Analyticity properties of 2d Ising Field Theories

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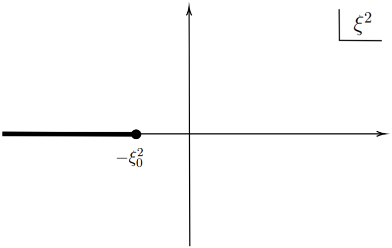
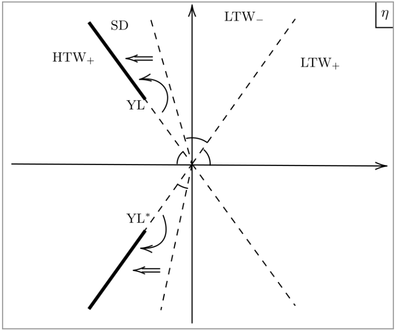
Outline

- Basic set up: 2D Ising field theories (IFTs):
  Definitions, space of theories, parametrization, and scaling functions.
- Yang-Lee Singularity of 2D IFT:
  Position, critical behaviours, effective action and singular expansions.
- Analyticity of the scaling functions:
  Analyticity and dispersion relation in the high-T space, and their extensions.
- Summary and outlooks.
- Appendixes.

The 2D Ising Scenario

- To conclude: In the complexified theory space of continuous 2D Ising models: There exist only the Yang-Lee singularity on the first sheet of parameter space, and it is described by the 2D Yang-Lee CFT.
- The analyticity of the fundamental scaling functions (i.e. free energy and correlation length) follows the above statement. One can establish the dispersion relations which represents the scaling functions via integrations along the Yang-Lee branch cut.





- Furthermore, such statements on analyticity can be extended, as describing both symmetry-preserved and symmetry-broken phase jointly.

Basics

- Ising model: classical spins (up and down) sit on sites of a lattice, with nearest site interactions ($J > 0$) and external magnetic field:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j + H \sum_i \sigma_i \,, \quad \mathcal{Z} = \sum_{\{\sigma_i\}} e^{-\beta \mathcal{H}} \,,$$

  usually we consider hypercube lattices, i.e.: square in 2D and cubic in 3D.

- Why Ising model? It describes the most important universality class in nature: Vapor/liquid phase transition, ferromagnetic transition with Curie point, etc.

- Ising model / universality class in various dimensions:
  $D = 4$: Landau-Ginzburg description of φ^4, $4 - \epsilon$ expansion and Wilsonian RG.
  $D = 3$: Numerical solutions at criticality (perturbative RG, Monte-Carlo, numerical conformal bootstrap, fuzzy sphere, etc).

- In 2D, Onsager gave the solution at $H = 0$ (Onsager 1944)[1].
  Yang and Lee established the theorem of circle and zeros (Yang & Lee 1952)[2].
  However, for generic J and H: no solution available in closed form.

[1] Lars Onsager. "Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition". In: *Phys. Rev.* 65 (3-4 1944), pp. 117–149. DOI: 10.1103/PhysRev.65.117. URL: https://link.aps.org/doi/10.1103/PhysRev.65.117.

[2] Chen-Ning Yang and Tsung-Dao Lee. "Statistical theory of equations of state and phase transitions. I. Theory of condensation". In: *Physical Review* 87.3 (1952), p. 404.

The Ising Field Theories

- In the continuous limit, Ising models at criticality $\Longrightarrow$ Ising conformal field theories (ICFTs), and Ising field theories are described by the action:

$$\mathcal{A}_{\text{IFT}} = \mathcal{A}_{\text{CFT}}^{\text{Ising}} + \frac{m}{2\pi} \int \varepsilon(x)\, d^2x + h \int \sigma(x)\, d^2x \,,$$

  as relevant deformations away from Ising CFT at UV.

- Ising CFT has $\mathbb{Z}_2$ symmetry, and with two local relevant scalar operators ($\Delta < d$).

- $\varepsilon(x) \sim \sigma_i \sigma_{i+1}$: energy density operator, $\mathbb{Z}_2$ even, with $\frac{m}{2\pi} = \tau \propto 1 - \frac{T}{T_c}$;

- $\sigma(x) \sim \sigma_i$: spin density operator, $\mathbb{Z}_2$ odd, with $h \propto H$.

- Both operators are normalized as:

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}}} \,,$$

  and $\Delta_{\mathcal{O}}$ is the conformal dimension of $\mathcal{O}$. Scheme of (m, h) are fixed.

- The conformal dimensions Δ_σ and Δ_ε determine the Ising critical behaviours.

2D Ising CFT as minimal model $(3,4)$

- In 2D, Ising CFT is described by a representation of Virasoro algebra, which is known as the minimal model $(3,4)$.

- The central charge and conformal dimensions of 2D ICFT are:

$$c_{\text{Ising}} = \frac{1}{2}\,, \quad (h_\varepsilon, \overline{h}_\varepsilon) = (\frac{1}{2}, \frac{1}{2}) \quad \text{and} \quad (h_\sigma, \overline{h}_\sigma) = (\frac{1}{16}, \frac{1}{16})\,.$$

- The scaling dimensions are $\Delta_\sigma = \frac{1}{8}$ and $\Delta_\varepsilon = 1$, thus $[m] = 1$ and $[h] = \frac{15}{8}$. Dimensionless combinations:

$$\eta = \frac{m}{h^{8/15}}\,, \quad \text{and} \quad \xi = \frac{h}{|m|^{15/8}}\,.$$

  are known as the scaling parameters, which labels the RG flows away from ICFT. They are related by $\xi = \eta^{-\frac{15}{8}}$ or $\eta = \xi^{-\frac{8}{15}}$ (up to signs), and both live in $\mathbb{C}$.

- With vanishing h, action of IFT is equivalent to the one of 2d Majorana fermions:

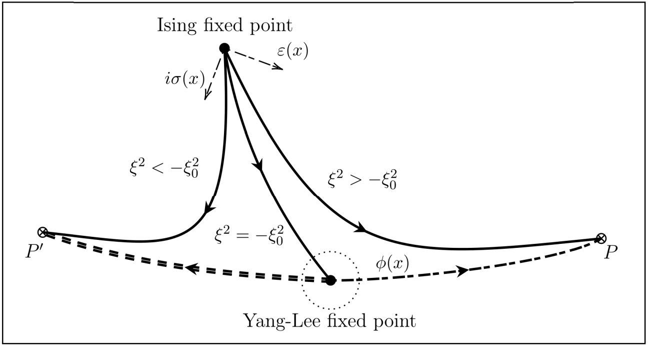
$$\mathcal{A}_{\text{FF}} = \frac{1}{2\pi}\int (\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} + im\bar{\psi}\psi)\, d^2x = \mathcal{A}^{\text{Ising}}_{\text{CFT}} + \frac{m}{2\pi}\int \varepsilon(x)\, d^2x\,.$$

  $|m|$ now is the fermion mass. Their Hilbert spaces are different by projection.

Big picture and the goal of the Project

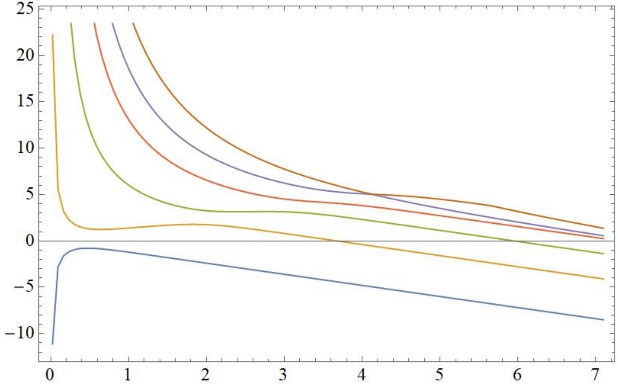
- Goal: find out the behaviours of IFTs in the theory space (parameterized by ξ or η).
- Methods: analyze the scaling functions (dimensionless functions of ξ or η).
- In this talk: focus on the inverse correlation length $M_1 = 1/R_c$, and define:

$$\hat{M}_1(\xi) = \frac{M_1(m,h)}{|m|}, \quad \mathcal{M}_1(\eta) = \frac{M_1(m,h)}{|h|^{8/15}}.$$



Numerical finite size spectrum and M_1

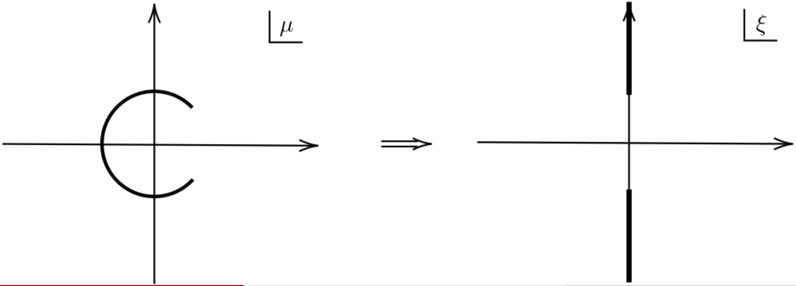
- Numerical finite size spectrum: put IFT hamiltonian on a finite size cylinder, truncate free fermion descendants, then do numerical diagonalization.
- The asymptotic gaps at $R \to \infty$ represent masses of particle states.



- The lightest mass M_1 is from $E_1 - E_0$, and $R_c = 1/M_1$ is the correlation length. R_c determines the exponential decay of 2-pt correlator.
- Compare to the higher masses, the physical meaning of M_1 is more universal.

The Yang-Lee circle in the High-T.

- At $T > T_c$: disordered phase with unbroken $\mathbb{Z}_2$ symmetry: $h \leftrightarrow -h$. Scaling functions are even in ξ, thus depend on $\xi^2 \sim h^2$.
- Yang-Lee theorem: the lattice partition function $\mathcal{Z}$ has zeros distributed on the unit circle of fugacity $\mu = e^{-2\beta H}$ plane. The circle becomes an arc in high-T phase.
- In the continuous limit, the zeros of $\mathcal{Z}$ condensed into a branch cut of $F = -\log \mathcal{Z}$, known as Yang-Lee (YL) branch cut. The edges become YL edge singularities.
- Thus, we expect that $\hat{M}_1(\xi)$ has two symmetric singularities at imaginary axis, they are the Yang-Lee singularities with certain critical behaviours.

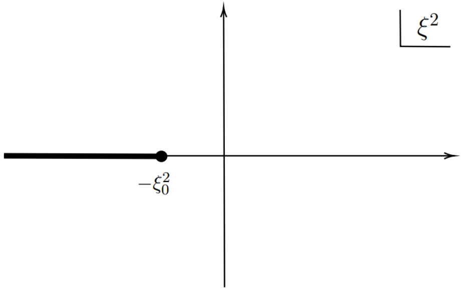


High-T IFT with a pure imaginary magnetic field.

- Unbroken $\mathbb{Z}_2$ symmetry at high-T. With pure imaginary h: “pseudo-hermiticity”.

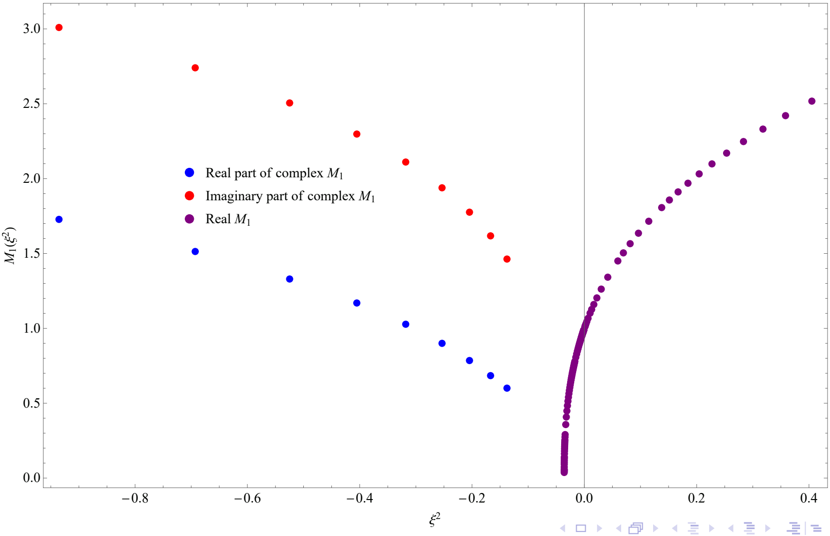
$$\exists\, S^2 = 1\,, \quad \text{so that} \quad H^\dagger = SHS\,, \quad \text{with} \quad S:\ \sigma \to -\sigma\,.$$

- The spectrum is bounded, with either unique real ground state or a pair of complex conjugate vacua $|0_\pm\rangle$. With imaginary h, M_1 can be real or complex.
- Position of the Yang-Lee singularity: $\xi^2 = -\xi_0^2$, where the correlation length $1/M_1$ grows to infinite, representing a continuous phase transition.



The numerical measurement of $\hat{M}_1(\xi^2)$.

- The data of $\hat{M}_1(\xi^2)$ can be measured from the finite size spectrum numerically (putting IFT on a cylinder), and here is the numerics:



The 2D Yang-Lee CFT

- The CFT representing 2D Yang-Lee criticality, is identified as the minimal model $(2,5)$. The Yang-Lee CFT is not unitary, and with the only relevant operator ϕ:

$$c_{\text{YL}} = -\frac{22}{5}\,, \quad (h_\phi, \overline{h}_\phi) = (-\frac{1}{5}, -\frac{1}{5})\,.$$

- Near the Yang-Lee singularity $\xi^2 = -\xi_0^2$, the critical behaviour is controlled by YLCFT, as described by the effective action:

$$\mathcal{A}_{\text{eff}} = \mathcal{A}_{\text{YLCFT}} + \lambda(\xi^2) \int d^2x\, \phi(x) + \sum_i g_i(\xi^2) \int d^2x\, \mathcal{O}_i(x)\,,$$

$$\text{with} \quad \lambda = \lambda_1 \cdot (\xi^2 + \xi_0^2) + \cdots\,, \quad g_i = g_i^{(0)} + g_i^{(1)} \cdot (\xi^2 + \xi_0^2) + \cdots\,.$$

  here $\mathcal{O}_i(x)$ denote irrelevant operators, as descendants of I and ϕ.

- Since λ has the mass dimension $[\lambda] = \frac{12}{5}$, the leading critical behaviour is given by mass-coupling relation:

$$\hat{M}_1(\xi^2) = (\xi^2 + \xi_0^2)^{\frac{5}{12}} \left(b_0 + \cdots\right).$$

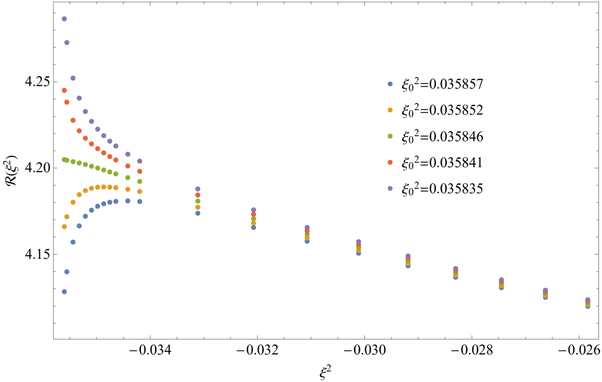
- Beyond leading ones: irrelevant perturbations and use dimensional analysis.

The position of Yang-Lee singularity.

- The regular behaviour of $\mathcal{R}_1 \Longrightarrow$ location of Yang-Lee critical point, as:

$$\mathcal{R}_1(\xi^2) = \hat{M}_1(\xi^2)/(\xi^2+\xi_0^2)^{\frac{5}{12}} = b_0 + b_1(\xi^2+\xi_0^2) + c_0(\xi^2+\xi_0^2)^{\frac{5}{6}} + \cdots$$

- Numerical result: $\xi_0^2 = 0.035846(4)$[3][4].



[3] Hao-Lan Xu and Alexander Zamolodchikov. “2D Ising Field Theory in a magnetic field: the Yang-Lee singularity”. In: *JHEP* 08 (2022), p. 057. DOI: 10.1007/JHEP08(2022)057. arXiv:2203.11262 [hep-th].

[4] Vladimir V. Mangazeev, Bryte Hagan, and Vladimir V. Bazhanov. “Corner Transfer Matrix Approach to the Yang-Lee Singularity in the 2D Ising Model in a magnetic field”. In: (Aug. 2023). arXiv:2308.15113 [hep-th].

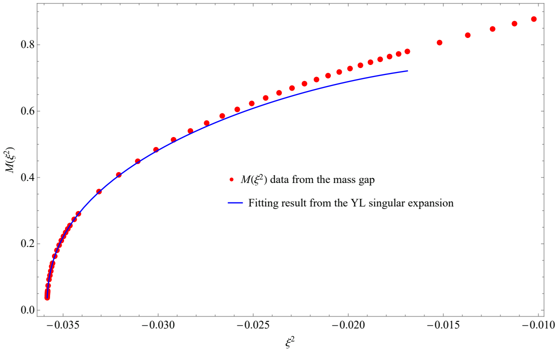
Subleading critical behaviours.

- Subleading critical behaviour: for operator $\mathcal{O}_i(x)$ with mass dimension Δ_i

$$\delta_i M(\xi^2)/(\xi^2+\xi_0^2)^{\frac{5}{12}} \propto (\xi^2+\xi_0^2)^{\frac{5}{12}(\Delta_i-2)} ,$$

from $g_i M_1^{\Delta_i-2}$ is dimensionless, while $M_1 \sim (\xi^2+\xi_0^2)^{\frac{5}{12}}$.
Example: the first irrelevant operator is $T\bar{T}$ with mass dimension 4, so that:

$$M_1(\xi^2) = (\xi^2+\xi_0^2)^{\frac{5}{12}}\left(b_0 + b_1(\xi^2+\xi_0^2) + c_0(\xi^2+\xi_0^2)^{\frac{5}{6}} + \cdots\right).$$



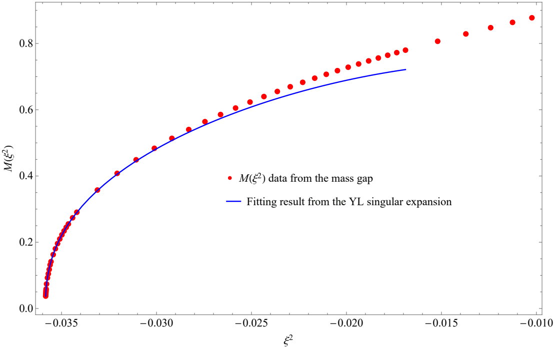
Critical amplitudes.

- Follow the singular expansion:

$$\hat{M}_1(\xi^2) = (\xi^2 + \xi_0^2)^{\frac{5}{12}} \left(b_0 + b_1(\xi^2 + \xi_0^2) + c_0(\xi^2 + \xi_0^2)^{\frac{5}{6}} + \cdots\right),$$

numerical fitting gives the critical amplitudes (leading and subleading):

$$b_0 = 4.228 \pm 0.005\,, \quad b_1 = 21.9 \pm 0.9\,, \quad c_0 = -14.4 \pm 0.6\,.$$



Coefficients in the effective action.

- Back to the effective action, which explicitly reads:

$$\mathcal{A}_{\text{eff}} = \mathcal{A}_{\text{YLCFT}} + \lambda(\xi^2) \int d^2x\, \phi(x) + \frac{\alpha(\xi^2)}{\pi^2} \int d^2x\, T\bar{T}(x) + \cdots ,$$

the couplings in the effective action are:

$$\lambda(\xi^2) = \lambda_1(\xi^2 + \xi_0^2) + \lambda_2(\xi^2 + \xi_0^2)^2 + \cdots , \quad \alpha(\xi^2) = \alpha_0 + \cdots .$$

- These couplings are related to the singular expansion of $\hat{M}_1$. Following the mass-coupling relation of Yang-Lee[5]:

$$M_{\text{YL}} = C_{\text{YL}} \lambda^{5/12} , \quad C_{\text{YL}} = \frac{2^{\frac{19}{12}} \sqrt{\pi}}{5^{\frac{5}{16}}} \frac{\left[\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})\right]^{\frac{5}{12}}}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})} = 2.64294463\ldots ,$$

the critical amplitudes are being represented as (here $f_{\text{YL}} = -\frac{\sqrt{3}}{12}$):

$$b_0 = C_{\text{YL}}\, \lambda_1^{5/12} , \quad b_1 = \frac{5}{12} C_{\text{YL}}\, \lambda_2\, \lambda_1^{-7/12} , \qquad c_0 = -\alpha_0\, f_{\text{YL}}\, b_0^3 . \tag{1}$$

- The numerical values in the effective couplings are:

$$\lambda_1 = 3.089 \pm 0.008 , \quad \lambda_2 = 38.4 \pm 1.6 , \quad \alpha_0 |m|^2 = -1.32 \pm 0.05 .$$

[5] Alexei B. Zamolodchikov. "Mass scale in the sine-Gordon model and its reductions". In: *Int. J. Mod. Phys. A* 10 (1995), pp. 1125–1150. DOI: 10.1142/S0217751X9500053X.

Properties of the massless flow.

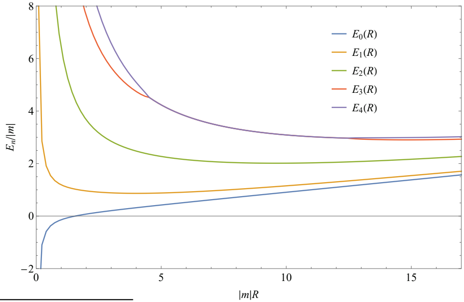
- Exactly at $\xi^2 = -\xi_0^2$, the effective action reads ($\Xi \sim L_{-4}\bar{L}_{-4} \cdot \phi$):

$$\mathcal{A}_{\text{eff}} = \mathcal{A}^*_{(2,5)} + \frac{\alpha_0}{\pi^2} \int (T\bar{T})(x) d^2x + \frac{\beta_0}{2\pi} \int \Xi(x) d^2x \,.$$

- Couplings α_0 and β_0 are measurable from finite size spectrum[6], as:

$$\alpha_0 |m|^2 = -1.32(5)\,, \quad \frac{\beta_0}{2\pi} |m|^{\frac{28}{5}} = +0.72 \pm 0.06\,.$$

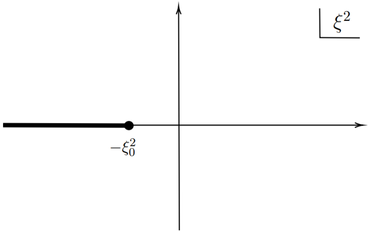
- $T\bar{T}$-deformation preserves integrability, while $\Xi(x)$ breaks integrability.



[6] Hao-Lan Xu and Alexander Zamolodchikov. "2D Ising Field Theory in a magnetic field: the Yang-Lee singularity". In: *JHEP* 08 (2022), p. 057. DOI: 10.1007/JHEP08(2022)057. arXiv:2203.11262 [hep-th].

Analyticity of $\hat{M}_1(\xi^2)$ at high-T.

- High-T standard analyticity conjecture: some scaling functions are analytic everywhere on the complex ξ^2-plane, except on the YL singularity and branch cut. Examples: mass $\hat{M}_1(\xi)$[7], free energy $\mathcal{G}(\xi)$[8], effective φ^3 coupling[9].



- Choice of the Yang-Lee branch cut:
  We take the natural choice, as the zeros on Yang-Lee circle "condense" into the branch cut in the scaling limit. Preserving PT symmetry on the branch cut.
- Other choice of the Yang-Lee branch cut: define via analytic continuation.

[7] Hao-Lan Xu and Alexander Zamolodchikov. "2D Ising Field Theory in a magnetic field: the Yang-Lee singularity". In: *JHEP* 08 (2022), p. 057. DOI: 10.1007/JHEP08(2022)057. arXiv:2203.11262 [hep-th].

[8] P Fonseca and A Zamolodchikov. "Ising field theory in a magnetic field: analytic properties of the free energy". In: *Journal of statistical physics* 110.3-6 (2003), pp. 527–590.

[9] Hao-Lan Xu and Alexander Zamolodchikov. "Ising Field Theory in a magnetic field: φ^3 coupling at $T > T_c$". In: (Apr. 2023). arXiv:2304.07886 [hep-th].

The mass dispersion relation (I).

- Following the high-T analyticity conjecture of $\hat{M}_1$, the dispersion relation follows (ξ^2):

$$M_1(\xi^2) = 1 + \xi^2 \int_{\xi_0^2}^{+\infty} \frac{dx}{\pi} \frac{\Im m\, M_1(-x+i0)}{x(x+\xi^2)} \,,$$

with representing $\hat{M}_1(\xi^2)$ via an integration on the Yang-Lee branch cut with the discontinuity:

$$\text{Disc}\, M_1(-x) = M_1(-x+i0) - M_1(-x-i0) = 2i\, \Im m\, M_1(-x+i0)\,.$$

In practice, we construct the discontinuity by interpolating expansions.

- The expansions we use are (at $\xi^2 = \xi_0^2$ and $\xi^2 \to -\infty$):

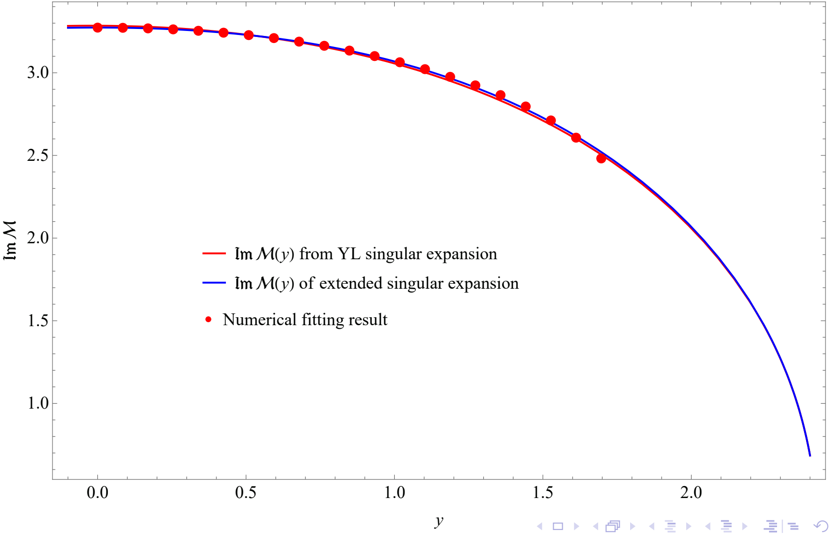
$$\hat{M}_1(\xi^2) = (\xi^2+\xi_0^2)^{5/12} \left[b_0 + b_1(\xi^2+\xi_0^2) + b_2(\xi^2+\xi_0^2)^2 + ...\right] + \\ (\xi^2+\xi_0^2)^{5/4}\left[c_0 + c_1(\xi^2+\xi_0^2) + ...\right] + (\xi^2+\xi_0^2)^{11/4}\left[d_0 + ...\right] + \ldots\,, \quad (2)$$

and ($\eta = -\xi^{-8/15}$):

$$M_1/|h|^{8/15} = \hat{M}_1(\xi^2)/(\xi^2)^{4/15} = M^{(0)} + M^{(1)}\eta + M^{(2)}\eta^2 + M^{(3)}\eta^3 + \ldots\,. \quad (3)$$

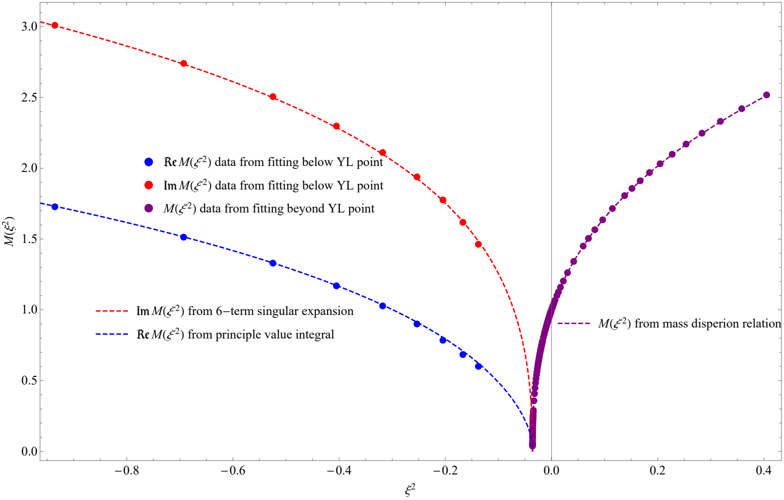
The mass dispersion relation (II).

- In η coordinate, the Yang-Lee point is located at $y = |\eta| = |\eta_0| \approx 2.4293$.
- The discontinuity $\Im m\, \mathcal{M}_1(y) = y\, \Im m\, \hat{M}_1$ is as shown:



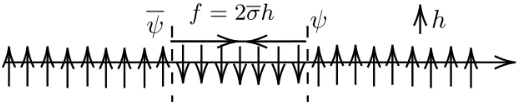
The mass dispersion relation (III).

- Using the constructed discontinuity, the mass dispersion relation can be verified numerically.



Phases and Scenarios: $T < T_c$

- At $T < T_c$: ordered phase with broken $\mathbb{Z}_2$ symmetry.
  Nonvanishing VEV of spin density operator:
  $\langle\sigma\rangle = \pm\bar{\sigma} = \pm\bar{s}|m|^{1/8}$, with $\bar{s} = 1.35783834\ldots$.
- Double degenerate vacuum at $h = 0$, degeneracy lifted at $h \neq 0$.
  At small h: stable vacuum (spins aline along h) and metastable vacuum.
- In $1 + 1$ d as a field theory: meson spectrum (McCoy-Wu scenario), fermions as domain walls and h provides binding force. String tension $f = 2\bar{\sigma}h$.
- Spectrum of McCoy-Wu mesons:
  computable using Bethe-Salpeter approximation[10].



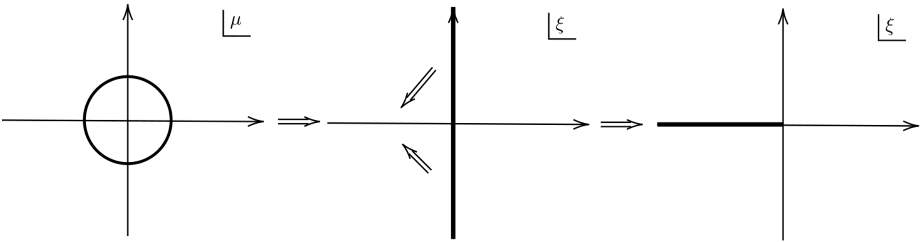
[10] Pedro Fonseca and Alexander Zamolodchikov. “Ising Spectroscopy I: Mesons at T¡ T_c”. In: *arXiv preprint hep-th/0612304* (2006).

The Yang-Lee theorem at low-T

- At $T < T_c$: Yang-Lee theorem gives the full circle.
  The zeros of lattice partition function sit on the unit circle in the fugacity plane.

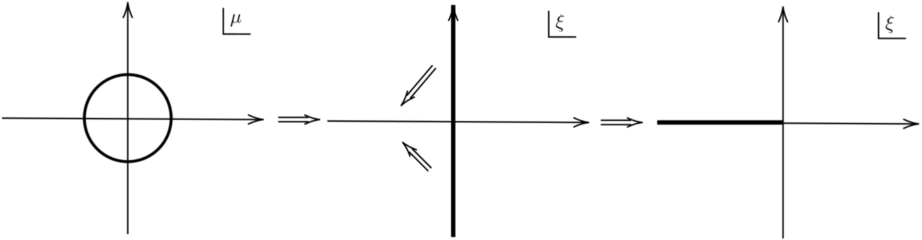
- $|\mu| < 1$ domain and $|\mu| > 1$ domain correspond to different choice of VEV.
  i.e.: $|\mu| > 1$ is the one with $\langle\sigma\rangle = +\bar{\sigma}$ and $|\mu| < 1$ is with $\langle\sigma\rangle = -\bar{\sigma}$.

- In the continuous limit, zeros of $\mathcal{Z}$ would condense into a natural bound of analyticity for thermodynamic functions. On the complex ξ-plane, the "wall" is the imaginary axis separating $\Re e\, \xi > 0$ and $\Re e\, \xi < 0$, and in each domain different functions M_1 , F , $\cdots$ can be defined. They are sitting in different phase.



Fisher-Langer's branch cut at low-T

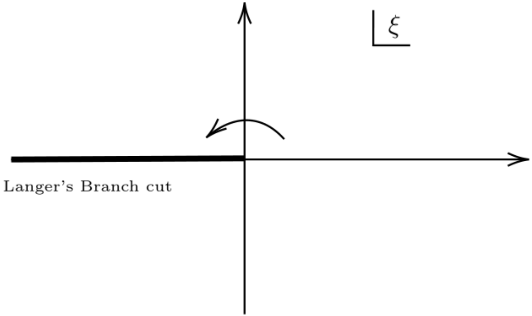
- As for the scaling functions, $\xi > 0$ and $\xi < 0$ define different branches.
- Functions defined on $\Re e\,\xi > 0$: possible to be continued to the full complex ξ-plane, leaving discontinuities along real negative axis.
  This is known as the Fisher-Langer's branch cut.
- $\xi = 0^{-}$: usually an essential singularity of the function, with form $\sim e^{-\pi/2\bar{s}\xi}$.



M_1 Analyticity at low-T

- By choosing $\langle\sigma\rangle = +\bar{\sigma}$ as vacuum: $\hat{M}_1(\xi)$ is a single valued functions at $\xi > 0$.
- Analyticity at low-T: $\hat{M}_1(\xi)$[11] can be continued to the complex ξ, except along the Fisher-Langer's branch cut: $-\infty < \xi < 0$, with the dispersion relation:

$$\hat{M}_1(\xi) = 2 + \xi \int_0^{+\infty} \frac{dt}{\pi} \frac{\Im m\, \hat{M}_1(-\xi + i0)}{t(t+\xi)} \, .$$

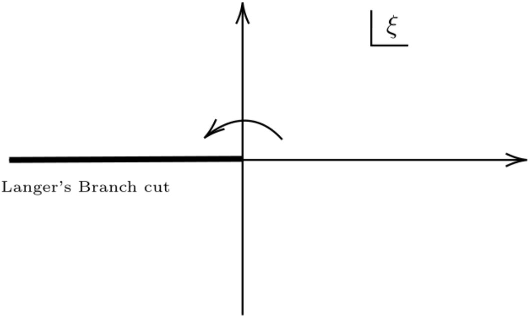


[11] Hao-Lan Xu. "On the analyticity of the lightest particle mass of Ising field theory in a magnetic field". In: (May 2024). arXiv:2405.09091 [hep-th].

Fisher-Langer's branch cut and essential singularities

- $\xi = 0^-$ is an essential singularity of $\hat{M}_1(\xi)$, because $\xi = -\epsilon \pm i0$ are in metastable vacuum, with the tunneling effects contribute.
- At finite T in metastable vacuum, thermal fluctuations generate "bubbles" of stable vacuum. The condensation of bubbles give[12][13]:

$$\Im m\, \hat{M}_1 \to (\text{Analytic terms}) + \frac{1}{\pi} e^{-\frac{\pi}{\lambda}} + \cdots\,, \quad \text{where} \quad \lambda = -2\bar{s}\xi > 0\,.$$



[12] M. B. Voloshin. "DECAY OF FALSE VACUUM IN (1+1)-DIMENSIONS". In: *Yad. Fiz.* 42 (1985), pp. 1017–1026.

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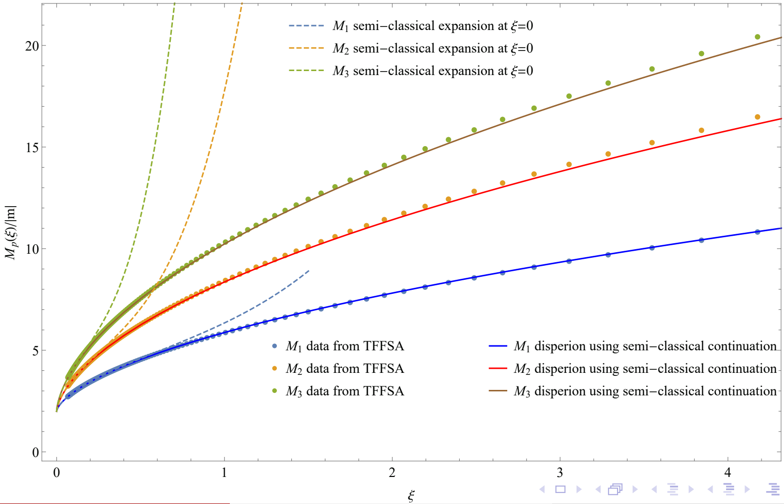
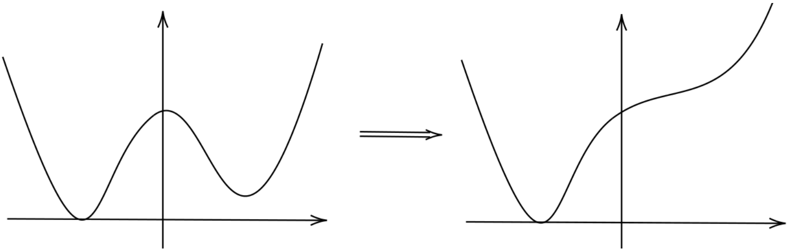
Low-T dispersion relations with Langer's branch cut.

- By approximating discontinuities, the low-T dispersion relation can be verified.
- For $\hat{M}_1(\xi)$, computation shows the non-analytic term is negligible.
  Also for $\hat{M}_2(\xi)$ and $\hat{M}_3(\xi)$ similar dispersion relations exist, and can be checked.



Spinodal point on Fisher-Langer's branch cut?

- $\xi \to 0^-$: decay of metastable vacuum, with thermal fluctuations and tunneling effects. At $\xi \to -\infty$ there exists one only true vacuum, and the phase transition happens classically.
- From mean field theory point of view, at negative ξ there exist a point where the metastable vacuum becomes classically unstable, and the picture changes.
- The point is known as spinodal point. However, no other singularity found on the Fisher-Langer branch cut[14] or at first sheet of ξ. Where is and what happened to the spinodal point?



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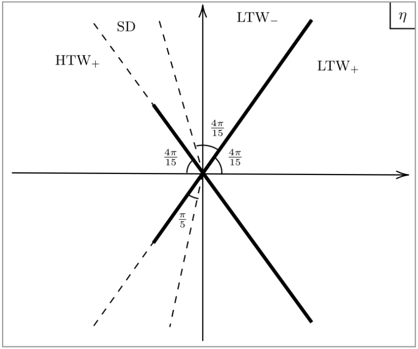
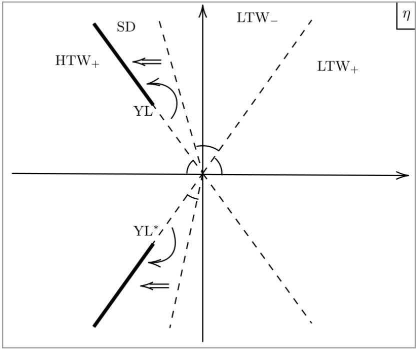
Extended analyticity conjectures: connecting all-T

- The question can be stated as: what are hiding under the FL branch cut? Instead, we switch to the analyticity on the complex $\eta = m/h^{\frac{8}{15}} = \xi^{-8/15}$ plane.
- $-\frac{8\pi}{15} \leq \text{Arg}\,\eta \leq +\frac{8\pi}{15}$: Low-T wedge (LTW), represents the full ξ-plane with $m > 0$.
- $-\frac{4\pi}{15} \leq \text{Arg}\,(-\eta) \leq +\frac{4\pi}{15}$: High-T wedge$_+$ (HTW$_+$), represents $\Re e\,\xi > 0$ for $m < 0$.
- In between: shadow domain (SD), which is under the FL branch cut.
- YL branch cut: $\eta = -ye^{\pm\frac{4\pi i}{15}}$ with $y \leq Y_0$, $Y_0 \approx 2.4293$.



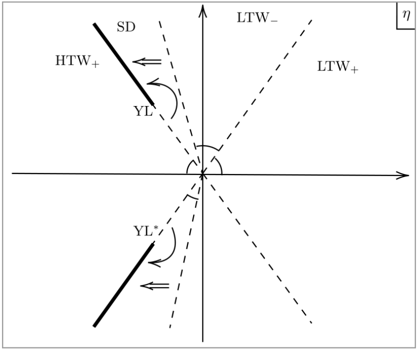
Extended analyticity conjectures: scaling functions on the η-plane.

- Focus on the function $\mathcal{M}_1(\eta) = M_1/|h|^{\frac{8}{15}} = |\eta|\,\hat{M}_1$, which is regular at $\eta = 0$.
- At complex η, we rotate the YL branch cuts, as extending from $-Y_0 e^{\pm\frac{4\pi i}{15}}$ to infinity.
- Question: what are the analytic structures within the SD? If $\mathcal{M}_1(\eta)$ is analytic within the SD, then the Yang-Lee point can be identified as the spinodal singularity.



Extended analyticity conjecture as minimal conjecture.

- Extended analyticity conjecture: the scaling function is analytical anywhere on the complex η-plane, except on the rotated YL branch cuts[15][16].
- No extra singularities within SD, YL point is the nearest one under FL branch cut.
- The extended analyticity conjecture is the most elegant conjecture. Meanwhile, if other singularities exist, then one must consider their physical interpretations.



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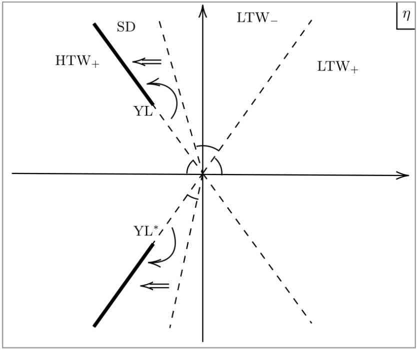
Extended dispersion relations.

- As a result, the mass extended dispersion relation can be formulated as[17]:

$$\mathcal{M}_1(\eta) = M_1^{(0)} + M_1^{(1)}\eta + \frac{2\eta^2}{\pi}\int_{Y_0}^{\infty}\frac{dy}{y^2}\,\frac{y\,\Re e\left(e^{-\frac{11\pi i}{15}}\Delta_1(y)\right) - \eta\,\Re e\,\Delta_1(y)}{y^2 - 2\cos\left(\frac{11\pi}{15}\right)\eta y + \eta^2}\,,$$

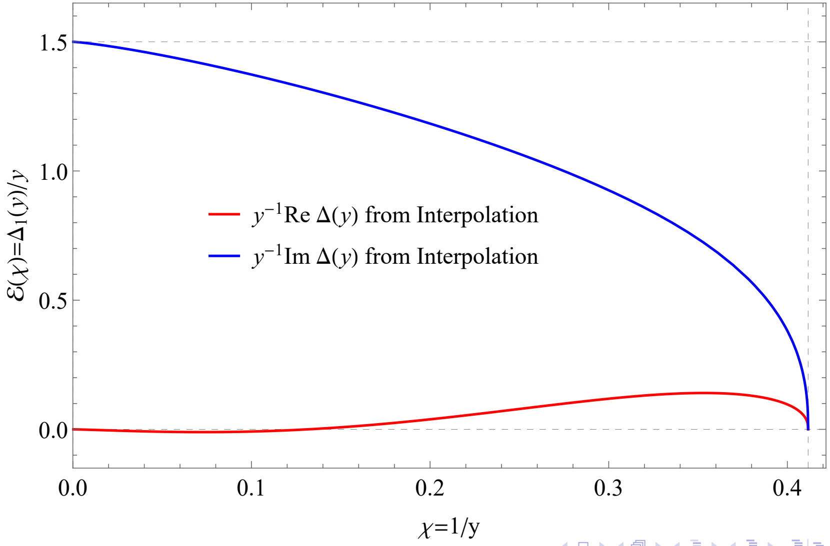
$$\text{where}\quad \Delta_1(y) = \frac{i}{2}e^{\frac{4\pi i}{15}}\left[\hat{\mathcal{M}}_1(ye^{\frac{11\pi i}{15}+i0}) - \hat{\mathcal{M}}_1(ye^{\frac{11\pi i}{15}-i0})\right].$$



[17] Hao-Lan Xu. "On the analyticity of the lightest particle mass of Ising field theory in a magnetic field". In: (May 2024). arXiv:2405.09091 [hep-th].

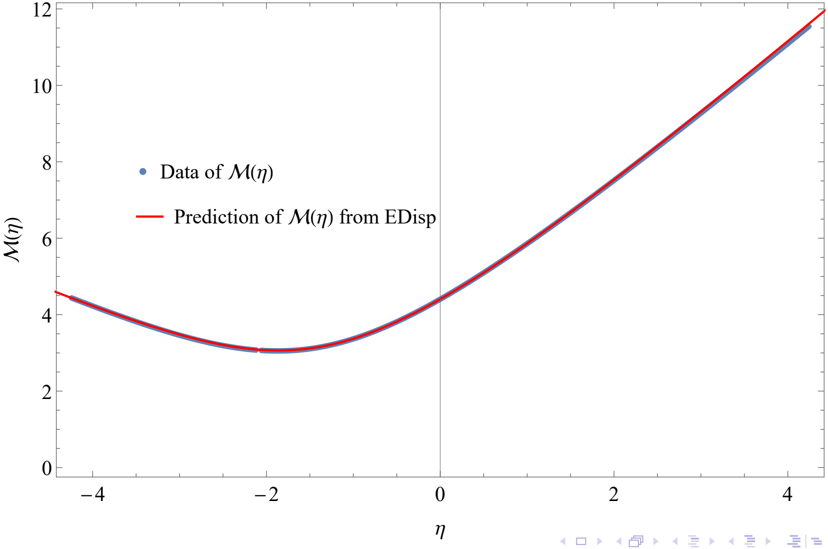
Numerical verification of extended dispersion relations of M_1 (I)

- The discontinuities $\Delta_1(y)$ of $\mathcal{M}_1(\eta)$ is now complex on the rotated YL branch cut.



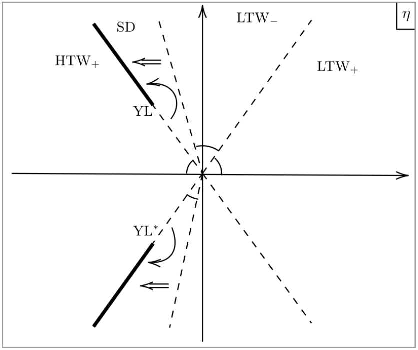
Numerical verification of extended dispersion relations of M_1 (II)

- For real η, numerical verification is straightforward.



Extended analyticity conjecture and true spinodal point.

- The low-T analyticity conjecture indicates that perturbations ”push down” the spinodal point inside the FL branch cut, and now we know that the YL point is the non-perturbative spinodal point, near which the scaling behaviours follow the Yang-Lee universality class.
- It is very interesting to explore the applicability of such picture in higher dimensions[18], and what analytic structures are behind the YL branch cut.



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Summary and Outlooks.

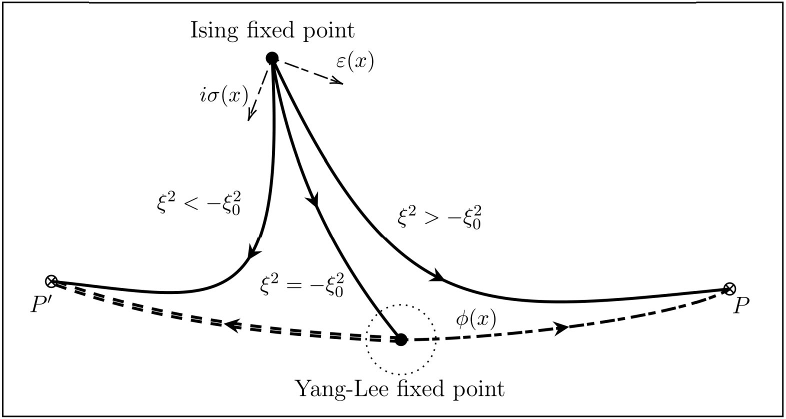
Summary: analyticity structure of 2d Ising field theories:

- 2D Ising field theory, basics and numerical approaches;
- The Yang-Lee critical point: position, critical behaviours and effective action;
- Analyticity of M_1: high-T and extended analyticity.

Outlook: unsolve questions and future directions:

- Goal: understand the structure of 2D IFT theory space in the non-perturbative regime, with both analytical and numerical methods.
- Difficulties: limitations of numerical methods & difficulties in analytical methods.
- Unsolved questions: (On going)
  (i): Singularities on higher sheets: positions, behaviours and universality classes?
  (ii): Spectrum and scattering behaviours in complex η?
  (iii): Numerical tools in higher dimensions (i.e. Fuzzy sphere in 3D)?
  (iv): Theory space of Ising field theories in higher dimensions.

Thank You for Listening!



The Ising Bethe-Salpeter equation

- When $T < T_c$ and $h = 0$, the excitations of 1+1d IFT are free Majorana fermions with mass $m \propto 1 - T/T_c$, as domain walls between "up" and "down" vacuum. These fermions are confined as pairs with nonvanishing h.

- The Ising Bethe-Salpeter equation in rapidity space reads:

$$\left(m_q^2 - \frac{M^2}{4\cosh^2\theta}\right)\Psi(\theta) = f\int_{-\infty}^{+\infty}\frac{d\theta'}{2\pi}G(\theta|\theta')\Psi(\theta')\,,$$

where $$G(\theta|\theta') = 2\frac{\cosh(\theta-\theta')}{\sinh^2(\theta-\theta')} + \frac{1}{4}\frac{\sinh\theta\sinh\theta'}{\cosh^2\theta\cosh^2\theta'}\,,$$

as bootstrap-like equation treating confining pair of fermions with $\sigma(x)$.

- m_q is the quark mass, and f is the string tension. They are usually approximated by m and $f_0 = 2\bar{\sigma}h$ when ξ is small. The dimensionless ration $\lambda_{\text{eff}} = f/m_q^2$ is a function of ξ. At small ξ, $\lambda_{\text{eff}} \approx \lambda = 2\bar{s}\xi$.

- Form of $G(\theta|\theta')$ follows the 4-pt form factor of $\sigma(x)$, and should be modified if consider multi-quark corrections.

The Ising Bethe-Salpeter equation at complex ξ

- In the Fourier space, the Ising Bethe-Salpeter equation becomes:

$$8\big(1+\lambda\,\nu\tanh\frac{\pi}{2}\nu\big)\psi(\nu)$$
$$=\int d\nu'\Big\{\frac{\lambda}{2}\frac{\nu\nu'}{\cosh\frac{\pi}{2}\nu\cosh\frac{\pi}{2}\nu'}+\frac{M^2}{m^2}\big[K(\nu-\nu')-K(\nu+\nu')\big]\Big\}\psi(\nu')\,,$$

  where:

$$\psi(\nu)=\int\frac{d\theta}{2\pi}\,\Psi(\theta)e^{-i\nu\theta}\,,\quad K(\nu)=\frac{\nu}{2\sinh\frac{\pi}{2}\nu}\,.$$

- At complex ξ and λ, there exist two kinds of singularities:
  (1): At some complex $\tilde{\lambda}_k$, the two roots of $1+\lambda\,\nu\tanh\frac{\pi}{2}\nu$ coincident, which can be understood as two poles of $\psi(\nu)$ collide. $\tilde{\lambda}_k$ satisfies:

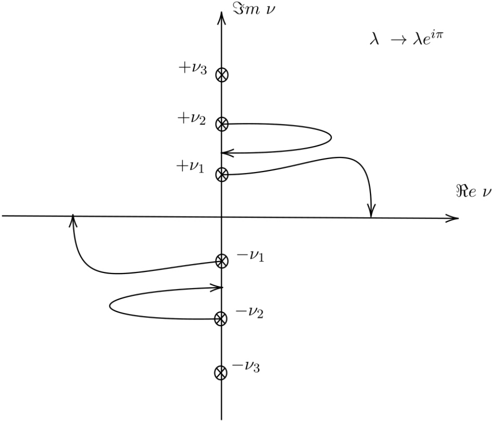
$$1+\tilde{\lambda}_k\,\tilde{\nu}_k\tanh\frac{\pi}{2}\tilde{\nu}_k=0\,,\quad \text{and}\quad \frac{d}{d\nu}\big(1+\tilde{\lambda}_k\,\tilde{\nu}_k\tanh\frac{\pi}{2}\tilde{\nu}_k\big)\Big|_{\nu=\tilde{\nu}_k}=0\,.$$

  (2): At some λ_k, the k'th eigenvalue M_k^2 becomes zero, implying criticality.

- Due to the small first term of RHS, $\tilde{\lambda}_k$ and λ_k are close to each other.

Singularities of Ising Bethe-Salpeter wave functions

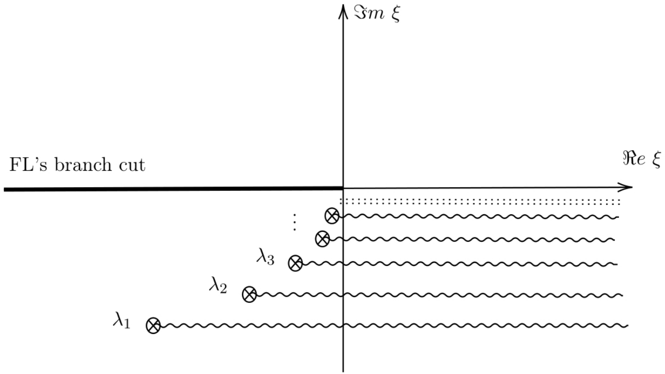
- At real λ, the roots of $1 + \lambda\,\nu\tanh\frac{\pi}{2}\nu$ (as poles of $\psi(\nu)$) sit on $\Im m\,\nu$. When the phase of λ varies, they rotate clockwise.



- After $\mathrm{Arg}\,\lambda > \pi$, roots ν_1 and ν_{k+1} can pinch at some complex $\tilde{\lambda}_k$, which are close to the higher complex critical points λ_k.

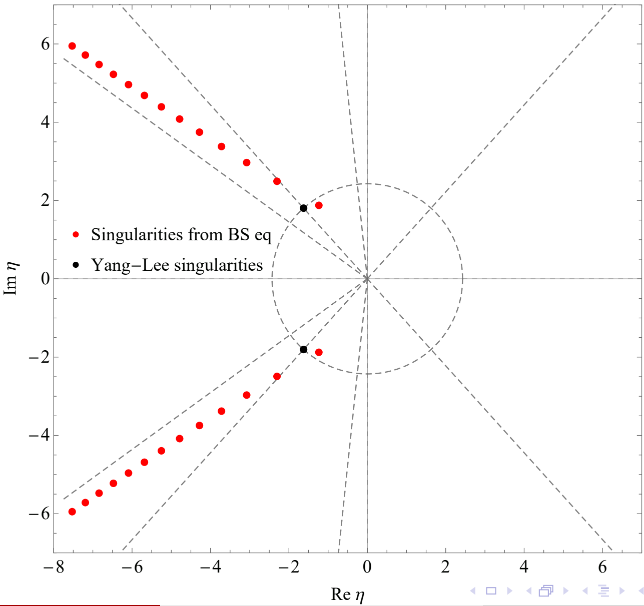
Singularities λ_k on the complex ξ plane: the Lasagne

- Singularities $\lambda_k \approx \tilde{\lambda}_k$ are behind the Fisher-Langer's branch cut on the complex ξ-plane. They are square root branching points of scaling functions, and we choose the branch cuts lie horizontally (like Lasagne).



- The leading singularity λ_1 is very close to the exact Yang-Lee point ($\sim 6\%$), and the rest are behind the Yang-Lee cut.

Singularities λ_k on the complex η plane



Free fermion Hilbert space and projections.

- At $h = 0$, the 2d IFT action is equivalent to the action of massive free fermion:

$$\mathcal{A}_{\text{FF}} = \mathcal{A}_{\text{CFT}}^{\text{Ising}} + \frac{m}{2\pi}\int \varepsilon(x)\, d^2x = \frac{1}{2\pi}\int \left(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} + im\bar{\psi}\psi\right) d^2x\,.$$

The fermionic field has mode expansion (on the cylinder $w = x + iy \sim w + R$):

$$\psi(w) = \left(\frac{-2\pi i}{R}\right)^{-1/2} \sum_r \left(a_r u_{-r}(w) + h.c.\right), \quad \text{with} \quad \{a_r, a_{-r'}\} = \delta_{r,r'}\,.$$

- Based on the B.C. on the cylinder, the Hilbert space of FF splits:

$$\text{Neveu-Schwarz sector:} \quad \psi(x+R) = -\psi(x) \quad \text{then} \quad a_k\,, \quad k \in \mathbb{Z} + \frac{1}{2}\,;$$

$$\text{Ramond sector:} \quad \psi(x+R) = +\psi(x) \quad \text{then} \quad a_n\,, \quad n \in \mathbb{Z}\,,$$

say $\mathcal{H}_{\text{FF}} = \mathcal{H}_{\text{NS}} \oplus \mathcal{H}_{\text{R}}$, and same for $\bar{\psi}$ sector (they are coupled).

- However, the Hilbert space of IFT requires $\sigma(x)$ local and single-valued, so that:

$$\begin{aligned} \text{Low-T:} \quad m > 0 \quad &: \quad N_{\text{NS}} \quad \text{even}\,, \quad N_{\text{R}} \quad \text{even}\,, \\ \text{High-T:} \quad m < 0 \quad &: \quad N_{\text{NS}} \quad \text{even}\,, \quad N_{\text{R}} \quad \text{odd}\,, \end{aligned}$$

or roughly, $\mathcal{H}_{\text{IFT}} = \mathcal{H}_{\text{FF}}/\mathbb{Z}_2$.

The truncated free-fermion space approach (I)

- As for the spectrum, since H_{FF} is diagonal as $E_0(R) + \sum \omega_i$. If we know the cylinder form-factor of $\sigma(x)$, then we can diagonalize $H_{\text{IFT}} = H_{\text{FF}} + h \int \sigma$ to find the full finite size spectrum of IFT.
- However, generic Hamiltonian is infinite dimensional and not possible to diagonalize. $\Longrightarrow$ *the truncated free-fermion space approach* (TFFSA):
- H_{IFT} now can be divided into blocks:

$$H_{\text{IFT}} = \left(\begin{array}{c|c} \left\{ E^{\text{NS}}(R) \right\} & \left\{ hR\langle\{k\}|\sigma|\{n\}\rangle \right\} \\ \hline \left\{ hR\langle\{n\}|\mu|\{k\}\rangle \right\} & \left\{ E^{\text{R}}(R) \right\} \end{array} \right). \quad (4)$$

- $E^{\text{NS}}(R)$ and $E^{\text{R}}(R)$ are diagonal in free fermion space.
- The explicit form for $\langle\{k\}|\sigma|\{n\}\rangle$ is given (see Appendix II).
- Truncation: keep only states with level no more than L:

$$L \geq \sum_i |n_i| \quad \text{or} \quad L \geq \sum_j |k_j|.$$

- Now the hamiltonian is finite dimensional $\Longrightarrow$ numerical diagonalization. To reduce inaccuracy from truncation effect, one can increase the truncation level L.

The truncated free-fermion space approach (II)

- Many information are readable from the finite size spectrum.

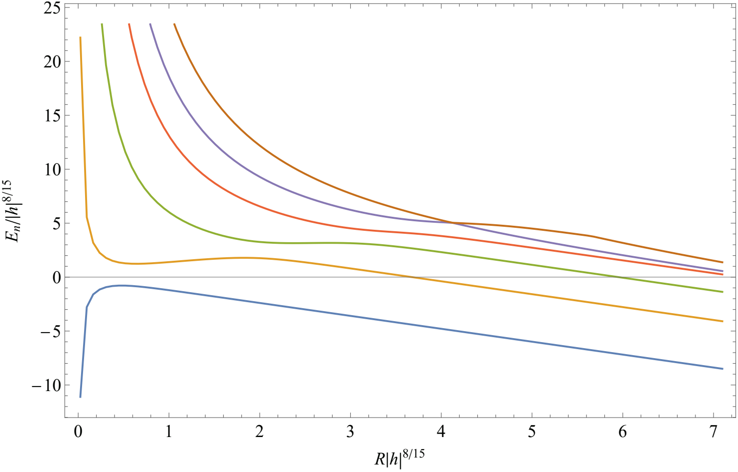


Figure: Finite size spectrum at $\eta = 0$, the integrable E_8 point.

The truncated free-fermion space approach (II)

- Many information are readable from the finite size spectrum.

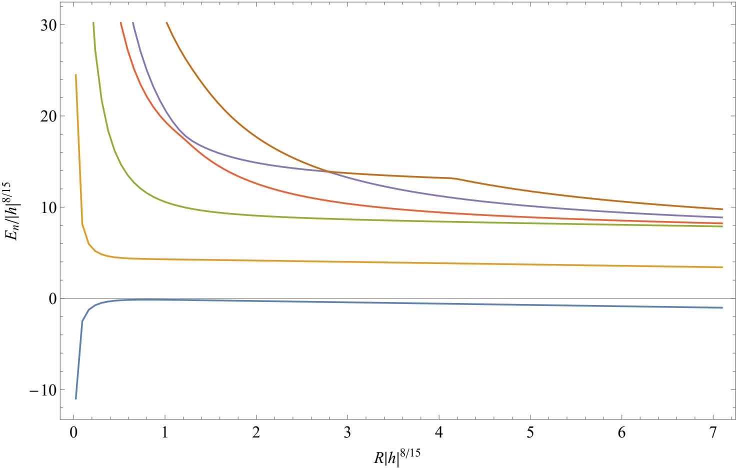


Figure: Finite size spectrum at $\eta = -10.0/(2\pi)^{7/15}$, with A_2 is no longer stable.

The truncated free-fermion space approach (II)

- At pure imaginary h, the spectrum become complex, however unbroken $\mathbb{Z}_2$ ensures the spectrum come as complex conjugate pairs.

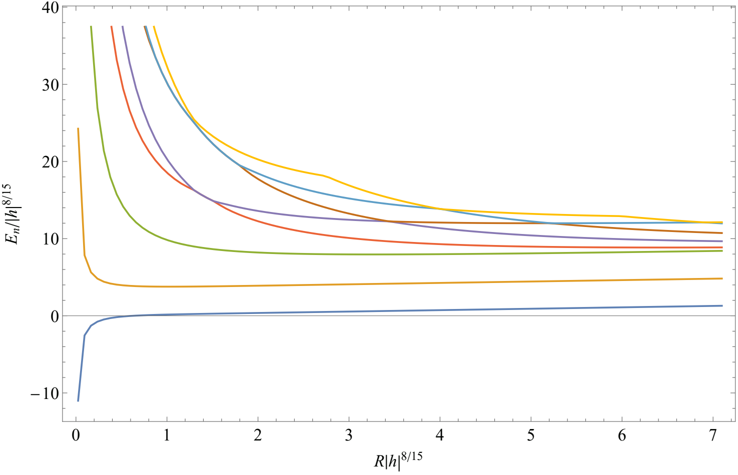


Figure: Finite size spectrum at $y = -9.0/(2\pi)^{7/15}$, with $\eta = -ye^{\pm\frac{4\pi i}{15}}$.

The truncated free-fermion space approach (II)

- The Yang-Lee point is located at $-Y_0 \approx -2.4293$.
  For $0 > y > -Y_0$, even the bulk energy become complex.

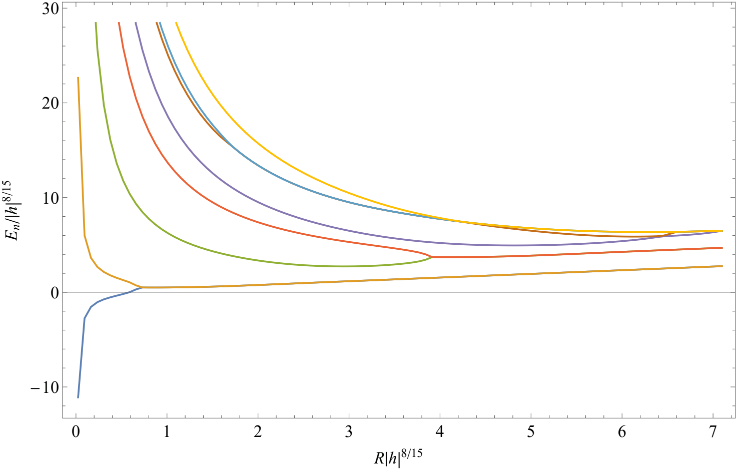


Figure: Finite size spectrum at $y = -2.0/(2\pi)^{7/15}$, now within the Yang-Lee branch cut.

The truncated free-fermion space approach (II)

- The Yang-Lee point is located at $-Y_0 \approx -2.4293$.
  For $0 > y > -Y0$, even the bulk energy become complex.

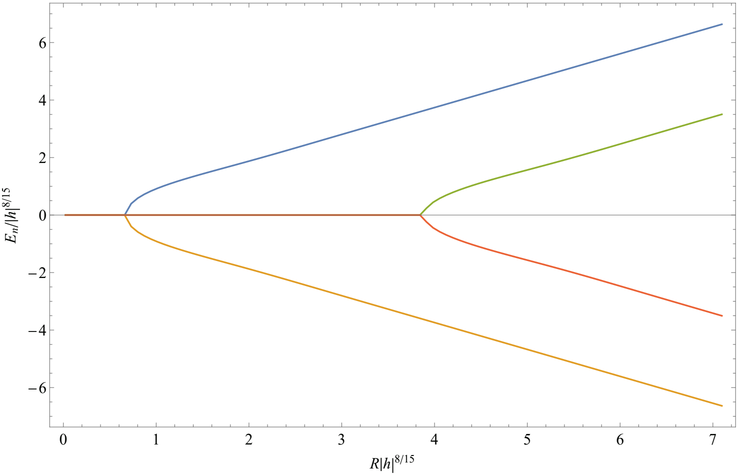


Figure: Imaginary part of finite size spectrum at $y = -2.0/(2\pi)^{7/15}$.

Critical amplitudes and effective couplings.

- The expansion coefficients determines the couplings of the effective action. i.e.: from Yang-Lee field theory, the mass coupling relation gives:

$$M_{\mathrm{YL}} = C_{\mathrm{YL}} \lambda^{\frac{5}{12}} , \quad \text{with} \quad C_{\mathrm{YL}} = 2.64294463 \ldots .$$

Thus the relevant coupling expansion reads:

$$\lambda_1 = \left(b_0 / C_{\mathrm{YL}}\right)^{\frac{12}{5}} = 3.089 \pm 0.008 \, , \quad \lambda_2 = \frac{12}{5} \frac{b_1}{C_{\mathrm{YL}}} \lambda_1^{\frac{7}{12}} = 38.4 \pm 1.6 \, .$$

- For more general terms, the proportional coefficient is related to the 2-pt form factor of the corresponding operator:

$$\frac{\delta_i M(\xi^2)}{(\xi^2 + \xi_0^2)^{\frac{5}{12}}} \propto (\xi^2 + \xi_0^2)^{\frac{5}{12}(\Delta_i - 2)} \times \langle \theta | \mathcal{O}_i(0) | \theta \rangle \, .$$

- However, the leading irrelevant contribution is generated by the $T\bar{T}$ operator, which is ”solvable”. As an result, c_0 is related to its perturbation explicitly. Denote the term in the action $\frac{\alpha}{\pi^2} \int (T\bar{T})$, we have:

$$\alpha_0 = -\frac{c_0}{f_{\mathrm{YL}} b_0^3} = -1.32 \pm 0.05 \, , \quad \text{where} \quad f_{\mathrm{YL}} = -\frac{\sqrt{3}}{12} \, .$$

The $T\bar{T}$ deformation and its application.

- How it comes? Consider the $T\bar{T}$ deformation of a theory:

$$\mathcal{A}_0 \to \mathcal{A}_\alpha = \mathcal{A}_0 + \frac{\alpha}{\pi^2} \int d^2x \, (T\bar{T})(x) \, ,$$

and based on the factorization property of $T\bar{T}$ operator, one can show the finite size spectrum satisfies:

$$\frac{\partial}{\partial \alpha} E^{(\alpha)}(R) + E^{(\alpha)}(R) \frac{\partial}{\partial R} E^{(\alpha)}(R) = 0 \, ,$$

which works for every energy level separately. The solution shows that the deformed spectrum is shifted:

$$E^{(\alpha)}(R) = E^{(0)}(R - \alpha E^{(\alpha)}(R)) \, .$$

- As an result, $T\bar{T}$ deformation leads to the transformation of F and M_1:

$$F^{(\alpha)} = \frac{F^{(0)}}{1 + \alpha F^{(0)}} \, , \quad M_1^{(\alpha)} = \frac{M_1^{(0)}}{1 + \alpha F^{(0)}} = M_1^{(0)} (1 - \alpha F^{(\alpha)}) \, .$$

- In our case, the undeformed theory is approximately nothing but massive Yang-Lee QFT, with $F = f_{\mathsf{YL}} M_1^2$. One can easily express c_0 in terms of α_0.

Details on the finite size spectrum at $T > T_c$

- When $\eta > 0$, the spectrum of IFTs can be explained by the McCoy-Wu scenario.
- At $\eta = 0$, the E_8 field theory has 8 stable particles. Other than A_2 and A_3:

A_4	A_5	A_6	A_7	A_8
$4\cos\frac{\pi}{5}\cos\frac{7\pi}{30}$	$4\cos\frac{\pi}{5}\cos\frac{2\pi}{15}$	$4\cos\frac{\pi}{5}\cos\frac{\pi}{30}$	$8\cos^2\frac{\pi}{5}\cos\frac{7\pi}{30}$	$8\cos^2\frac{\pi}{5}\cos\frac{2\pi}{15}$
2.405	2.956	3.218	3.891	4.783

- The eight mass ratios of IFT at T_c with nonvanishing h are the same as the eight components of Frobenius vector of the E_8 Cartan.
- Also, the E_8 field theory can be constructed via affine Toda field theory, using the E_8 Dynkin diagram.
- $M_3/M_1 \approx 1.989$ and $M_2/M_1 \approx 1.618$.
- Stability threshold $2M_1$, so once $\eta \neq 0 \implies A_4 \sim A_8$ become unstable.
- Along with η decrease, M_3 and M_2 would increase.
- At $\eta_3 \approx -0.326(8)$, $M_3 \approx 2M_1$ and become unstable.
- At $\eta_2 \approx -4.908(7)$, $M_2 \approx 2M_1$ and become unstable.
- Finally at $\eta \to -\infty$, back to free-fermion theory.

The truncated free-fermion space approach (I)

- However, H_{IFT} is infinite dimensional! $\Longrightarrow$ Unable to diagonalize with $h, m \neq 0$!
- Numerical method: *the truncated free-fermion space approach* (TFFSA): Truncate H_{IFT} to some level L, then do numerical diagonalization to $H_{\text{IFT}}^{(L)}$.
- H_{IFT} can be divided as free-fermion part and spin interactions:

$$H_{\text{IFT}} = H_{\text{FF}} + hV = \frac{1}{2\pi}\int_0^R \left[\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} + im\bar{\psi}\psi\right]dx + h\int_0^R \sigma(x)dx.$$

- Different B.C. $\Longrightarrow$ R-sector (PBC) and NS-sector (ABC):

NS-sector: $$\psi(w) = \left(\frac{-2\pi i}{R}\right)^{-1/2} \sum_{k\in\mathbb{Z}+\frac{1}{2}} \left(a_k e^{-\frac{2\pi i k}{R}w} + a_k^{\dagger} e^{+\frac{2\pi i k}{R}w}\right),$$

R-sector: $$\psi(w) = \left(\frac{-2\pi i}{R}\right)^{-1/2} \sum_{n\in\mathbb{Z}} \left(a_n e^{-\frac{2\pi i n}{R}w} + a_n^{\dagger} e^{+\frac{2\pi i n}{R}w}\right).$$

- Anticommutating relations:

$$\{a_k, a_{k'}^{\dagger}\} = \delta_{k,k'}, \quad \{a_n, a_{n'}^{\dagger}\} = \delta_{n,n'}.$$

The truncated free-fermion space approach (II)

- Notions of states:

NS-sector: $$|k_1,\cdots,k_N\rangle_{\text{NS}} = a^\dagger_{k_1}\cdots a^\dagger_{k_N}|\Omega\rangle_{\text{NS}} \quad k_1,\cdots,k_N \in \mathbb{Z}+\frac{1}{2},$$

R-sector: $$|n_1,\cdots,n_N\rangle_{\text{R}} = a^\dagger_{n_1}\cdots a^\dagger_{n_N}|\Omega\rangle_{\text{R}} \quad n_1,\cdots,n_N \in \mathbb{Z},$$

- H_{IFT} now can be divided into blocks:

$$H_{\text{IFT}} = \left(\begin{array}{c|c} \left\{E^{\text{NS}}(R)\right\} & \left\{hR\langle\{k\}|\sigma|\{n\}\rangle\right\} \\ \hline \left\{hR\langle\{n\}|\mu|\{k\}\rangle\right\} & \left\{E^{\text{R}}(R)\right\} \end{array}\right), \tag{5}$$

where $E^{\text{NS}}(R)$ and $E^{\text{R}}(R)$ are diagonal in free fermion space.

- N-particle state energy:

$$E_N(R) = E_0(R) + \sum_i^N \omega_{r_i}(R), \quad \omega_r(R) = \sqrt{m^2 + \left(\frac{2\pi r}{R}\right)^2}; \tag{6}$$

$$E_0^{\text{NS,R}}(R) = RF(m,0) - |m|\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\cosh\theta\,\log\left(1\pm e^{-|m|R\cosh\theta}\right), \tag{7}$$

The form of $E_0(R)$ is given by thermodynamic Bethe ansatz.

The truncated free-fermion space approach (III)

- Rules for particle #:
  N_{NS} must be even; N_{R} is even for low-T, and odd for high-T
- Off diagonal part? Need $\langle k_1 \cdots k_M|\sigma(0)|n_1, \cdots n_N\rangle$ on the cylinder!
- General formula:

$$_{\mathrm{NS}}\langle k_1, \cdots, k_K|\sigma(0,0)|n_1, \cdots, n_N\rangle_{\mathrm{R}} = S(R)\prod_{j=1}^{K}\tilde{g}(\theta_{k_j})\prod_{i=1}^{N}g(\theta_{n_i})F_{K,N}\big(\{\theta_{k_j}\}|\{\theta_{n_i}\}\big),$$

where the spin-field form factor in infinite space:

$$F_{K,N}(\theta_1, \cdots, \theta_K|\theta'_1, \cdots, \theta'_N) = i^{[\frac{K+N}{2}]}\bar{\sigma}\cdot$$
$$\prod_{0<i<j\leq K}\tanh\left(\frac{\theta_i-\theta_j}{2}\right)\prod_{0<p<q\leq N}\tanh\left(\frac{\theta'_p-\theta'_q}{2}\right)\prod_{0<s\leq K, 0<t\leq N}\coth\left(\frac{\theta_s-\theta'_t}{2}\right),$$

and the normalization of finite size vacuum-vacuum matrix element ($r = mR$):

$$S(R) = \exp\left\{\frac{r^2}{2}\int\frac{d\theta_1 d\theta_2}{(2\pi)^2}\frac{\sinh\theta_1\sinh\theta_2}{\sinh(r\cosh\theta_1)\sinh(r\cosh\theta_2)}\log\left|\coth\frac{\theta_1-\theta_2}{2}\right|\right\},$$

which is defined by the vev of σ or μ.

The truncated free-fermion space approach (IV)

- The finite size leg factors: $g(\theta_n) = \frac{{}_{\text{NS}}\langle 0|\mu|n\rangle_{\text{R}}}{{}_{\text{NS}}\langle 0|\sigma|0\rangle_{\text{R}}}$ and $\tilde{g}(\theta_k) = \frac{{}_{\text{NS}}\langle k|\mu|0\rangle_{\text{R}}}{{}_{\text{NS}}\langle 0|\sigma|0\rangle_{\text{R}}}$:

$$g(\theta), \tilde{g}(\theta) = \frac{\exp \pm\kappa(\theta)}{\sqrt{|m|R\cosh\theta}}, \quad \kappa(\theta) \int \frac{d\beta}{2\pi} \frac{1}{\cosh(\theta-\beta)} \log\left(\frac{1-e^{-|m|R\cosh\beta}}{1+e^{-|m|R\cosh\beta}}\right).$$

- Derivation: too complicated, based on the explicit form of $\widehat{SL}(2)$ generator and corresponding Ward identities.
- Truncation: keep only states with level no more than L:

$$L \geq \sum_i |n_i| \quad \text{or} \quad L \geq \sum_j |k_j|.$$

- Now the hamiltonian is finite dimensional:

$$H_{\text{IFT}}^{(L)} = \left(\begin{array}{c|c} \left\{E^{\text{NS}}(R)\right\} & \left\{hR\langle\{k\}|\sigma|\{n\}\rangle\right\} \\ \hline \left\{hR\langle\{n\}|\mu|\{k\}\rangle\right\} & \left\{E^{\text{R}}(R)\right\} \end{array}\right),$$

and numerical diagonalization approach is available.