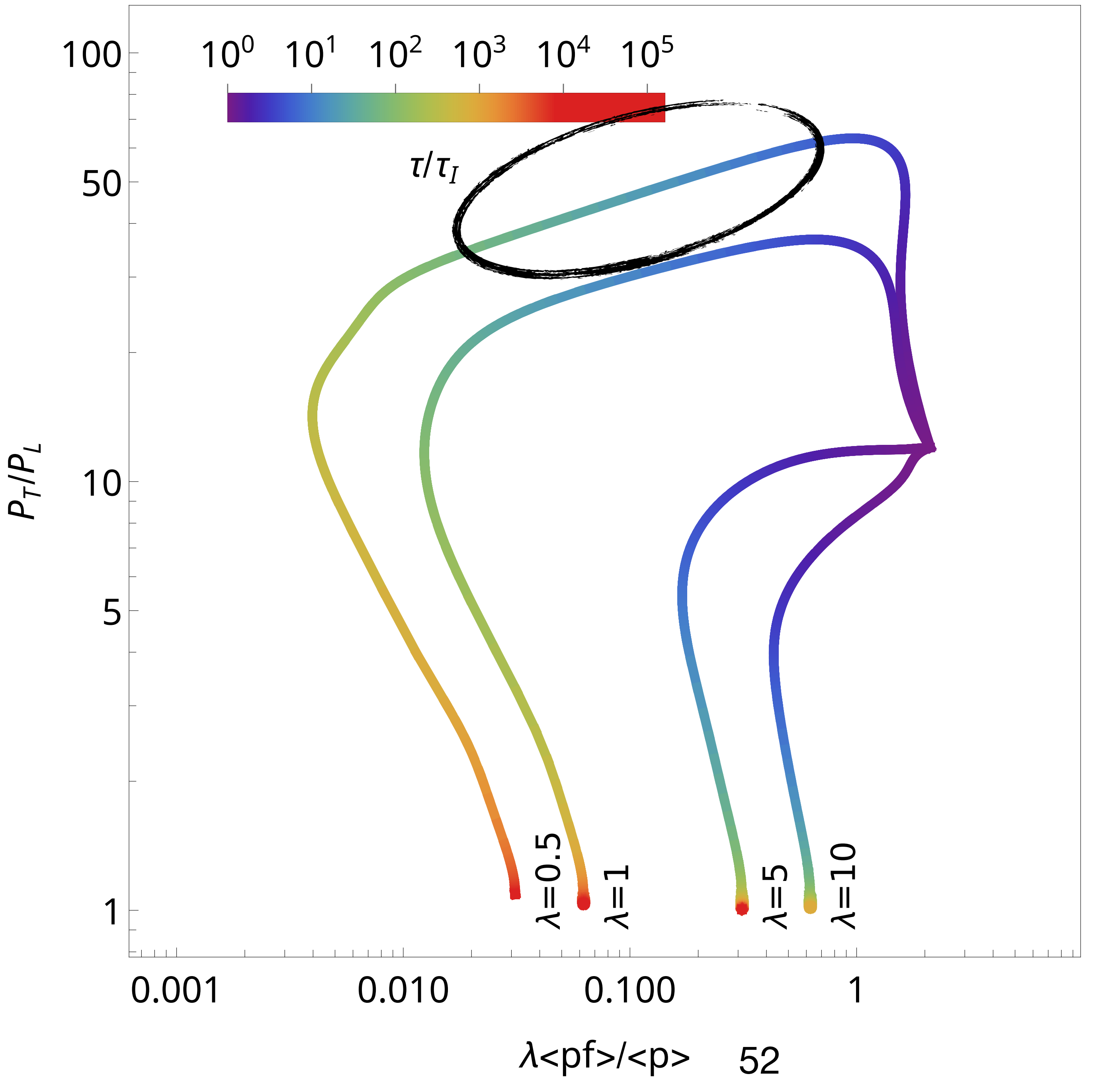


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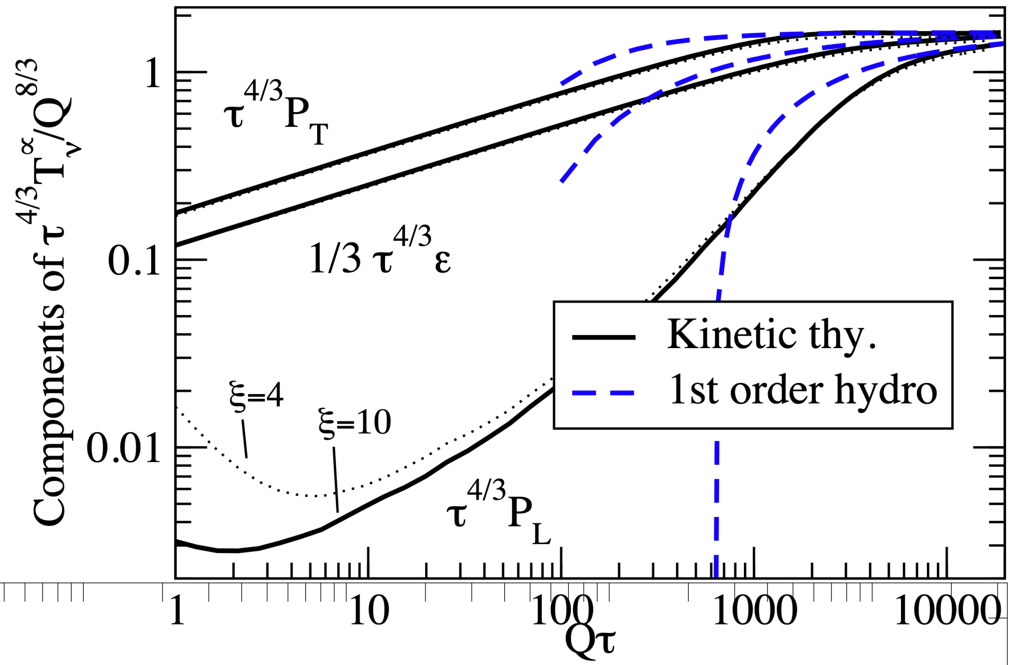


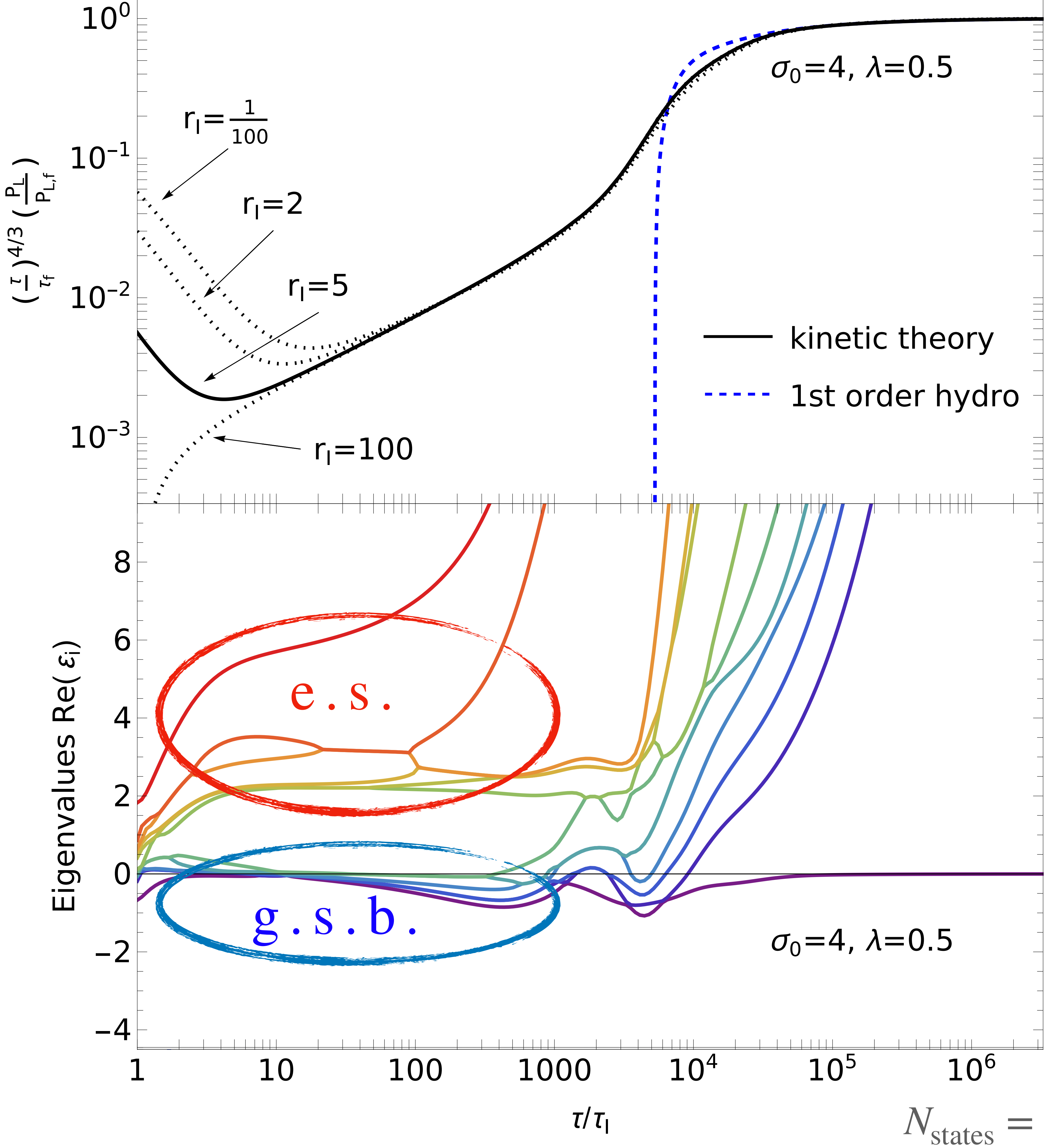

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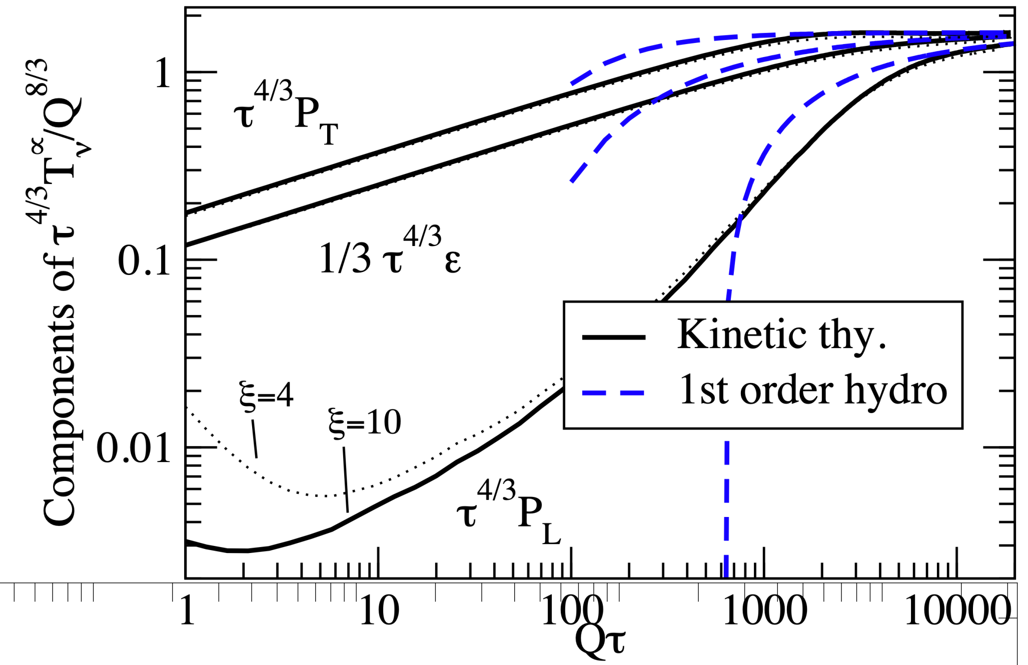


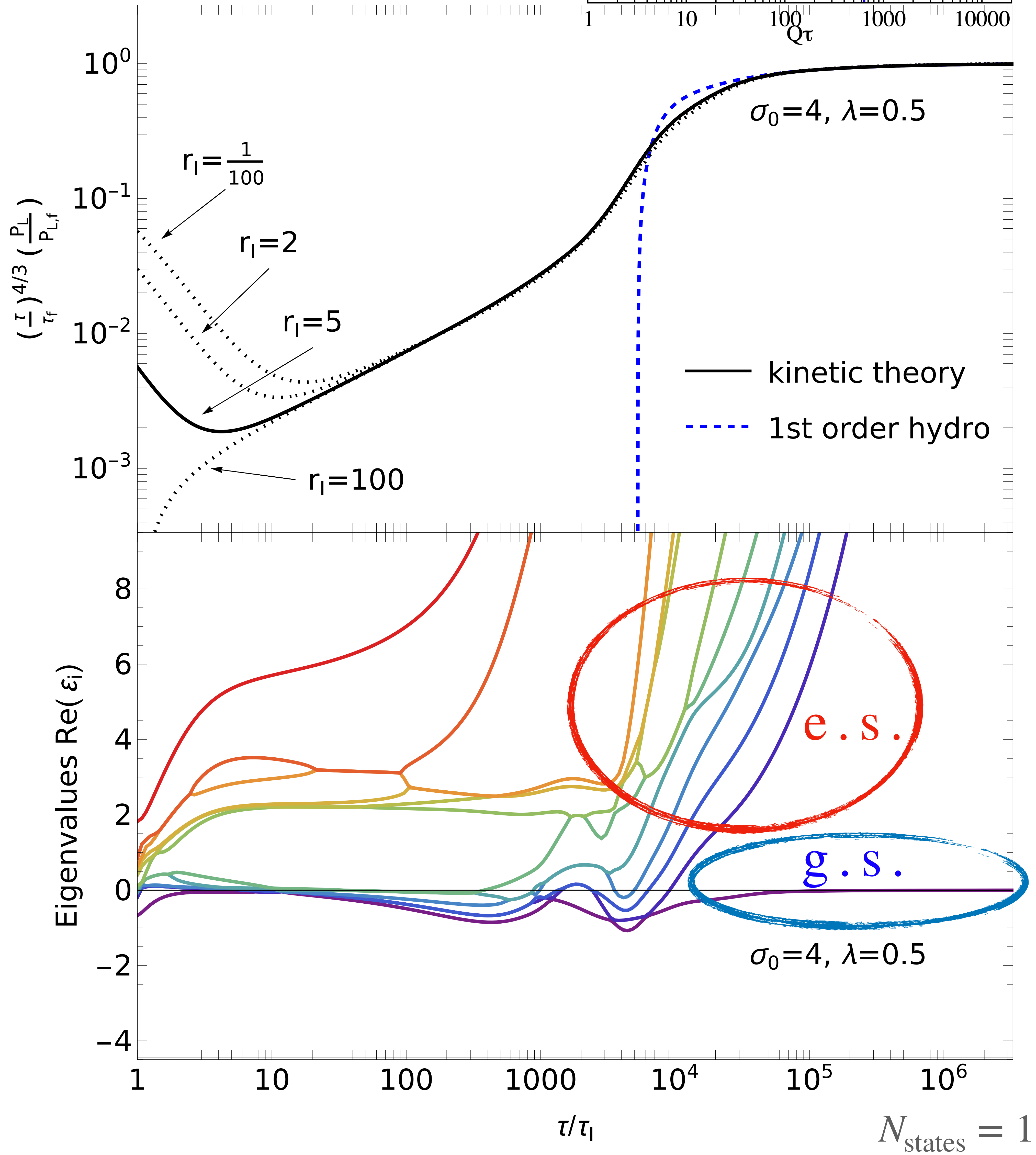


$N_{\text{states}} = 12$

see Florian's talk Tue 16:30

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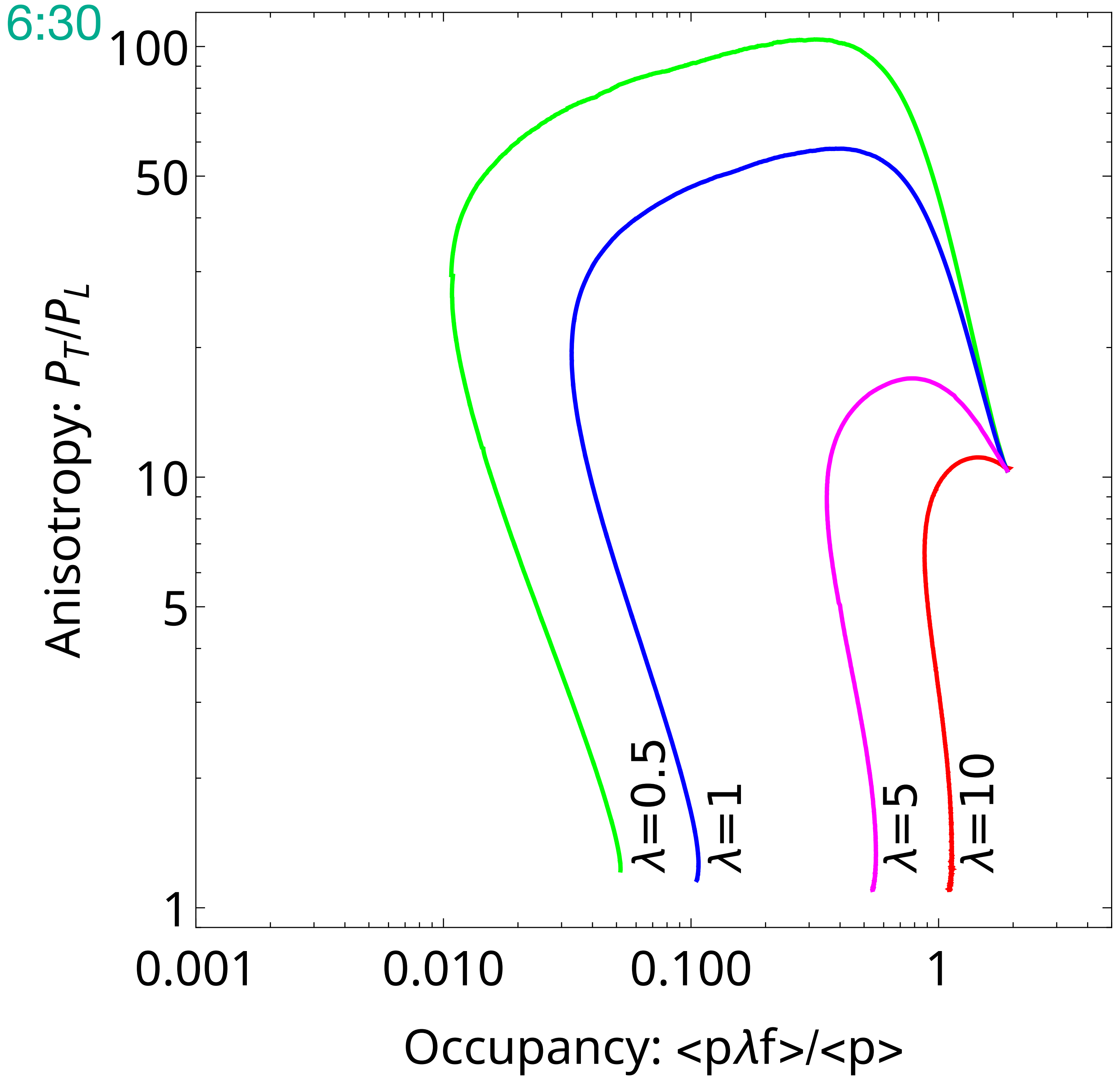


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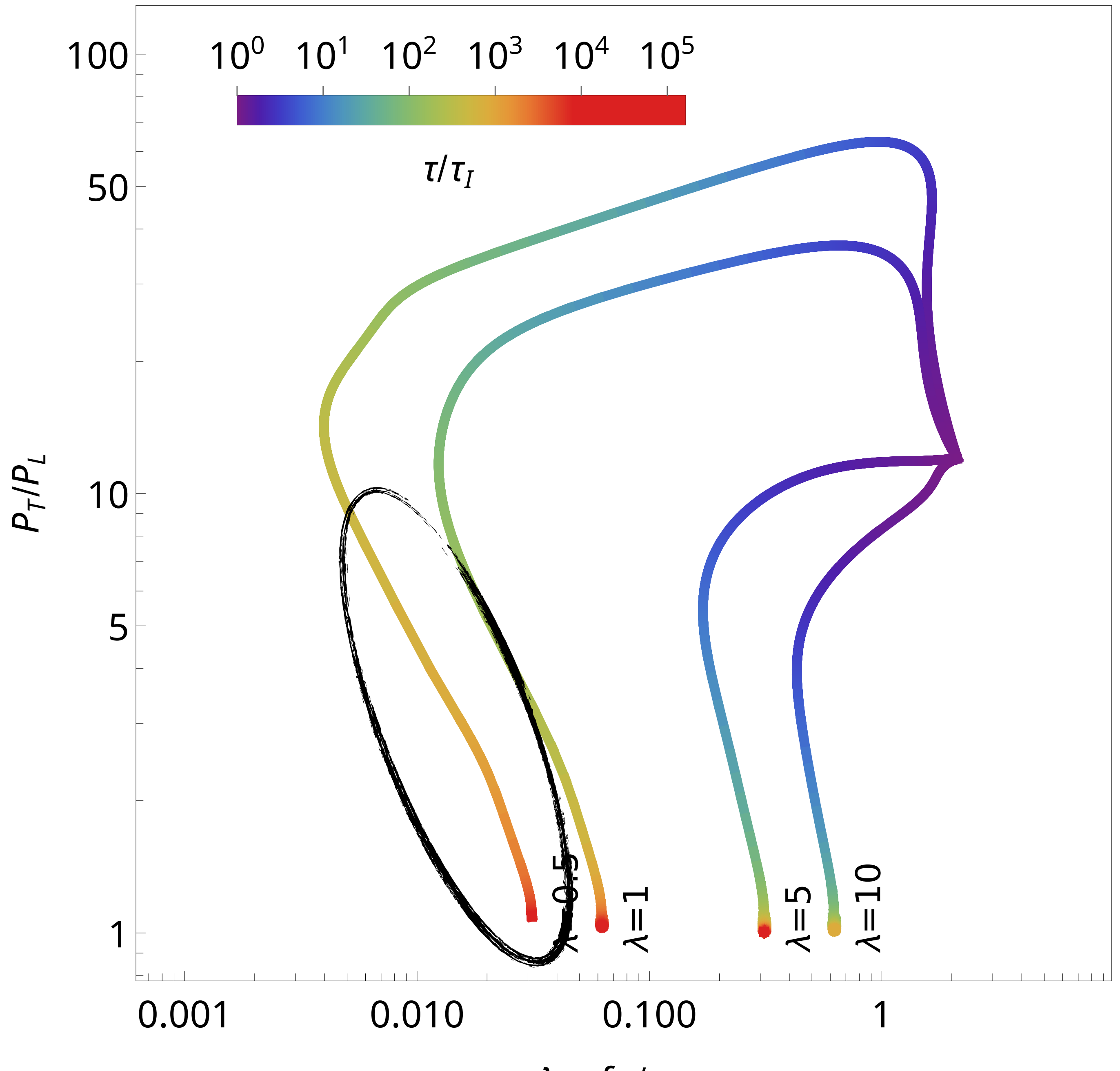

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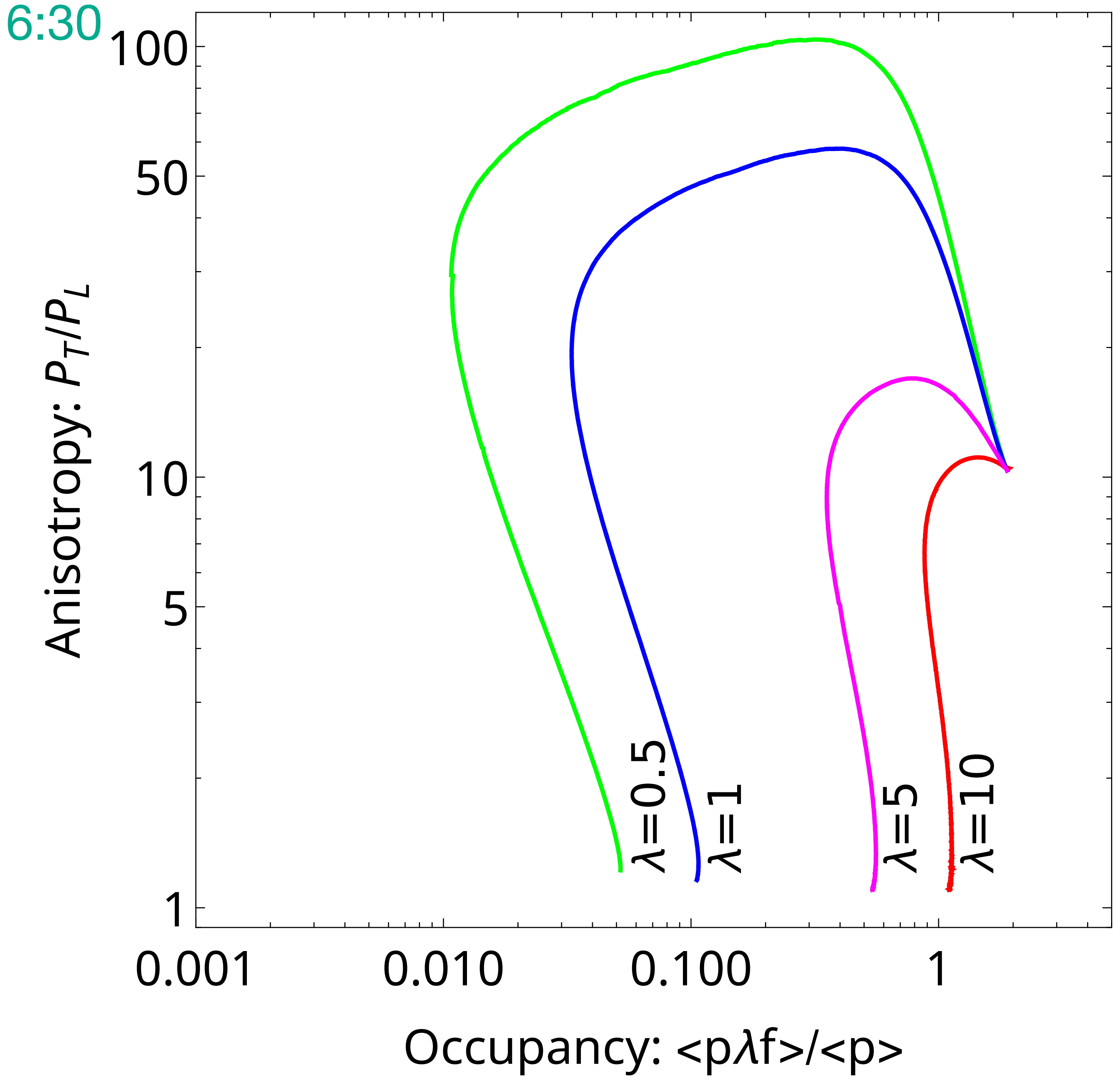
Tue 16:30

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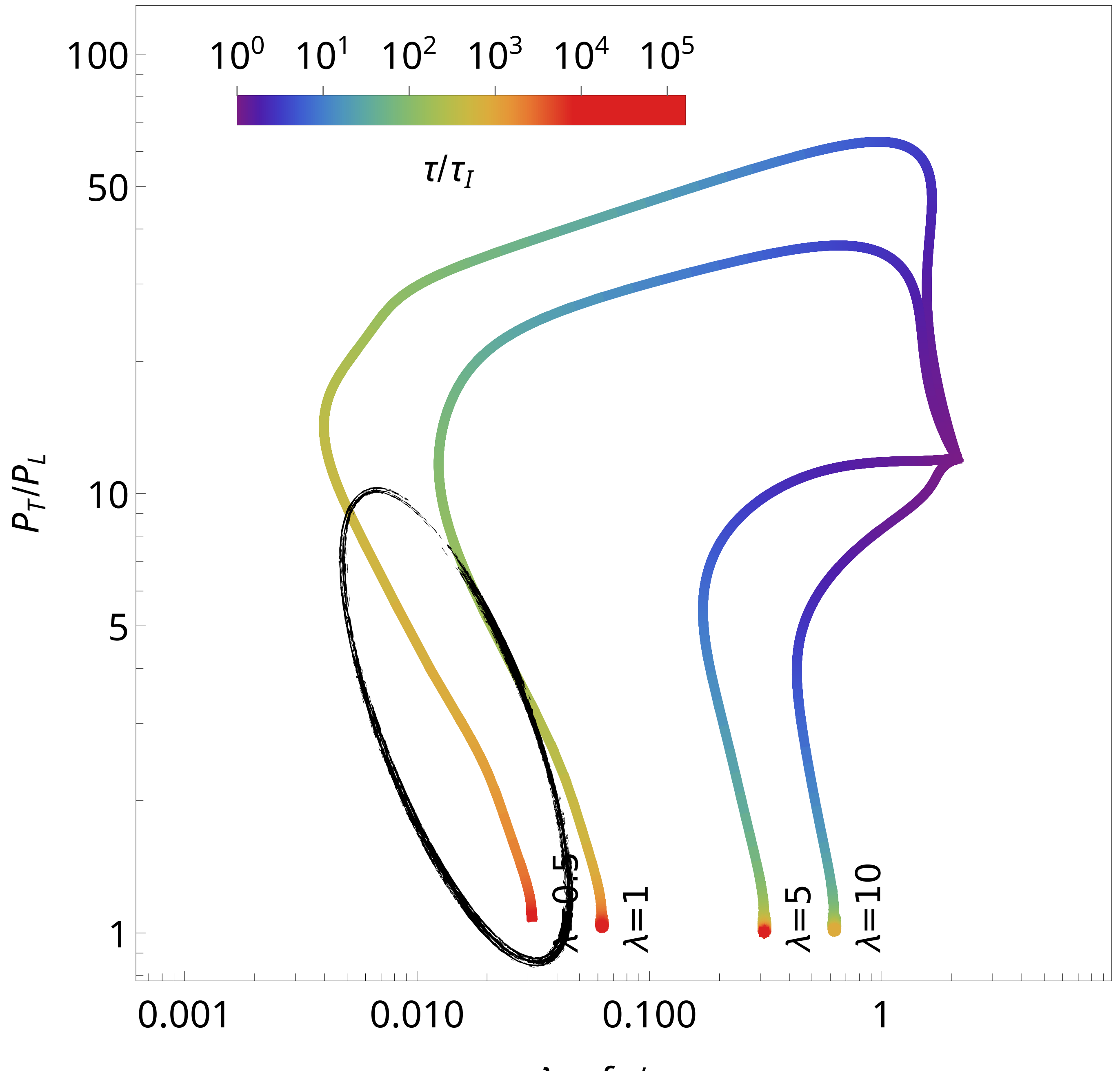


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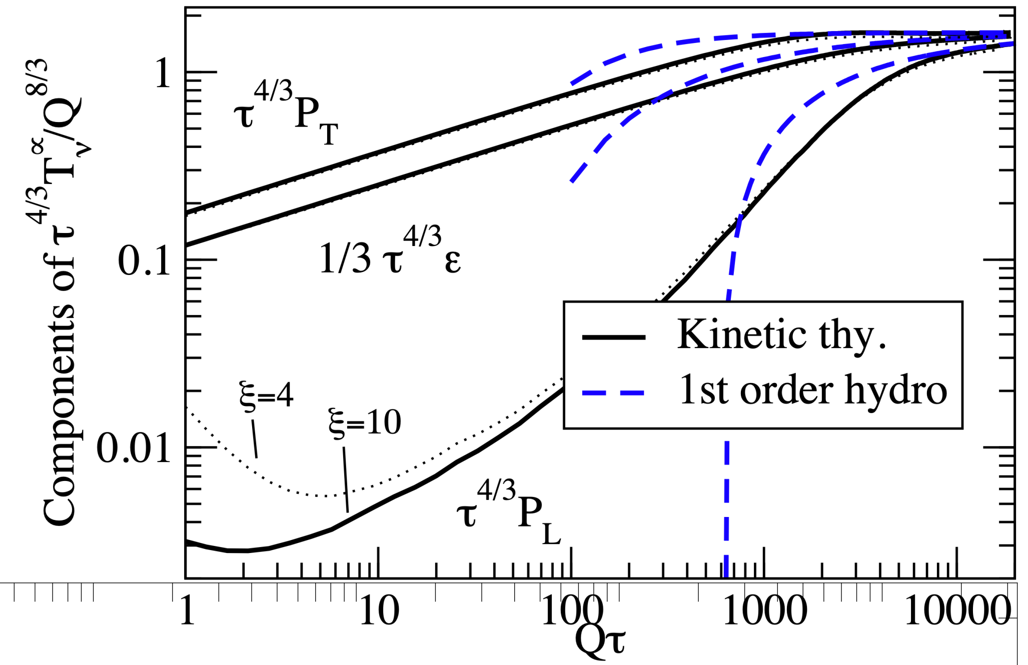


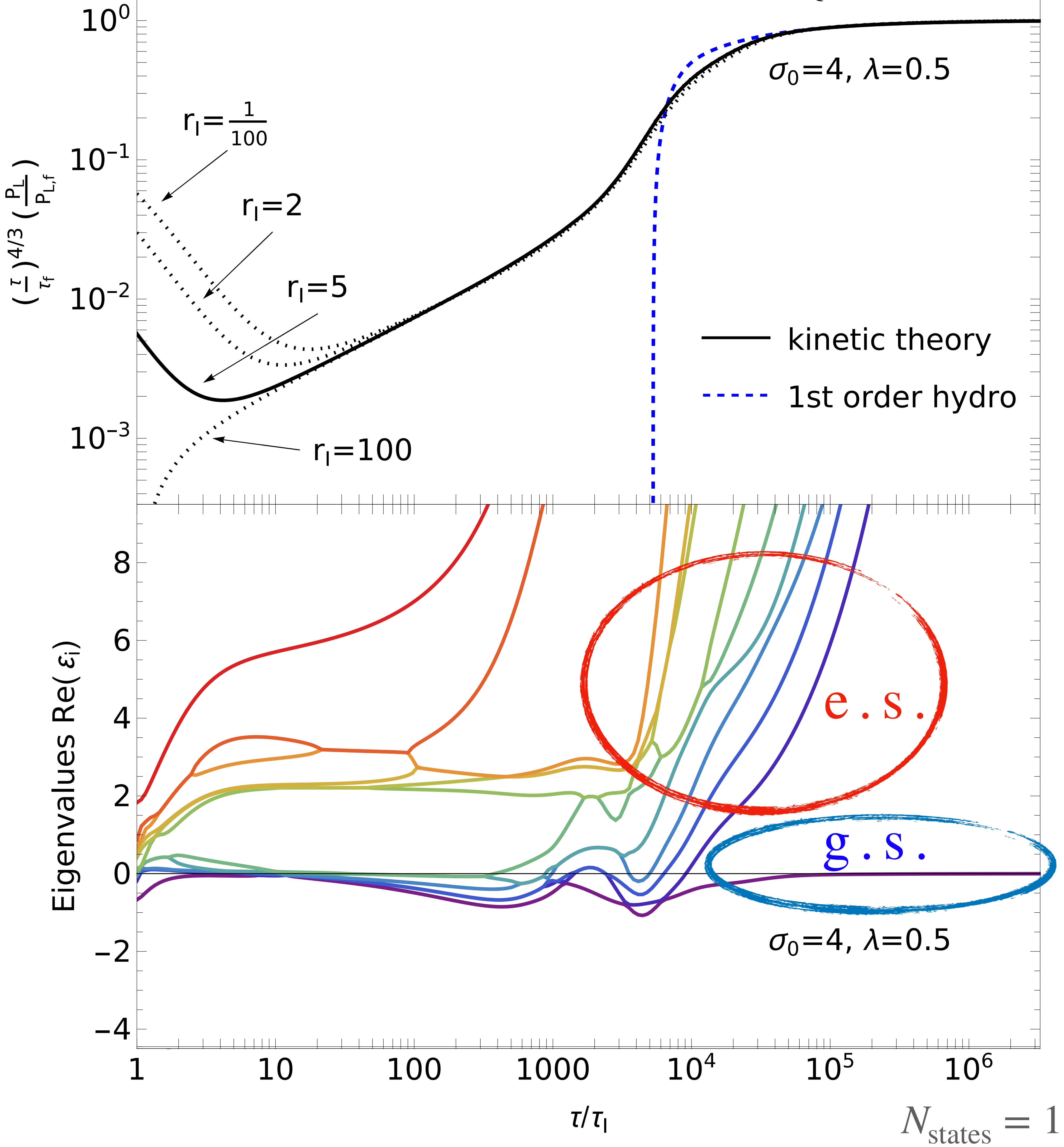


$N_{\text{states}} = 12$

# Adiabatic Hydrodynamization

## the stages of the bottom-up scenario

**Sequential memory loss!**

v:2507.2132 AH
work with BEDA +
—> 2)* - (Bose
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c.f. Kurkela & Zhu (2015)
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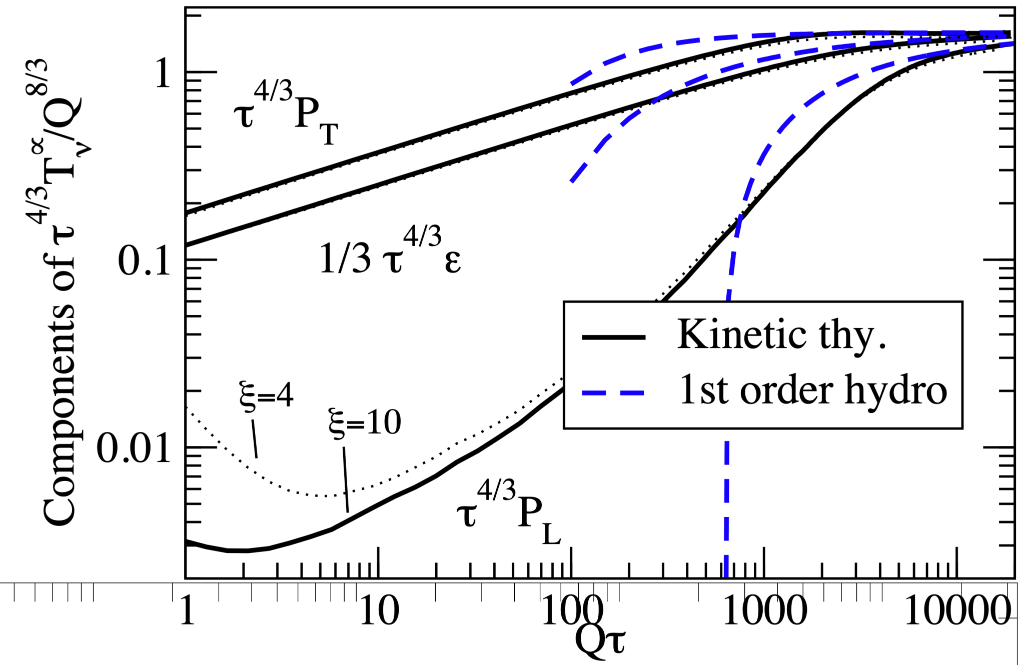


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At late times, $|0(\tau)\rangle$ describes a thermal distribution with a time-dependent temperature

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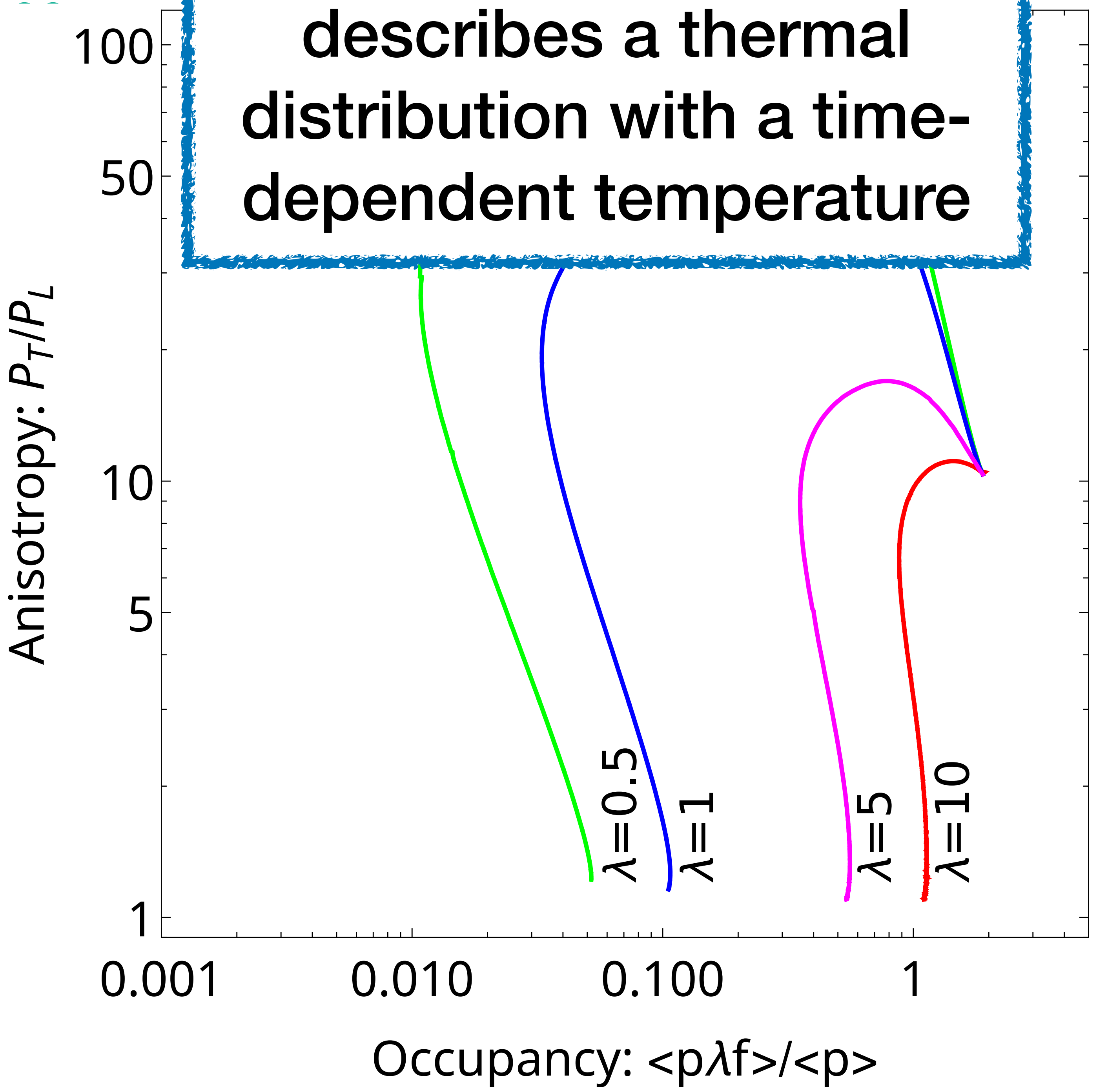
Tue 16

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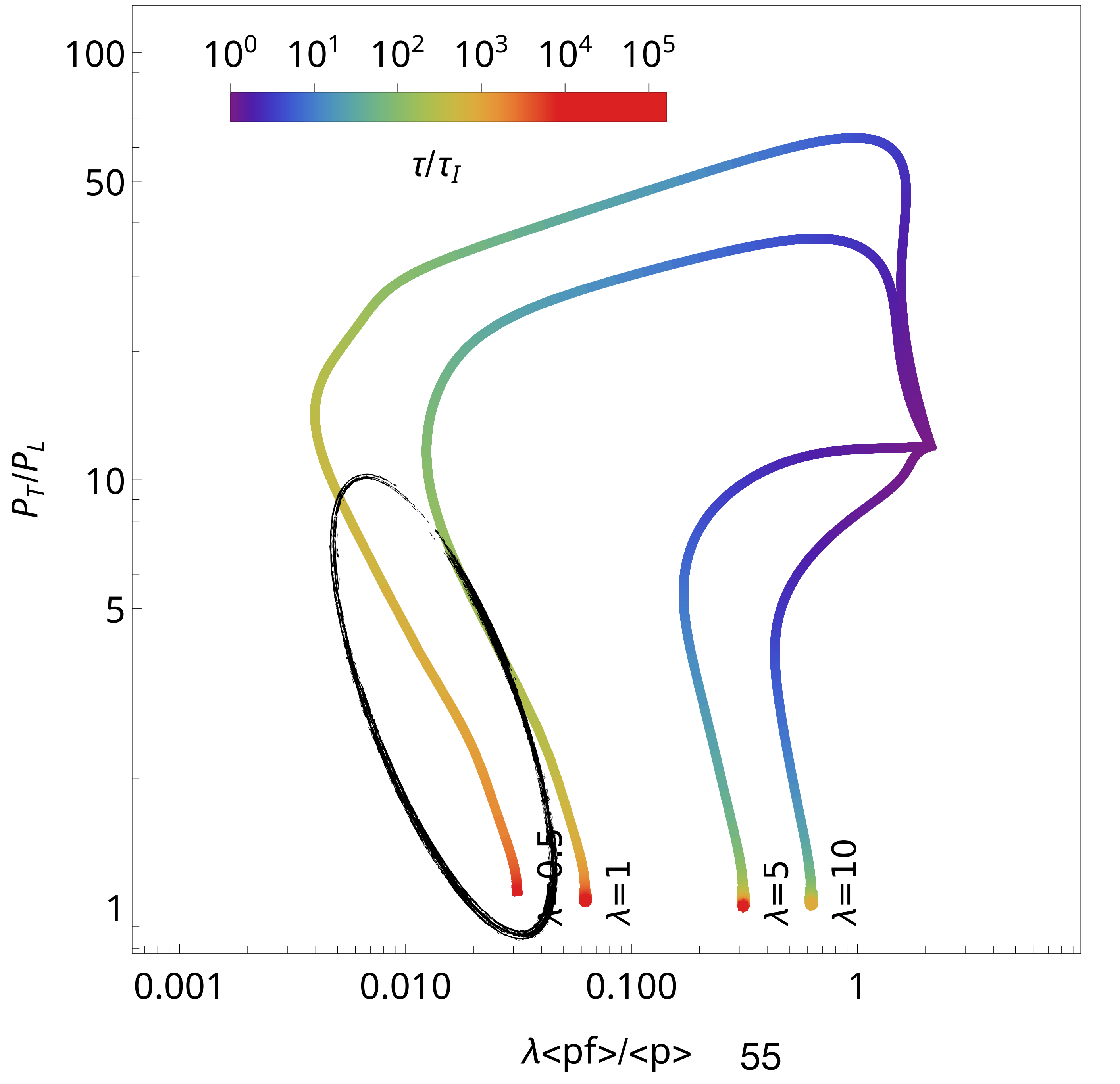
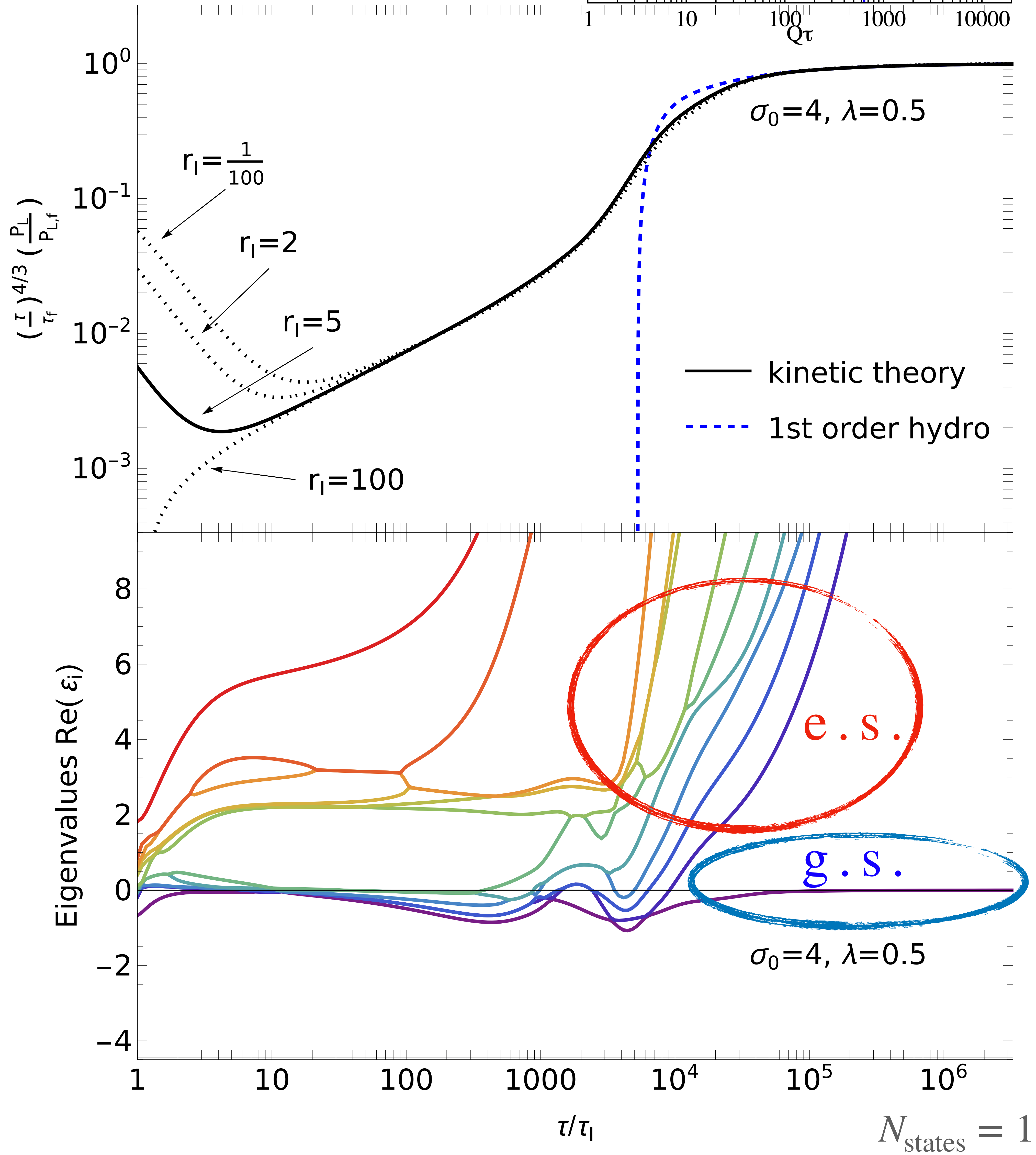



$N_{\text{states}} = 12$

*with respect to $y = \ln(\tau/\tau_I)$

# Approach to Hydrodynamics

## a robust feature of the spectrum of $\mathscr{H}$ at late times

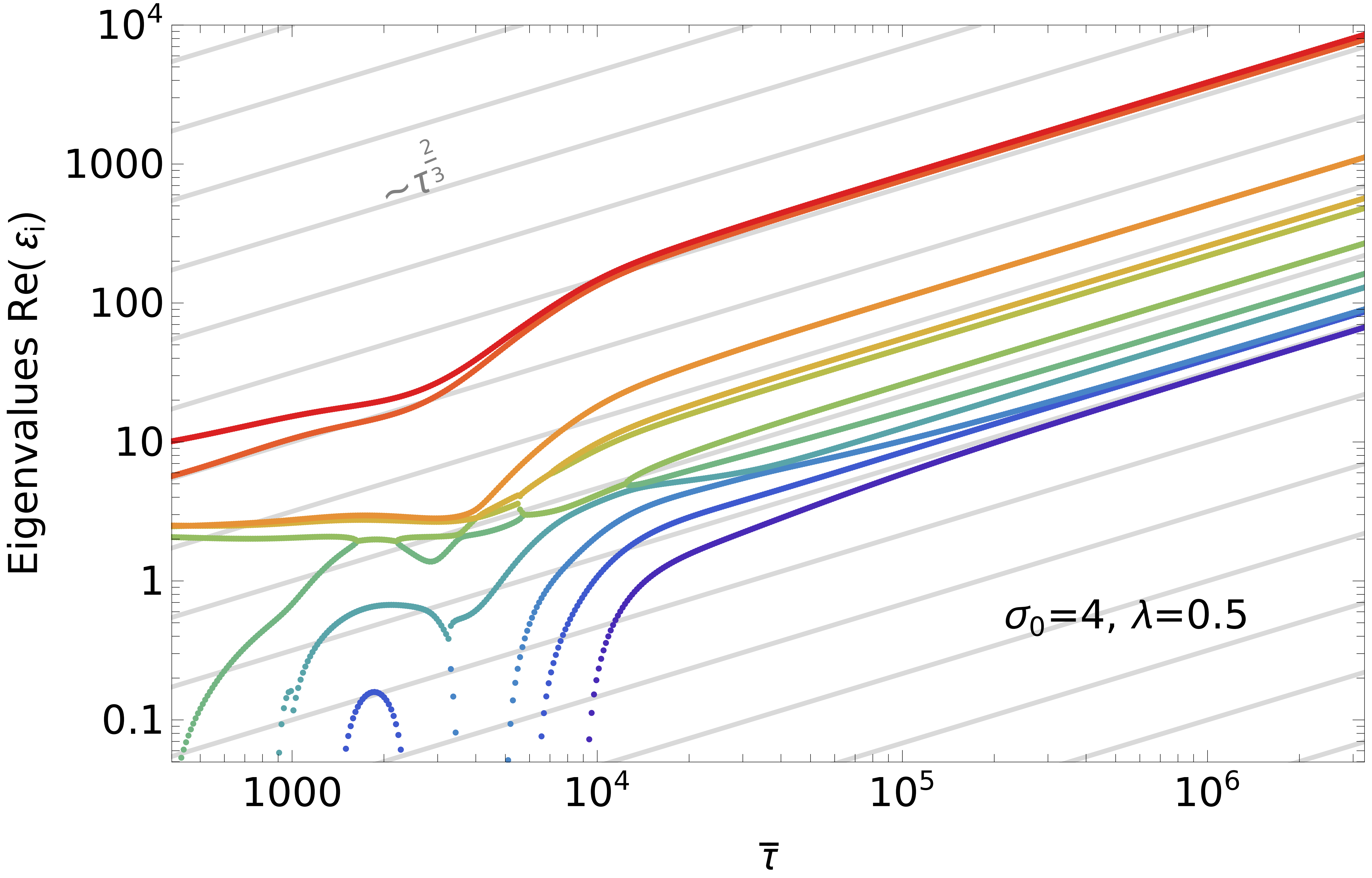
as a consequence of rescaling the momentum $p$ by $T(\tau)$, i.e., write the evolution in terms of $\chi = p/T(\tau)$

- If we look at the late-time energy spectrum*, we see that the energies grow $\propto \tau^{2/3}$ (actually $\tau T$).

$$\Longrightarrow \quad a_n^{\text{non-hydro}} \propto e^{-\#\tau^{2/3}}$$

(faster than hydro evolution!)

- Compare with Du, Heller, Schlichting & Svensson *Phys.Rev.D* 106 (2022) 1, 014016
  hydro transseries?
- This holds for any scaleless collision kernel [2507.21232]
  (i.e., if all dimensionful quantities are derived from $f$)






$N_{\text{states}} = 12$

# Conclusions from this study

## new insights into the process of hydrodynamization

- We have shown, in a not too simple kinetic theory, that:
    - Loss of memory of the initial condition can be understood in terms of the opening of energy gaps that make the information in excited states decay.
    - In each scaling regime, the ground state(s) evolve adiabatically, either by themselves or as a set, and “high-energy” modes effectively decouple from the dynamics.
    - With $1 \leftrightarrow 2$ processes in the collision kernel, we were able to apply the AH framework in a setting where hydrodynamization is rapid.
- Future work:
    - Include a nontrivial profile in position space, emulating the fireball formed in a HIC.

# Outlook

## for the Adiabatic Hydrodynamization framework

- AH provides an organizing principle to:
  - Identify attractors, regardless of whether they exhibit scaling phenomena
  - Explain memory loss of the initial condition by explicitly characterizing the decay of rate of information outside the attractor
- I have only discussed kinetic theory applications today. However,
  - Nothing stops us from using this framework for any equation that looks like $\partial_t f = -Hf$.
    - The main task for a practitioner is to cast the dynamics in this form.

# Extra slides

# Finding the adiabatic frame

- Putting together $\dfrac{\partial f}{\partial t} = x\dfrac{\partial f}{\partial x} + D[f;t]\dfrac{\partial^2 f}{\partial x^2}$ and $f(x,t) = A(t)\,w\big(x/B(t),t\big)$, we get

$$\frac{\partial w}{\partial t} = -\alpha w + (1-\beta)\left[\xi\frac{\partial w}{\partial \xi} + \frac{D}{B^2(1-\beta)}\frac{\partial^2 w}{\partial \xi^2}\right]$$

$\alpha \equiv \dot{A}/A$
$\beta \equiv -\dot{B}/B$

- This is valid for any choice of $A(t), B(t)$. Then, let me *choose* $B$ such that

$$\frac{D}{B^2(1-\beta)} = 1, \quad \text{which is to say} \quad \frac{\dot{B}}{B} = -1 + \frac{D}{B^2}.$$

- With this choice,

$$\frac{\partial w}{\partial t} = -\alpha w + (1-\beta)\left[\xi\frac{\partial w}{\partial \xi} + \frac{\partial^2 w}{\partial \xi^2}\right].$$

# ‘Optimizing’ adiabaticity

## rescaling the degrees of freedom

- From the previous discussion, we see that scaling plays a crucial role in this problem.

- This gives us a very useful tool to ‘optimize’ adiabaticity. For instance, if we have a distribution function evolving as

$$f(p_\perp, p_z, \tau) = A(\tau)\, w\Big(p_\perp/B(\tau), p_z/C(\tau); \tau\Big),$$

then we can look for the choice of $A, B, C$ that maximize the degree to which the dynamics of $w$ is adiabatic.

- We take $|\psi\rangle \leftrightarrow w(\zeta, \xi; \tau)$.

$q[f;\tau] = 4\pi\alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f] I_a[f]\tau$

# 'Optimizing' adiabaticity

## in practice

- The original kinetic equation has the form

$$\tau\partial_\tau f - p_z\partial_{p_z} f = q[f;\tau]\,\nabla_{\mathbf{p}}^2 f\,.$$

- This is a linear equation of motion, except for the non-linear dependence through $q[f;\tau]$.

- Nothing prevents us from making the replacement $q[f;\tau] \to q(\tau)$, solve the equation for an arbitrary $q(\tau)$, and in the end replace the resulting distribution $f[q(\tau)]$ in the definition of $q$ and solve self-consistently:

$$q(\tau) = q[f[q(\tau)];\tau].$$

# ‘Optimizing’ adiabaticity

## in practice

- One can then write the kinetic equation for $w$ as

$$\partial_y w = -\mathscr{H} w \,,$$

with $$\mathscr{H} = \alpha - (1-\gamma)\left[\tilde{q}\,\partial_\xi^2 + \xi\,\partial_\xi\right] + \beta\left[\tilde{q}_B\,(\partial_\zeta^2 + \frac{1}{\zeta}\partial_\zeta) + \zeta\,\partial_\zeta\right].$$

For brevity, we have denoted

$$\tilde{q} = \frac{q}{C^2(1-\gamma)}\,, \quad \tilde{q}_B \equiv -\frac{q}{B^2\beta}\,.$$

What is the advantage of this?

- Because $A, B, C$ are a choice of coordinates (a “gauge” choice to describe the system), we can choose them such that $\tilde{q} = \tilde{q}_B = 1$.

How?

Note that

$$\tilde{q}(\tau) = \frac{q(\tau)}{C^2(\tau)(1-\gamma(\tau))} \implies \gamma(\tau) = \boxed{-\frac{\tau\partial_\tau C}{C} = 1 - \frac{q(\tau)}{\tilde{q}(\tau)C^2},}$$

Differential equation for $C(\tau)$

$\implies$ we can choose $\tilde{q}$ by “fixing the gauge” and choosing $C(\tau)$.

$\tilde{q} = 1$ corresponds to fixing $C(\tau)$ by solving: $-\dfrac{\tau\partial_\tau C}{C} = 1 - \dfrac{q(\tau)}{C^2}$ . Same for $\beta$ and $\tilde{q}_B$.

# Results

## low-lying energy states

- We can choose $A$ such that $\alpha = \gamma + 2\beta - 1$ to set the ground state energy $\mathscr{E}_{0,0} = 0$.
- The eigenvalues of $\mathscr{H}$ are $\mathscr{E}_{n,m} = 2n(1-\gamma) - 2m\beta \ , \quad n, m = 0, 1, 2, \ \dots$
- The left and right eigenstates are:

$$\phi^{L}_{n,m} = \mathrm{He}_{2n}\big(\xi\big)\ {}_1F_1\left(-2m,1,\frac{\zeta^2}{2}\right),$$

$$\phi^{R}_{n,m} = \frac{1}{\sqrt{2\pi}\,(2n)!}\mathrm{He}_{2n}\big(\xi\big)\ {}_1F_1\left(-2m,1,\frac{\zeta^2}{2}\right)\exp\left(-\frac{\xi^2}{2}-\frac{\zeta^2}{2}\right)$$

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**Gapped energy levels!**

**$\Longrightarrow$ Ground state will dominate after a transient time**

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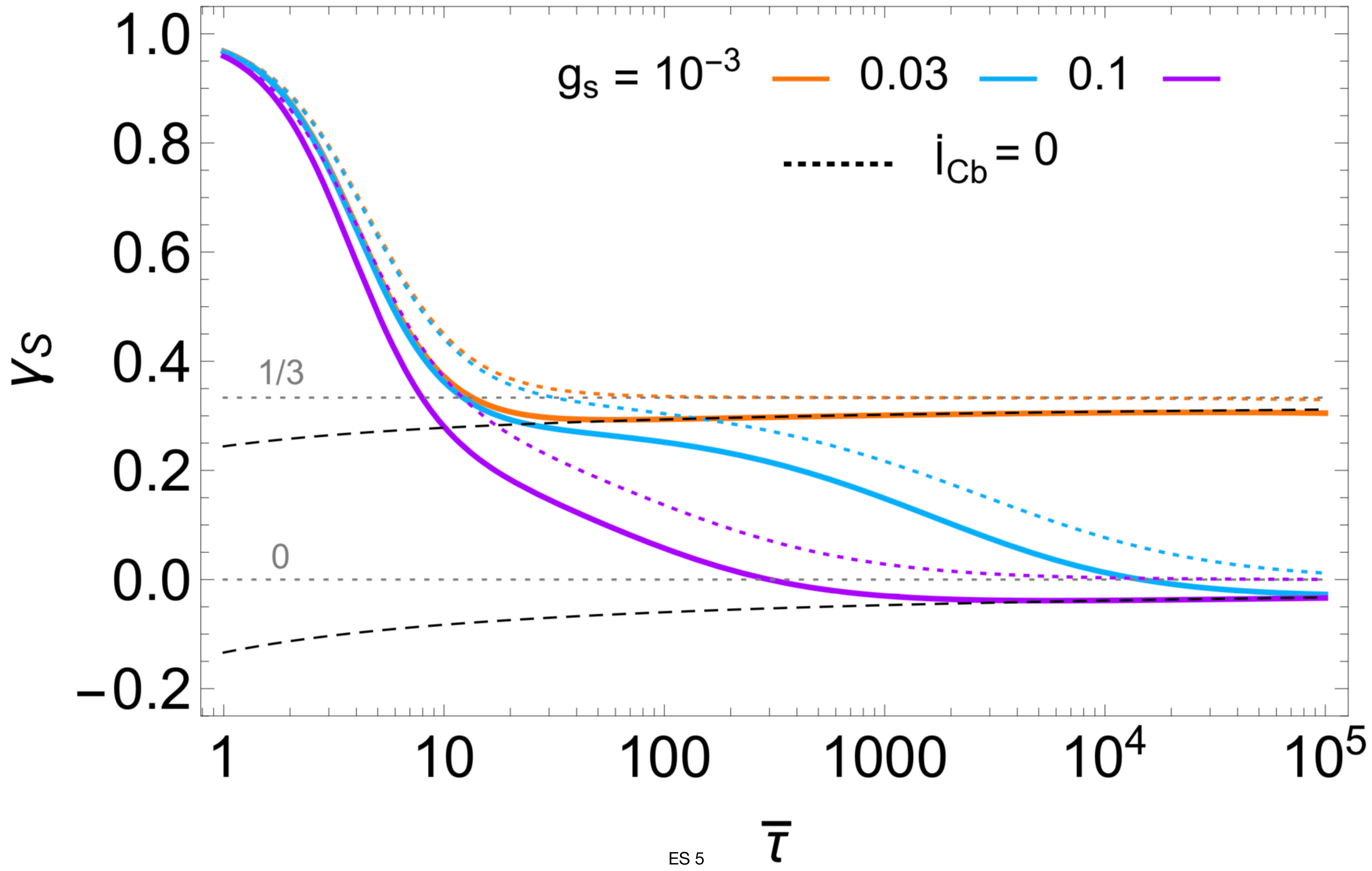
Left and right eigenstates differ because $\mathscr{H}$ is not hermitian

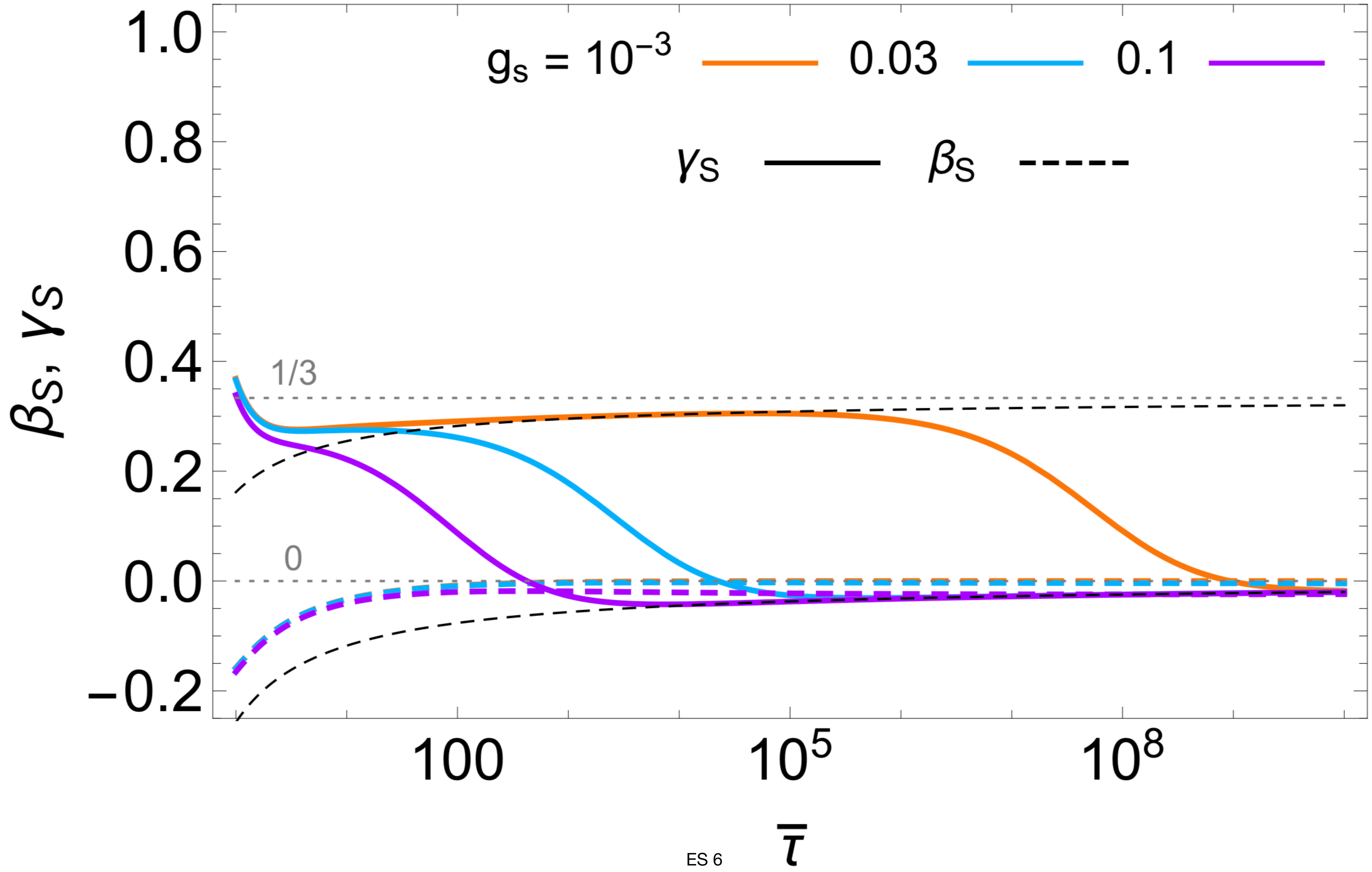
# Evolution of the exponents for different coupling strengths






$\sigma_0 = 0.1$

# Evolution of the exponents for different coupling strengths






$\sigma_0 = 0.6$

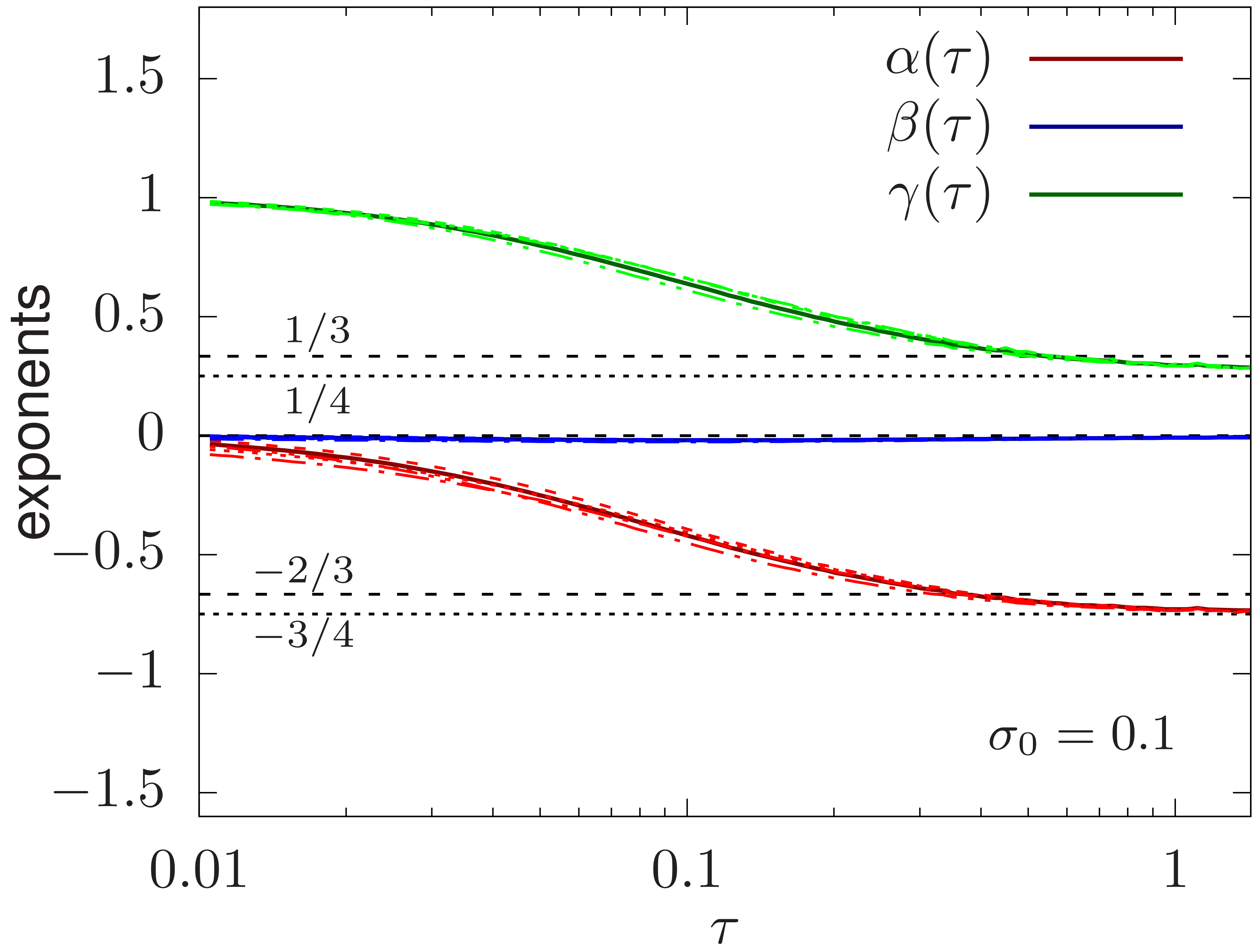
$$f(\tau_I) = \frac{\sigma_0}{g_s^2} \exp\left(-\frac{p_\perp^2 + \xi^2 p_z^2}{Q_s^2}\right); \xi = 2, Q_s\tau_I = 70, g_s = 10^{-3}$$

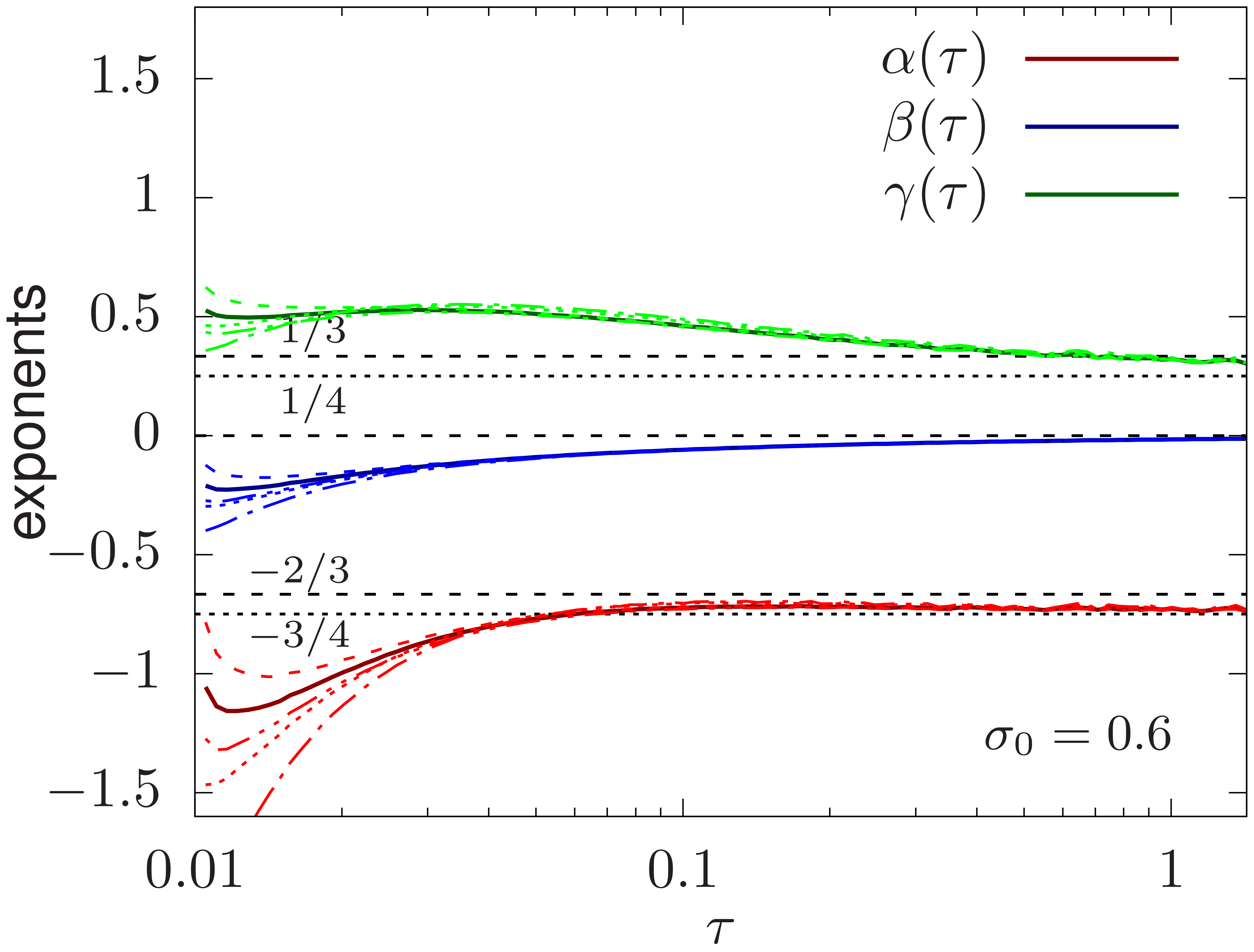
# Evidence for AH in QCD effective kinetic theory

## by A. Mazeliauskas, J. Berges [6]

- After a transient time, [6] observed that $f_g$ took a time-dependent scaling form

$$f(p_\perp, p_z, \tau) = e^{\int^\tau \alpha(\tau')\,\mathrm{d}\ln\tau'} f_S\left(e^{\int^\tau \beta(\tau')\,\mathrm{d}\ln\tau'} p_\perp, e^{\int^\tau \gamma(\tau')\,\mathrm{d}\ln\tau'} p_z\right).$$




[6] A. Mazeliauskas, J. Berges, "Prescaling and far-from-equilibrium hydrodynamics in the quark-gluon plasma" Phys. Rev. Lett. 122, 122301 (2019)

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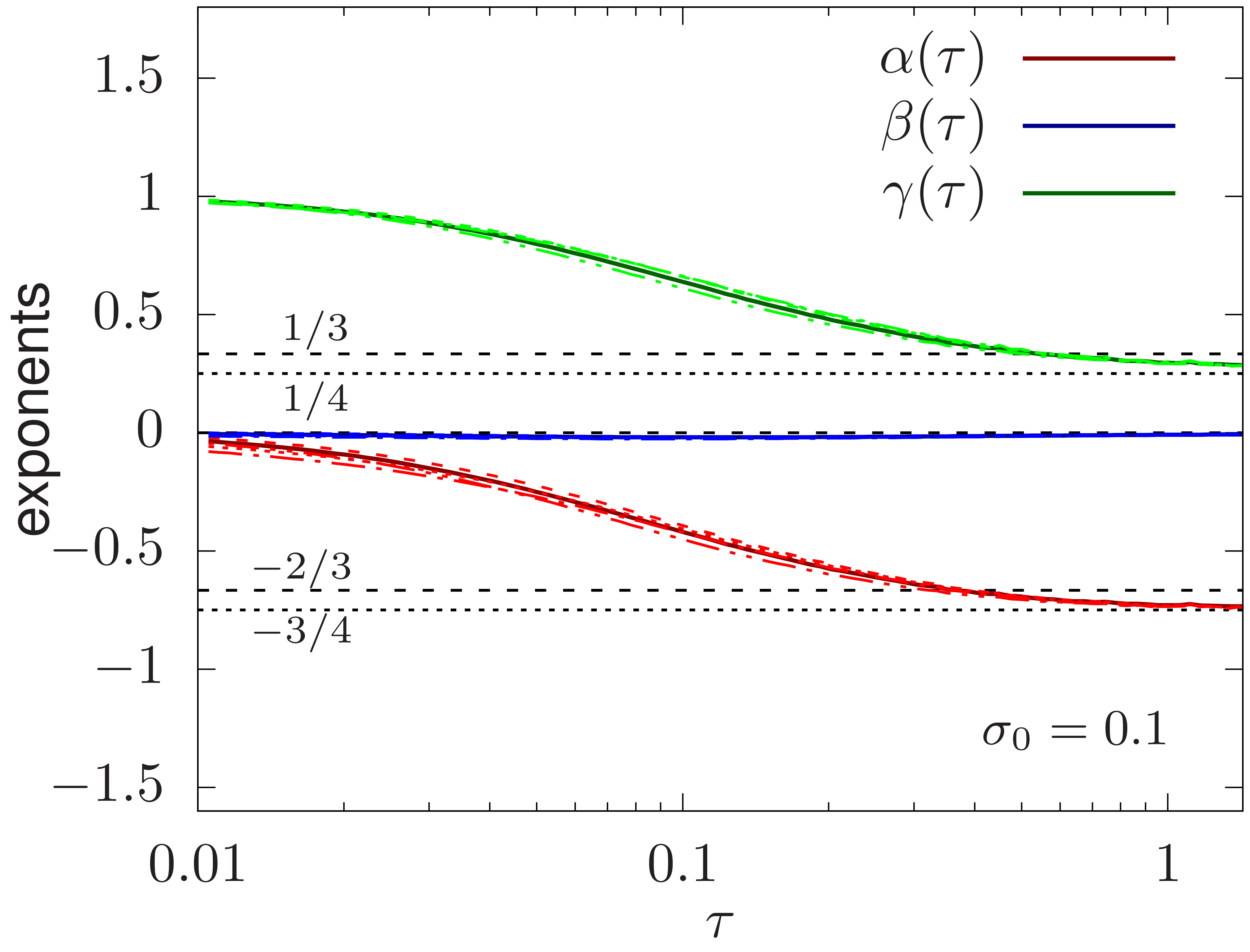


In the plots, the exponents were obtained by taking moments of the distribution function:

$$\mathrm{n}_{m,n}(\tau) = \int_{\mathbf{p}} p_\perp^m |p_z|^n f(p_\perp, p_z; \tau),$$

and using that, if scaling takes place,

$$\frac{\partial_\tau \ln \mathrm{n}_{m,n}}{\partial \ln \tau} = \alpha(\tau) - (m+2)\,\beta(\tau) - (n+1)\,\gamma(\tau)$$

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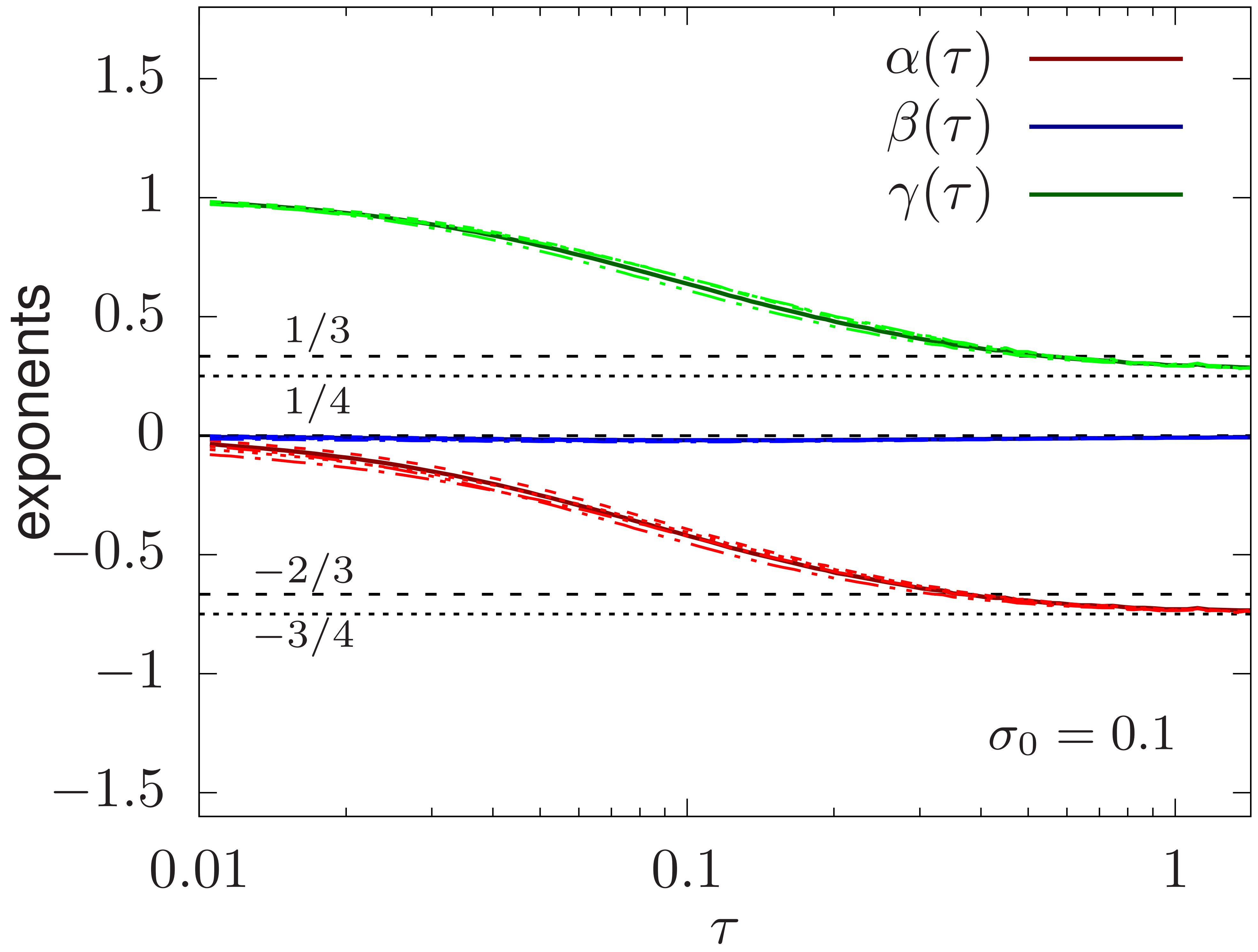


Then, one can use triads of moments to obtain $\alpha, \beta, \gamma$. For example, if we use $\mathrm{n}_{0,0}, \mathrm{n}_{1,0}, \mathrm{n}_{0,1}$,

$$\alpha = 4\partial_{\ln\tau}\ln \mathrm{n}_{0,0} - 2\partial_{\ln\tau}\ln \mathrm{n}_{1,0} - \partial_{\ln\tau}\ln \mathrm{n}_{0,1}\,,$$

$$\beta = \partial_{\ln\tau}\ln \mathrm{n}_{0,0} - \partial_{\ln\tau}\ln \mathrm{n}_{1,0}\,,$$

$$\gamma = \partial_{\ln\tau}\ln \mathrm{n}_{0,0} - \partial_{\ln\tau}\ln \mathrm{n}_{0,1}\,.$$

If every triad of moments gives the same $\alpha, \beta, \gamma$, then the distribution has the above scaling form.

Curves in the figure $\Longleftrightarrow$ different triad choices.

[6] A. Mazeliauskas, J. Berges, "Prescaling and far-from-equilibrium hydrodynamics in the quark-gluon plasma" Phys. Rev. Lett. 122, 122301 (2019)

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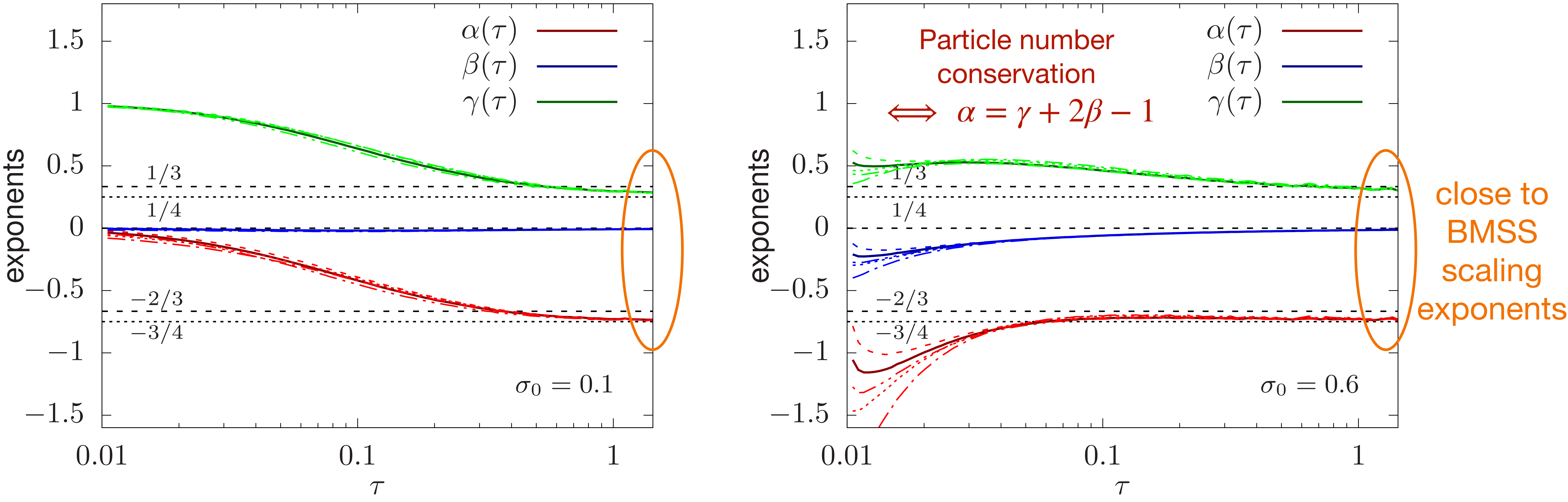


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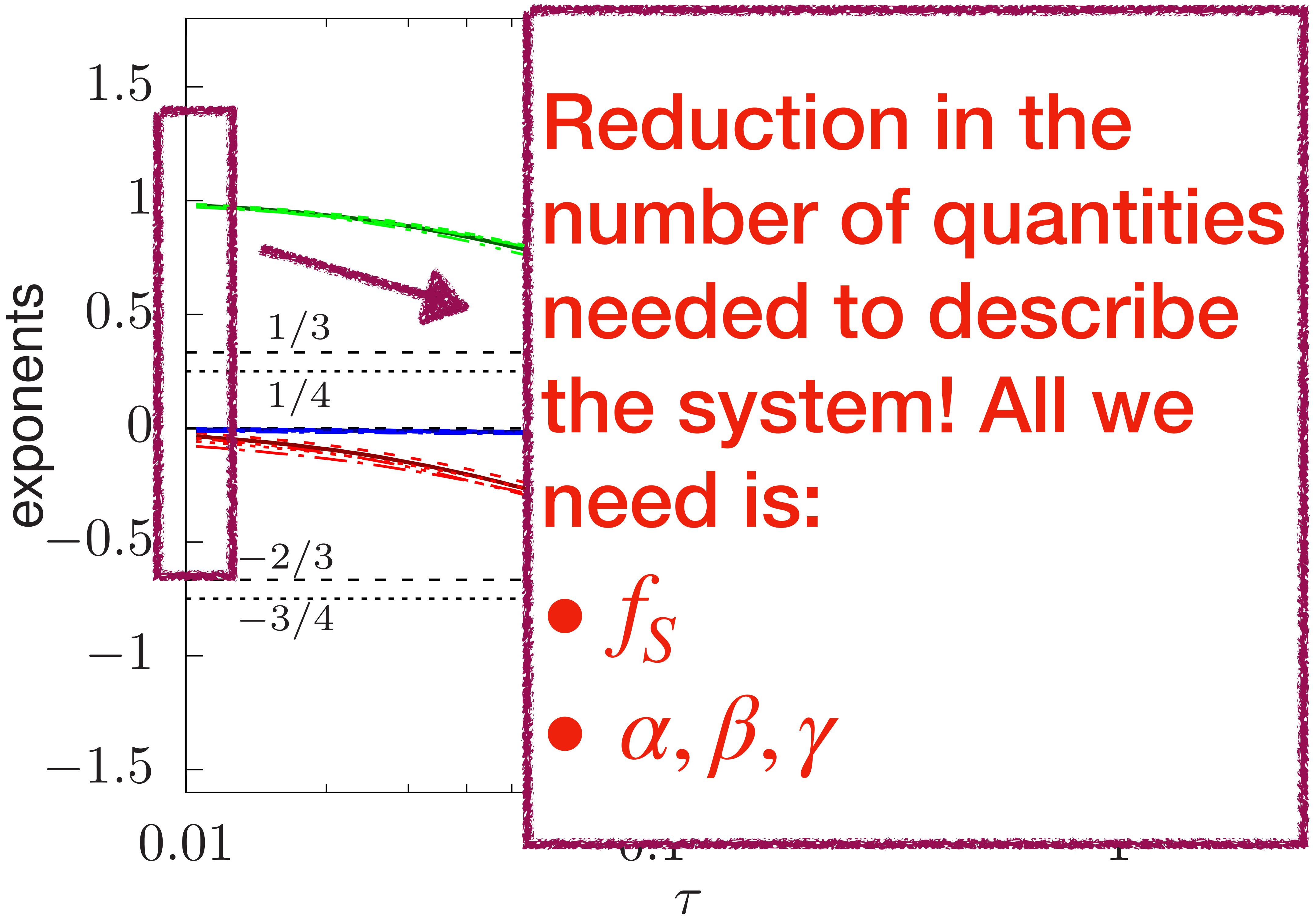
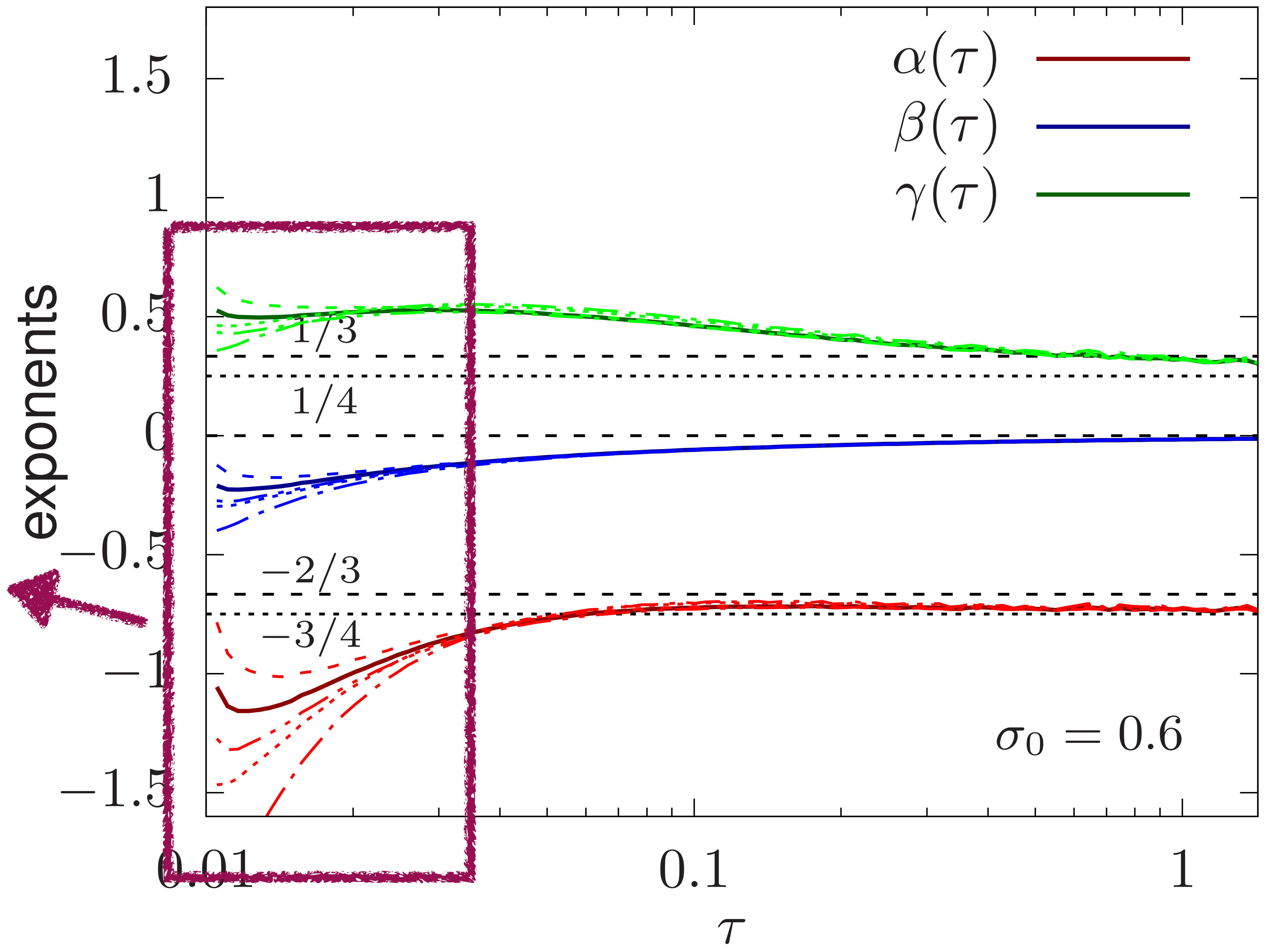



Reduction in the number of quantities needed to describe the system! All we need is:

- $f_S$
- $\alpha, \beta, \gamma$

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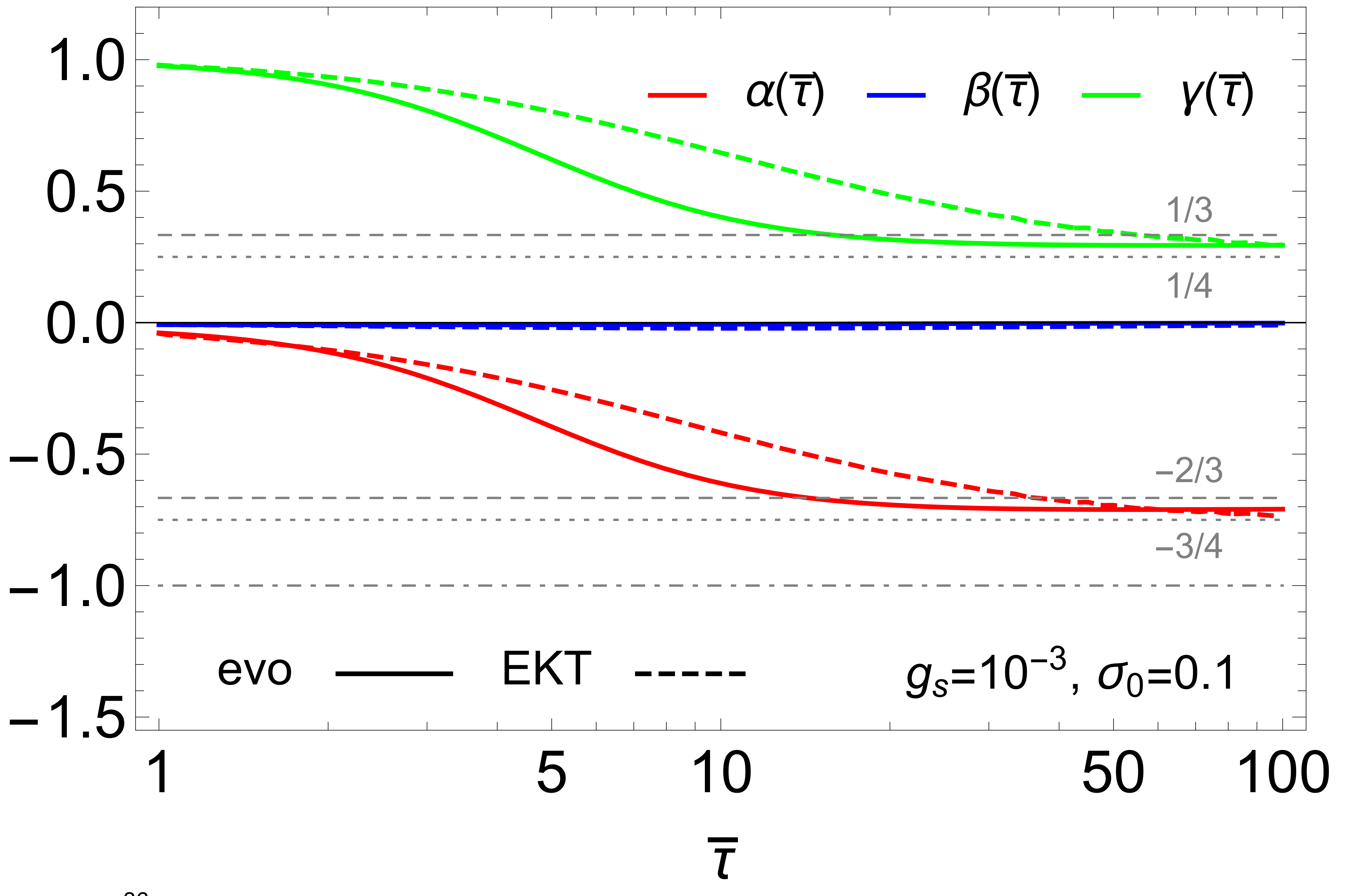
# Recapitulation: Results of the previous section

## low-lying energy states

- Recall that the eigenvalues of $\mathscr{H}$ in the early time regime are $\mathscr{E}_{n,m} = 2n(1-\gamma) - 2m\beta$, for $n, m = 0, 1, 2, \ \ldots.$

- But, $\beta \to 0$ on the BMSS fixed point (late times on the plot on the right).

$\Longrightarrow$ No substantial memory loss for the $p_\perp$ dependence of $f$. That is to say, no thermalization.



# Breaking the scaling regime

## restoring terms in the collision kernel

- To make the approach to hydrodynamics possible, we need to restore the terms we dropped:

$$\partial_\tau f - \frac{p_z}{\tau}\partial_{p_z} f = 4\pi\,\alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f] I_a[f]\,\nabla_{\mathbf{p}}^2 f$$

$$\Downarrow$$

$$\partial_\tau f - \frac{p_z}{\tau}\partial_{p_z} f = 4\pi\,\alpha_s^2 N_c^2 l_{\mathrm{Cb}}[f]\left[I_a[f]\,\nabla_{\mathbf{p}}^2 f + I_b[f]\,\nabla_{\mathbf{p}}\cdot\left(\hat{p}(1+f)f\right)\right]$$

- We will neglect the explicit Bose enhancement in the last term in what follows. The equilibrium distribution will thus be Boltzmann instead of Bose-Einstein.

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# Adiabaticity beyond scaling

## how to choose a frame with adiabatic ground state evolution

- We evolve $r(y)$ and $D(y)$ according to

$$\frac{\partial_y D}{D} = \rho \left( 1 - D \left\langle \frac{2}{p} \right\rangle \right) ,$$

$$\partial_y r = -\frac{1}{r} \frac{J_0}{J_4 J_0 - J_2^2} \left[ -2(J_2 - J_4) + \frac{\tau \lambda_0 \ell_{\mathrm{Cb}} I_a}{D^2} (J_0 - 3J_2) \right] ,$$

where

$$J_n(r) = \int_{-1}^{1} du\, u^n e^{-u^2 r^2/2}$$, and we set $\rho = 10$.

# Scaling exponents in the new basis

- We plot

$$\beta_{p_T^2} = -(1/2)\partial_y \log\langle p_\perp^2\rangle\,,$$

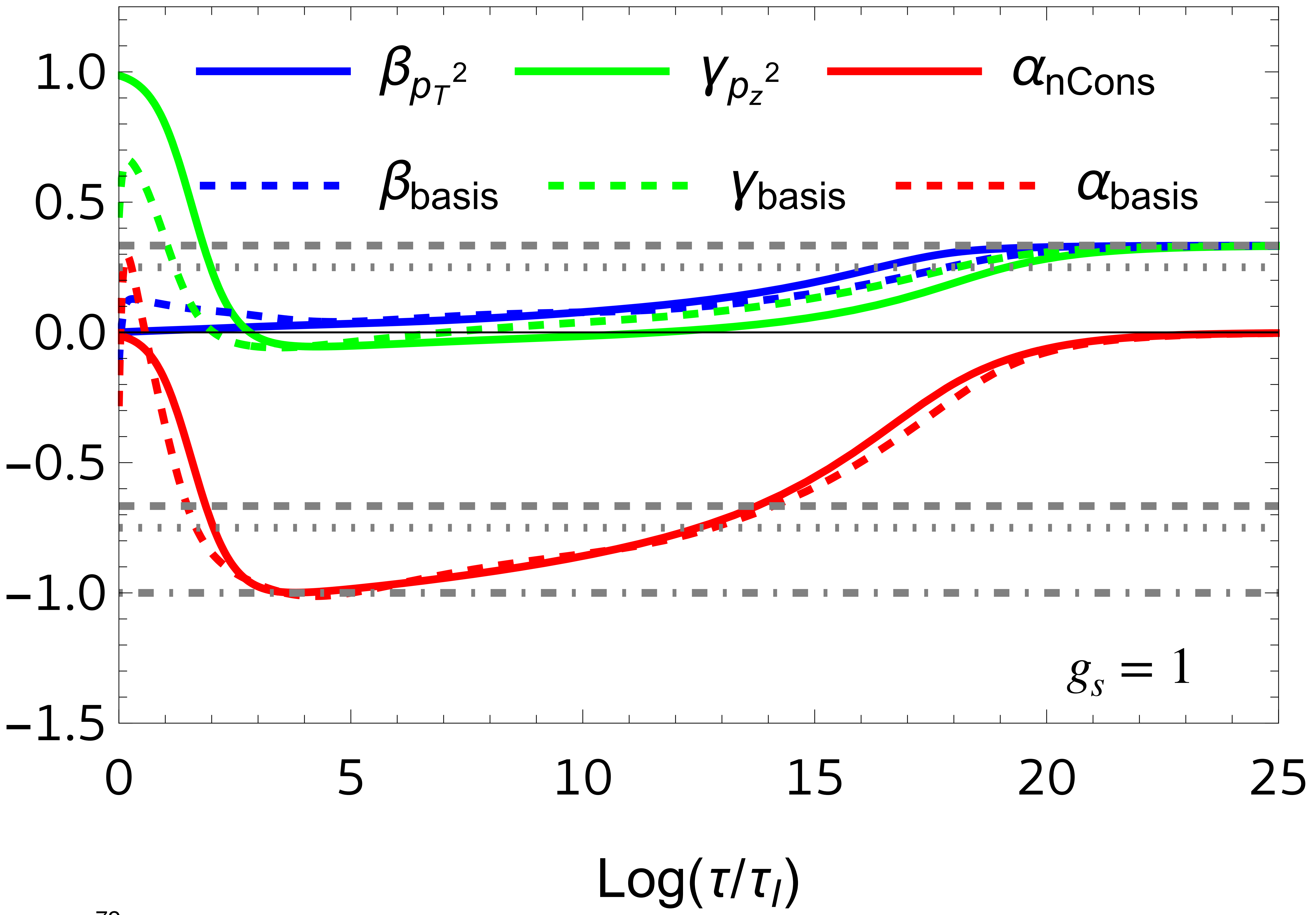
$$\gamma_{p_z^2} = -(1/2)\partial_y \log\langle p_z^2\rangle\,,$$

$$\alpha_{\mathrm{nCons}} = \gamma_{p_z^2} + 2\beta_{p_T^2} - 1\,,$$

from the solution to the kinetic equation, and also from the first basis state $\beta_{\mathrm{basis}}, \gamma_{\mathrm{basis}}, \alpha_{\mathrm{basis}}$.

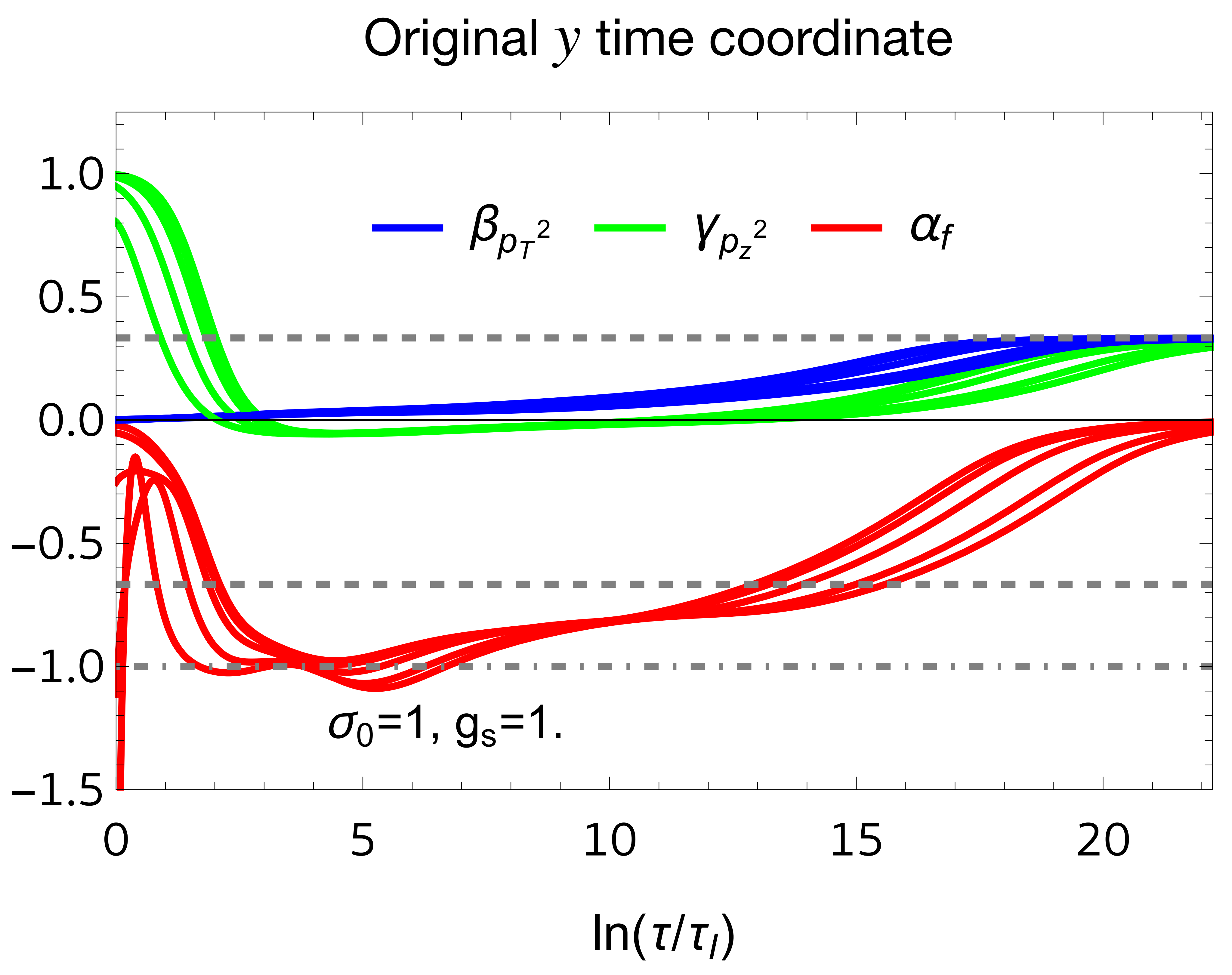
- At early times (up to $\log(\tau/\tau_I) \sim 10$) we see the dilute fixed point.

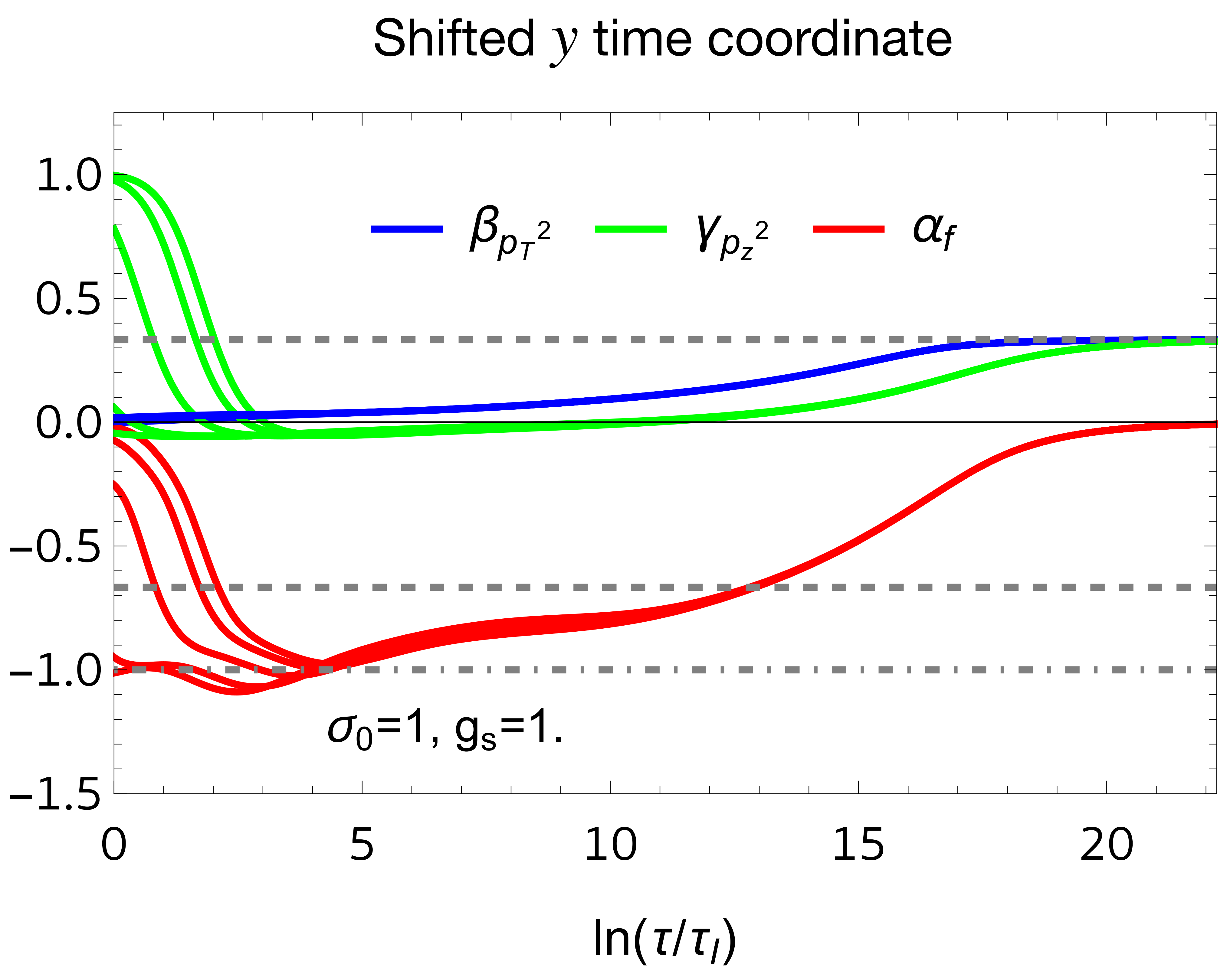
- At late times, hydrodynamics.



# Evidence for an attractor

## starting from different initial conditions



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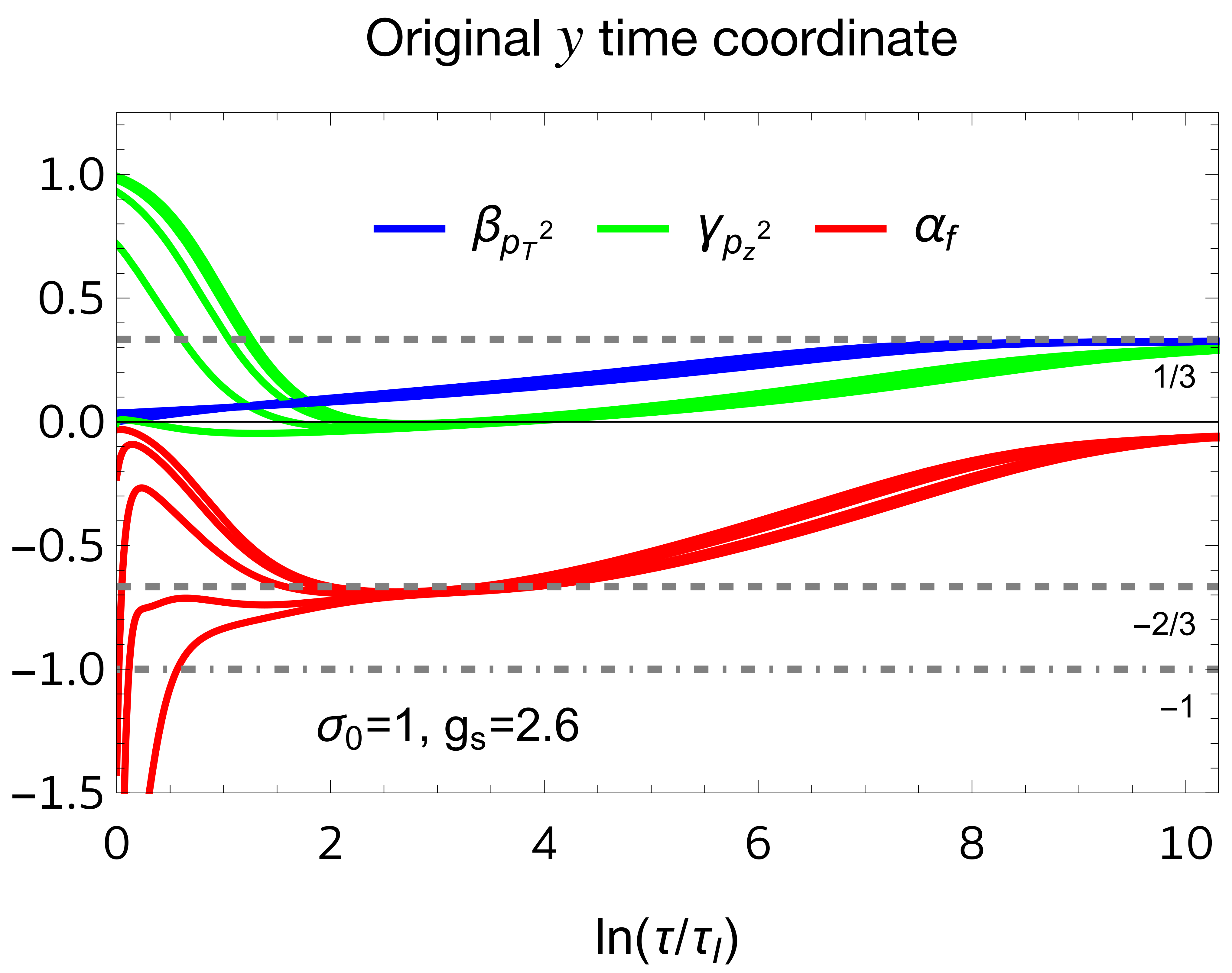
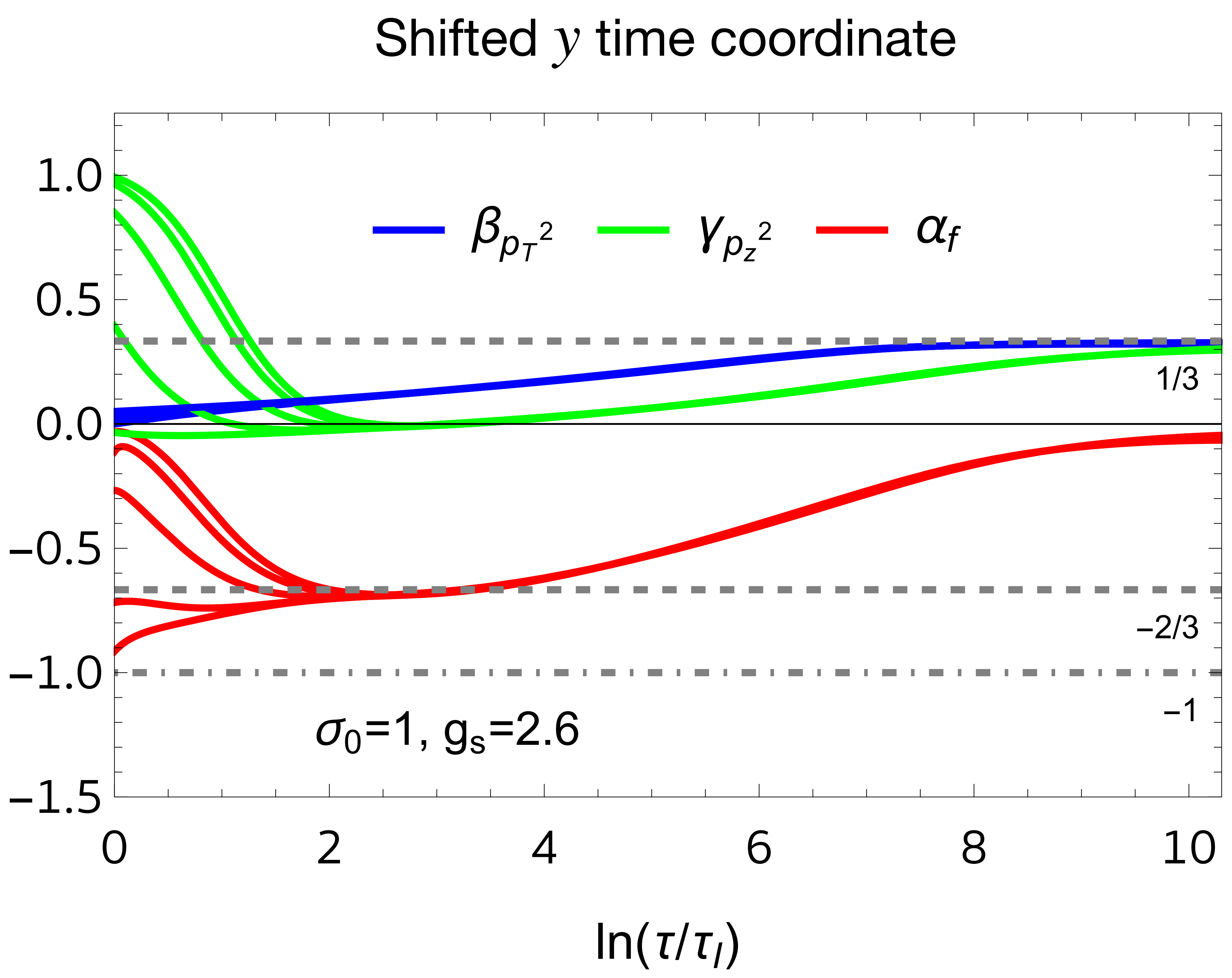


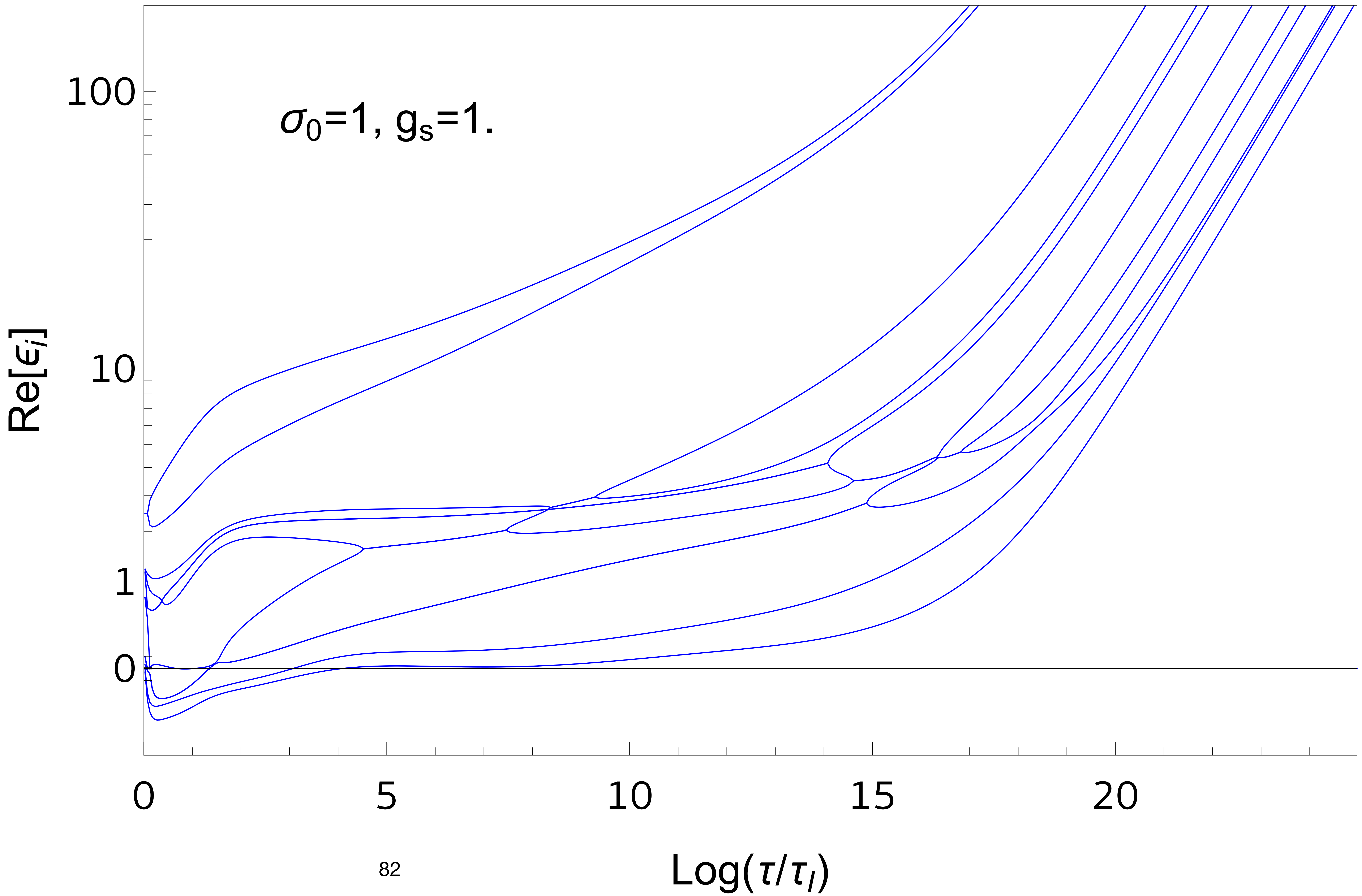



# Energy levels

## from early times to late times

$$f(\mathbf{p}, \tau = \tau_I) = \frac{\sigma_0}{g_s^2} e^{-\sqrt{2}p/Q_s} e^{-r_i^2 u^2/2} Q_0(u; r)$$

- We see that up until $\log(\tau/\tau_I) \sim 10$, the ground state is approximately degenerate.
- When the system approaches hydrodynamics, a gap opens and a unique ground state remains.



# Eigenstate coefficients

$$f(\mathbf{p}, \tau = \tau_I) = \frac{\sigma_0}{g_s^2} e^{-\sqrt{2}p/Q_s} e^{-r_i^2 u^2/2} Q_0(u; r)$$

## from early times to late times

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