Nonthermal fixed points beyond the transport paradigm

ECT* Workshop “Attractors and thermalization in nuclear collisions and cold quantum gases”

Aleksandr N. Mikheev

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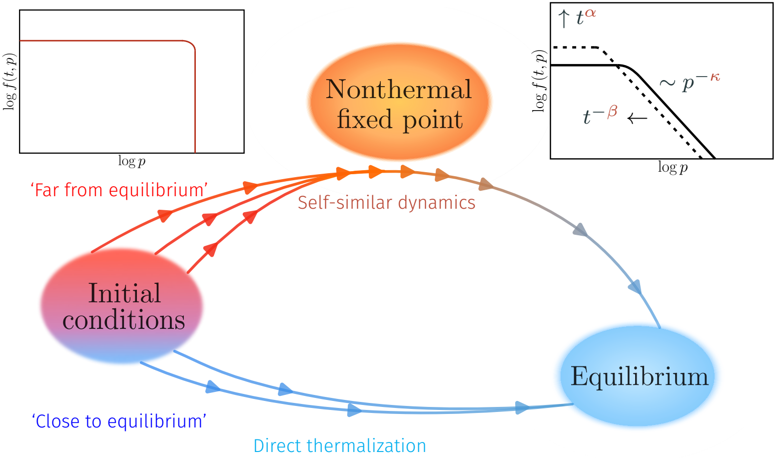
Department of Physics,
University of Konstanz

Dynamics of closed quantum systems out of equilibrium

$$f(t,p) = t^{\alpha} f_{\mathrm{s}}(t^{\beta} p)$$



J. Berges, A. Rothkopf, J. Schmidt, *Phys. Rev. Lett.* **101**, 041603 (2008)

Dynamics of closed quantum systems out of equilibrium

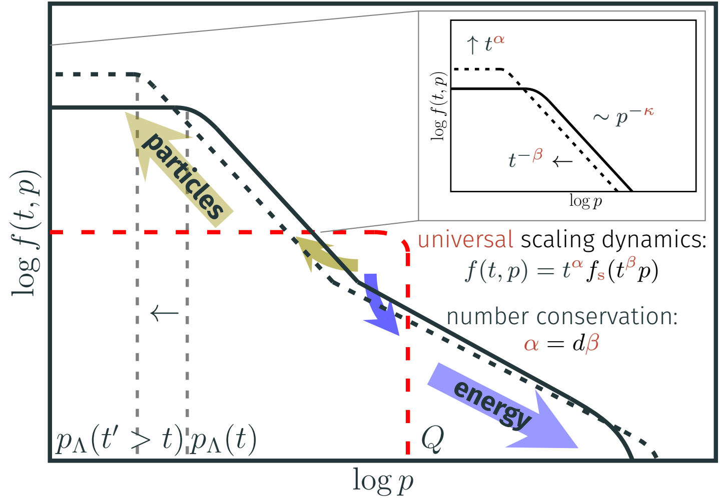


Figure taken from ANM, I. Siovitz, T. Gasenzer, *Eur. Phys. J. Spec. Top.* ***232***, *3393-3415 (2023)*

Dynamics of closed quantum systems out of equilibrium

Experimental observations:

- M. Prüfer, P. Kunkel, H. Strobel, S. Lannig, D. Linnemann, C.-M. Schmied, J. Berges, T. Gasenzer, M. K. Oberthaler, *Nature* ***563***, *217 (2018)*
- S. Erne, R. Bücker, T. Gasenzer, J. Berges, J. Schmiedmayer, *Nature* ***563***, *225 (2018)*
- J.A.P. Glidden, C. Eigen, L.H. Dogra, T.A. Hilker, R.P. Smith, Z. Hadzibabic, *Nat. Phys.* ***17***, *457 (2021)*
- A.D. García-Orozco, L. Madeira, M.A. Moreno-Armijos, A.R. Fritsch, P.E.S. Tavares, P.C.M. Castilho, A. Cidrim, G. Roati, V.S. Bagnato, *Phys. Rev. A* ***106***, *023314 (2022)*
- S. Huh, K. Mukherjee, K. Kwon, J. Seo, S.I. Mistakidis, H.R. Sadeghpour, J. Choi, *Nat. Rhys.* ***20***, *402 (2024)*

Countless numerical examples, e.g.:

- A. Piñeiro Orioli, K. Boguslavski, J. Berges, *Phys. Rev. D* ***92***, *025041 (2015)*
- M. Karl, T. Gasenzer, *New. J. Phys.* ***19***, *093014 (2017)*
- ANM, C.-M. Schmied, T. Gasenzer, *Phys. Rev. A* ***99***, *063622 (2019)*
- C.-M. Schmied, ANM, T. Gasenzer, *Phys. Rev. Lett.* ***122***, *170404 (2019)*
- K. Boguslavski, A. Piñeiro Orioli, *Phys. Rev. D* ***101***, *091902 (2020)*

Common theoretic perspective: Particle/energy transport $\implies$ kinetic theory
Solve numerically/semi-analytically, study scaling properties of the scattering integral $I[f](t, \boldsymbol{p})$, ...

Kinetic theory perspective

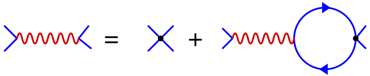
- Consider a $\mathrm{U}(N)$-symmetric Bose gas with quartic contact interactions:

$$\hat{H}_{\mathrm{U}(N)} = \int_{\boldsymbol{x}} \left[\hat{\Phi}_a^\dagger \left(-\frac{\nabla^2}{2m} \right) \hat{\Phi}_a + \frac{g}{2} \left(\hat{\Phi}_a^\dagger \hat{\Phi}_a \right)^2 \right]$$

- Quantum Boltzmann equation (2-to-2, $f \gg 1$):

$$\partial_t f(t, \boldsymbol{p}) = \underbrace{\int_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{r}} |T_{\boldsymbol{pkqr}}|^2 \left(f_{\boldsymbol{p}} f_{\boldsymbol{q}} f_{\boldsymbol{r}} \pm \text{perm}^s \right) \delta\left(\sum \boldsymbol{p}_i \right) \delta\left(\sum \omega(\boldsymbol{p}_i) \right)}_{I[f](t,\boldsymbol{p})}$$

- Each loop brings one coupling g and one propagator G $\implies$ expansion in $\sim g\,G \sim g f$ $\implies$ fails in the IR region once $f \sim p^{-\kappa} \gtrsim g^{-1}$
- Collective effects have to be taken into account $\implies$ standard solution: $1/N$ expansion (resummation of an infinite number of bubble diagrams)



resummed effective coupling

$$\implies \quad g_{\mathrm{eff}}(\boldsymbol{p}) \sim p^2$$

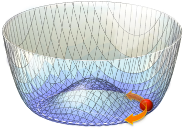
G. Aarts, D. Ahrensmeier, R. Baier, J. Berges, J. Serreau, *Phys. Rev. D* **66**, 045008 (2002)
R. Walz, K. Boguslavski, J. Berges, *Phys. Rev. D* **97**, 116011 (2018)
I. Chantesana, A. Piñeiro Orioli, T. Gasenzer, *Phys. Rev. A* **99**, 043620 (2019)

Kinetic theory perspective

- Quantum Boltzmann equation (2-to-2, $f \gg 1$):

$$\partial_t f(t,\boldsymbol{p}) = \underbrace{\int_{\boldsymbol{k},\boldsymbol{q},\boldsymbol{r}} |T_{\boldsymbol{pkqr}}|^2 \left(f_{\boldsymbol{p}} f_{\boldsymbol{q}} f_{\boldsymbol{r}} \pm \text{perm}^s\right) \delta\left(\sum \boldsymbol{p}_i\right) \delta\left(\sum \omega(\boldsymbol{p}_i)\right)}_{I[f](t,\boldsymbol{p})}$$

- Each loop brings one coupling g and one propagator $G \implies$ expansion in $\sim g\,G \sim g f \implies$ fails in the IR region once $f \sim p^{-\kappa} \gtrsim g^{-1}$
- Alternative: describe IR dynamics in terms of d.o.f. that are relevant in the IR $\implies$ low-energy effective theory



U(N)-symmetric Bose gas $\implies$ N Goldstones:
$N-1$ quadratic ones (phase diffs), with $\omega_Q(\boldsymbol{p}) \sim p^2$,
and a linear Bogoliubov (total phase), with $\omega_B(\boldsymbol{p}) \sim p$

The effective coupling emerges again via momentum-dependent interactions between the Goldstone modes

ANM, C.-M. Schmied, T Gasenzer, *Phys. Rev. A* ***99***, *063622 (2019)*
C.-M. Schmied, ANM, T. Gasenzer, *Phys. Rev. Lett.* ***122***, *170404 (2019)*

Results comparison

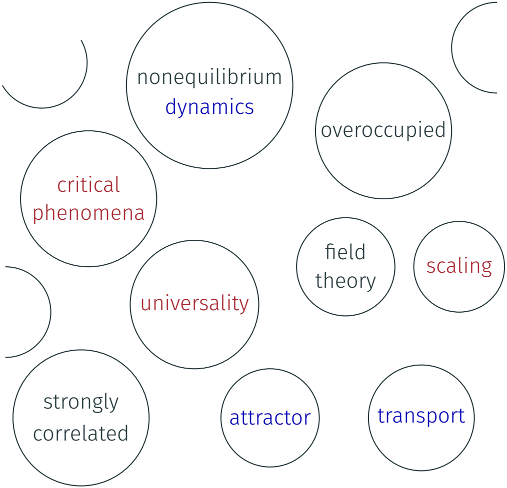
Remarkable agreement between next-to-leading order $1/N$ resummation, two-loop EFT, and $N = 3$ classical-statistical simulations

	EFT [1]	$1/N$ expansion [2]	Numerics [1,3]	Experiment [4]
sym	$\mathrm{U}(N \to \infty)$	$\mathrm{U}(N \to \infty)$	$\mathrm{U}(3)$	$\mathrm{SO}(2) \times \mathrm{U}(1)$
dim	d	d	3	1
α	$d/2$	$d/2$	1.62 ± 0.37	0.54 ± 0.06
β	$1/2$	$1/2$	0.53 ± 0.09	0.33 ± 0.08
κ	$d+1$	$d+1$	$\simeq 4$	$\simeq 2.6$

Kinetic theory framework has proven incredibly successful in describing nonthermal fixed points! But is it how we usually think about scaling?

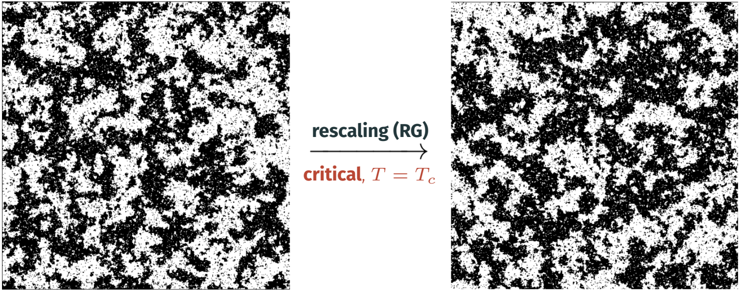
[1] ANM, C.-M. Schmied, T. Gasenzer, *Phys. Rev. A* ***99***, *063622 (2019)*
[2] I. Chantesana, A. Piñeiro Orioli, T. Gasenzer, *Phys. Rev. A* ***99***, *043620 (2019)*
[3] C.-M. Schmied, ANM, T. Gasenzer, *Phys. Rev. Lett.* ***122***, *170404 (2019)*
[4] M. Prüfer, P. Kunkel, H. Strobel, S. Lannig, D. Linnemann, C.-M. Schmied, J. Berges, T. Gasenzer, M.;K. Oberthaler, *Nature* ***563***, *217 (2018)*

NTFPs “tag cloud”



(Nonequilibrium) scaling: RG perspective

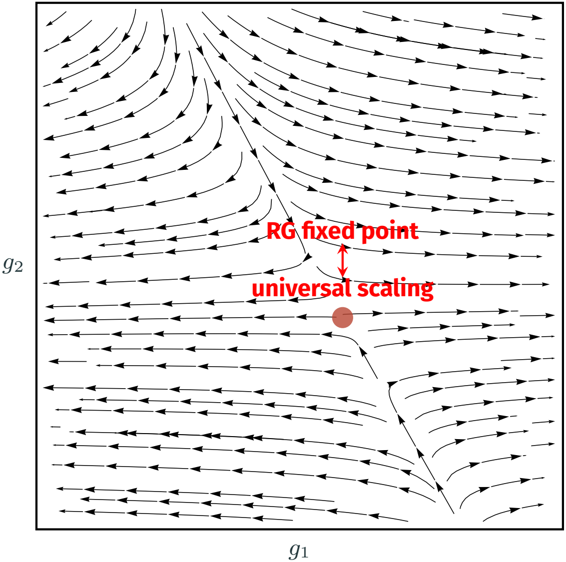
- Critical phenomena/scaling $\iff$ RG fixed points. Example from statistical field theory: 2D Ising model



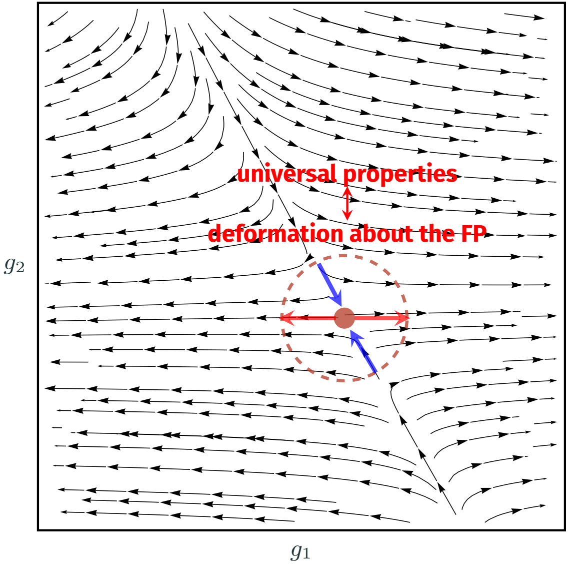
- Self-similarity/scale-invariance!
- Substantial progress also in stationary nonequilibrium:

J. Berges, G. Hoffmeister, *Nucl. Phys. B* **813**, 383 (2009)
J. Berges, D. Mesterházy, *Nucl. Phys. B Proc. Suppl.* **228**, 37 (2012)
S. Mathey, J.M. Pawlowski, T. Gasenzer, *Phys. Rev. A* **92**, 023635 (2015)
...

RG fixed points and universal scaling



RG fixed points and universal scaling



(Equilibrium) fixed-point equation

- The regularized (inverse) propagator in the vicinity of a fixed point:

$$\Gamma_k^{(2)}(p) = \underbrace{\Gamma^{(2)}(p)}_{\text{scaling}} \left[1 + \delta Z_k(p)\right], \quad \Gamma^{(2)}(p) \sim p^{\kappa}$$

  Since the RG scale k is the only remaining scale $\implies$ δZ_k should only depend on k through $p^2/k^2 \equiv x$

- k-dependence of $\Gamma_k^{(2)}$ is governed by the Wetterich flow equation

$$k\partial_k \Gamma_k^{(2)}(\boldsymbol{p}) = \mathrm{Flow}_2\left[\Gamma_k^{(m\leq 4)}\right](\boldsymbol{p}) = \int_{\boldsymbol{q}} I_k \left[\Gamma_k^{(m\leq 4)}\right](\boldsymbol{p}, \boldsymbol{q})$$

- For a 2PI-like truncation scheme, $k\partial_k \Gamma_k^{(2)} = k\partial_k \Sigma_k[G_k]$, the integrated flow takes the form

$$\delta Z(x) = \lambda_d \int_x^{\infty} \frac{\mathrm{d}x'}{x'^{\chi(\kappa)}} f(x'; \delta Z, \kappa)$$

J.M. Pawlowski, D.F. Litim, S. Nedelko, L. von Smekal, *Phys. Rev. Lett.* ***93***, *152002 (2004)*
S. Mathey, T. Gasenzer, J.M. Pawlowski, *Phys. Rev. A* ***92***, *023635 (2015)*

Asymptotic behavior

- For $x \to \infty \iff k \to 0$, we should recover $\Gamma^{(2)} \implies \delta Z(x \to \infty) = 0$
- For $x \to 0 \iff k \to \infty$, the regulator dominates, so the scaling must be canceled:

$$\delta Z(x \to 0) = -1 + \frac{\Gamma^{(2)}_{k\to\infty}(\boldsymbol{p})}{\Gamma^{(2)}(\boldsymbol{p})} = -1 + \frac{\hat{S}^{(2)}(x)}{\hat{\Gamma}^{(2)}(x)}$$

- The second term consists of something finite and/or diverging, so

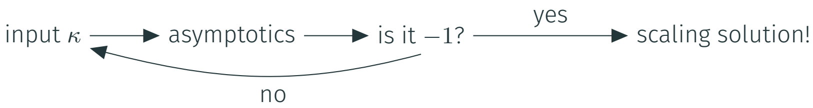
$$\delta Z(x \to 0) = \underbrace{-1}_{\text{the key!}} + \hat{m}^2 x^{-\#} + \text{subdominant}$$

- For $x \ll 1$ the integrated flow may be simplified as

$$\delta Z(x \ll 1) = \lambda_d \int_x^\infty \frac{\mathrm{d}x'}{x'^{\chi(\kappa)}} f(x'; \delta Z \to 0, \kappa) = -1 + \dots$$

$\implies$ vary κ and see if you get -1. If so, proper scaling solution!

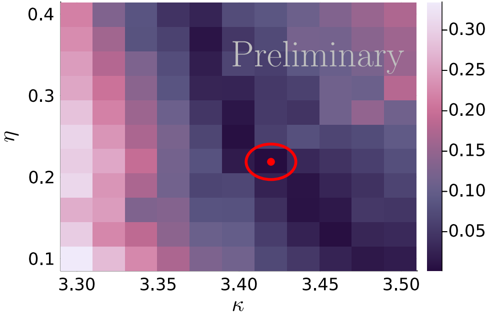
- Recipe for fixed points:



Searching for NTFP scaling solutions

- Nonequilibrium $\Longrightarrow$ two propagators (F and ρ) $\Longrightarrow$ two asymptotics
- Two scaling exponents: $\beta \equiv 1/(2-\eta)$ and κ
- It's convenient to introduce an auxiliary constraint function

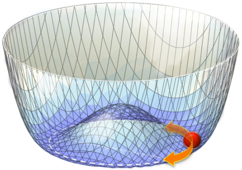
$$Q(\hat{\tau}, \hat{\omega}, x) \equiv \left| \frac{\delta Z_F^W(\hat{\tau} \to \infty, \hat{\omega} \to 0, x \to 0)}{\delta Z_\rho^W(\hat{\tau} \to \infty, \hat{\omega} \to 0, x \to 0)} - 1 \right| \to 0$$

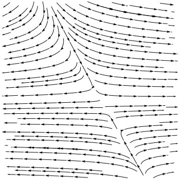


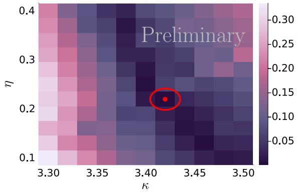
ANM, J.M. Pawlowski, T. Gasenzer, *work in progress*

Summary and outlook

- NTFPs are often viewed from the dynamical perspective through the lens of formalisms such as kinetic theory and hydrodynamics
- Complementary perspective: NTFPs = self-consistent scaling solutions of nonequilibrium quantum field theories
- A natural framework for scaling is renormalization group: universal properties $\iff$ RG fixed-point equations
- Other "alternative perspectives": self-consistency condition for the Dyson equation (work with V. Noel, C. Huang, J. Berges), holography, ...
- Un(der)explored directions: relevant/irrelevant initial-condition terms, timescales for approach/departure from attractor as $V \to \infty$, ...
  Perhaps more natural to address through frameworks beyond transport?







Backup slides

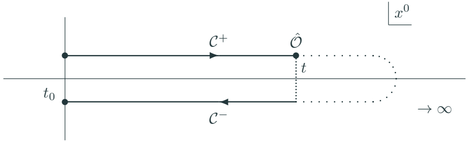
Nonequilibrium case: parametrization

- Nonequilibrium $\Longrightarrow$ double the number of fields/propagators:

$$\underbrace{F(x,y) = \big\langle\{\hat\varphi(x),\hat\varphi(y)\}\big\rangle - \langle\hat\varphi(x)\rangle\langle\hat\varphi(y)\rangle}_{\text{statistical function} \leftrightarrow \text{occupancies}}, \quad \underbrace{\rho(x,y) = \big\langle[\hat\varphi(x),\hat\varphi(y)]\big\rangle}_{\text{spectral function} \leftrightarrow \text{spectrum}}$$

- In Keldysh basis:

$$\hat\varphi_{\mathrm{cl}} = \frac{\hat\varphi_+ + \hat\varphi_-}{2}$$
$$\hat\varphi_{\mathrm{q}} = \hat\varphi_+ - \hat\varphi_-$$



- Accordingly, two δZ's:

$$\Gamma^{(2)}_{k,\mathrm{clq}}(t,t',\boldsymbol{p}) = \Big[\Gamma^{(2)}_{\mathrm{clq}} - \mathrm{i}\Gamma^{(2)}_{\mathrm{clq}} \circ (G_{\mathrm{qcl}}\delta Z_{k,\,\rho}) \circ \Gamma^{(2)}_{\mathrm{clq}}\Big](t,t',\boldsymbol{p})$$
$$\Gamma^{(2)}_{k,\mathrm{qq}}(t,t',\boldsymbol{p}) = \Big[\Gamma^{(2)}_{\mathrm{qq}} + \mathrm{i}\Gamma^{(2)}_{\mathrm{qcl}} \circ (F\delta Z_{k,F}) \circ \Gamma^{(2)}_{\mathrm{clq}}\Big](t,t',\boldsymbol{p})$$

ANM, J.M. Pawlowski, T. Gasenzer, *work in progress*

Model and vertices

- Consider phase fluctuations (phonons) in a single-component Bose gas. To leading order:

$$\mathcal{L}_{\text{eff}}^{\text{LO}} = P\left(D_t\theta\right), \quad D_t\theta = \partial_t\theta - \frac{1}{2m}\left(\nabla\theta\right)^2 \equiv D_t\varphi - \mu$$

$\Longrightarrow$ momentum-dependent vertices!

- Simplest approximation = truncate at P_2
- Galilei Slavnov-Taylor identities control the ratios between the vertices

[L. Canet, H. Chaté, B. Delamotte, N. Wschebor, *Phys. Rev. E* ***86***, *061128 (2012)*]

$$\mathcal{L}_{\text{int}} = \frac{g_{\text{cl}}}{4}\partial_t\varphi_{\text{q}}\left(\nabla\varphi_{\text{cl}}\right)^2 - \frac{g_{\text{cl}}}{2}\partial_t\varphi_{\text{cl}}\nabla\varphi_{\text{cl}}\nabla\varphi_{\text{q}} + \frac{g_{\text{cl}}^2}{4}\left(\nabla\varphi_{\text{cl}}\right)^3\nabla\varphi_{\text{q}} - \cancel{\frac{g_{\text{q}}}{16}\partial_t\varphi_{\text{q}}\left(\nabla\varphi_{\text{q}}\right)^2} + \cancel{\frac{g_{\text{q}}g_{\text{cl}}}{16}\nabla\varphi_{\text{cl}}\left(\nabla\varphi_{\text{q}}\right)^3}$$

leaving only one coupling, g_{cl}, in the classical-statistical approximation

- Simplest choice for vertices: $\Gamma_k^{(n>2)} \propto S^{(n>2)}$

D. T. Son, M. Wingate, *Annals Phys.* ***321***, *197 (2006)*
M. Greiter, F. Wilczek, E. Witten, *Mod. Phys. Lett. B* ***03***, *903 (1989)*
Y. Takahashi, *Fortschr. Phys.* ***36***, *83 (1988)*
Y. Takahashi, *Fortschr. Phys.* ***36***, *63 (1988)*

Scaling forms and the regulator

- Full propagators $F, \rho = \underbrace{\text{scaling functions}}_{\text{input}} + \underbrace{\text{scaling exponents } \boldsymbol{\kappa}}_{\text{to be determined}}$

  Scaling forms:

$$\rho(\tau, \sigma, \boldsymbol{p}) = \frac{\sin[\omega(\tau, \boldsymbol{p})\, \sigma]}{\omega(\tau, \boldsymbol{p})}\,, \qquad F(\tau, \sigma, \boldsymbol{p}) = \frac{\tau^{-\gamma}}{p^{\kappa}} \cos[\omega(\tau, \boldsymbol{p})\, \sigma]$$

  where $\omega(\tau, \boldsymbol{p}) = \tau^{-\beta(z-z_0)}\, |\boldsymbol{p}|^{z_0}$, $z_0 \to 1$, and $z = 1/\beta \equiv 2 - \eta$

- Simplest choice for the regulator concerns only spatial fluctuations:

$$R_{k,\mathrm{ab}}(t, t', \boldsymbol{p}) = \Gamma^{(2)}_{\mathrm{ab}}(t, t', \boldsymbol{p})\, r_{\mathrm{ab}}\left(\boldsymbol{p}^2/k^2\right)$$

  with $r_{\mathrm{clq}} = r_{\mathrm{qcl}} = r_{\mathrm{qq}} \equiv r$, $r_{\mathrm{clcl}} \equiv 0$. Doesn't explicitly change the scaling & symmetries preserved by construction!

- With this, we have all the ingredients to solve for the deformation functions δZ

K. Boguslavski, A. Piñeiro Orioli, *Phys. Rev. D* ***101***, *091902 (2020)*
A. Piñeiro Orioli, J. Berges, *Phys. Rev. Lett.* ***122***, *150401 (2019)*