Hydrodynamic Attractor in Ultracold Atoms

KF & Y. Nishida, Phys. Rev. A **98**, 063634 (2018);
KF & T. Enss, Phys. Rev. Lett. **133**, 173402 (2024)

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Attractors and thermalization in nuclear collisions and cold quantum gases

September 2025

UNIVERSITÄT
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STRUCTURES
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collaborators


Yusuke
Nishida


Tilman
Enss

Plan of this talk

1. Introduction : Realization in ultracold atoms

- Hydrodynamics & Hydro attractor
- Our idea: Using scattering-length driving

2. Time-dependent scattering length in hydrodynamics

KF & Y. Nishida, PRA **98**, 063634 (2018).

3. Our proposed driving protocol and attractor

KF & T. Enss, Phys. Rev. Lett. **133**, 173402 (2024)

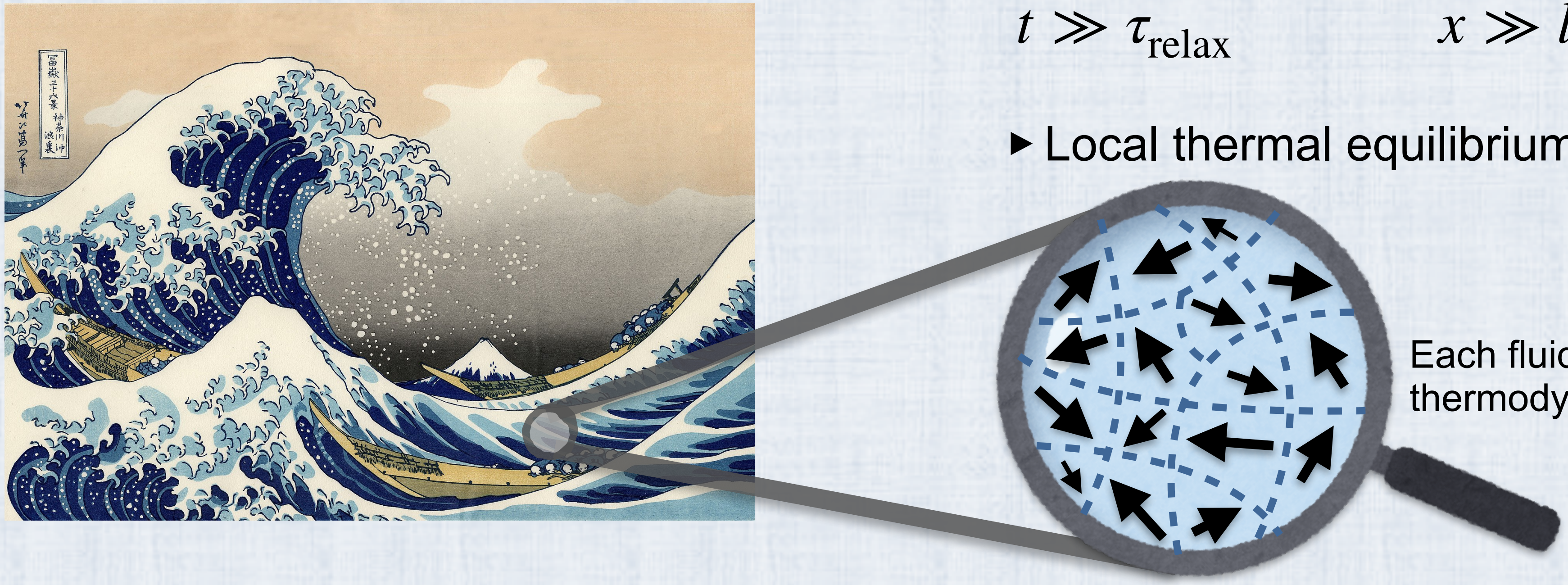
4. Summary

Hydrodynamics

Hydrodynamics universally describes long-time and long-distance dynamics.

$t \gg \tau_{\mathrm{relax}}$ $\qquad$ $x \gg l_{\mathrm{mfp}}$

- Local thermal equilibrium assumption



Each fluid cell is described by thermodynamic densities and fluid velocities

Hydrodynamic equations are constructed based on **the derivative expansion** w.r.t ∂_t and ∂_x.

$\sim \tau_{\mathrm{relax}}/t$ $\qquad$ $\sim l_{\mathrm{mfp}}/x$

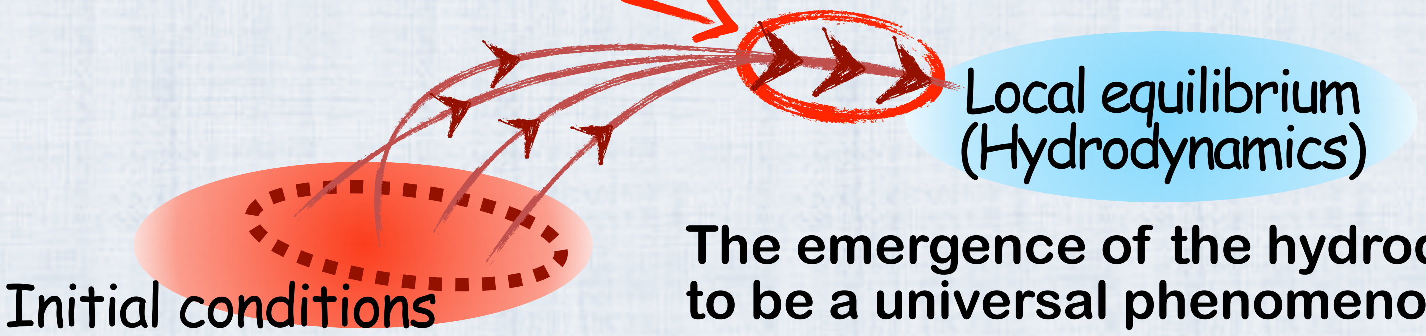
For example,

Stress tensor in hydrodynamics : $T_{ij} = \underset{\text{0th order}}{\underline{p\delta_{ij} + \rho v_i v_j}} + \underset{\text{1st order}}{\underline{(\text{viscous terms})}} + \underset{\text{higher order}}{\underline{\cdots}}$

Observation in real-time dynamics

✓ Hydrodynamic Attractor

☑ Alexander's overview talk



The emergence of the hydrodynamic attractors is expected to be a universal phenomenon, not limited to the QGP.

- Model calculations show the emergence of attractors in various fluid expansions.

Can we observe this attractor behavior in real-time dynamics?

😢 In heavy-ion collisions, we can only access the final-state particle momenta as observables.

👍 **Ultracold atom experiments offer well-controlled time-resolved measurements.**

➡ **Proposal of "a fluid expansion" leading to the attractor, realizable in ultracold atoms**

Our key idea: KF & T. Enss Phys. Rev. Lett. **133**, 173402 (2024)

Realizing phenomena equivalent to fluid expansions by driving the scattering length.

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1. Introduction : Realization in ultracold atoms

Proposal of "a fluid expansion" realizable in ultracold atoms leading to the attractor

Our key idea:

Realizing phenomena equivalent to fluid expansions by driving the scattering length.

2. Time-dependent scattering length in hydrodynamics

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Time-dep. scattering length in hydrodynamics

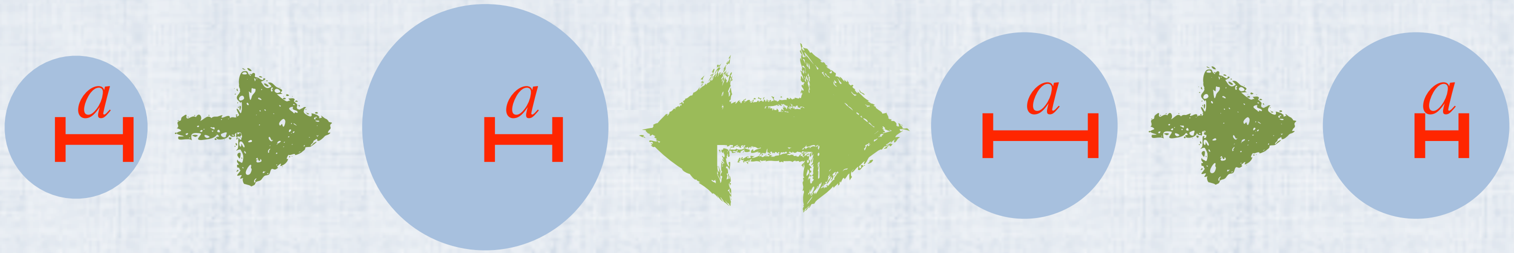
Ultracold atomic gases :

- Their inter-particle interaction is characterized only by the (*s*-wave) scattering length a.
- The scattering length a can be tuned via the Feshbach resonance.
  Its spatiotemporal modulation is also possible.

➡ **Time-dependent scattering length $a(t)$**

Hydrodynamically, this $a(t)$ results in the same effect as isotropic fluid expansion.

KF & Y. Nishida, PRA **98**, 063634 (2018)



In terms of $\dfrac{\text{Fluid Size}}{\text{Scattering Length}}$ these two phenomena are equivalent.

Isotropic expansion & contraction $=$ **Shrinking & stretching of the scattering length**

Time-dep. scattering length in hydrodynamics

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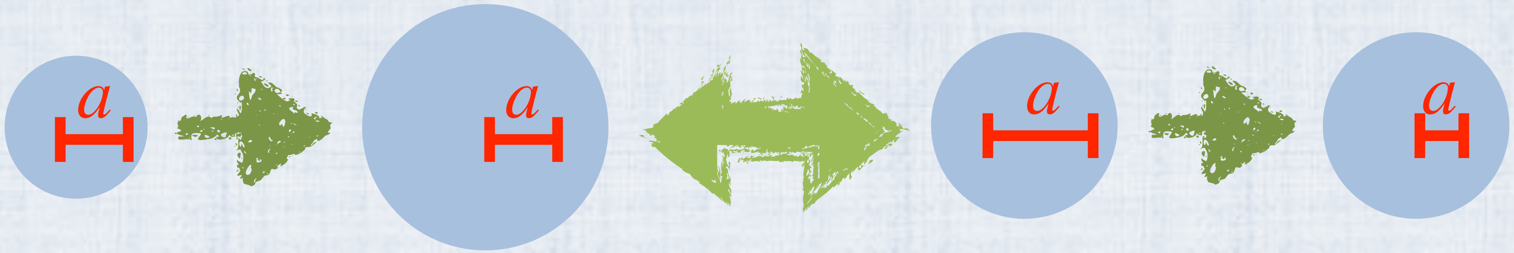

In terms of $\dfrac{\textbf{Fluid Size}}{\textbf{Scattering Length}}$ these two phenomena are equivalent.

✓ Driving the scattering length allows us to **emulate arbitrary isotropic fluid expansion,** while the system remains uniform and at rest.

Dynamics with time-dep. scattering length

Focusing on two-component Fermi gases in the normal phase (uniform, 3-dim)

- Energy density production (Dynamic sweep theorem) S. Tan, Ann. Phys. (2008)

$$\frac{d}{dt}\mathcal{E}(t) = \frac{C(t)}{4\pi m a^2(t)}\frac{d}{dt}a(t)$$

$\mathcal{E}(t)$: Energy density

$C(t)$: Contact density

- Contact density (conjugate quantity to the scattering length)

$$C(t) = C_{\text{eq}}[a(t)] + \underline{12\pi m a(t)\pi(t)}$$

Dissipative correction

$\pi(t)$ represents **the pressure deviation** from its equilibrium value.

cf. Pressure relation $P = \frac{2}{3}\mathcal{E} + \frac{C}{12\pi m a}$

- Hydrodynamic relaxation dynamics for $\pi(t)$

$$\boxed{\tau_{\text{relax}}\partial_t\pi(t) + \pi(t) = -\underline{\zeta[a(t)]}V_a(t)} \quad \text{with} \quad \boxed{V_a(t) = -3\frac{\partial_t a(t)}{a(t)}}$$

Bulk viscosity at $a(t)$

The bulk viscosity is given by the contact-contact correlation

☑ Tilman's talk

Bulk strain rate tensor

In general situation with fluid velocity $\vec{v}(t,\vec{x})$

$$V_a(t,\vec{x}) = \nabla\cdot\vec{v}(t,\vec{x}) - 3\left[\frac{\partial_t a(t,\vec{x})}{a(t,\vec{x})} + \vec{v}(t,\vec{x})\cdot\frac{\nabla a(t,\vec{x})}{a(t,\vec{x})}\right]$$

KF & Y. Nishida, PRA **98**, 063634 (2018).

✓ **The consequence of the equivalence**

Dynamics of the pressure deviation

▸ Hydrodynamic relaxation dynamics for $\pi(t)$

$$\boxed{\tau_{\mathrm{relax}}\partial_t\pi(t)+\pi(t)=-\underline{\zeta[a(t)]}V_a(t)} \quad \text{with} \quad \boxed{V_a(t)=-3\frac{\partial_t a(t)}{a(t)}}$$

- cf. Muller-Israel-Stewart theory in relativistic hydrodynamics
- One can derive this equation from the linear-response theory with exponential relaxation.

$$\delta C(t)=\int_{-\infty}^{t}dt'\frac{\partial C(t)}{\partial a^{-1}(t')}\delta a^{-1}(t') \qquad \text{with} \quad \frac{\partial C(t)}{\partial a^{-1}(t)}\simeq\left(\frac{\partial C}{\partial a^{-1}}\right)_{\mathrm{eq}}e^{-(t-t')/\tau_{\mathrm{relax}}} \quad \longleftrightarrow \quad \zeta(\omega)=\frac{i\chi}{\omega+i\tau_{\mathrm{relax}}^{-1}}$$

(valid for long times)

- **Effectively capture hydrodynamic corrections up to infinite order**

  Since the equation explicitly contains τ_{relax}, its solution can be directly expanded w.r.t. τ_{relax}.

  ➡ **The gradient expansion w.r.t. τ_{relax}/t underlying hydrodynamics**

$$\pi(t)=\underbrace{-\zeta V_a(t)}_{\text{Navier-Stokes hydro. Result}}+O(\tau_{\mathrm{relax}}/t)$$

Navier-Stokes hydro. Result

▸ Higher-order hydrodynamic corrections can be obtained from the expansion as needed.

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2. Time-dependent scattering length in hydrodynamics

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- Hydrodynamic relaxation dynamics for the pressure deviation $\pi(t)$

$$\tau_{\text{relax}}\partial_t\pi(t)+\pi(t)=-\zeta[a(t)]V_a(t)$$ with $$V_a(t)=-3\frac{\partial_t a(t)}{a(t)}$$

3. Our proposed driving protocol and attractor

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Power-law driving for hydro attractor

- Hydrodynamic relaxation dynamics for $\pi(t)$

$$\boxed{\tau_{\mathrm{relax}}\partial_t\pi(t) + \pi(t) = -\underline{\zeta[a(t)]}V_a(t)} \quad \text{with} \quad \boxed{V_a(t) = -3\frac{\partial_t a(t)}{a(t)}}$$

☑ Protocol for driving the scattering length :

Initially push the system out of equilibrium, then let it gradually approach thermal equilibrium

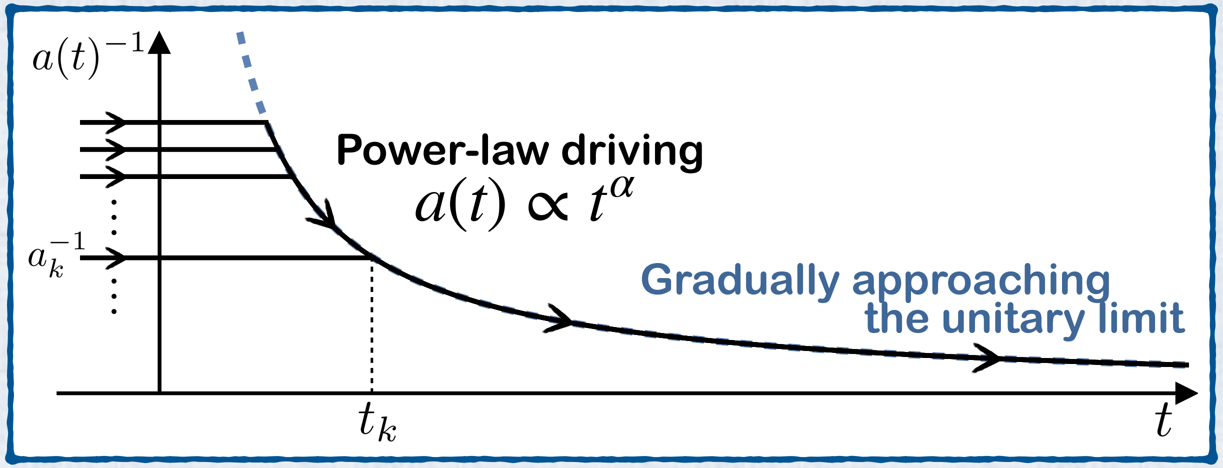
Example for two-comp. Fermi gases close to the unitary limit

$$a^{-1}(t) = \begin{cases} a_k^{-1} & (t < t_k) \\ a_k^{-1}(t/t_k)^{-\alpha} & (t > t_k) \end{cases}$$

Keep the scattering length fixed at a value a_k until $t = t_k$, then start the power-law driving, i.e., $a(t) \propto t^\alpha$.

- The system approaches equilibrium because of $V_a(t) = -3\alpha/t$.

By varying a_k & t_k, various initial states can be realized.



To make the late-time driving the same, we fix $\tilde{a} := a_k(t_k/\tau_\zeta)^\alpha$.

→ The driven scattering length follows a single curve.

Resulting attractor behavior

- Hydrodynamic relaxation dynamics for $\pi(t)$

$$\boxed{\tau_{\text{relax}}\partial_t\pi(t)+\pi(t)=-\underline{\zeta[a(t)]V_a}(t)} \quad \text{with} \quad \boxed{V_a(t)=-3\frac{\partial_t a(t)}{a(t)}}$$

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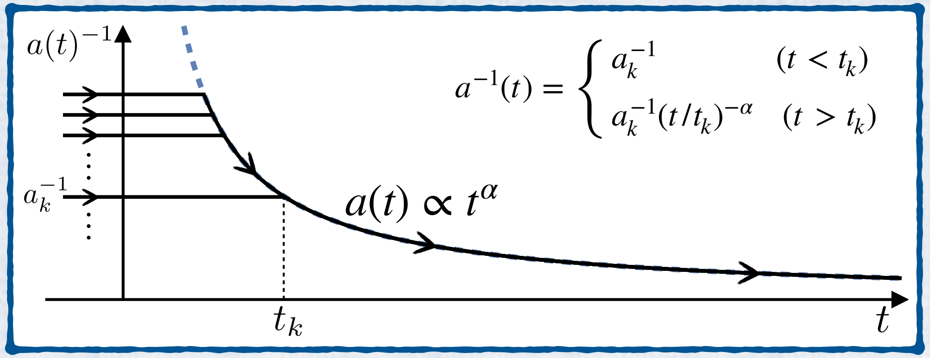
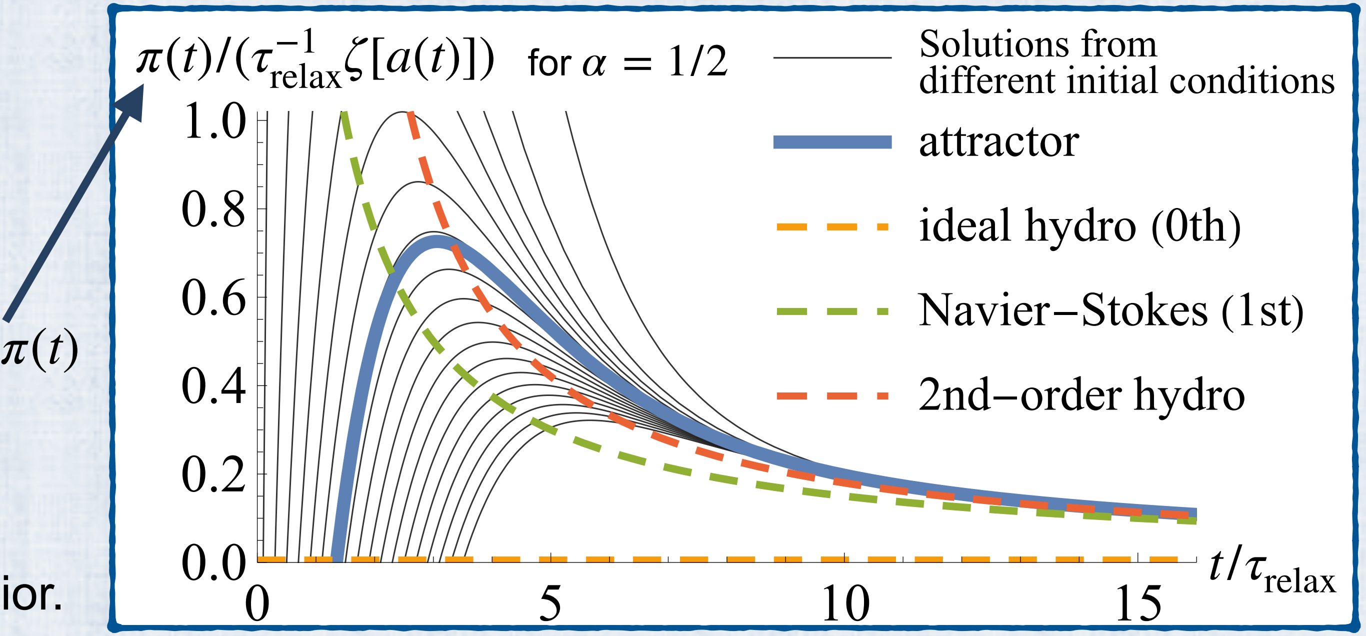


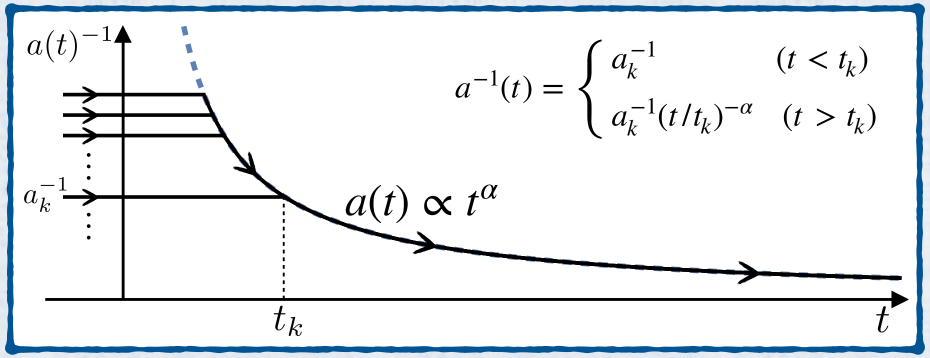
(Assuming $\zeta[a]\propto a^{-2}$ around the unitary limit)

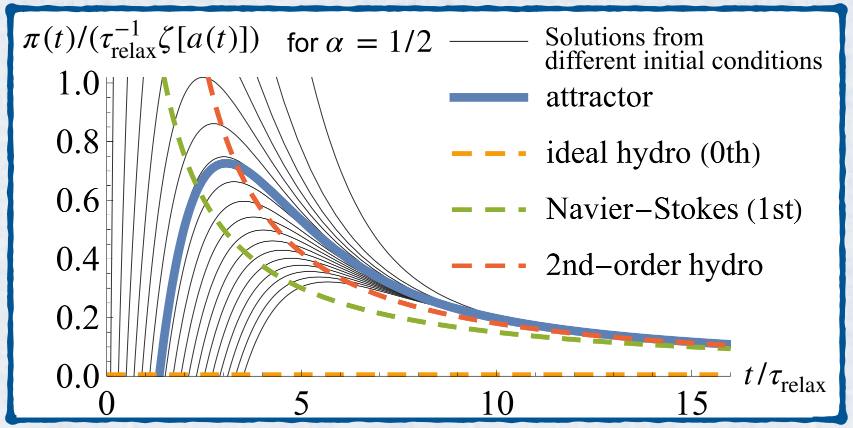
Dimensionless $\pi(t)$



- Solutions first converge to the attractor.
- Afterwards, the attractor approaches the hydrodynamic behavior.

Analytic results : divergence of the expansion





▸ **Analytical solution**

$$\pi(t) = \pi_{\mathrm{ini}} e^{-(t-t_{\mathrm{ini}})/\tau_{\mathrm{relax}}} + \pi_{\mathrm{att}}(t)$$

Non-hydrodynamic mode

- Depend on initial conditions a_k & t_k
- Cannot be expanded w.r.t τ_{relax}/t

✓ **Attractor solution** $\pi_{\mathrm{att}}(t) = \dfrac{3\alpha\zeta[\tilde{a}]}{\tau_{\mathrm{relax}}}(-1)^{2\alpha+1} e^{-t/\tau_{\mathrm{relax}}}\Gamma(-2\alpha, -t/\tau_{\mathrm{relax}})$

- Does NOT depend on a_k & t_k separately ➡ **Universal!!**
- Can be expanded with respect to τ_{relax}/t.

$$\pi_{\mathrm{att}}(t) \sim (\tau_{\mathrm{relax}}/t)^{2\alpha+1}\left[1 + (2\alpha+1)(\tau_{\mathrm{relax}}/t) + \cdots\right]$$: **divergent series**

Navier-Stokes

2nd-order hydro

n th-order coefficient $\propto (n + 2\alpha)!$ ⟶ **factorial divergent**
(Borel summable)

In the attractor solution, the gradient expansion is an asymptotic, divergent series.

▸ The attractor solution cannot be obtained even if higher-order fluid corrections are summed up.

▸ Origin of time-scale separation: non-hydro → attractor, attractor → hydro.

Summary & Outlook

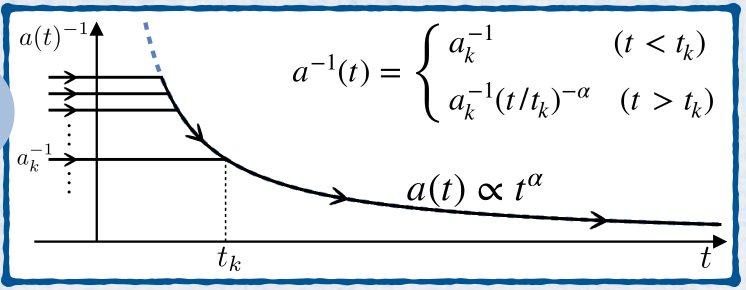
KF & T. Enss, Phys. Rev. Lett. **133**, 173402 (2024)

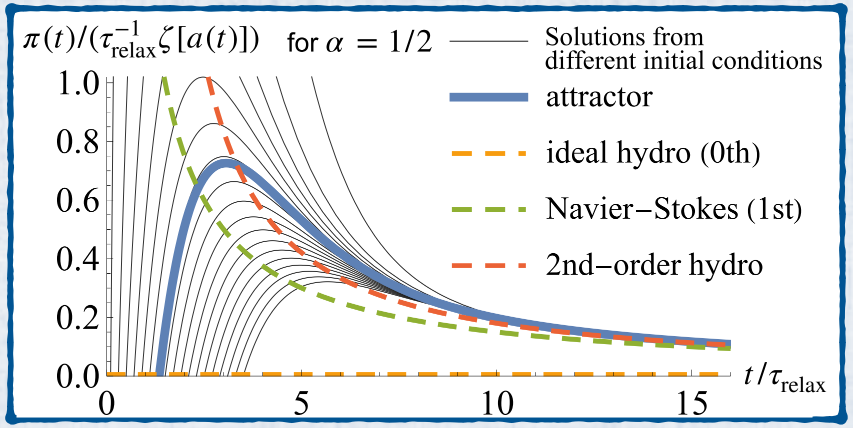
Our key idea:

Equivalence between isotropic expansion and contraction of the scattering length

➡ Emulating fluid expansion that exhibits a hydro attractor, using scattering-length driving

KF & Y. Nishida, PRA **98**, 063634 (2018).





▸ For example, in a two-component Fermi gas of ^{40}K,
the time window $t/\tau_{\rm relax} \sim 5-10$ will be visible.

- Contact dynamics was measured with 0.1ms time resolution. ($E_F \sim h \times 20$ kHz, $T/T_F \sim 0.25$ ➡ $\tau_{\rm relax} \sim 0.15$ ms)

Thywissen group, Science (2014), PRL (2017)

✓ Future directions

- Further analysis under the same driving, including details such as the energy density dynamics.

☑ **Clemens's talk**

- Other driving protocol of the scattering length ☑ **Aleksas's talk**
- Exploration of other exciting developments in nuclear physics with ultracold atoms ??

Let's discuss !!

Thank you!!