On the analytic structure of Bethe-Salpeter equation for ‘t Hooft and Ising model

ECT* Workshop – Analytic structure of QCD and Yang-Lee edge singularity

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IFT Basics

- We are only considering **2D** Ising model.

$$H_{\text{Ising}} = -\sum_{\langle ij \rangle} J\sigma_i\sigma_j + \sum_i H\sigma_i. \quad \sigma_i \pm 1.$$

- The Ising model at the critical points $T = T_c$ and $h = 0$ could be described by the minimal CFT $\mathcal{M}_{(4,3)}$.
- The **relevant** primary fields are: $I(x)$ $(0,0)$, $\sigma(x)$ $(\frac{1}{16}, \frac{1}{16})$, and $\epsilon(x)$ $(\frac{1}{2}, \frac{1}{2})$.
- The IFT by definition is

$$\mathcal{A}_{\text{IFT}} = \mathcal{A}_{(3,4)} + \tau \int \epsilon(x)\, d^2x + h \int \sigma(x)\, d^2x$$

- τ and h are related to in T and H through.

$$\tau = C_\tau \Delta T \left(1 + O(\Delta T, H^2)\right) \quad h = C_h H \left(1 + O(\Delta T, H^2)\right),$$

$\Delta T = 1 - T_c/T$.

- By the simple fact of dimensional analysis, the theory space is parameterized by $\xi = \frac{h}{|2\pi\tau|^{15/8}}$.
- **Our goal is to study the analytic structure of $M(\xi)$ and $G(\xi)$.**

$$F(\xi) = \frac{m^2}{8\pi} \log m^2 + m^2 G(\xi)$$

The Yang-Lee edge singularity

- The Yang-Lee critical point at $\xi^2 = -\xi_0^2 = -0.03583\cdots$, is described by the effective minimal model [Cardy, 1985].

$$\mathcal{A} = \mathcal{A}_{(2,5)} + \lambda(\xi^2)\int \phi(x)d^2x + \sum_i a_i(\xi^2)\int \mathcal{O}_i d^2x$$

- Both $M(\xi)$ and $G(\xi)$ have branch cuts at $\xi = i\xi_0$.

$$G(\xi^2) = G_{\mathrm{reg}}(\xi^2) + (\xi^2 + \xi_0^2)^{5/6} G_A(\xi^2) + \cdots$$
$$M(\xi^2) = (\xi^2 + \xi_0^2)^{\frac{5}{12}} M_A(\xi^2) + \cdots$$

Discussed By HaoLan

Free fermion regime

- $h = 0, \tau \neq 0$ ($\xi = 0$), IFT could be described by free Majorana Fermions [McCoy and Wu, 1978].

$$\mathcal{A}_{IFT} = \mathcal{A}_{FF} + h \int \sigma(x) d^2 x$$

$$\mathcal{A}_{FF} = \frac{1}{2\pi} \int (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} + im \psi \bar{\psi}), \quad m = 2\pi\tau.$$

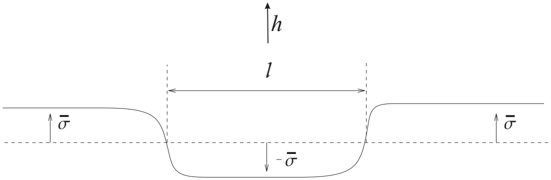
- For the **High temperature regime** ($\tau \geq 0$), it is possible to do perturbation analysis.

$$G_{\mathrm{High}}(\xi^2) = G_2 \xi^2 + G_4 \xi^4 + G_6 \xi^6 + \cdots$$

G_2	-1.8452280	G_4	8.33370
G_6	-95.1689(4)	G_8	1457.55(11)
G_{10}	-25884.(13)	G_{12}	$5.03(1)\times 10^5$

[Fonseca and Zamolodchikov, 2001]

- In the **In the low-T domain, the perturbative analysis cannot work because of confinement.**



- There is a **Essential singularity** at $\xi = 0$ For low T. i.e. Fisher-Langer singularity [Langer, 1967].

Bethe-Salpeter equation

- For $h = 0$, IFT is free fermion. We define its creation and annihilation operators

$$\{\mathbf{a}(p),\ \mathbf{a}^\dagger(p')\} = 2\pi\delta(p - p'), \quad \{\mathbf{a}(p),\ \mathbf{a}(p')\} = \{\mathbf{a}^\dagger(p),\ \mathbf{a}^\dagger(p')\} = 0.$$

- **We can view the full IFT as Free fermion + a perturbation in $\boldsymbol{\sigma}$.**

$$H = H_0 + hV, \quad H_0 = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, \omega(p)\mathbf{a}^\dagger(p)\mathbf{a}(p), \quad V = -\int_{-\infty}^{\infty} \sigma(\mathbf{x})d\mathbf{x}.$$

- Instated taking the finite size limit, take **2-particle approximation** instead: The Hilbert space is: $|\,\Psi^{(2)}\,\rangle = \frac{1}{2}\int \frac{dp_1}{2\pi}\frac{dp_2}{2\pi}\Psi(p_1,p_2)|\,p_1,p_2\,\rangle$..

- The Bethe-Salpeter equation.

$$(\omega(p_1)+\omega(p_2)+\Delta E)\Psi(p_1,p_2)=f_0\int_{-\infty}^{\infty}\delta(p_1+p_2-q_1-q_2)\mathcal{G}(p_1,p_2|q_1,q_2)\Psi(q_1,q_2)\frac{dq_1}{2\pi}\frac{dq_1}{2\pi},$$

$$\mathcal{G}(p_1,p_2|q_1,q_2)=\frac{1/4}{\sqrt{\omega(p_1)\omega(p_2)\omega(q_1)\omega(q_2)}}\Big[\frac{\omega(p_1)+\omega(q_1)}{p_1-q_1}\frac{\omega(p_2)+\omega(q_2)}{p_2-q_2}+$$
$$\frac{\omega(p_1)+\omega(q_2)}{p_1-q_2}\frac{\omega(p_2)+\omega(q_1)}{p_2-q_1}+\frac{p_1-p_2}{\omega(p_1)+\omega(p_2)}\frac{q_1-q_2}{\omega(q_1)+\omega(q_2)}\Big]$$

- In the infinite momentum frame, define $u = 2p/P$.
  $\phi(u) = \lim_{P\to\infty} \Psi_P(uP/2)$[Fonseca and Zamolodchikov, 2006].

$$\left(\frac{m^2}{1-u^2} - \frac{M^2}{4}\right)\phi(u) = f_0 \int_{-1}^{+1} F(u|v)\,\phi(v)\,\frac{dv}{2\pi}.$$

$$F(u|v) = \frac{1}{\sqrt{(1-u^2)(1-v^2)}}\left[\frac{2(1-uv)}{(u-v)^2} - \frac{uv}{4}\right].$$

- In important part is that $F \sim \frac{1}{(u-v)^2}$ singularity gives linear forces at large distance. (Confinement)
- Pros: Works very well with analytic continuation of f_0 ($\propto \xi$). Cons: Only an approximation for small h.

Intro to the ‘t Hooft model

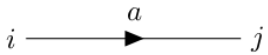
- The ‘t Hooft model is **$1+1D$ QCD, fundamental matter, $N \to \infty$ with gN^2 fixed**.

$$\mathcal{L} = \frac{1}{4}\mathrm{tr} F_{\mu\nu}F^{\mu\nu} - q_a \left(i\not{D} - m_a\right)\overline{q}^a, \quad a = 1, \cdots m.$$

- We can choose a gauge such that $\mathcal{L} = -\frac{1}{2}\mathrm{tr}\left(\partial_- A_+\right)^2 + \cdots$.

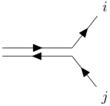
The Feynman rules are:

$-\frac{i}{k_-^2}$ Gluon propagator



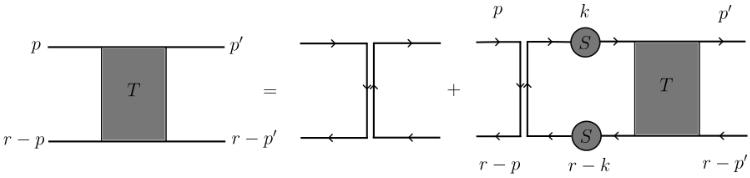
$\frac{i}{\gamma_+ p_- + \gamma_- p_+ - m_a - i\epsilon}$

$= i\frac{\gamma_+ p_- + \gamma_- p_+ + m_a}{2p_+ p_- - m_a^2 - i\epsilon}$ Fermion propagator



$ig\gamma_-$ Gluon-quark interaction

- In the large N limit, only planar graphs contribute.
- The 4-point quart propagator could be exactly resummed. $\Rightarrow$ Bethe-Salpeter equation.



- **The 't Hooft equation**

$$\left[\frac{\alpha}{1-u^2}\right]\phi_k(u)+\int_{-1}^{1}\frac{\phi_k(v)}{(u-v)^2}\,dv=\mu_k^2\phi_k(x).$$

$$u=2p_-/r_-,\quad v=2p'_-/r_-,\quad \alpha=\frac{m^2}{gN^2}-1.$$

- Take the following integral transformation on the wave function $\phi(u)$.

$$\psi(\nu) = \int_{-1}^{1} \frac{du}{1-u^2} \phi(u) \left(\frac{1+u}{1-u} \right)^{i\nu/2},$$

- The Bethe-Salpeter equation for *Ising model* becomes.

$$\boxed{f_I(\nu)\psi(\nu) = \mu^2 \hat{K}\psi(\nu) + \lambda \frac{\nu}{\mathrm{ch}\,(\pi\nu/2)} \overline{\psi}.}$$

- The Bethe-Salpeter equation for *'t Hooft model* becomes.

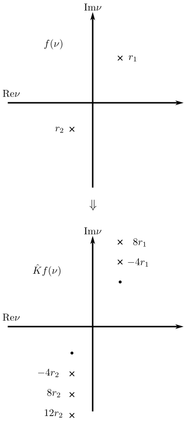
$$\boxed{f_H(\nu)\psi(\nu) = \mu^2 \hat{K}\psi(\nu).}$$

where

$$\overline{\psi} \equiv \int dv \frac{\nu}{\mathrm{ch}\left(\frac{\pi\nu}{2}\right)}, \quad \hat{K}\psi(\nu) \equiv \int d\nu' \frac{\pi(\nu-\nu')}{\sin \pi(\nu-\nu')/2} \psi(\nu'), \quad \begin{cases} f_I(\nu) = 1 + \lambda\, \nu \tanh\left(\frac{\pi\nu}{2}\right) \\ f_H(\nu) = 1 + \lambda\, \nu \coth\left(\frac{\pi\nu}{2}\right) \end{cases}$$

The analytic properties of wave function

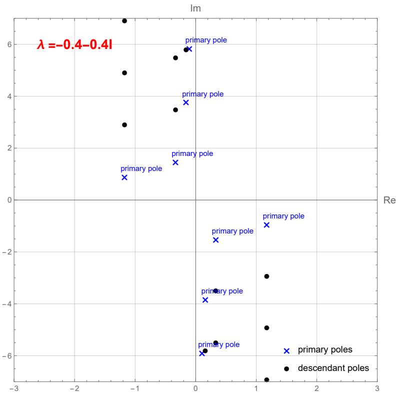
- $\psi(\nu)$ has infinite number of simple poles.
- $\psi(\nu)$ has simple poles at $f_l(\nu) = 0$.
- If $f(\nu)$ has poles has poles at ν_a, $\hat{K}f$ has poles at $\nu_a + 2i, \nu_a + 4i, \cdots$ for $Im\,\nu_a \geq 0$, $\hat{K}f$ has poles at $\nu_a - 2i, \nu_a - 4i, \cdots$ for $Im\,\nu_a \leq 0$.



- The minimal consistent poles are $\nu_{n,k}(\lambda) = \pm(\nu_k(\lambda) + 2ni)$, $\pm\nu_k$ is the root of $f_l(\nu)$. (Primary poles).
- The poles and residues characterized expansion in large transverse momentum.

$$\psi(\nu) \sim \frac{r}{v - \nu_k} \quad \longrightarrow \quad \phi(u) \sim r(1-u^2)^{\nu_k}.$$

- How the poles behave in λ-plane also determine the analyticity properties of $M^2(\lambda)$.



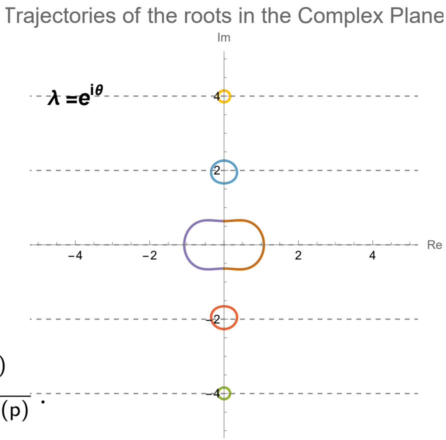
Pinching singularities

- Consider $\lambda = |\lambda| e^{i\theta}$, ν_1 and $-\nu_1$ exchanged. i.e. a branch cut at $\lambda = 0$.
- More generally, we could see that the point **where two poles collide is a square root singularity**.

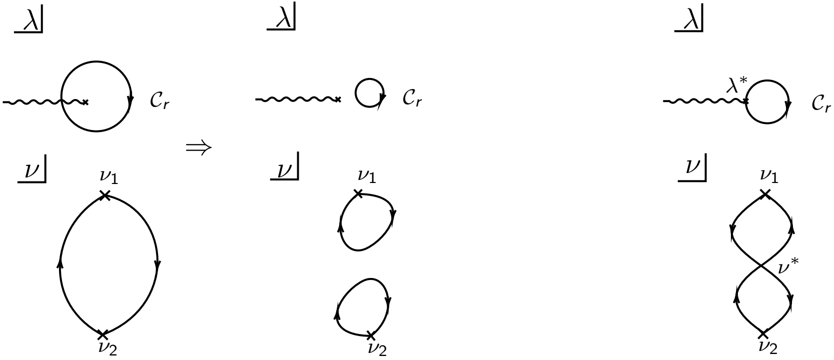
$$\boxed{f_l(\nu) = 0, \quad f_l{}'(\nu) = 0}$$

or

$$\pi\nu^{(\mathrm{p})} + \sinh \pi\nu^{(\mathrm{p})} = 0\,, \quad \text{and} \quad \lambda^{(\mathrm{p})} = -\frac{\cosh \frac{\pi}{2}\nu^{(\mathrm{p})}}{\nu^{(\mathrm{p})} \sinh \frac{\pi}{2}\nu^{(\mathrm{p})}}\,.$$

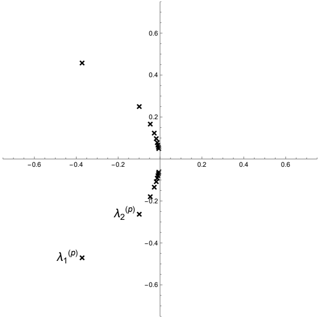


Pinching singularities

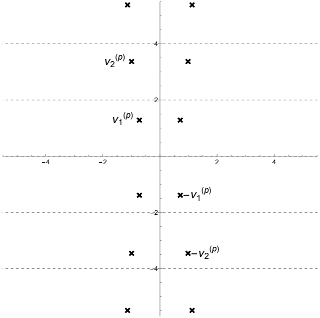


Pinching singularities

For the Ising model:



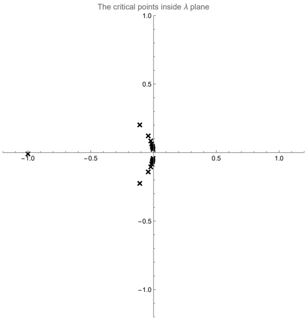
(a) The critical points of λ. We can see that there is an accumulation point at $\lambda = 0$.



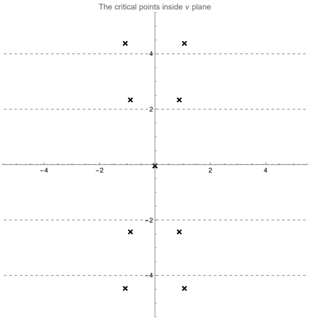
(b) The critical points of ν. The gray dashed line denotes boundaries $\mathrm{Im}\nu = 2n$.

Pinching singularities

For the 't Hooft model:



(a) The critical points of λ. We can see that there is an accumulation point at $\lambda = 0$.



(b) The critical points of ν. The gray dashed line denotes boundaries $\text{Im}\nu = 2n$.

Poles trajectory

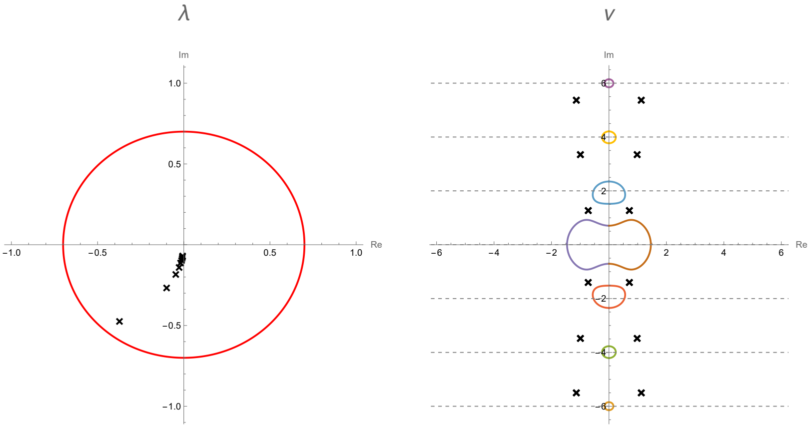


Figure: The trajectories of the poles ν_n, when λ evolves along the red circle in λ-space, where $|\lambda| = 0.7$.

Poles trajectory

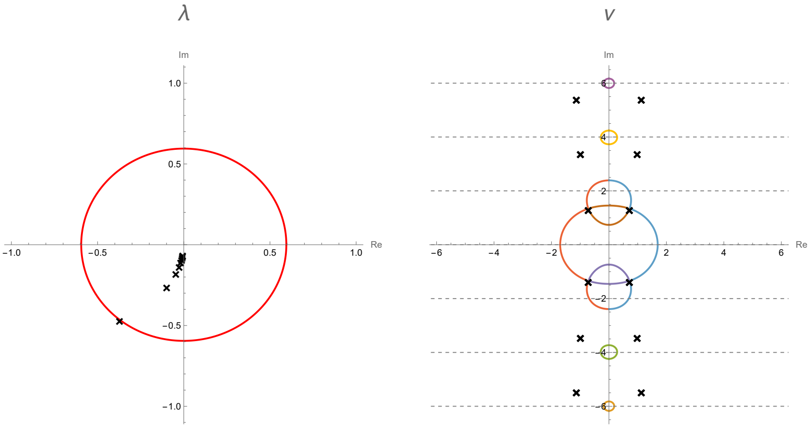


Figure: The trajectories of poles passed through the first branching point. $|\lambda| = 0.595065$.

Poles trajectory

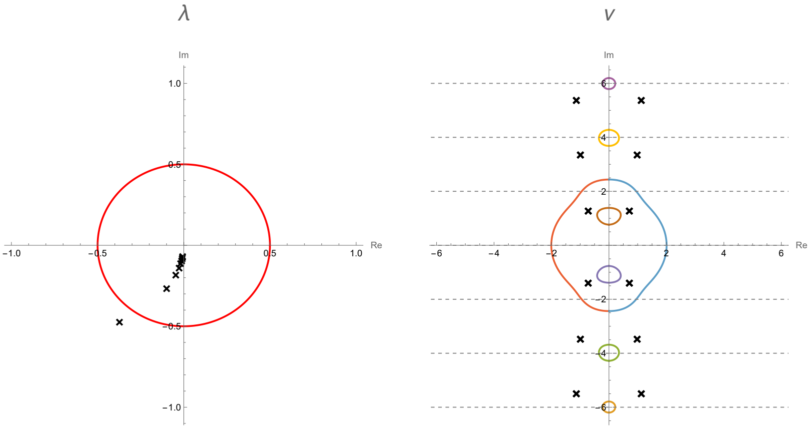


Figure: $|\lambda| = 0.5$. The second pair of poles exchange instead.

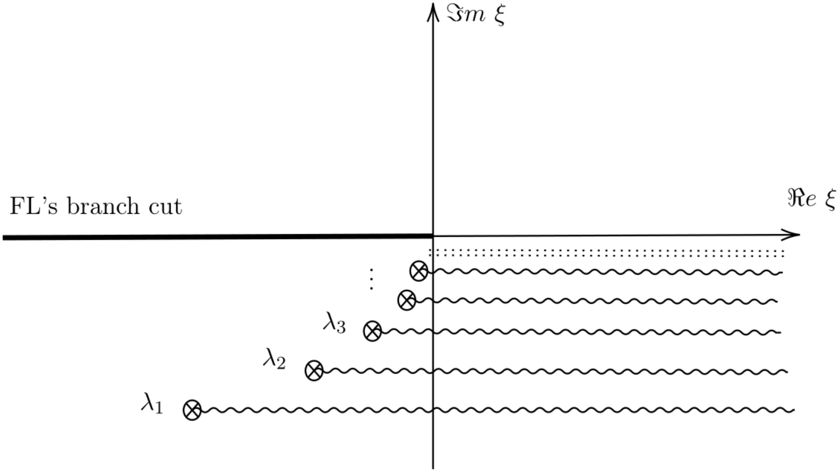


Figure: The branch structure of $M^2(\xi)$, i.e. The Lasagne.

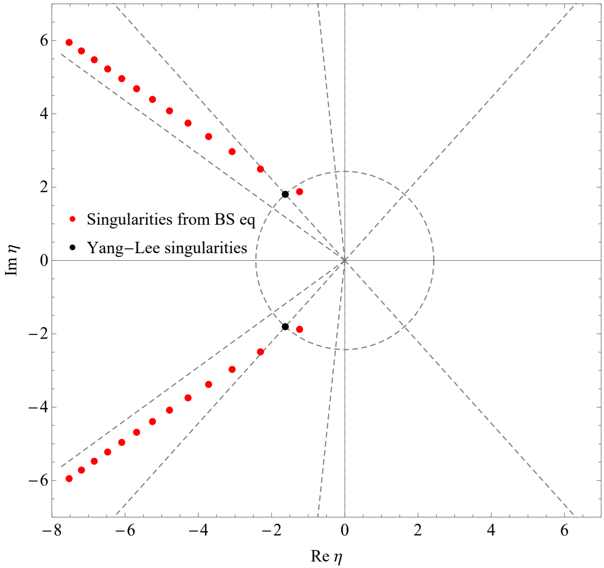


Figure: The branch points in the extended η-plane.

Deformed Beth-Salpeter Equation

- Consider analytic continuation of λ. $\lambda(\theta) = \lambda_0 e^{i\theta}$.
- For sufficiently small λ, ν_1 and $-\nu_1$, exchanged while all other poles return is its origin.
- The Bethe-Salpeter equation is deformed.

$$\overline{\psi} \to \overline{\psi} - 2\pi i r_1 \frac{\nu_1}{\mathrm{ch}(\pi\nu_1/2)}, \quad \hat{K}\psi(\nu) \to \hat{K}\psi(\nu) - 2\pi i r_1 K(\nu_1, \nu),$$

- The deformed equation

$$f(\nu)\psi(\nu) - \frac{\lambda}{8}\frac{\nu}{\mathrm{ch}\frac{\pi\nu}{2}}\left(\overline{\psi} - 2\pi i r_1 \frac{\nu_1}{\mathrm{ch}\frac{\pi\nu_1}{2}}\right) = M^2\left[\hat{K}\psi(\nu) - 2\pi i r_1 K(\nu, \nu_1)\right].$$

Deformed Beth-Salpeter Equation.

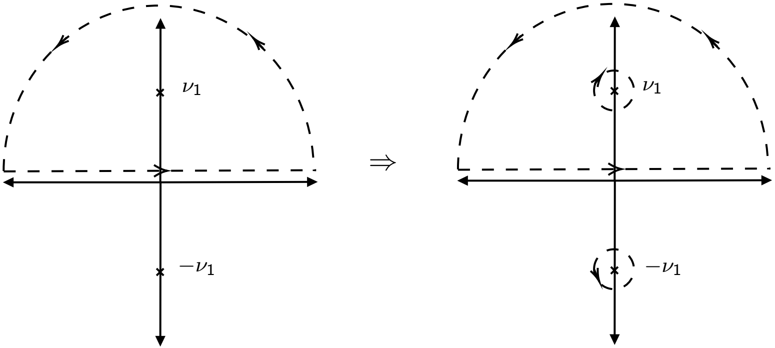


Figure: The contour deformed after a pole goes to the lower half plane.

Numerical spectrum

- The deformed Bethe-Salpeter equation could be solved numerically in the region where **only one pole crosses the real line.**
- The discretized $\psi(x_i)$ and the residue r_1 is the unknown in:

$$f(\nu)\psi(\nu)-\frac{\lambda}{8}\frac{\nu}{\mathrm{ch}\frac{\pi\nu}{2}}\left(\overline{\psi}-2\pi i r_1\frac{\nu_1}{\mathrm{ch}\frac{\pi\nu_1}{2}}\right)=M^2\left[\hat{K}\psi(\nu)-2\pi i r_1 K(\nu,\nu_1)\right].$$

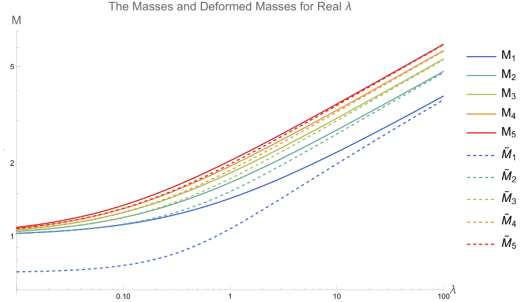
- We are essentially solving a generalized eigenvalue systems for $\vec{v}=\left(\ \psi_i\Delta x\ \ \frac{r_1}{2\pi i}\ \right)$.

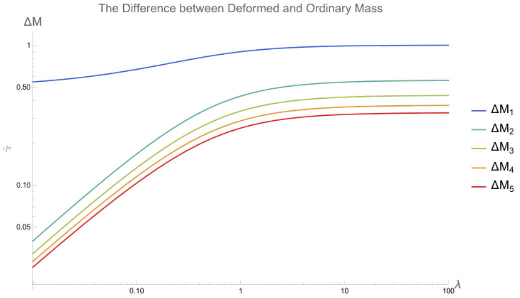
$$A\cdot\vec{v}=M^2 B\cdot\vec{v}.$$

where

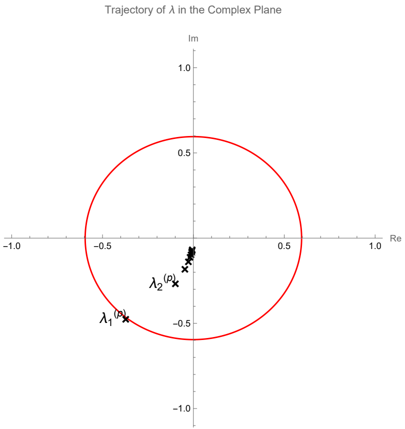
$$A=\begin{array}{c} \\ x_j \\ \nu_1 \end{array}\begin{array}{c|c} \Delta x\psi_i & r_1/(2\pi i) \\ \hline f(x_i)\delta_{ij}-\lambda\frac{x_i}{\mathrm{ch}(\pi x_i/2)}\frac{x_j}{\mathrm{ch}(\pi x_j/2)} & \frac{\lambda}{8}\frac{x_i}{\mathrm{ch}(\pi x_i/2)}\frac{\nu_1}{\mathrm{ch}(\pi\nu_1/2)} \\ \hline -\frac{\lambda}{8}\frac{x_j}{\mathrm{ch}(\pi x_j/2)}\frac{\nu_1}{\mathrm{ch}(\pi\nu_1/2)} & \frac{f'(\nu_1)}{4\pi i}+\left(\frac{\nu_1}{\mathrm{ch}(\pi\nu_1/2)}\right)^2 \end{array}\qquad B=\begin{array}{c} \\ x_j \\ \nu_1 \end{array}\begin{array}{c|c} \Delta x\psi_i & r_1/(2\pi i) \\ \hline K(x_i,x_j) & -K(\nu_1,x_i) \\ \hline -K(\nu_1,x_j) & -K(\nu_1,\nu_1) \end{array}$$

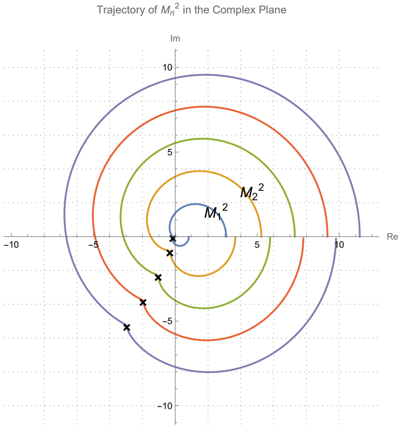
Real spectrum

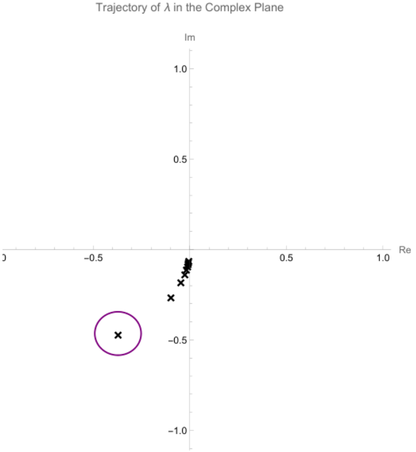


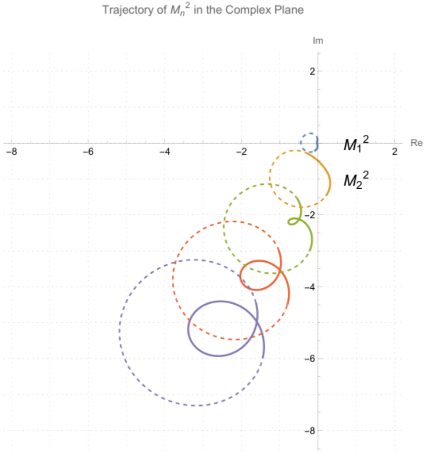


Complex spectrum






Trajectory of λ in the Complex Plane
Im
Re
1.0
0.5
-0.5
-1.0
-0.5
0.5
1.0


Trajectory of M_n^2 in the Complex Plane
Im
Re
M_1^2
M_2^2
-8
-6
-4
-2
2
2
-2
-4
-6
-8

Massless Points

- For 't Hooft model, the pinching singularity $\lambda^{(p)}$ also correspond to the **massless points** $\lambda^{(z)}$ where at lest one mass vanishes $M(\lambda^{(z)}) = 0$.
- For the Ising model, these two points don't coincide **Because of the $\bar{\psi}$ term**.
- The consistency condition

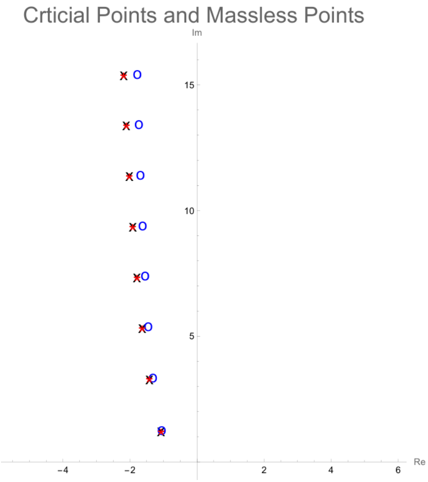
$$f_I(\nu)\psi(\nu) = \frac{\lambda}{16} \frac{\nu}{\mathrm{ch}\,(\pi\nu/2)} \overline{\psi} \quad \Rightarrow \quad F(\lambda) = \frac{16}{\lambda}.$$

where $F(\lambda) \equiv \int_{-\infty}^{\infty} \left(\frac{\nu}{\mathrm{ch}(\pi\nu/2)} \right)^2 \frac{1}{f_\lambda(\nu)} d\nu..$

- When ν_k passes the real line, one needs to add a residue term

$$F(\lambda) = \overline{F}(\lambda) - 4\pi i \left(\frac{\nu_k}{\mathrm{ch}\,(\pi\nu_k/2)} \right)^2 \frac{1}{f'_\lambda(\nu_k)}.$$

Figure: Plotted in the $\alpha = 1/\lambda$ plane.



Remarks

- We found an infinite number of square root branching points. With the accumulation at $\lambda = 0$. (**Lasagna structure**)
- The first branching point is very close to the Yang-Lee singularity.
- Numerical observation find that $\lambda = \lambda_1^{(p)}$, M_1^2 is very small.
- **Cojecture**: $\lambda = \lambda_1^{(p)}$ correspond to the Yang-Lee singularity. While other pinching points correspond to unknown critical non-unitary CFTs.
- A more sophisticated method is needed to probe critical points beyond Yang Lee.

Thank you for listening

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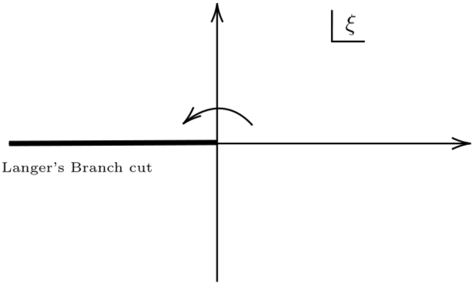
Low temperature Analyticity

- $G_{\rm low}(\xi)$ has only one branch cut in $(-\infty, 0]$
- There is a *essential singularity* at $\xi = 0$. The physical interpretation: **Nucleation process of metastable states**.
- This branch cut could be estimated by Langers's theory of nucleation. $Im\, G_{\rm low}(\xi) \sim \frac{\xi}{4\pi} e^{-\frac{\pi}{\xi\bar{\sigma}}}$ [Langer, 1967].

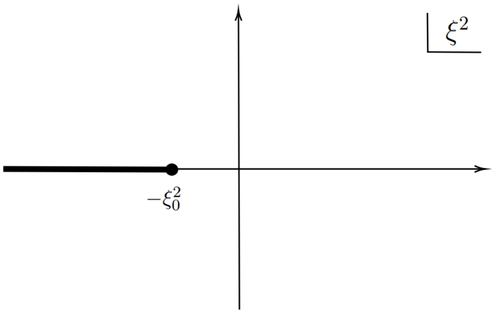


The dispersion relation [Fonseca and Zamolodchikov, 2001].

$$G_{\rm low}(\xi) = \tilde{G}_1 \xi - \xi^2 \int_0^\infty \frac{Im\, G_{\rm low}(-t + i0)}{t^2(t+\xi)} \frac{dt}{\pi}$$

High temperature Analyticity

- $G_{\text{high}}(\xi)$ is an even function of ξ it has branch cuts at $\xi^2 \in (-\infty, \xi_0^2]$. $\xi_0^2 \sim 0.18930$.
- The branching point corresponds to the Yang-Lee singularity. Described by non-unitary $\mathcal{A}_{(2,5)}$ minimal model.



The dispersion relation:

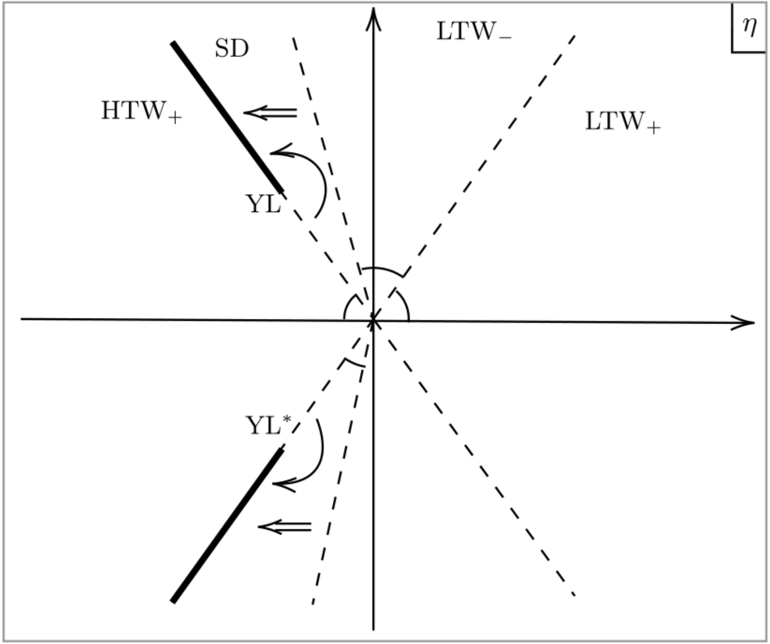
$$G_{\text{high}}(\xi^2) = -\xi^2 \int_0^\infty \frac{2Im\, G_{\text{high}}(-t+i0)}{t(t^2+\xi^2)} \frac{dt}{\pi}.$$

Extended Analyticity

- The low temperature phase and the high temperature phase are connected by analytic continuation.
- The continuation become explicit when expressed in terms of $\boldsymbol{\eta = (2\pi\tau)/h^{\frac{8}{15}}}$.

$$\Psi(\eta) = \begin{cases} \eta^2 G_{\mathrm{low}}(\eta^{-15/8}) & \eta > 0 \\ \eta^2 G_{\mathrm{high}}((-\eta)^{-15/8}) & \eta < 0. \end{cases}.$$

- Three distinct domains in the η-plane.
  1. Low-T domain: $-\frac{8}{15}\pi < \mathrm{Arg}\eta < \frac{8}{15}\pi$.
  2. High-T domain: $-\frac{4}{15}\pi < \mathrm{Arg} - \eta < \frac{4}{15}\pi$.
  3. The shadow domain: $\frac{9}{15}\pi < \mathrm{Arg}\eta < \frac{11}{15}\pi$, $-\frac{11}{15}\pi < \mathrm{Arg}\eta < \frac{8}{15}\pi$.


η
SD
LTW−
HTW+
LTW+
YL
YL*