# ANALYTIC METHODS FOR UNCOVERING ATTRACTORS

**_Inês Aniceto_**

Attractors and thermalisation in
nuclear collisions and cold quantum gases

*ECT* Workshop, 23rd September 2025*

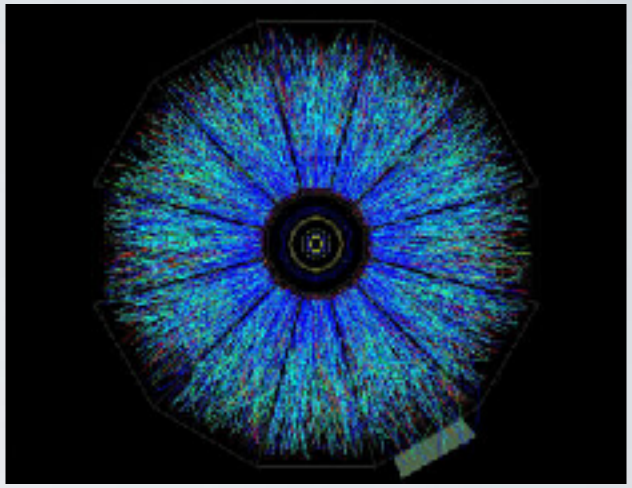
$$\sum_{n=0}^{\infty} E_n\, g^n \quad e^{-A/g}$$

University of Southampton

Engineering and Physical Sciences Research Council

# FROM ATTRACTORS TO ASYMPTOTICS

# STRONGLY COUPLED SYSTEMS



[RHIC]

- Motivation from **heavy-ion collisions** in particle accelerators (CERN, RHIC)

- Collisions give rise to **strongly coupled** fluids such as **quark-gluon plasma**

- Plasma quickly goes through an **thermalization process**: described by ***relativistic hydrodynamics***

***Relativistic hydrodynamics***: **effective field theory** describing the **slow evolution of averaged conserved currents** of fluid close to equilibrium

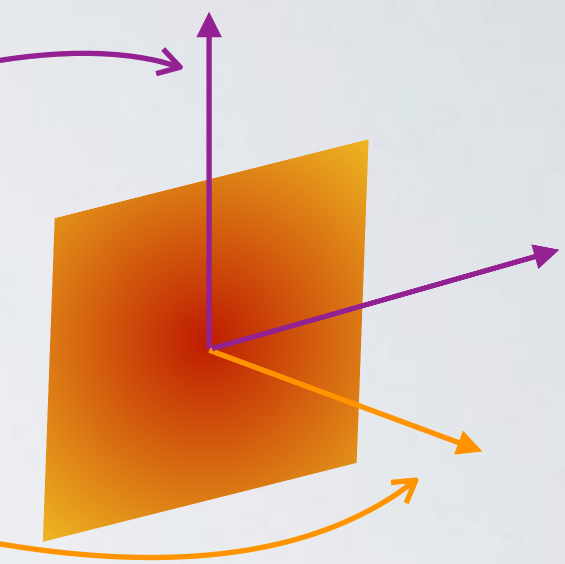
- Very well described by a **perfect fluid + dissipative terms**
- **Memory loss**: evidence of highly non-equilibrium initial conditions?

# RELATIVISTIC HYDRODYNAMICS

***Simplify problem***: expanding plasma, **conformal Bjorken flow**

(conformal invariance, transversely homogeneous,
invariance under longitudinal Lorentz boosts

[Bjoken '83]



$$T^{\mu\nu} = \mathcal{E}\, u^\mu u^\nu + \mathcal{P}(\mathcal{E})(\eta^{\mu\nu} + u^\mu u^\nu) + \Pi^{\mu\nu}$$

$\mathcal{P}(\mathcal{E}) = \mathcal{E}/3 \sim T^4$

dissipative effects

From symmetries: ***All physics encoded in*** $\mathcal{E}(\tau)$.

Analysis: perform a ***large time expansion*** $\tau \gg 1$ .

# ENERGY DENSITY AT LATE TIMES

*Symmetries*: late-time behaviour highly constrained

$$\varepsilon(\tau) \propto \tau^{-4/3}\left(1 + \sum_{k=0}^{+\infty} \frac{\epsilon_k}{\tau^{2k/3}}\right), \quad \tau \gg 1$$

Leading behaviour:
**perfect fluid**

Subleading terms:
**dissipative effects, gradient expansion**

But:

- How to **effectively determine this series**?
- Information of **non-hydrodynamic d.o.f.**?

# EMERGENCE OF HYDRODYNAMICS

**Effective evolution** towards hydrodynamics

***Theories of viscous hydrodynamics***

Truncate gradient expansion of dissipative terms

Non-linear ODEs

[Müller'67;Israel, Stewart'76; Baier,Romatschke,Son,Starinets,Stephanov'07]

*Microscopic **strongly coupled CFT***

Linearised Einstein eqs around black hole solution

Linear PDEs

[Janik,Peschanski'05]

***Kinetic theory** of particle interactions*

Particle distribution function in RTA Boltzmann equation

Integral equation

[Anderson ,Witting'74]

# HYDRODYNAMIC ATTRACTORS

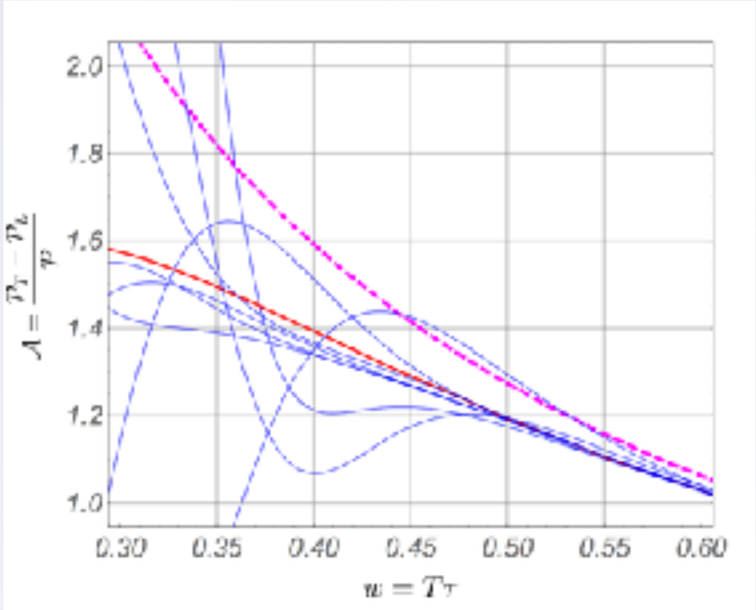
Hydrodynamic description accurate at earlier times than expected

- The hydrodynamic model encodes ***non-hydrodynamic degrees of freedom***, non-perturbative in nature

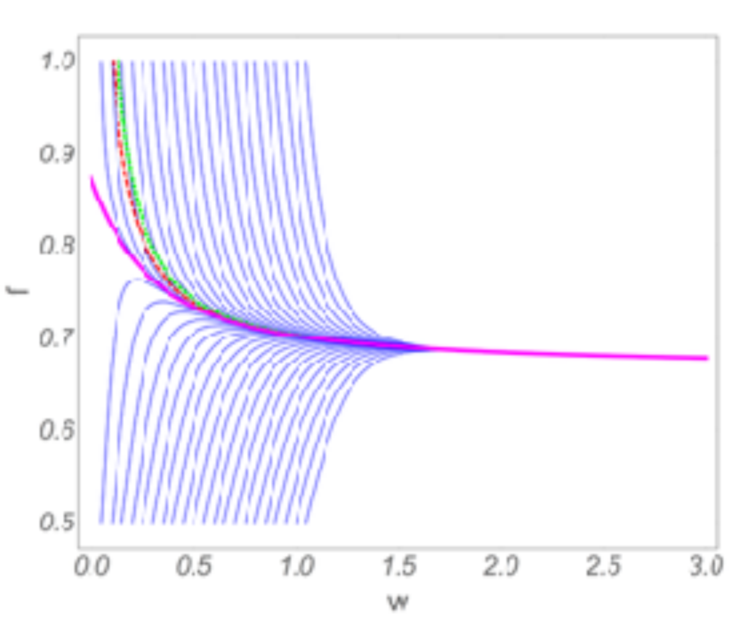
- These modes play a major role during the early times of the expanding plasma, very sensitive to initial conditions

- Still far from equilibrium, the different initial solutions become exponential close to each other

- Evolution of the system towards equilibrium effectively described by viscous hydrodynamics




$\mathcal{N} = 4SYM$ [Spalinski'17]




MIS [Heller,Spalinski'15]

# LATE TIME ASYMPTOTICS

Expectations from asymptotics?

$$\varepsilon(\tau) \propto \tau^{-4/3}\left(1+\sum_{k=0}^{+\infty}\frac{\epsilon_k}{\tau^{2k/3}}\right), \quad \tau \gg 1$$

$$\epsilon_k \sim \Gamma(k)$$

factorially divergent!

***Hydrodynamic attractor:***

- described by a divergent, asymptotic perturbative series;
- asymptotic properties encode all the information about the exponentially small non-hydrodynamic modes;
- initial conditions uniquely encoded in a set of parameters determining the strength of the non-hydrodynamic modes

# ON ASYMPTOTIC SERIES AND STOKES PHENOMENA

# ASYMPTOTIC SERIES

Start with a function $f(g)$ with perturbative expansion

$$f(g) \simeq \sum_{n=0}^{\infty} f_n \, g^n$$

## Asymptotic expansion

$$f_n \, g^n \to \infty$$

no matter how small we choose $g$ to be

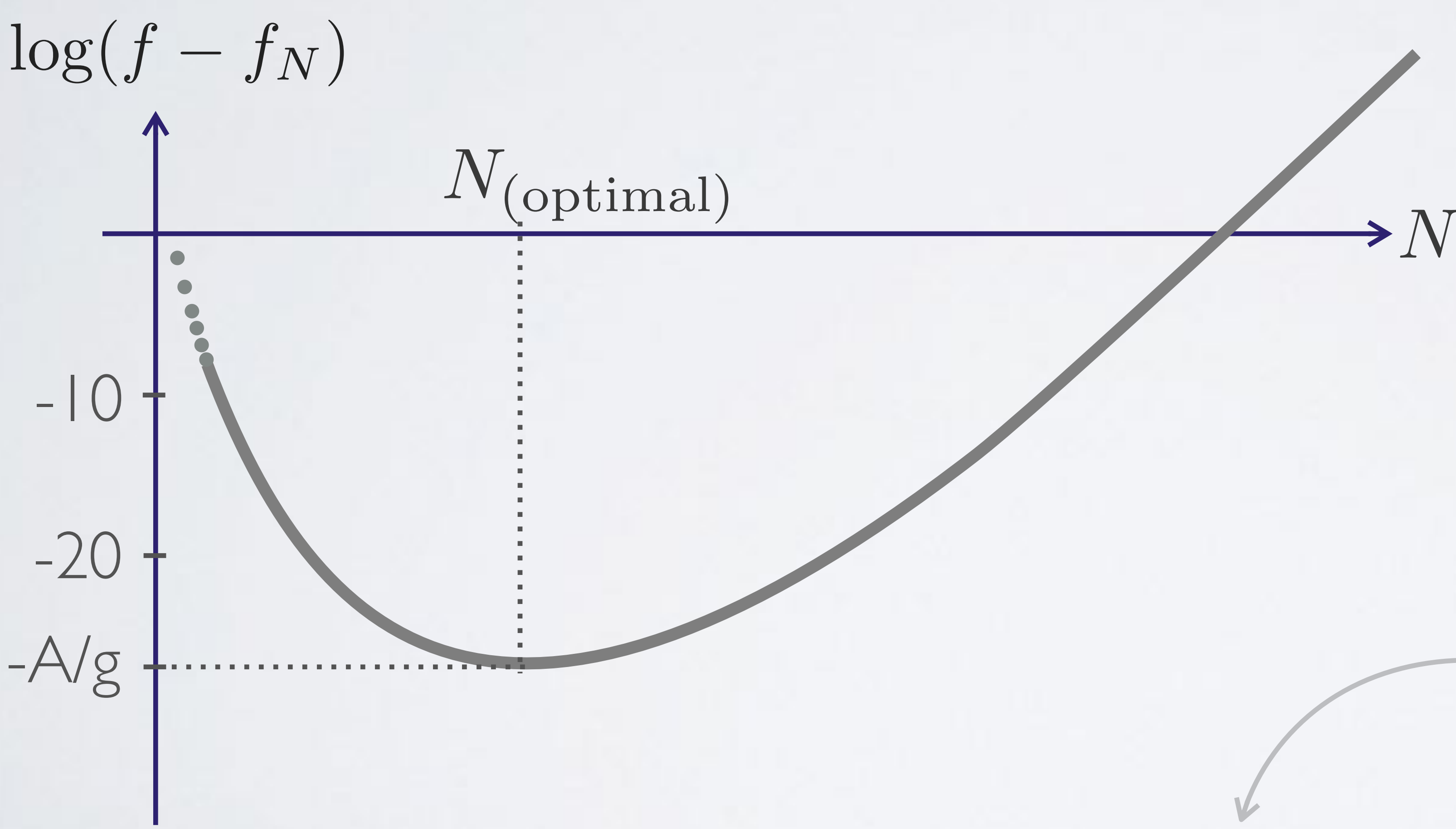
$$\sum_{n=0}^{N} f_n \, g^n$$

Truncating the series: good approximation to the function, with an optimal $N$

# ASYMPTOTIC SERIES

$$f(g) \simeq \sum_{n=0}^{\infty} f_n\, g^n$$

- Assume $g$ fixed and small
- Define $f_N(g) = \sum_{n=0}^{N} f_n\, g^n$




$$N_{(\mathrm{optimal})} \approx A/g$$

Optimal error:

$$(f - f_N)(g) \sim \mathrm{e}^{-A/g}$$

for some value A

Non-perturbative effect!

Exponentially small

## “塵も積もれば、山となる”

*“Even dust, when piled up, will become a mountain.”*

**Uncovering non-perturbative phenomena:**

- Divergent, **asymptotic series** reveal existence of **exponentially small terms**

- ***Stokes phenomena:*** **these small terms appear** and can grow to **dominate** the system

$$\sum_{n\geq 0} a_n\, \varepsilon^n$$

$$\downarrow$$

$$\mathrm{e}^{-A/\varepsilon}$$

How does it work?

# STOKES PHENOMENA & GLOBAL SOLUTIONS

**Local solution:** $$y_0 \sim \sum_{n=0}^{\infty} a_n(x)\,\varepsilon^n \quad x,\,\varepsilon \in \mathbb{C}$$

$|\varepsilon| \ll 1$ $$a_n(x) \sim n!\,A(x)^{-n}$$

$$\mathcal{S}_\theta y(\varepsilon, x, \sigma_i) = \mathcal{S}_\theta y_0 + \sigma_1\,\mathrm{e}^{-\frac{A(x)}{\varepsilon}}\,\mathcal{S}_\theta y_1 + \cdots$$

$\mathcal{B}$




$\sigma_i$

$\mathcal{S}_\theta y_k$

$\mathcal{O}(1)$

$$y \sim y_0 + \sigma_1\,\mathrm{e}^{-\frac{A(x)}{\varepsilon}}\,y_1 + \sigma_2\,\mathrm{e}^{-\frac{2A(x)}{\varepsilon}}\,y_2 + \cdots$$

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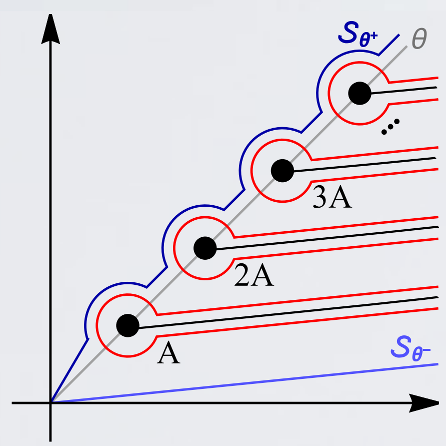
**Integral transform $\mathcal{B}$ (Borel)**




Analysis of singularity structure:

- Emergence of non-perturbative terms

$$\mathcal{S}_\theta y(\varepsilon, x, \sigma_i) = \mathcal{S}_\theta y_0 + \sigma_1\, \mathrm{e}^{-\frac{A(x)}{\varepsilon}}\, \mathcal{S}_\theta y_1 + \cdots$$

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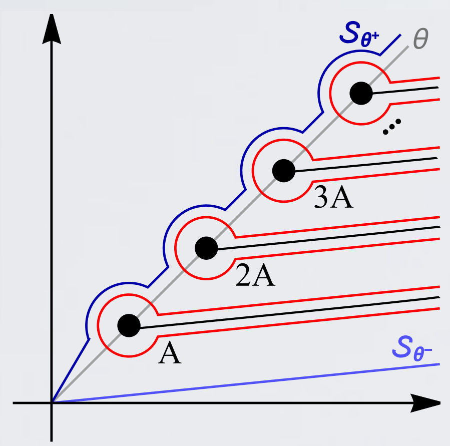


$\mathcal{S}_\theta y_k$

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$\mathcal{O}(1)$

Full description of the system: **Transseries**

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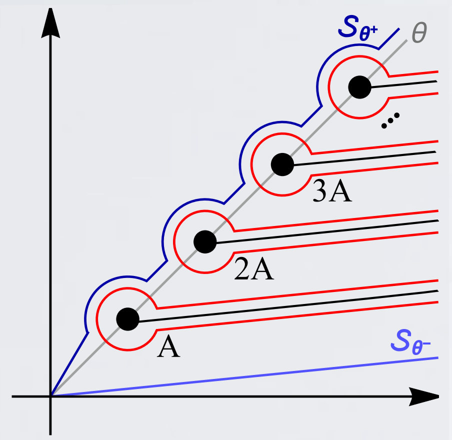


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May appear/grow to become $\mathcal{O}(1)$

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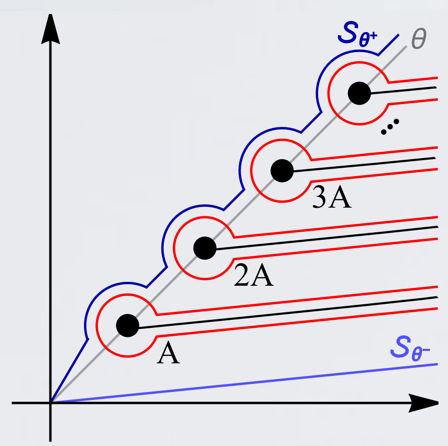


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General **summation** procedures for *each* series $\mathcal{S}_\theta y_k$

$\sigma_i$

$$\mathcal{S}_\theta y(\varepsilon, x, \sigma_i) = \mathcal{S}_\theta y_0 + \sigma_1 \mathrm{e}^{-\frac{A(x)}{\varepsilon}} \mathcal{S}_\theta y_1 + \cdots$$

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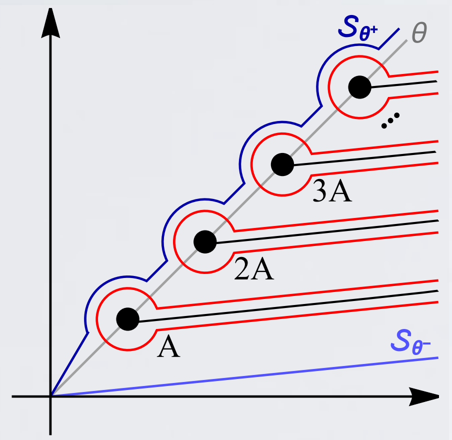


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**Global *analytic* solution:** summed **Transseries** including ***all*** contributions

$$\mathcal{S}_\theta y(\varepsilon, x, \sigma_i) = \mathcal{S}_\theta y_0 + \sigma_1 \mathrm{e}^{-\frac{A(x)}{\varepsilon}} \mathcal{S}_\theta y_1 + \cdots$$

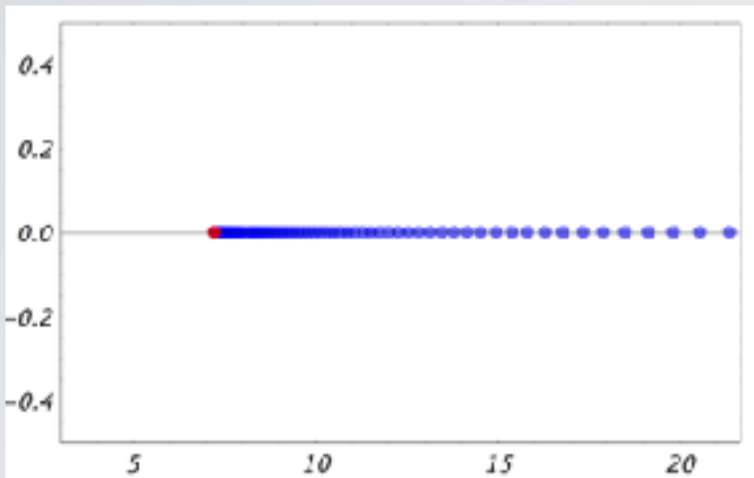
$\sigma_i$ initial data

General **summation** procedures for *each* series $\mathcal{S}_\theta y_k$

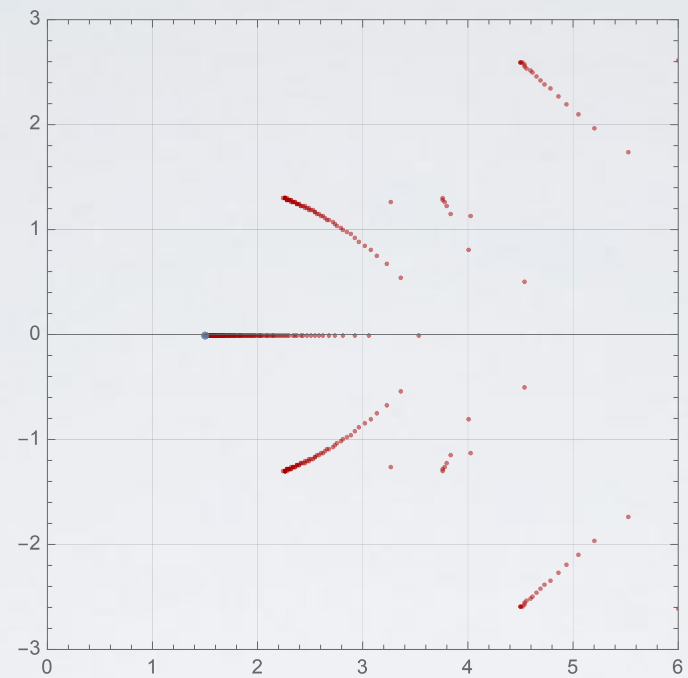
**Stokes phenomena:**

May appear/grow to become $\mathcal{O}(1)$

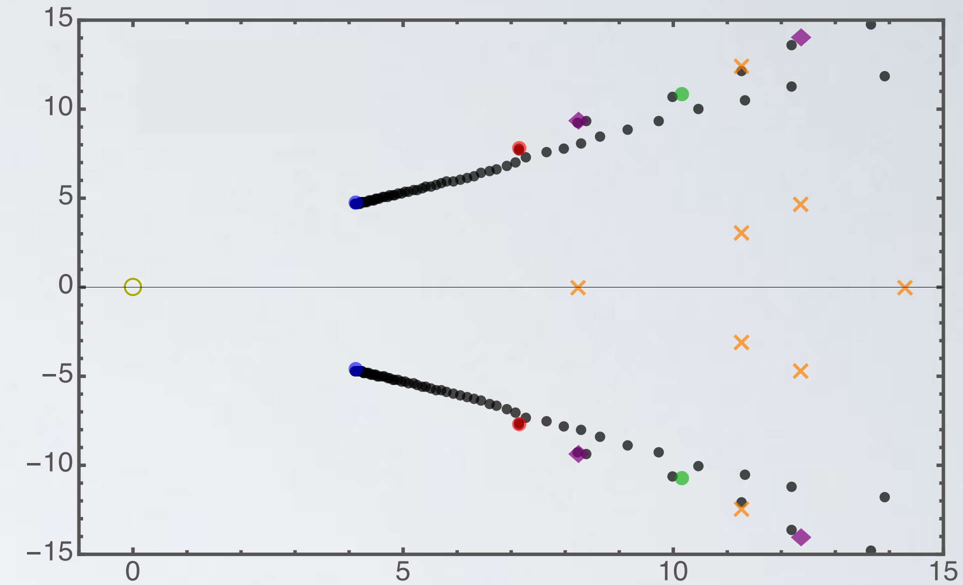
# NON-HYDRO MODES AS SINGULARITIES



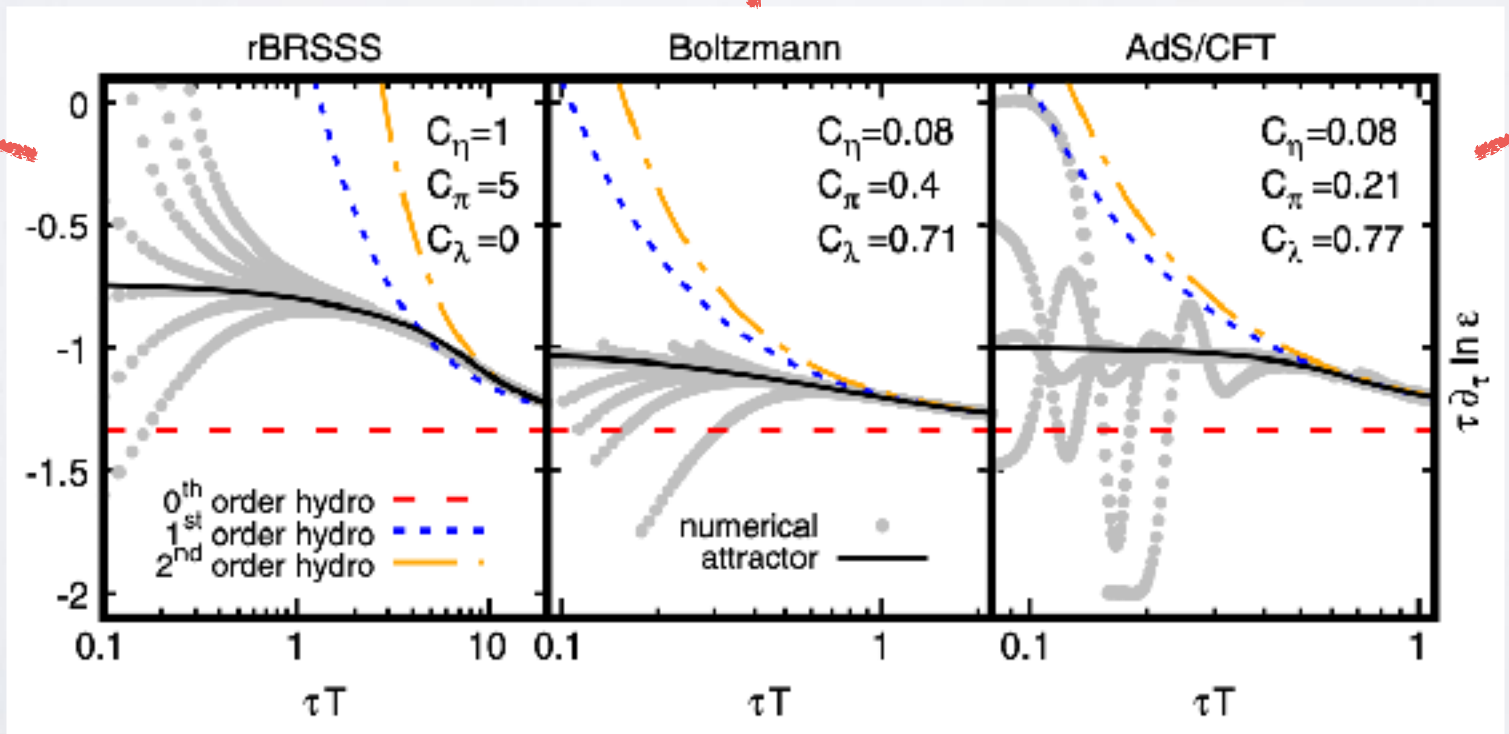

[Heller,Spalinski'15]




[Heller,Kurkela,Spalinski,Svensson'16]




[Heller,Janik,Witaszczyk '15][IA *et al* '18]




[Romatschke'18]

# ASYMPTOTICS IN MIS

# MIS HYDRODYNAMICS

*Non-linear ODE describing the energy density/anisotropy:*

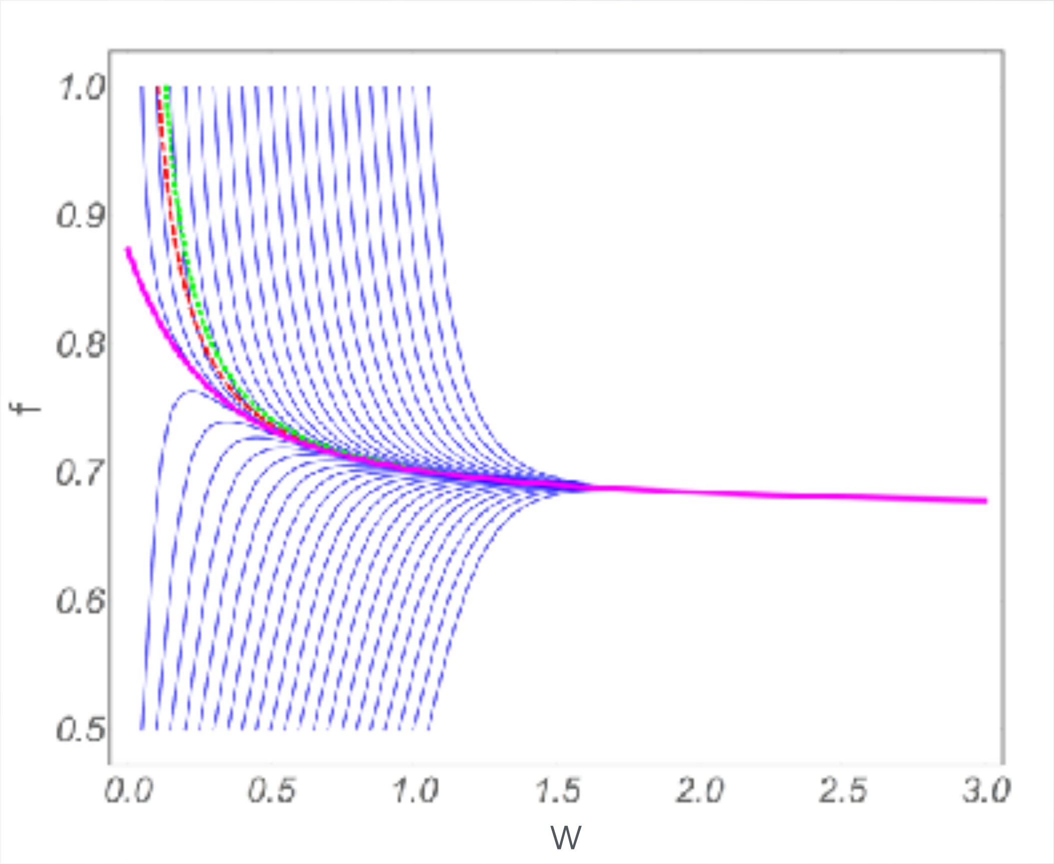
$$w\, C_{\tau\Pi}\, f f' + 4C_{\tau\Pi}\, f^2 + \left(w - \frac{16 C_{\tau\Pi}}{3}\right) f - \frac{4C_\eta}{9} + \frac{16 C_{\tau\Pi}}{9} - \frac{2w}{3} = 0$$

phenomenological parameters

$w = \tau T$ dimensionless parameter

***Attractor solution:*** stable solution, converging to a finite value at early times

***Generic solution***: divergent at early times, decays rapidly towards the attractor solution




[Heller, Spalinski '15; Basar, Dunne '15; IA, Spalinski '15]

# MIS HYDRODYNAMICS

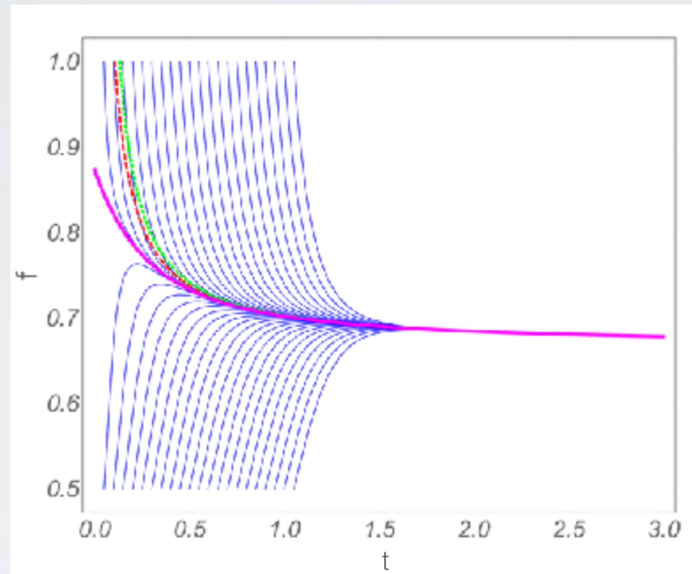
**Early time:** convergent series

***Finite:*** attractor

$$f_{\text{Att}}(w) = \frac{2}{3} + \frac{1}{3}\sqrt{\frac{C_\eta}{C_{\tau\Pi}}} + \mathcal{O}(w)$$

***Generic:*** 1-parameter family

$$f_C(w) = \frac{2C}{3w^4} + \frac{4}{3} + \mathcal{O}(w)$$



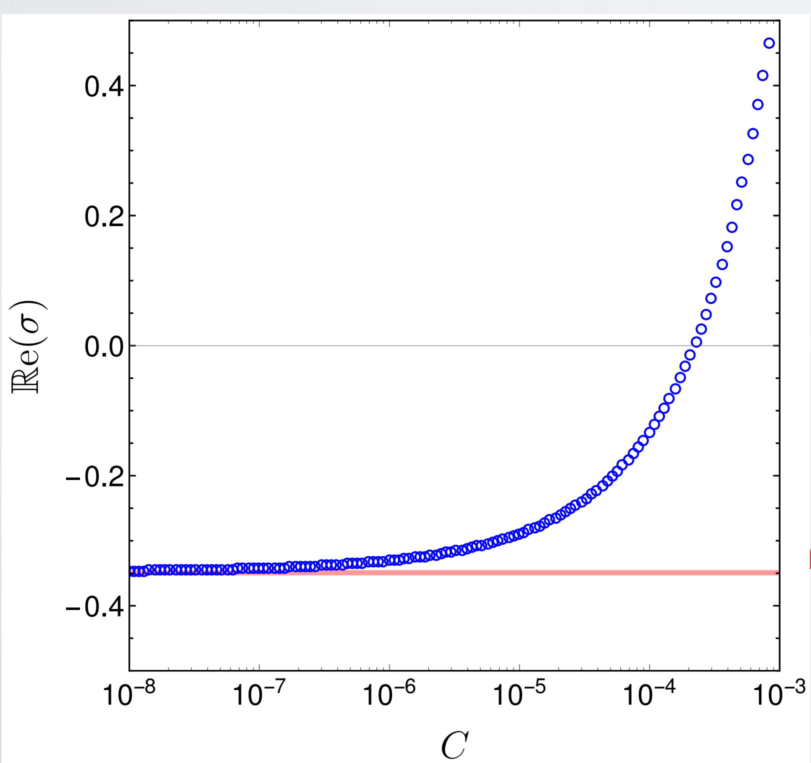

**Late time:** asymptotic series

$$f(w,\sigma) = \sum_{n=0}^{+\infty} \left(\sigma w^{-\beta} \mathrm{e}^{-Aw}\right)^n \Phi_n(w)$$

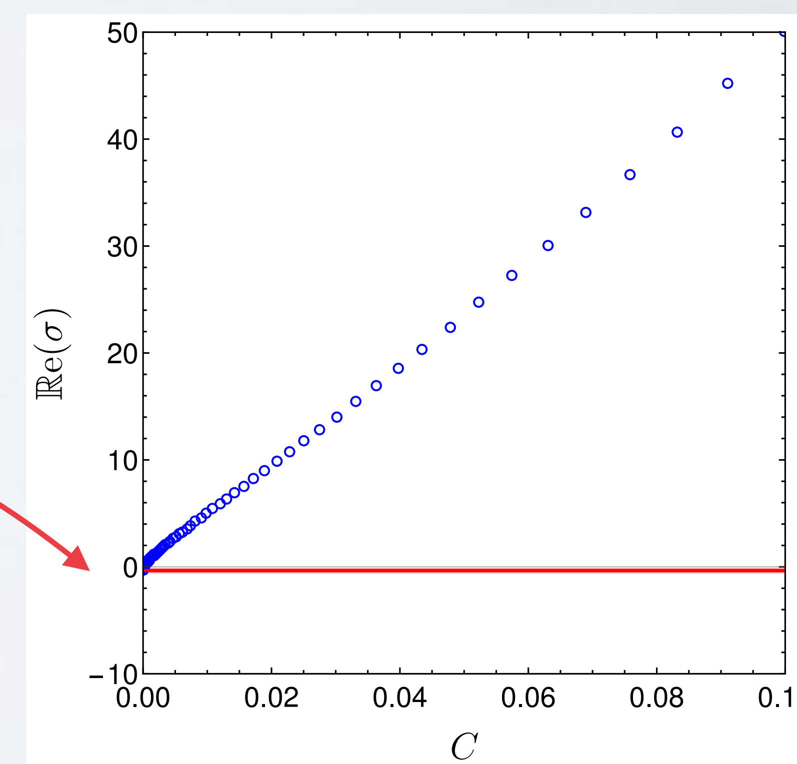
$$\Phi_n(w) \simeq \sum_{k=0}^{+\infty} a_k^{(n)} w^{-k}$$

transseries with single, decaying non-hydro mode

Map $\sigma \leftrightarrow C$




$\mathrm{Re}\left(\sigma_{\text{Attr}}\right) \sim -0.3493$




[IA,Hasenbichler,Olde Daalhuis '22]

# RTA BOLTZMANN ATTRACTOR FROM EARLY TIMES

# BOLTZMANN RTA ATTRACTOR

Boltzmann equation RTA (Bjorken flow) $$\frac{\partial f}{\partial \tau} = \frac{f_{\rm eq} - f}{\tau_R}, \quad \tau_R = \gamma T(\tau)^{-1}$$

Study the early time attractor using dimensionless moments:

$$\mathscr{M}_n \equiv \frac{\mathscr{L}_n}{\mathscr{L}_0}, \quad n \geq 1, \quad \text{where} \qquad \mathscr{L}_n \equiv \int \frac{d^3p}{(2\pi)^3 \tau} p_0 \, P_{2n}(\cos\psi) \, f(\tau, p_0, p_\varsigma)$$

*Analysis of moments via generating function:* $$G_{\mathscr{M}} = \sum_{n \geq 0} x^n \mathscr{M}_n(w)$$

- Obeys a PDE in $x, w$
- Very efficient calculation at early late 'times' $w \sim \tau/\tau_R$

[IA,Noronha,Spalinski'23]

# CONTINUATION FROM EARLY TO LATE TIMES

Pressure anisotropy at early times: $\mathscr{A} = -\frac{\mathscr{M}_1}{3} = \sum_{n\geq 0} a_n w^n$

Convergent series: **Analytically continue to late times**

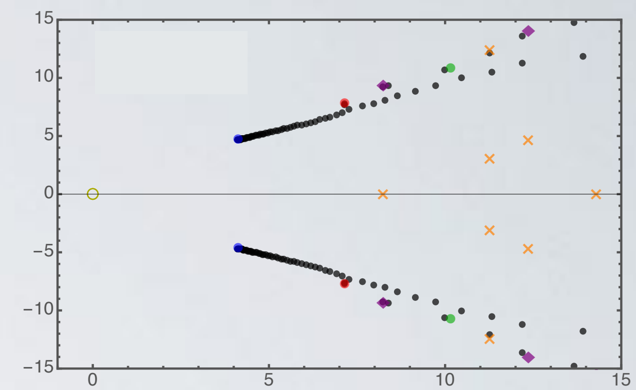
$$\mathrm{P}\mathscr{A}_{(70,71)} \sim \frac{1.60004}{w} + \frac{0.27348}{w^2}$$

Padé approximant: *excellent agreement with hydrodynamic behaviour!*

Predicts value of shear viscosity for RTA Kinetic theory

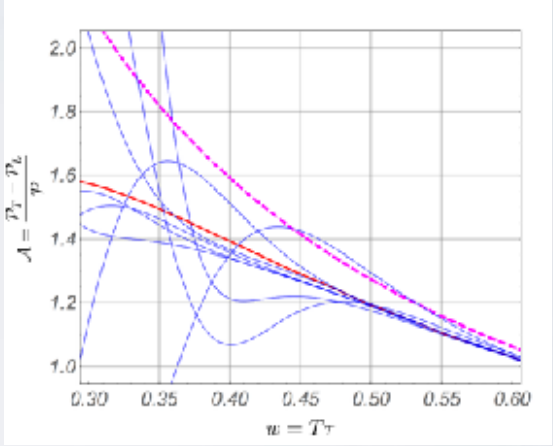
$$\mathscr{A}_{w\gg 1} \sim \frac{8\eta/s}{w} = \frac{8/5}{w}$$

# SUMMARY/OUTLOOK




## Hydrodynamic gradient expansion

- accurate description after decay of non-hydro modes
- asymptotics encodes all "lost" information of non-hydrodynamic modes




## Analytic continuation

- interpolation between initial conditions and hydrodynamic regime
- can be done for convergent and asymptotic series

## Further analytic behaviour in parameter space:

- summation of non-hydro modes: analytic structures in parameter space (branch points, zeros…)
- evidence of phase transitions (transient modes dominating)

## Applications to other (non) hydrodynamic attractors?

THANK YOU!

$$\sum_{n=0}^{\infty} E_n \, g^n \qquad e^{-A/g}$$