LYEs in association to the Chiral phase transition and Roberge-Weiss transition

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fQCD collaboration :

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Lee Yang Edge Singularities (LYE) are branch cuts of the partition functions on the complex plane of the μ:

- When the LYE singularities pinch the μ real axis, the singularities correspond to the critical end point;
- When the branch cut crosses the real axis, the first order phase transition occurs.

Yang, C. N., Lee T. D. Phys. Rev. 87, 404-409 (1952); Lee T. D. Yang, C. N., Phys. Rev. 87, 410-419 (1952);

M. Stephanov, Phys.Rev. D73 (2006) 094508

Second order Phase transition and Scaling analysis:

- 3d O(4) scaling expanded by $(T, \mu_B) = (T_c^0 \sim 130 - 140, 0)MeV$.
- Z(2) scaling expanded by CEP of chiral PT.
- Roberge-Weiss scaling which is a scaling behavior in YM sector.

Universality class analysis

One may considering the general form of symmetry breaking pattern at finite temperature and magnetic field:

- Order parameter M: magnetization
- Two directions: t and h are the general scaling variables with t being the reduced temperature, and h being the reduced magnetic field.
- The LYE singularities locate at $z = t/h^{1/\beta\delta} = |z_c|e^{i\pi/2\beta\delta}$.

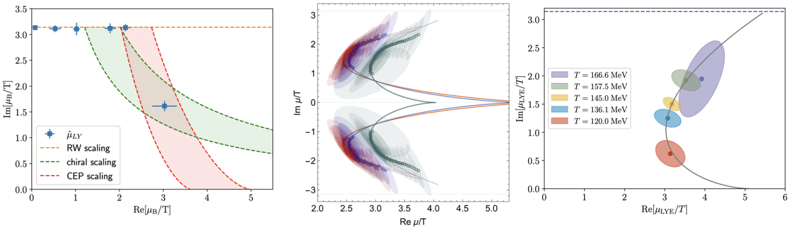
The $|z_c|$, β and δ are universal scaling parameters which can be determined in the universality class analysis:

	β	δ	$\lvert z_c\rvert$
$3d\ Z(2)$	0.33	4.79	2.43
$3d\ O(4)$	0.38	4.82	1.69
Mean field	0.5	3	1.89

A. Connelly, G. Johnson, F. Rennecke, V. Skokov, PRL 125(19), 191602 (2020);
G. Johnson, F. Rennecke, V. Skokov, PRD107(11), 116013 (2023);
F. Rennecke, V. Skokov, Annals Phys. 444, 169010 (2022)

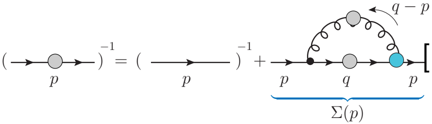
Obtain the LYEs at high T and small μ_B; Estimate the CEP by extrapolation.

What is the shape of the LYEs trajectory towards CEP? and What is the proper quantity for extrapolating?

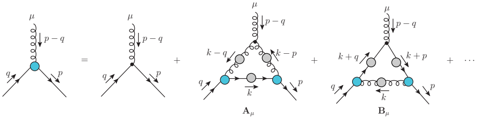


P. Dimopoulos et al, PRD 105, 034513 (2022);
G. Basar, PRC110, 015203 (2024);
D. A. Clarke et al, arXiv:2405.10196(2024)

A closed set of DSEs that can well describe the QCD property in vacuum and also at finite temperature and chemical potential. (**FG**, *J. Papavassiliou, J. Pawlowski, PRD 103 (2021) 9, 094013*)



$$[S^{-1}(p)] = S_0^{-1} + g_s \int_q G_A^{\mu\nu}(q-p)\,(i\gamma_\mu)\,S(q)\,\Gamma_\nu^{A\bar{q}q}(q,-p)$$



$$\Gamma_\mu^{A\bar{q}q}(q,-p) = g_s(i\gamma_\mu) + \mathbf{A}_\mu(q,-p) + \mathbf{B}_\mu(q,-p)$$

$$\mathbf{A}_\mu(q,-p) = \frac{Z_1 N_c}{2}\int_k \Gamma^{(0)}_{\mu\alpha\beta}\,G_A(k-q)\,G_A(k-p)\,\Gamma_\alpha^{A\bar{q}q}(k,-p)\,S(k)\,\Gamma_\beta^{A\bar{q}q}(q,-k)\,,$$

$$\mathbf{B}_\mu(q,-p) = -\frac{Z_1^f}{2N_c}\int_k G_A(k)\,\Gamma_\alpha^{A\bar{q}q}(k+p,-p)S(k+p)\,(i\gamma_\mu)\,S(k+q)\Gamma_\alpha^{A\bar{q}q}(q,-k-q)\,.$$

Here for the first step, we apply a modeling vertex to study LYEs.

The Cornwall–Jackiw–Tomboulis (CJT) effective potential:

$$\Gamma(S) = -Tr[\ln(S_0^{-1}S) - S_0^{-1}S + 1] + \Gamma_2(S),$$

where S_0 and S stands for the bare and full quark propagator,Γ_2 is the 2PI contribution. Calculating the variation respective to quark propagator, we have:

$$\frac{\partial^2\Gamma}{\partial S^2} = S^{-2} + \frac{\partial\Gamma_2(S)}{\partial S}.$$

combing with the derivative on the quark propagator DSE as:

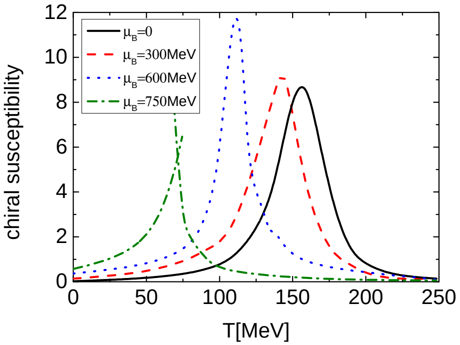
$$-S^{-2}\frac{\partial S}{\partial T} = 1 + \frac{\partial\Gamma_2(S)}{\partial S}\frac{\partial S}{\partial T},$$

The criterion is then given by[1]:

$$\frac{\partial S}{\partial T} = -\frac{1}{\partial^2\Gamma/\partial S^2}.$$

[1] **Fei Gao**, Yu-xin Liu. Phys. Rev. D 94 (2016) 7, 076009.

Scanning the susceptibility in the whole T- μ_B plane:



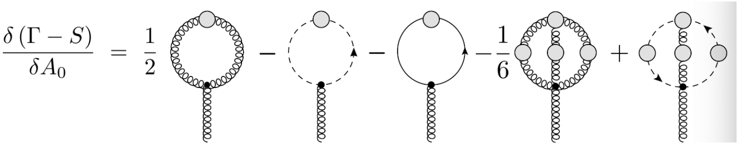
- Low chemical potential, crossover
- Large chemical potential, first order phase transition
- For larger imaginary chemical potential, the susceptibility becomes divergent.

Both are related to the Polyakov loop, and in functional approaches, one can apply the background field method.

Polyakov loop in background field method is related to A_4^a condensate as:

$$\mathcal{L}(A_4) = \frac{1}{N_c} \mathrm{tr}\, \mathcal{P} e^{ig\int dx_4 A_4} = \frac{1}{3}\left[1 + 2\cos\left(g\beta A_4/2\right)\right]$$

One can obtain A_4 condensate by solving the The DSE of A_4^a as $\frac{\delta(\Gamma - S_A)}{\delta A_4} = 0$.
The diagrammatic representation is:



A_4^a condensate is equivalent to the colored imaginary chemical potential:

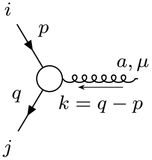
$$S_q^{-1}(p) = \mathrm{i}(\tilde{\omega}_n + gA_4)\gamma_4 \, Z_q^E(\boldsymbol{p}, \tilde{\omega}_n) + \mathrm{i}\gamma \cdot \boldsymbol{p} \, Z_q^M(\boldsymbol{p}, \tilde{\omega}_n) + Z_q^E(\boldsymbol{p}, \tilde{\omega}_n) \, M_q(\boldsymbol{p}, \tilde{\omega}_n)$$

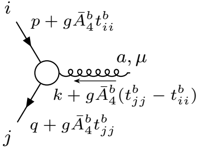
If there is discontinuity in the A_4 condensate, there will be discontinuity in the quark propagator and number density.

Moreover, when solving the quark gap equation, we put the condensate also in gluon propagator:

$$[D_0^{-1}]_{ab}^{\mu\nu} = z_a \left[\left(\frac{1}{\zeta} - 1 \right) \overline{D}_\mu^{ca} \overline{D}_\nu^{cb} + \delta_{\mu\nu} \overline{D}_\lambda^{ca} \overline{D}_\lambda^{cb} \right],$$

and also the quark gluon vertex:

$\Gamma_\mu^a(p, q; k) =$ 

, $\bar{\Gamma}_\mu^a(p, q; k) =$ 

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QCD thermodynamic properties

The quark propagator is the central element to compute the EoS of QCD in the functional QCD approaches.

The number density is calculated from the propagator by:

$$n_q^f(T,\mu_B) \simeq -N_c Z_2^f\, T \sum_n \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{tr}_D \left[\gamma_4 S^f(p)\right]$$

One may then incorporate the lattice QCD simulation at $\mu = 0$ here to combine the advantages of the two methods. The pressure is given by:

$$P(T,\boldsymbol{\mu}) = P_{Latt.}(T,\mathbf{0}) + \sum_q \int_0^{\mu_q} n_q(T,\mu)\, d\mu$$

[1] P. Isserstedt, C.S. Fischer and T. Steinert, PRD103 (2021) 054012
[2] **FG**, Yuxin Liu, PRD 94 (2016) 9, 094030
[3] H. Chen, M. Baldo, G. F. Burgio, and H.-J. Schulze, PRD86(2012)045006

Quark propagator with chiral and deconfinement phase transition:

$$S_q^{-1}(p) \simeq \mathrm{i}(\omega_n + \mathrm{i}\mu_q + gA_0)\gamma_4 + \mathrm{i}\gamma \cdot \boldsymbol{p} + M_q$$

with the dynamical quark mass M_q and the gluon condensate $gA_0 = 2\pi T\phi\tau^3$. A_0 is also related to the Polyakov loop $\mathcal{L}$:

$$\mathcal{L} = [1 + 2\cos(\pi\varphi)]/3, \quad \text{with} \quad \phi_{\text{fund}} = \{\pm\varphi/2, 0\}.$$

If one neglects the momentum dependence of M_q and A_0, then the number density can be expressed analytically:

$$\rho_q(T, \mu_q) = 2N_f N_c \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}[f(\epsilon, \mathcal{L}, \mu_q) - f(\epsilon, \mathcal{L}, -\mu_q)],$$

$$f(\epsilon, \mathcal{L}, \mu) = \frac{\mathcal{L}e^{2(\epsilon-\mu)/T} + 2\mathcal{L}e^{(\epsilon-\mu)/T} + 1}{e^{3(\epsilon-\mu)/T} + 3\mathcal{L}e^{2(\epsilon-\mu)/T} + 3\mathcal{L}e^{(\epsilon-\mu)/T} + 1},$$

$$\epsilon(\boldsymbol{k}, M_q) = \sqrt{\boldsymbol{k}^2 + M_q^2}.$$

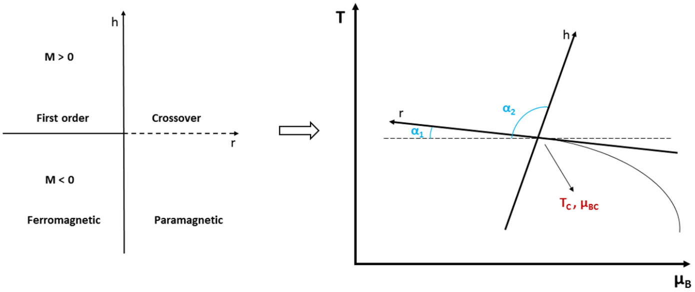
Yi. Lu, **FG**, et al, Phys. Rev. D.109.114031(2024)

Chiral phase transition:
The quark mass is parameterized from the Ising order parameter:

Parotto, Bluhm, Mroczek et.al. PRC 101, 034901 (2020);

$$\mathcal{M}_{\rm Ising} = \mathcal{M}_0 R^\beta \theta, \quad h = h_0 R^{\beta\delta} \tilde{h}(\theta), \quad r = R(1-\theta^2),$$

Map between the Ising parameters and the QCD phase diagram: $(T, \mu_B) \leftrightarrow (r, h)$.



Mapping the phase diagram

However, a global map of the QCD phase transition line to the Ising variables is required:

$$\frac{\mu_B}{\mu_B^E} - 1 = -r\omega\rho\cos\alpha_1 - h\omega\cos\alpha_2, \quad \frac{T}{T^E} - 1 = f_{\mathrm{PT}}(r) + h\omega\sin\alpha_2.$$

Phase transition happens at $h = 0$, which gives the constraint on the map function f_{PT}:

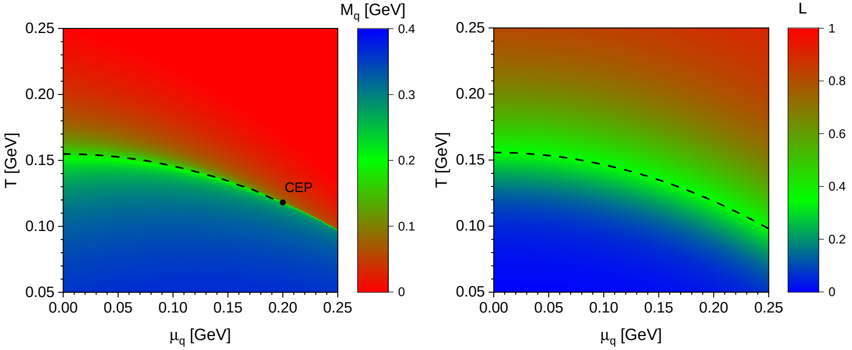
$$f_{\mathrm{PT}}(r) = \frac{T_c(\mu_B)}{T^E} - 1, \quad \mu_B = \mu_B^E\left(1 - r\omega\rho\cos\alpha_1\right).$$

Finally, the Ising mapping starts from a critical point. Its position is set from the predictions of functional QCD studies:

$$\mu_B^E = 3\,\mu_q^E = 600\,\mathrm{MeV},\, T^E = T_c(\mu_B^E) = 118\,\mathrm{MeV}.$$

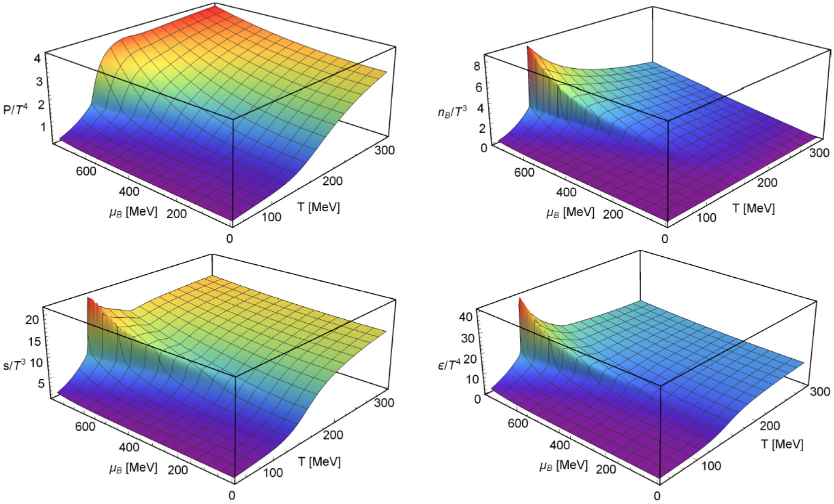
Deconfinement transition:
The Polyakov loop data at zero μ_B is taken from the fQCD result; at finite μ_B, the temperature scaling is suggested in Refs. :
(Fu and Pawlowski, PRD 92, 116006 (2015); S. Borsanyi et.al. (WB-collaboration), PRL. 126, 232001 (2021))

$$\mathcal{L}(T,\mu_q) = \mathcal{L}_{\text{fRG}}(T',0), \qquad \frac{T'}{T_c(0)} = \frac{T}{T_c(0)} + \kappa\left(\frac{3\,\mu_q}{T_c(0)}\right)^2 .$$

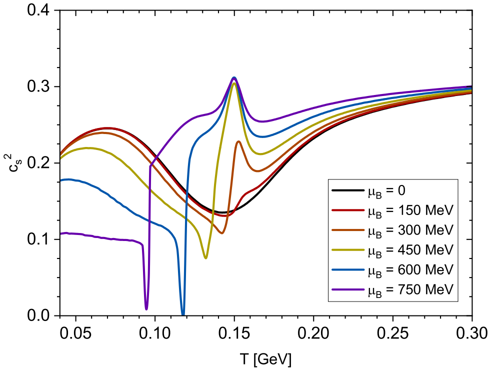


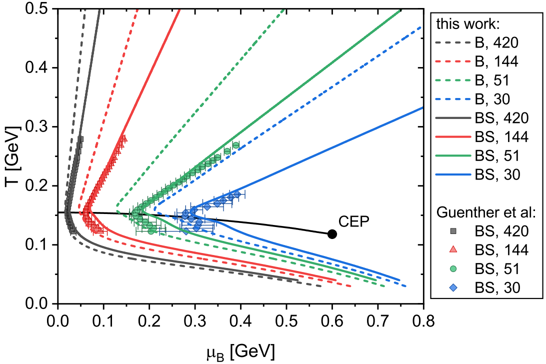
EoS from phase diagram mapping

$$s = \partial P/\partial T, \quad \epsilon = Ts - P + \mu_B n_B, \quad c_s^2 = \partial P/\partial \epsilon, \ldots$$



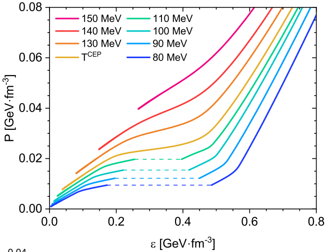
The speed of sound for different μ_B

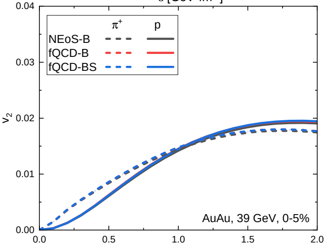


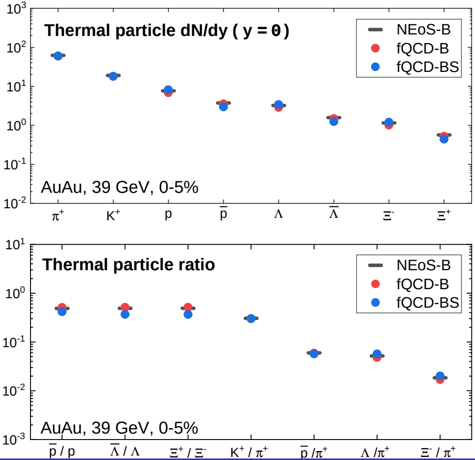


Combine with hydrodynamic simulation

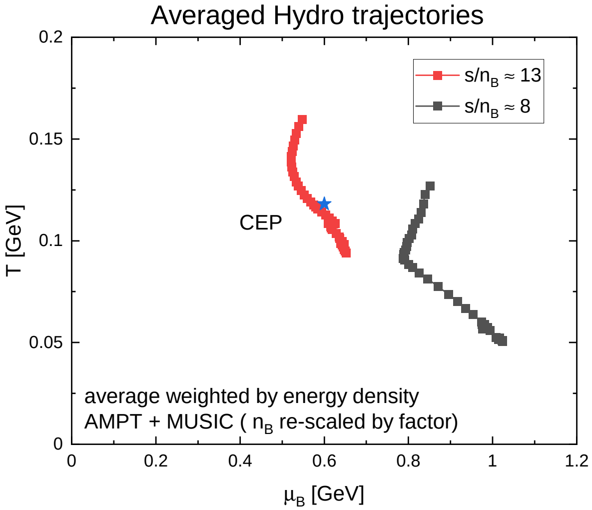
The EoS can be applied in hydrodynamic simulations after mapping the data into (ϵ, n_B) plane together with Maxwell construction.







The hydrodynamic simulation can be extended to first order region within the current EoS:



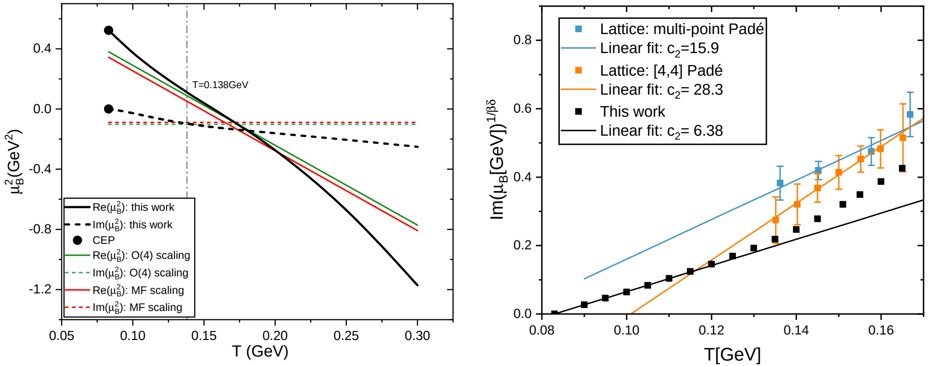
Towards the complex chemical potential:

$$S_q^{-1}(p) = \mathrm{i}(\tilde{\omega}_n + gA_4)\gamma_4 Z_q^E(\boldsymbol{p}, \tilde{\omega}_n) + \mathrm{i}\gamma \cdot \boldsymbol{p}\, Z_q^M(\boldsymbol{p}, \tilde{\omega}_n) + Z_q^E(\boldsymbol{p}, \tilde{\omega}_n)\, M_q(\boldsymbol{p}, \tilde{\omega}_n)$$

with $\tilde{\omega}_n = \omega_n + i\mu$

	$\mathrm{Re}\,\phi_3$	$\mathrm{Im}\,\phi_3$	$\mathrm{Re}\,\phi_8$	$\mathrm{Im}\,\phi_8$
$\mu = 0$	✓	0	0	0
$\mathrm{Re}\,\mu \neq 0,\ \mathrm{Im}\,\mu = 0$	✓	0	0	✓
$\mathrm{Im}\,\mu \neq 0,\ \mathrm{Re}\,\mu = 0$	✓	0	✓	0
$\mathrm{Im}\,\mu \neq 0,\ \mathrm{Re}\,\mu \neq 0$	✓	✓	✓	✓

LYE singularities in QCD with considering only the chiral phase transition



- no 3d O(4) scaling, and a trivial scaling around CEP works well
- A linear relation between $Im(\mu_B^{1/\beta\delta})$ and T
- $\beta\delta \sim 1.5 \pm 0.1$ in different classes is indistinguishable here
- In fact, the non universal quantity, the slope c_2 is useful

Ziyan Wan, Y. Lu, FG, Yuxin Liu, Eur.Phys.J.C 84 (2024) 9, 899

CEP location within the scaling analysis

Considering the mapping between the Ising parameters to T and μ_B,

$$\begin{pmatrix} t \\ h \end{pmatrix} = \mathbb{M} \begin{pmatrix} T - T_{\rm cep} \\ \mu_B - \mu_{B,CEP} \end{pmatrix}, \quad \mathbb{M} = \begin{pmatrix} t_T & t_\mu \\ h_T & h_\mu \end{pmatrix},$$

$$\mu_{LYE} \sim \mu_{B,CEP} - c_1(T - T_{\rm cep}) + \mathrm{i} c_2 (T - T_{\rm cep})^{\beta\delta}, c_1 = \frac{h_T}{h_\mu}, c_2 = x_{LY} \frac{{t_\mu}^{\beta\delta}}{h_\mu} \left(\frac{t_T}{t_\mu} - \frac{h_T}{h_\mu} \right)^{\beta\delta}$$

A global mapping form incorporating with the PT line(Yi Lu, FG, et al, PRD 109 (2024) 11, 114031):

$$\frac{\mu_B - \mu_{B,CEP}}{\mu_{B,CEP}} = -t\omega\rho\cos\alpha_1 - h\omega\cos\alpha_2, \frac{T - T_{\rm CEP}}{T_{\rm CEP}} = f_{\rm PT}(t) + h\omega\sin\alpha_2,$$

$$f_{\rm PT}(t) = \frac{\mu_{B,CEP}}{2T_{\rm CEP}} \left(2 - t\omega\rho\cos\alpha_1\right) t\omega\rho\sin\alpha_1$$

CEP location

c_2 determines the CEP location completely:

$$\overline{c}_2 = x_{LY} \frac{(\frac{T_{CEP}}{\mu_{B,CEP}})^{\beta\delta}\omega \sin\alpha_1}{(\omega\rho\sin\alpha_1)^{\beta\delta}} \left(\frac{T_{CEP}}{\mu_{B,CEP}} \cot^2\alpha_1 + 1\right)$$

The extracted $c_2 = 6.38$ GeV$^{1-\beta\delta}$, together with the correct PT line together with the typical Ising parameter,

$$\overline{c}_2 = (\frac{T_{CEP}}{\mu_{B,CEP}})^{\beta\delta} c_2 = 0.28, (T_{CEP}, \mu_{B,CEP}) = (118, 606)\mathrm{MeV}$$

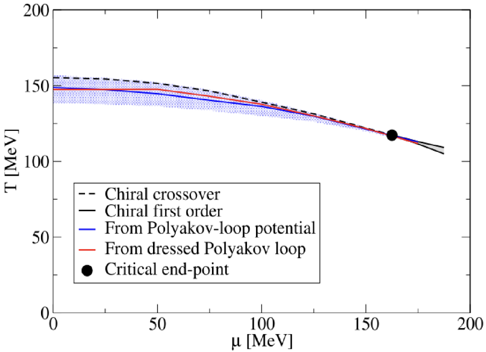
Note that the value of dimensionful quantity c_2 is with the wrong CEP location, the corrected value should be:

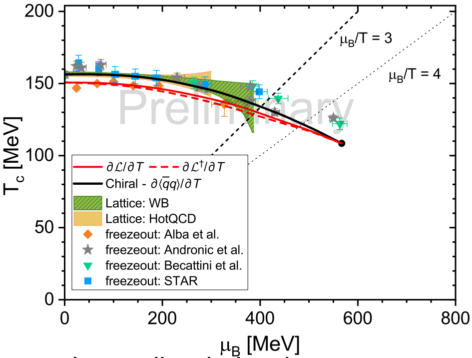
$$c_2' = 5GeV^{1-\beta\delta}$$

This is confirmed after incorporating the background gluonic field in the gap equation as will be discussed below.

A first check on deconfinement phase transition at high temperature

The two CEPs are at the same location. And the deconfinement is in precise agreement with Chiral PT in the whole first order phase transition region.



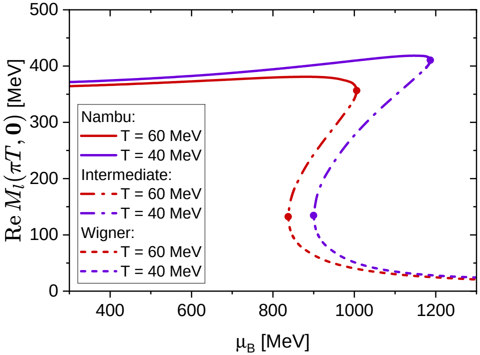


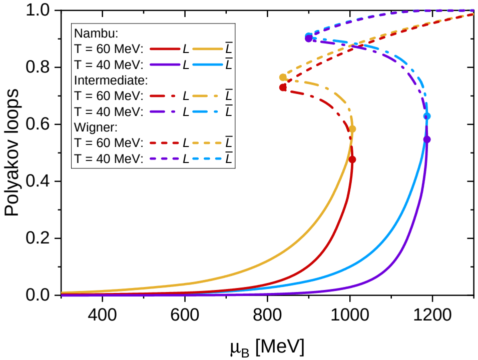
- Because a discontinuity of quark propagator immediately leads to a discontinuity in Polyakov loop

C. S. Fischer et al, PLB 732, 273(2014); Yi Lu, FG, J. Pawlowski, Yuxin Liu, arXiv: 2504.05099

Coexistence in first order phase transition region

Coexistence region: stable, meta stable and unstable phase.

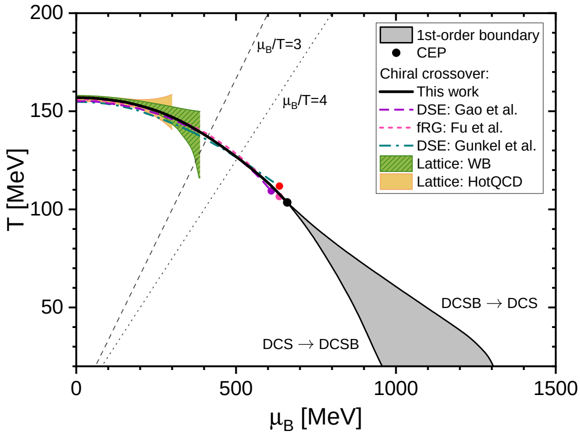




- Stable and Metastable phase: carefully choose the initial guess.
- More difficult for unstable phase, one requires Homotopy method:

$$G_q^{\text{init.}}(p;\eta) = \eta G_q^N(p) + (1-\eta) G_q^W(p)$$

Phase diagram in temperature-chemical potential region for 2+1 flavour QCD

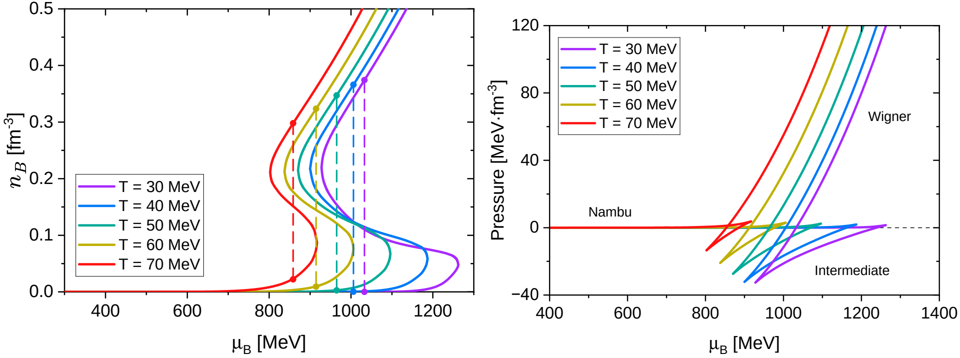


- Coexistence region slightly above liquid gas transition: stable and meta stable phase (**nucleation**); unstable phase (**spinodal decomposition**)
- Ideal phase transition for both Chiral PT and deconfinement at $\mu_B \approx 1100$ MeV

[1] Yi Lu, **FG**, Yu-xin Liu. arXiv:2509.02974.

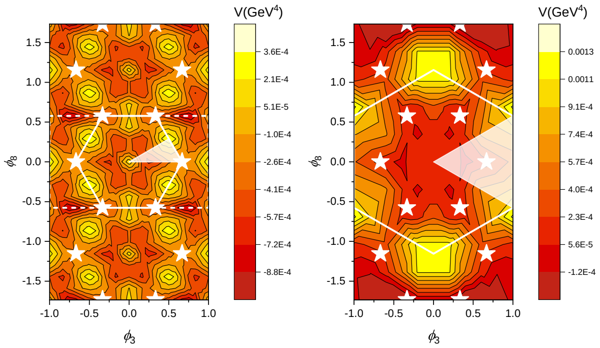
Maxwell construction from thermodynamic quantities

One can also determine the phase transition line from Maxwell construction of thermodynamic quantities:



- Maxwell construction: $\mu_N = \mu_W$ and $P_N = P_W$
- Unstable phase: bending over in number density which means the susceptibility becomes negative.

Symmetry analysis of Polyakov loop potential: Center symmetry



- Rotation and Translation in (ϕ_3, ϕ_8) plane with $\phi^a = g\beta A_4^a/(2\pi)$
- The symmetry transformation leads to a transformation in Polyakov Loop in the group $L = \{l, zl, z^2 l\}$ with $z = e^{2\pi i/3}$

U. Reinosa, Lecture Notes in Physics, Vol. 1006 (2022),

J. Braun, H. Gies, and J. M. Pawlowski, PLB684, 262 (2010)

RW Symmetry

A symmetry in the complex chemical potential and is determined by the background symmetry structure of (ϕ_3, ϕ_8)

- For $\mathrm{Im}\,\mu = -\pi T\theta = \frac{2\pi T}{3}n, \quad n \in \mathbb{Z}$,

and considering the minima of (ϕ_3, ϕ_8) shifts to another center sector, like $\phi_8 = \phi_8' + \sqrt{3}\theta$. The zeroth momentum in Lagrangian then becomes (three colors):

$$\omega_n + i\mathrm{Re}\,\mu + \phi_3\pi T - \phi_8'\frac{1}{\sqrt{3}}\pi T,$$

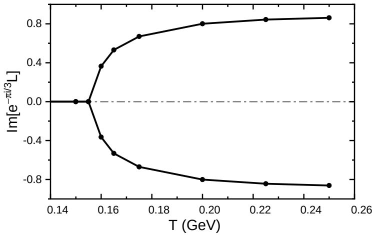
$$\omega_n + i\mathrm{Re}\,\mu - \phi_3\pi T - \phi_8'\frac{1}{\sqrt{3}}\pi T,$$

$$\omega_n + i\mathrm{Re}\,\mu + 3\theta\pi T + \phi_8'\frac{2}{\sqrt{3}}\pi T.$$

- For RW symmetry transformation $Im\,\mu' = Im\,\mu - \frac{2}{3}\pi T$, it is equivalent to the center symmetry transformation:

$$L \to zL$$

At $Im\,\mu = \frac{1}{3}\pi T$, the center symmetry transformation is equivalent to a complex conjugate transformation in the Lagrangian.



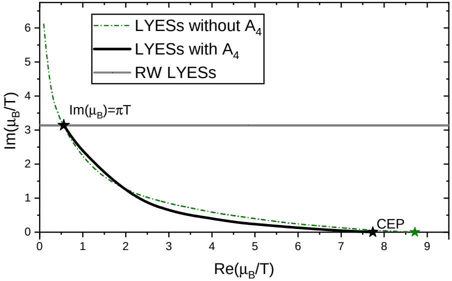
- If the physical states of Polyakov loop are unchanged under the above transformation: $L^* = z^{-1}L = e^{-2\pi i/3}L$, and the states are located at $Im\,\mu = (1+2n)/3\pi T$,
- Otherwise, it is complex conjugated to the axis: $Im\,\mu = (1+2n)/3\pi T$,

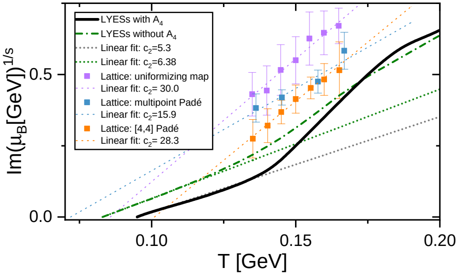
For every temperature and chemical potential where the RW phase transition can happen, it happens at a certain imaginary chemical potential as:

$$Im\,\mu = (1+2n)/3\pi T, n \in \mathbb{Z}.$$

Ziyan Wan, Y. Lu, FG, Yuxin Liu, arXiv:2504.12964(2025)

LYEs of QCD: Two distinct lines for chiral and RW PT





- LYEs for RW PT, appears at $Im[\mu_B] = \frac{1}{3}\pi T$ for any $T > 155$ MeV and μ_B, for smaller T, appears above a lower bound μ_B^c
- For $T < 230$ MeV, the LYEs for chiral and RW PT are mixed, the chiral ones are the leading order contribution with $Im[\mu_B] < \frac{1}{3}\pi T$
- The LYEs for chiral PT terminate at $T = 230$ MeV. Because when one goes above for $Im[\mu_B] > \frac{1}{3}\pi T$, A_4 condensate jumps first, which pulled the real singularity back to $Im[\mu_B] < \frac{1}{3}\pi T$. Only LYEs for RW are left then.

The conclusions and discussions:

- The scaling analysis of LYE singularities also suggest similar results for CEP (3d O(4) scaling at physical mass is ruled out).
- A trivial scaling near CEP and the slope of the linear relation between $Im[\mu_B]^{\beta\delta}$ and T can be used to extrapolate CEP.
- LYEs for RW PT always at $Im[\mu_B] = \frac{1}{3}\pi T$, and for every $Re[\mu_B]$.
- LYEs for Chiral PT are monotonous along T axis, terminate at $T = 230$ MeV.

Thank you!