Estimates on the convergence of expansions at finite baryon chemical potential

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Based on: Phys. Rev. D 110 (2024) 1, 016008

Authors: Rui Wen (main author), SY and Wei-jie Fu

Trento, 11.09.2025

JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

Alexander von HUMBOLDT STIFTUNG

CRC–TR 211 Strong-interaction matter under extreme conditions

fQCD

Outline

1. Introduction
    - QCD phase structure
    - Functional renormalization group
2. Baryon number fluctuations from fRG
    - QCD-assisted LEFT
    - Predictions
3. Comparison between direct calculation and expansion
    - Polyakov-Quark-Meson (PQM) Model
    - Comparison of different Expansion methods
4. Summary and conclusion

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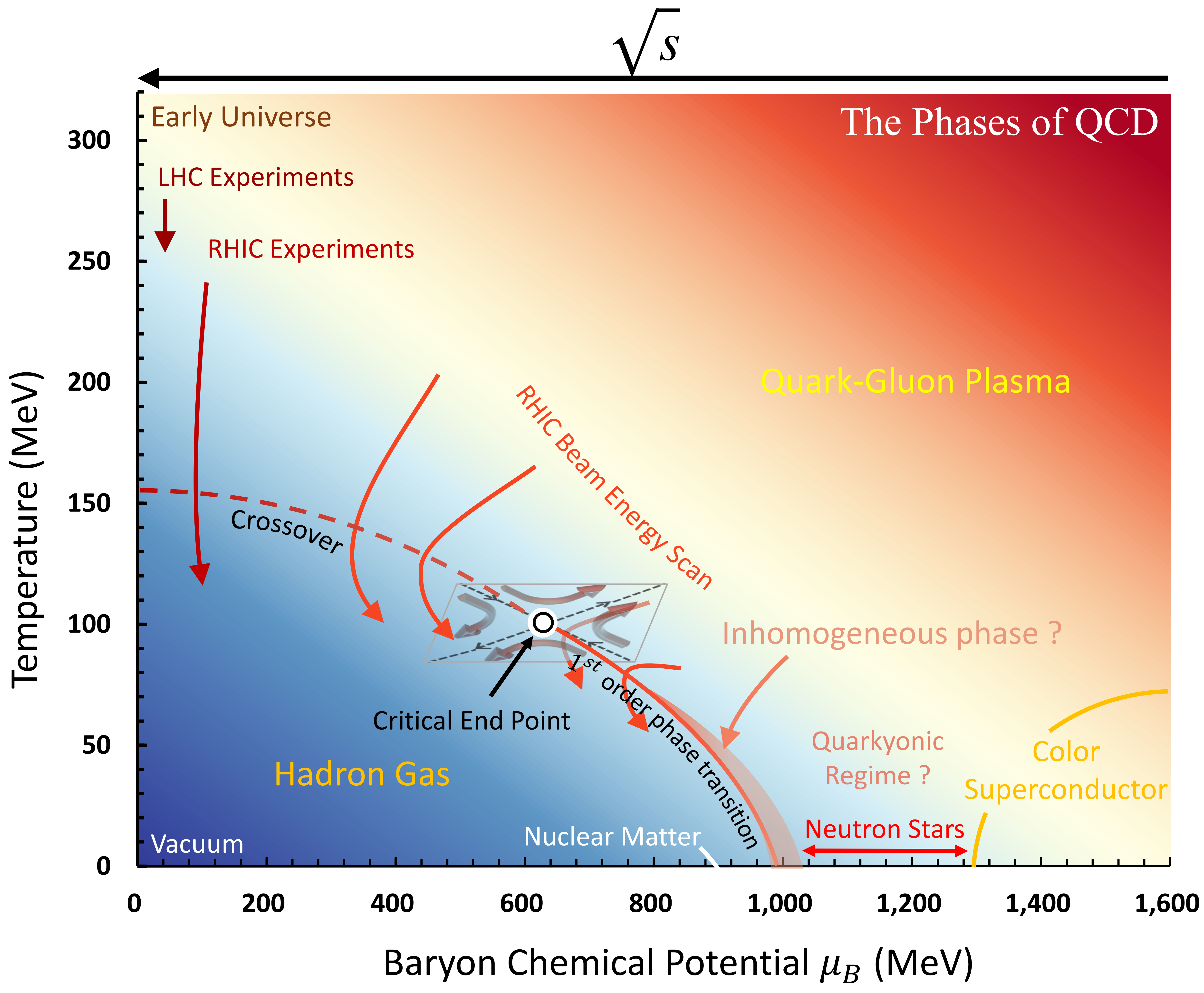
QCD phase structure

Experiments:

- LHC Experiments at Low density
- RHIC Beam Energy Scan at finite density
- Fixed target at high density
- …

Goals:

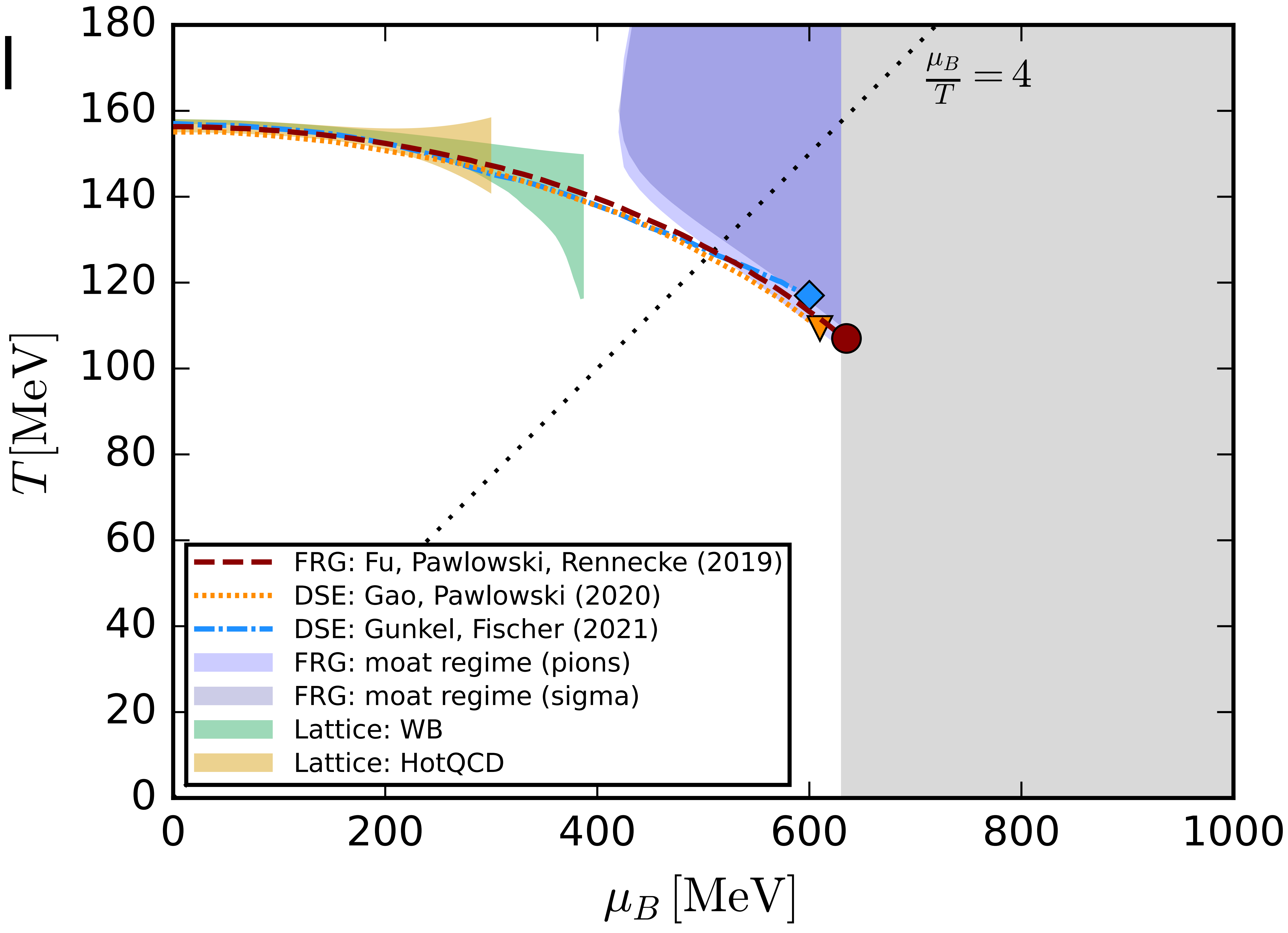
- Location of Critical End Point (CEP)
- New phase at high density region
- EoS
- …



SY, Tan, Fu (2023)

Theoretical predictions

- Lattice QCD (at vanishing chemical potential)
- Dyson-Schwinger Equations (DSE)
- Functional renormalization group (fRG)
- …



fQCD 2025

More complete version see Jan's talk

Functional renormalization group

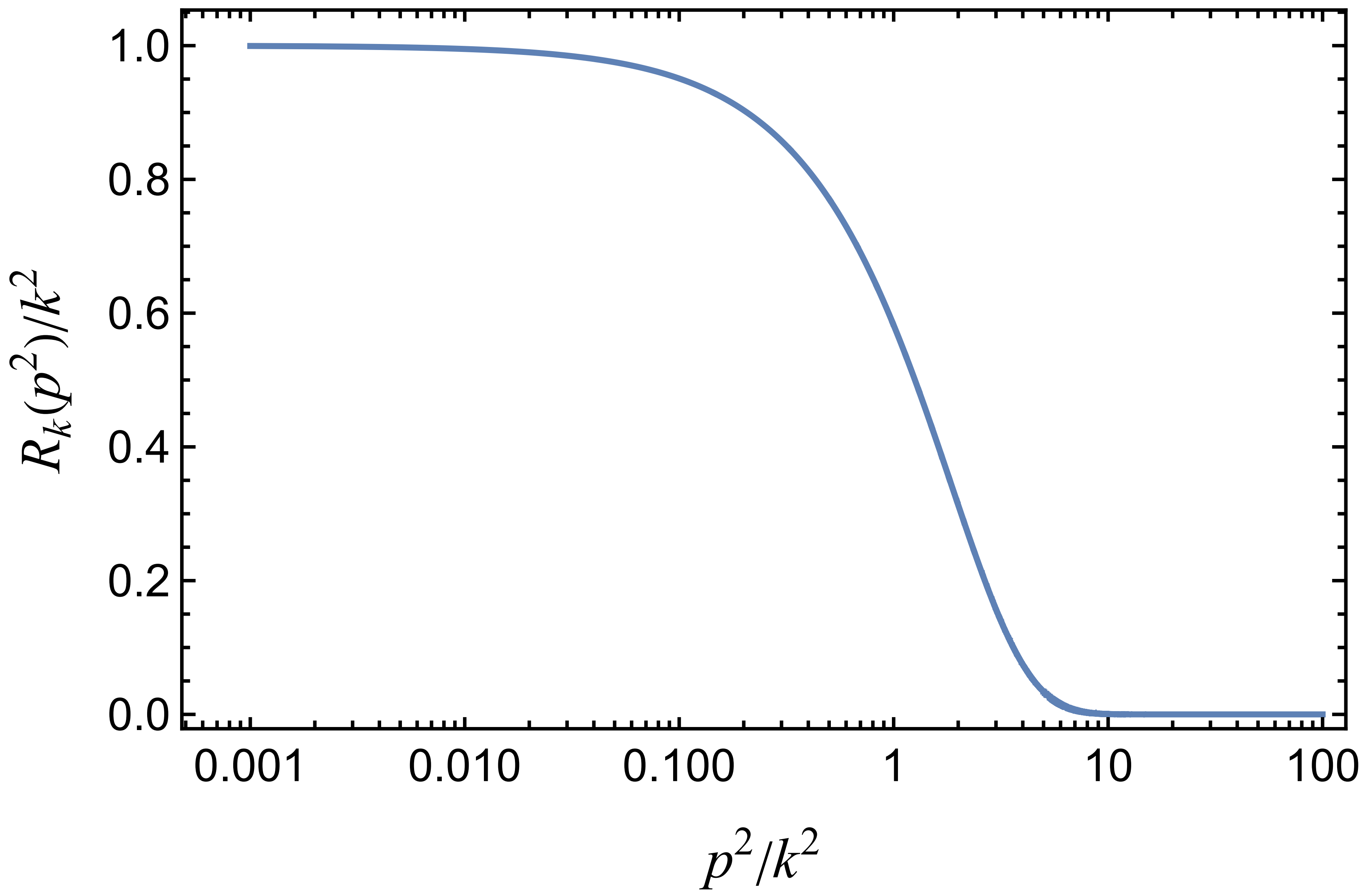
Generating functional

$$\mathcal{Z}_k[J] = e^{W_k[J]} = \int_\Lambda \mathcal{D}\chi \; e^{-S[\chi]-\Delta S_k[\chi]+J\,\chi}$$

$$\Delta S_k[\phi] = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\phi(-q)R_k(q)\phi(q)$$

A cutoff in two-point function

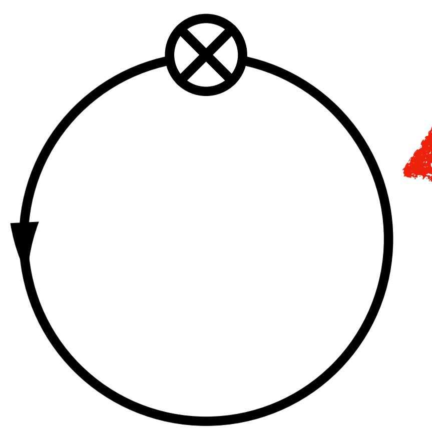
- The modes for $p^2 \lesssim k^2$ are suppressed
- The modes for $p^2 \gtrsim k^2$ are unchanged



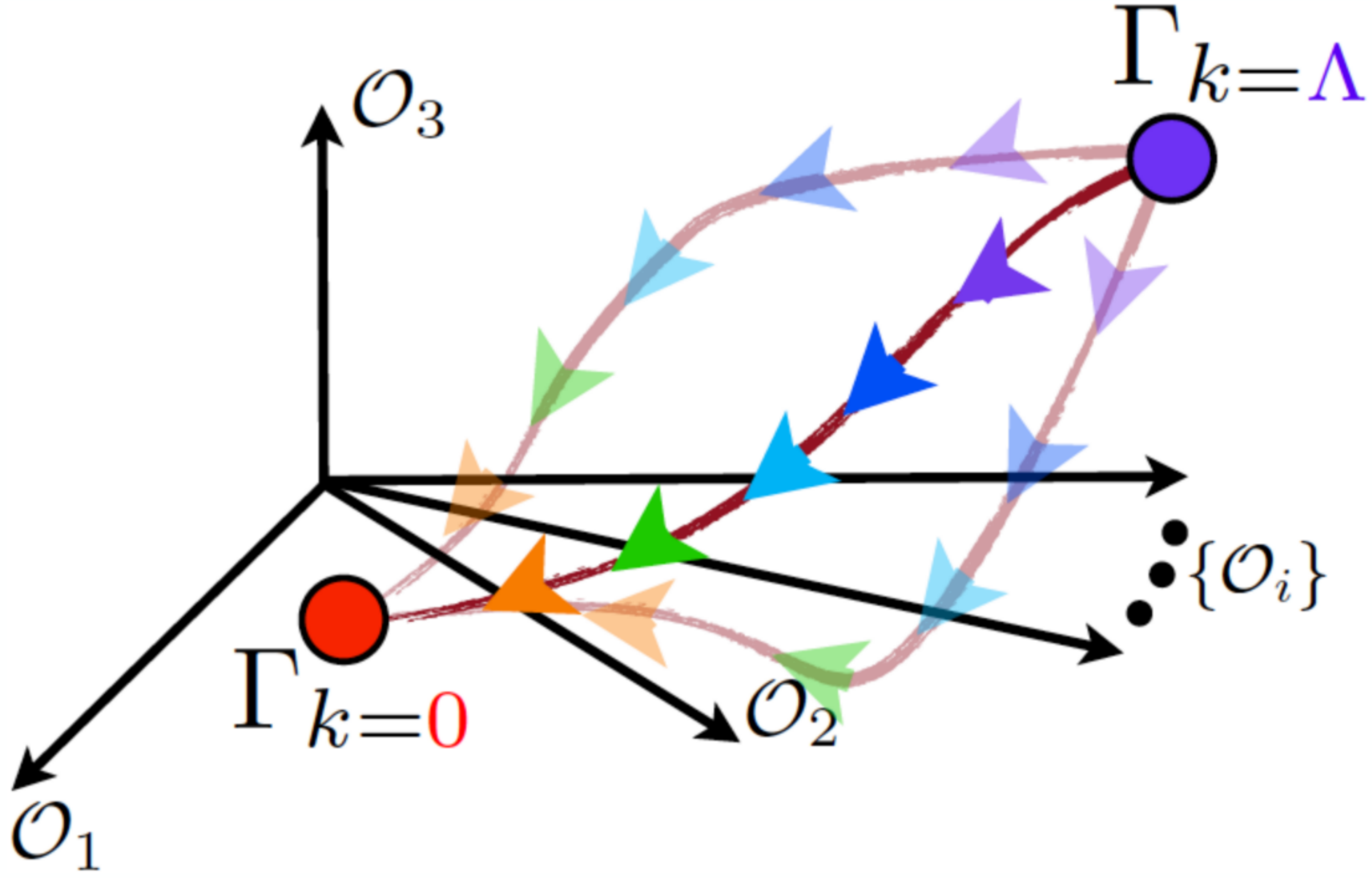
Functional renormalization group

Wetterich Equation

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\frac{1}{\Gamma_k^{(2)}[\phi] + R_k}\partial_t R_k \qquad t = \ln(\frac{k}{\Lambda})$$

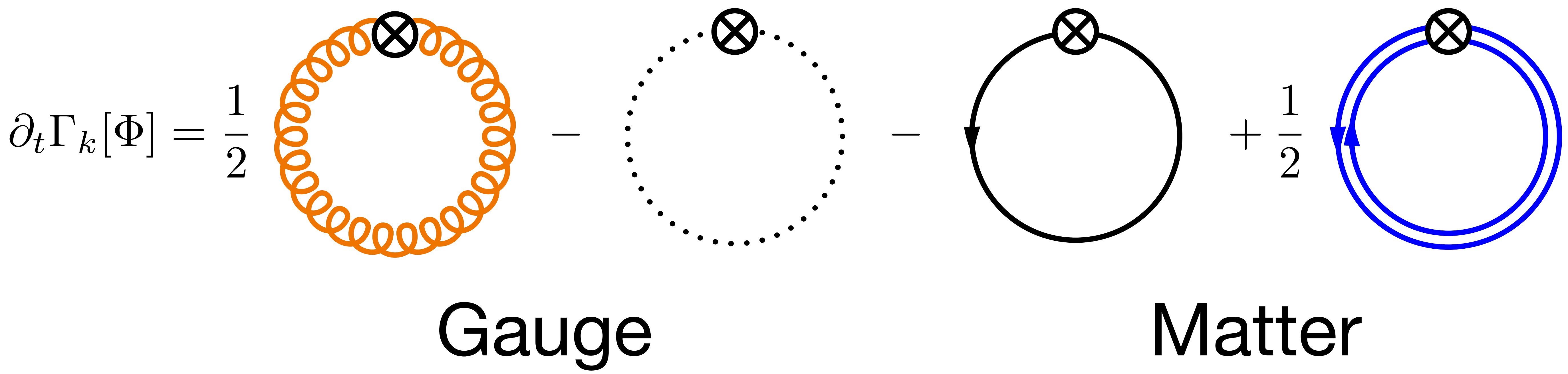


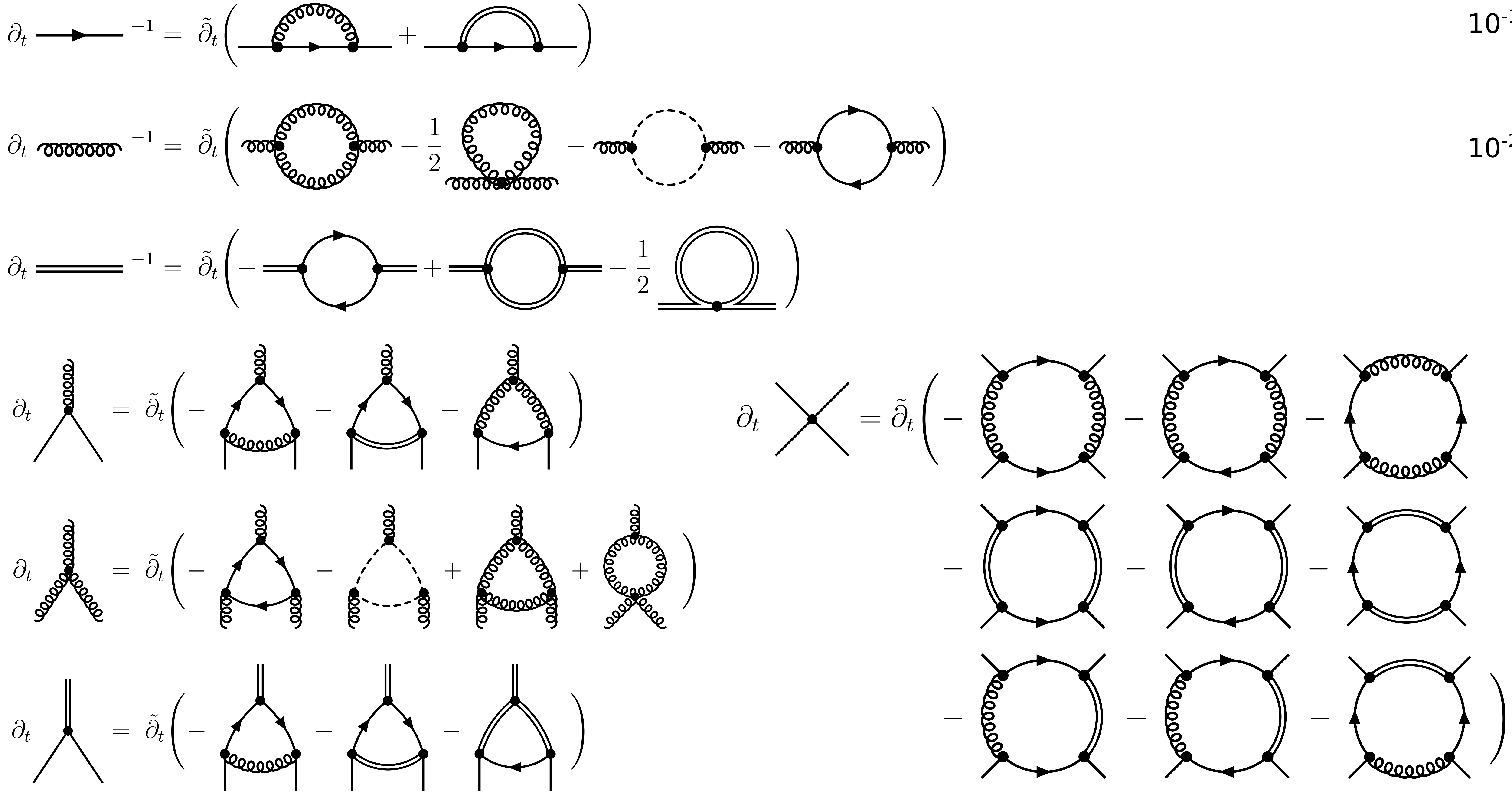
- Different paths correspond to different regulation functions
- Regulators will not influence the infrared results

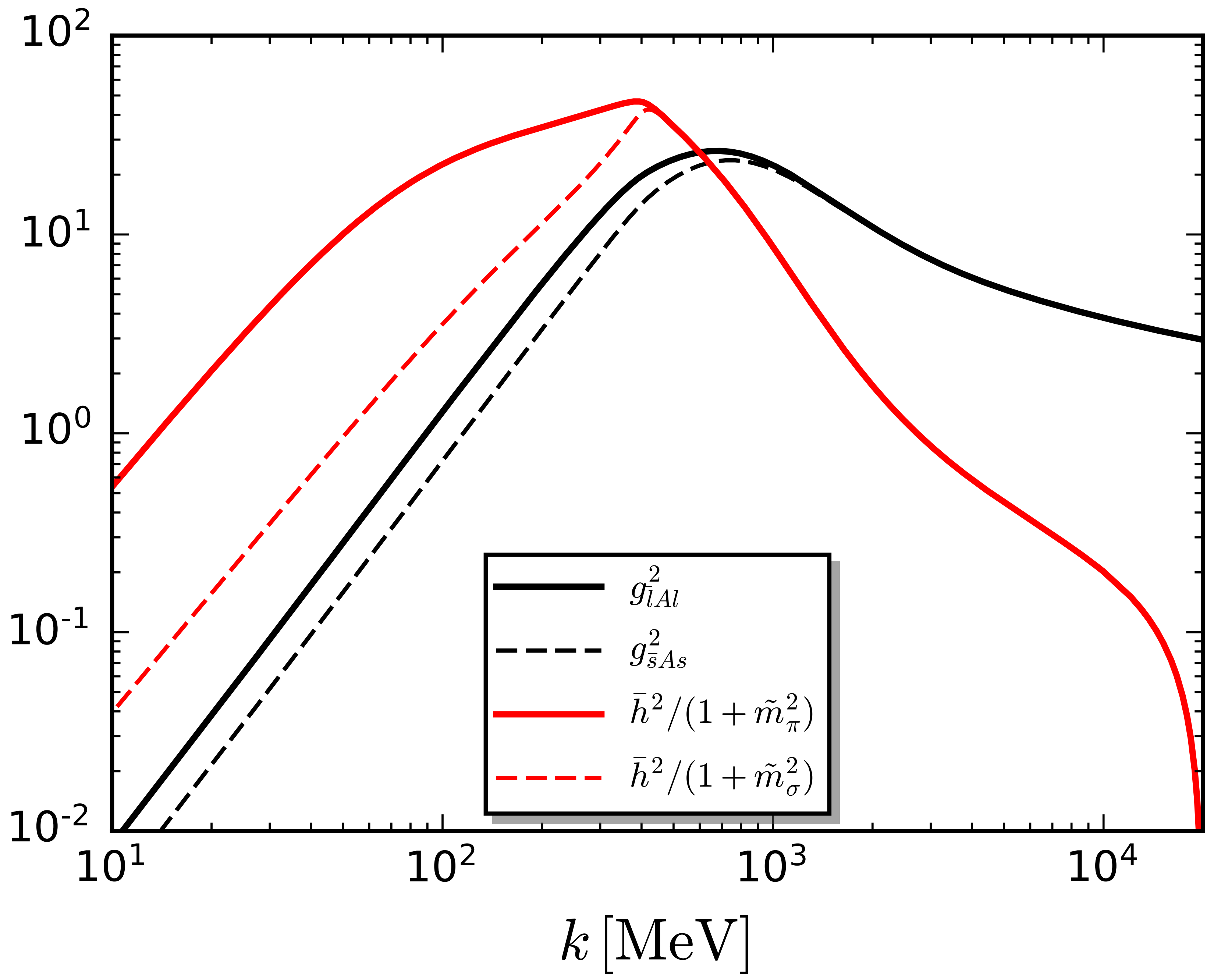


Fabian's talk

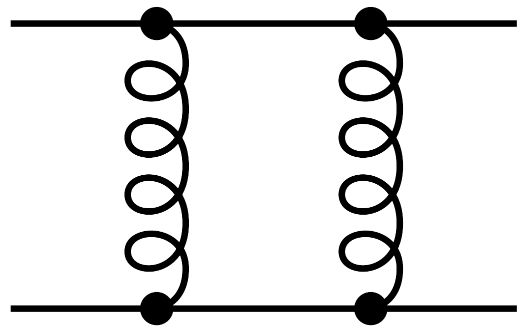
Functional renormalization group for QCD phase structure





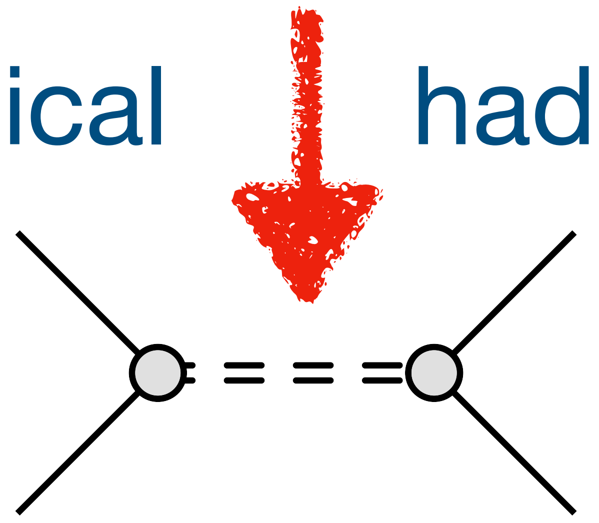


Fu, Rennecke, Pawlowski (2019)



Four-quark interaction

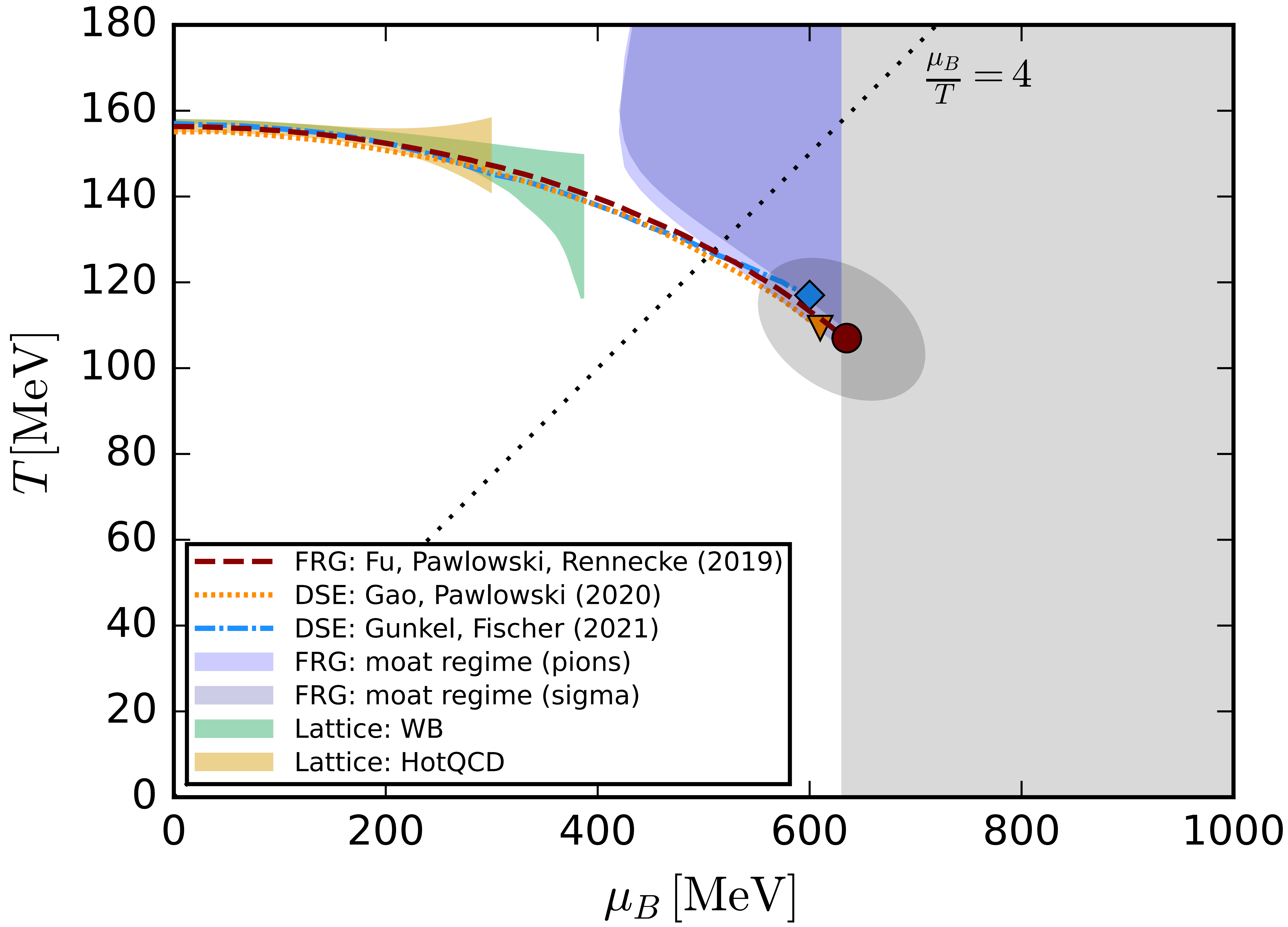
Dynamical hadronization



Yukawa coupling

Functional renormalization group for QCD phase structure

- Give the prediction of the phase boundary
- Try to have a better truncation at high density region (has little impact on the position of the CEP)
- High accuracy needed to have high-order thermodynamic quantities
- …



fQCD 2025

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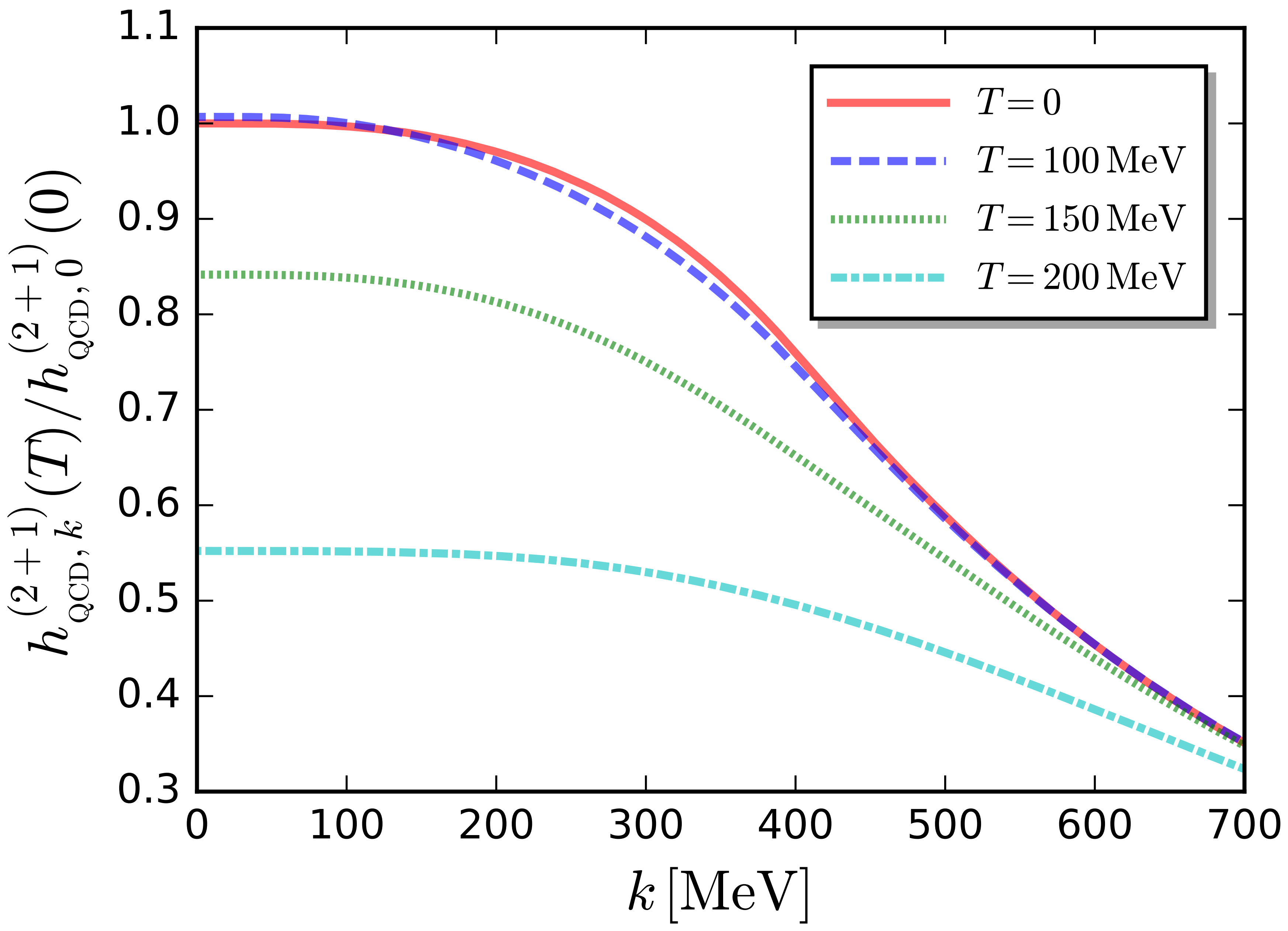
QCD-assisted LEFT

Effective action:

$$\Gamma_k = \int_x \Big\{ Z_q\, \bar q \big[\gamma_\mu \partial_\mu - \gamma_0(\mu + igA_0)\big] q + \frac{1}{2} Z_\phi\, (\partial_\mu \phi)^2 + h\, \bar q (T^0 \sigma + i\gamma_5 \vec{T}\cdot\vec{\pi}) + V_k(\rho) - c\sigma + V_{\rm glue}(L, \bar L) \Big\}$$

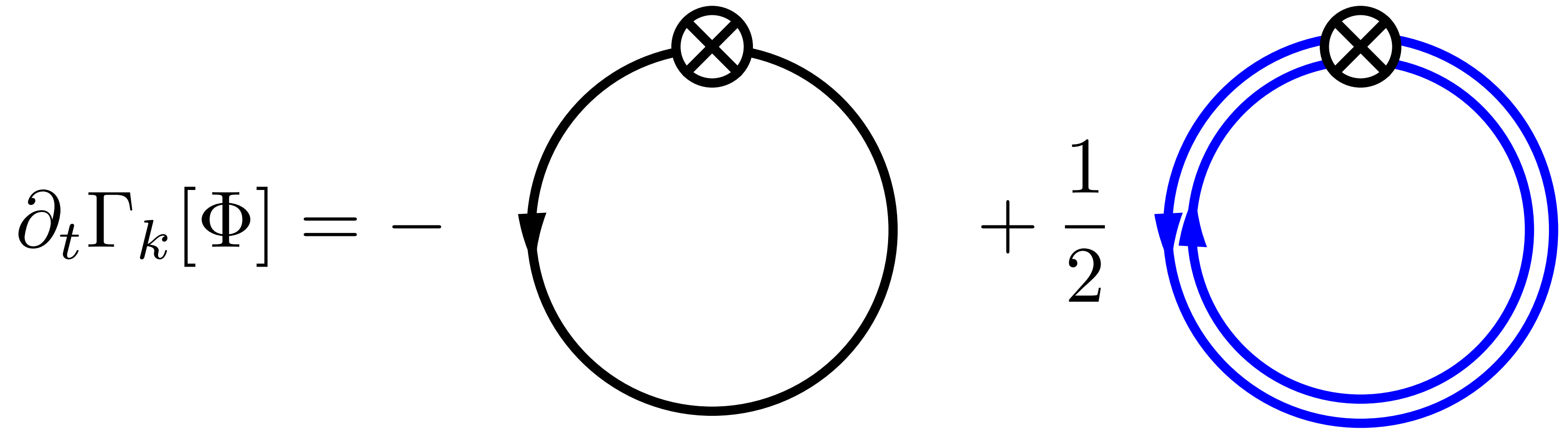
Together with some information from fRG-QCD:

Input from fRG-QCD



Fu, Luo, Pawlowski, Rennecke, SY (2024)

Flow equations:



Baryon number fluctuations

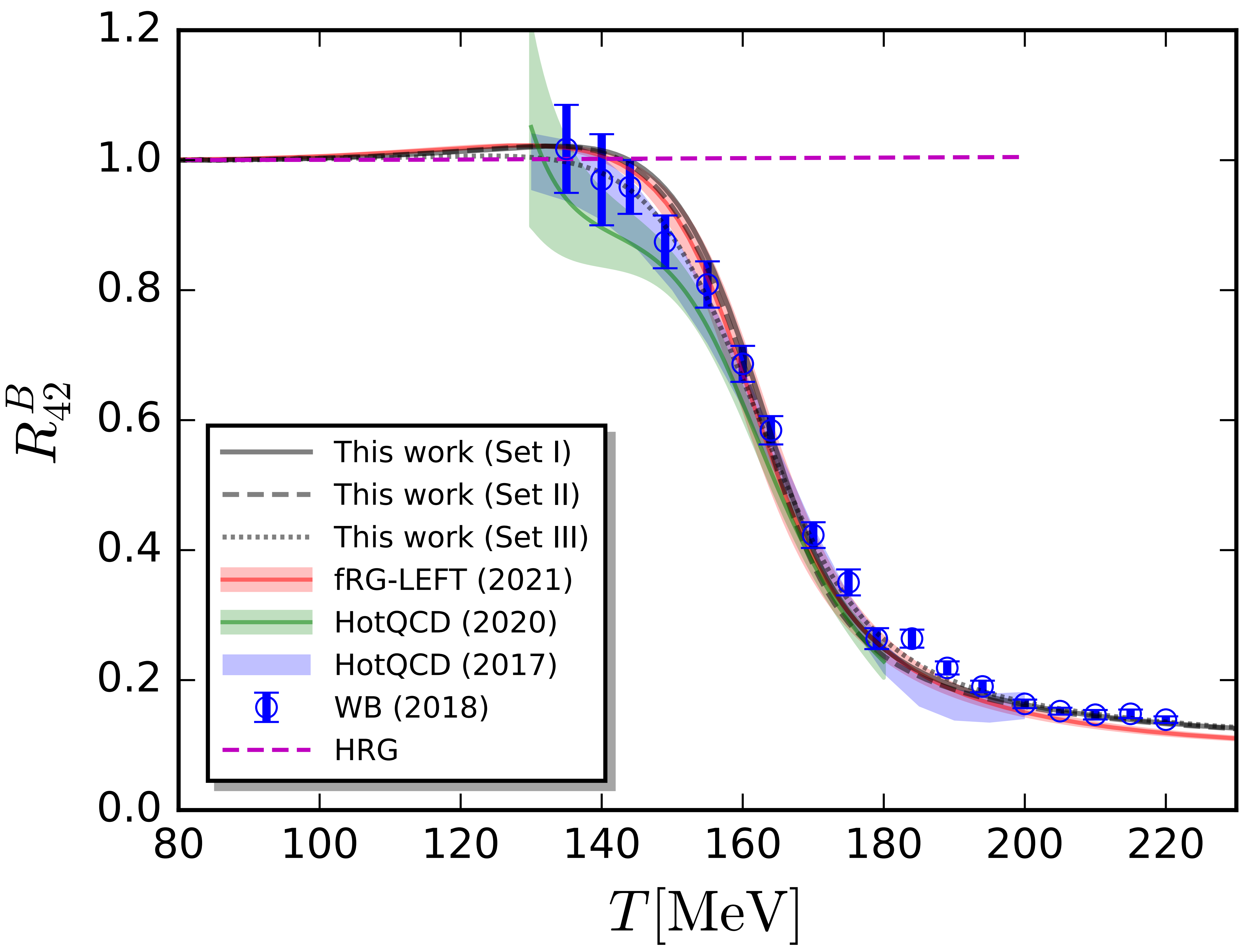
Pressure and Baryon number fluctuations:

$$p(T,\mu) = -\,\Omega(\mu,T)$$

$$\chi_n^B = \frac{\partial^n}{\partial \hat{\mu}_B^n}\frac{p}{T^4}, \qquad \hat{\mu}_B = \frac{\mu_B}{T}$$

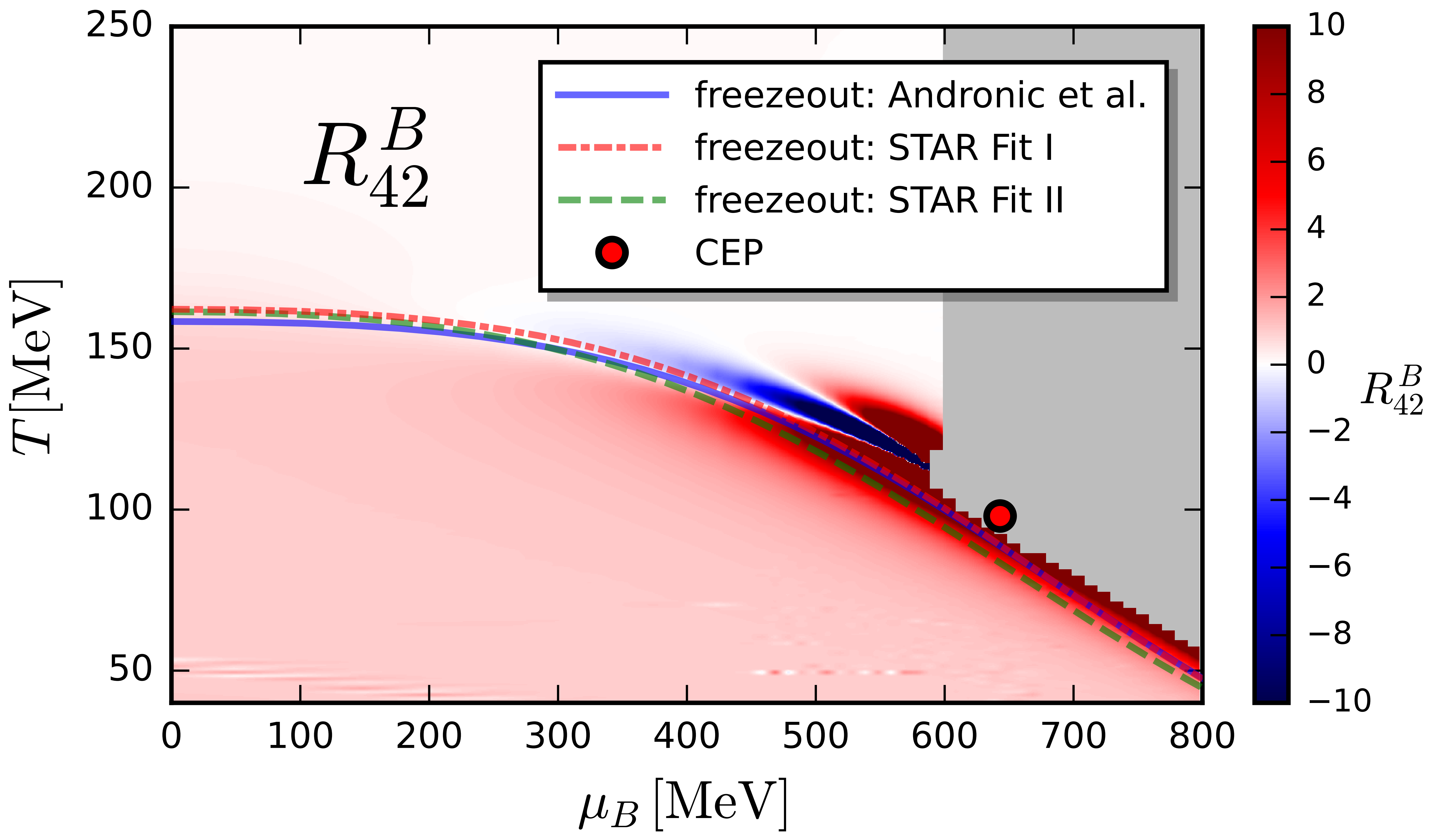
$$R_{m,n}^B = \frac{\chi_m^B}{\chi_n^B}$$

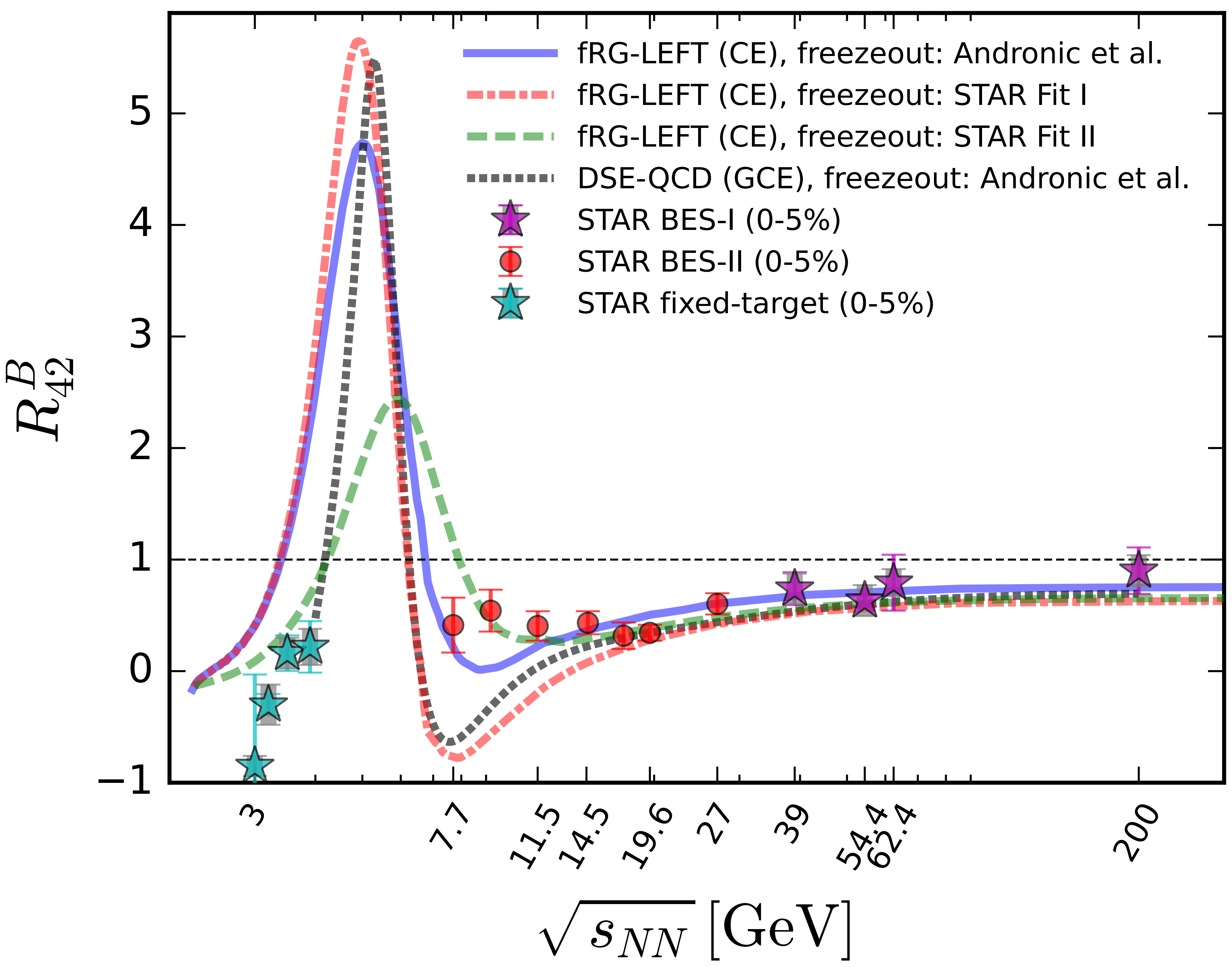
- The baryon number fluctuations at vanishing chemical potential can be used as Taylor coefficients



Fu, Luo, Pawlowski, Rennecke, SY (2024)

Predictions





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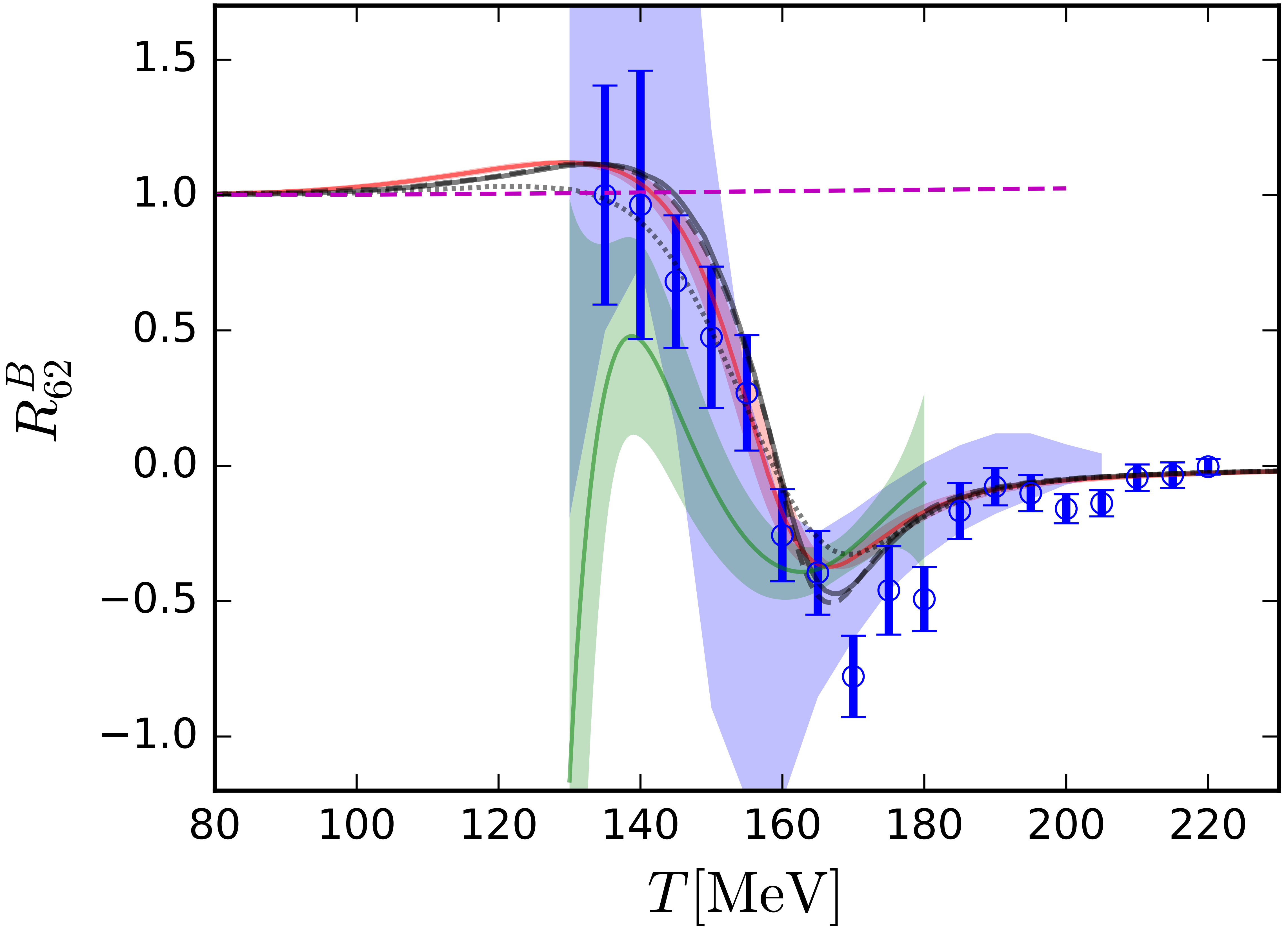
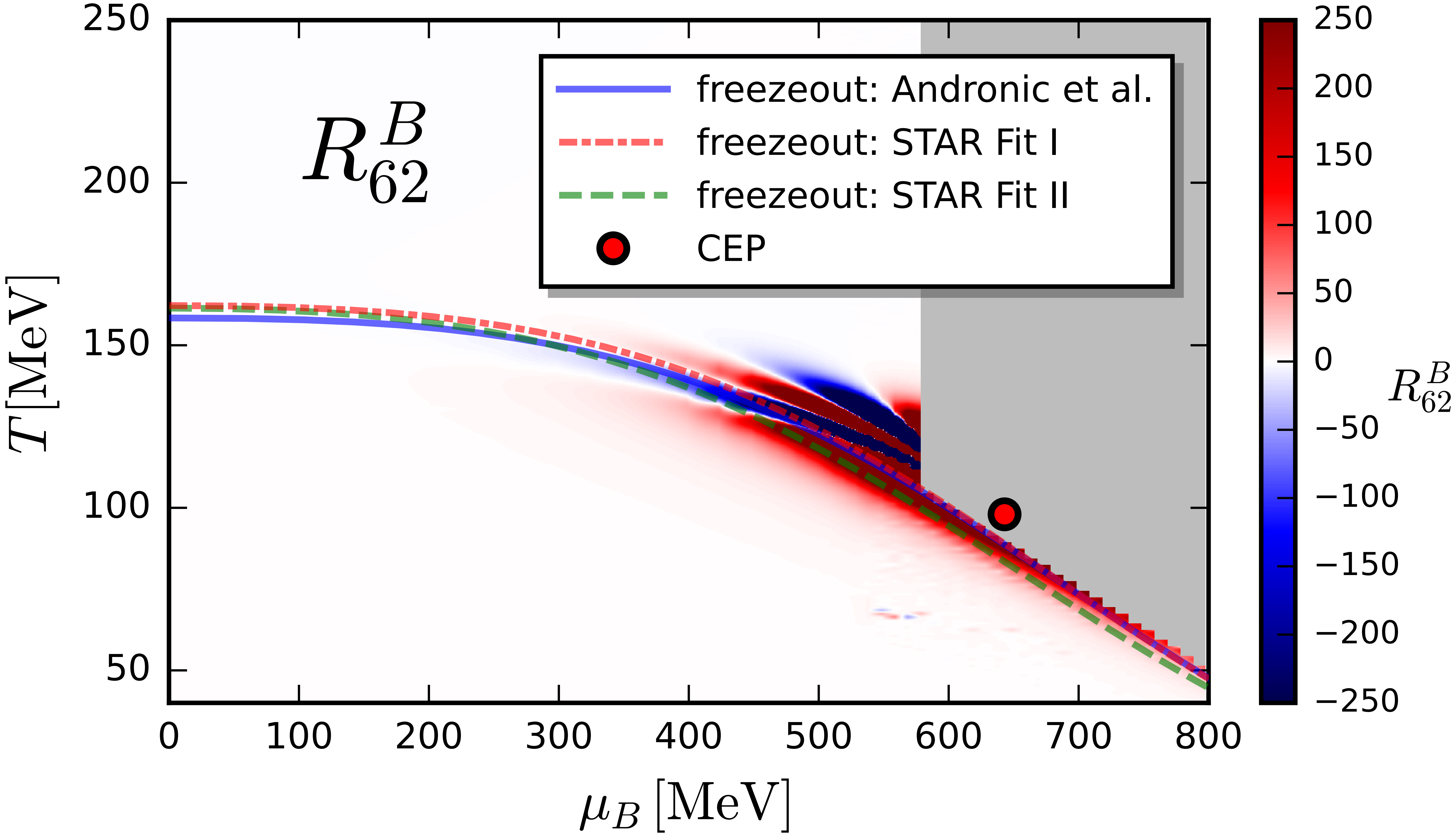
Predictions

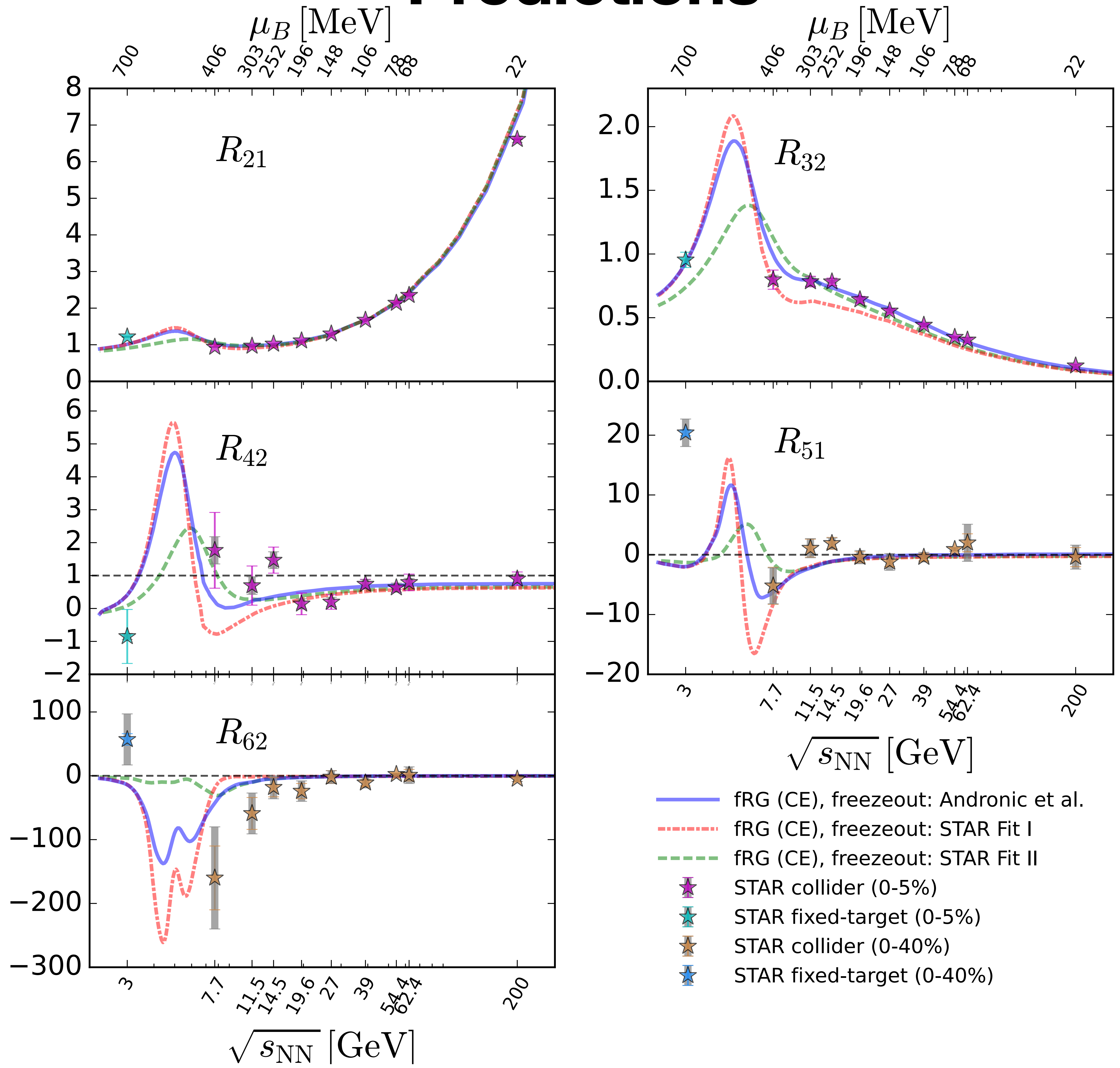
$\mu_B = 0$





Fu, Luo, Pawlowski, Rennecke, SY (2024)

Predictions



Fu, Luo, Pawlowski, Rennecke, SY (2024)

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Idea of this work

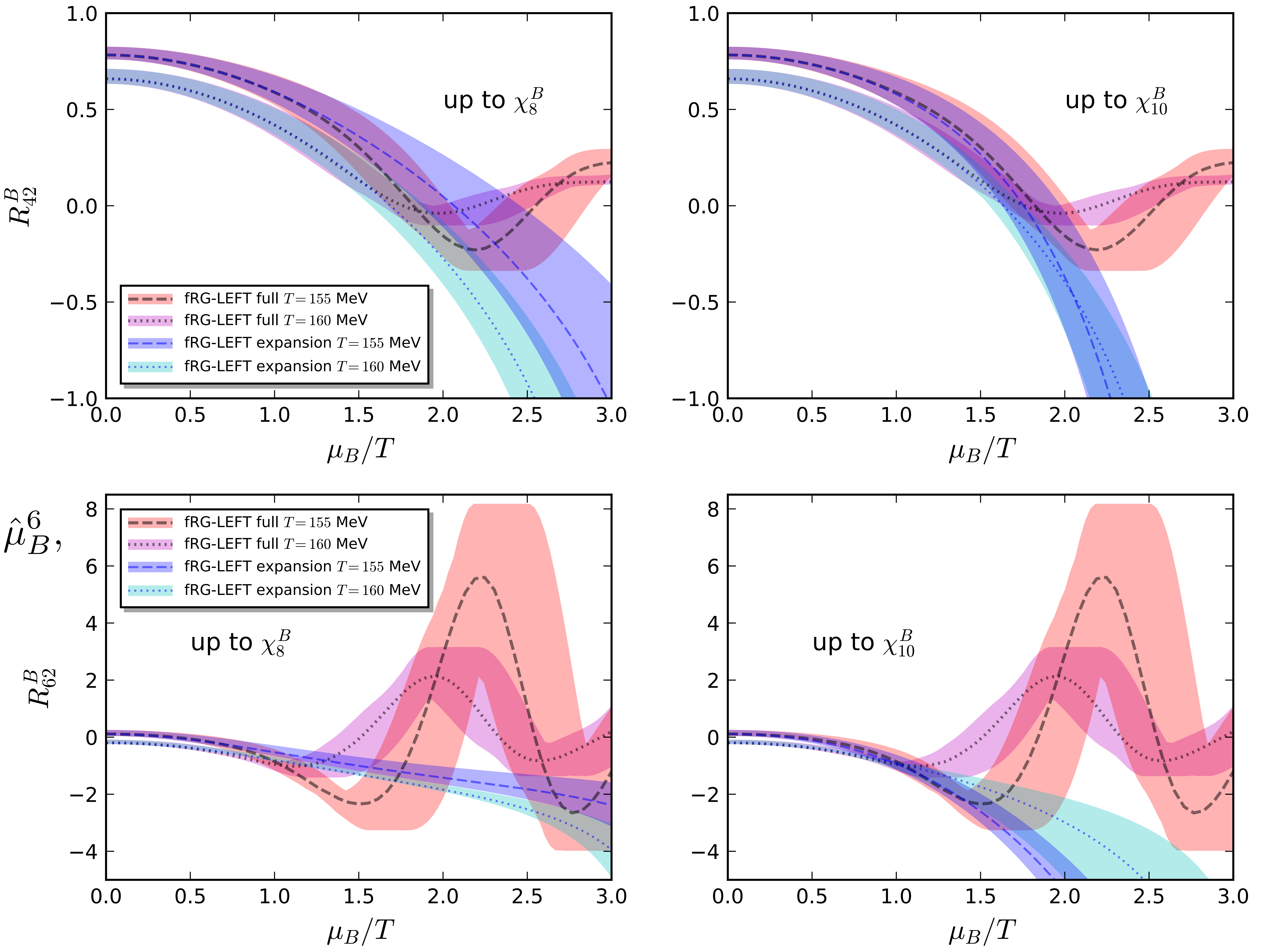
Taylor Expansion:

$$\frac{p(\mu_B)}{T^4} = \frac{p(0)}{T^4} + \sum_{i=1}^{\infty} \frac{\chi_{2i}^B(0)}{(2i)!}\hat{\mu}_B^{2i}$$

$$\chi_2^B(\mu_B) \simeq \chi_2^B(0) + \frac{\chi_4^B}{2!}\hat{\mu}_B^2 + \frac{\chi_6^B}{4!}\hat{\mu}_B^4 + \frac{\chi_8^B(0)}{6!}\hat{\mu}_B^6,$$

$$\chi_4^B(\mu_B) \simeq \chi_4^B(0) + \frac{\chi_6^B}{2!}\hat{\mu}_B^2 + \frac{\chi_8^B}{4!}\hat{\mu}_B^4,$$

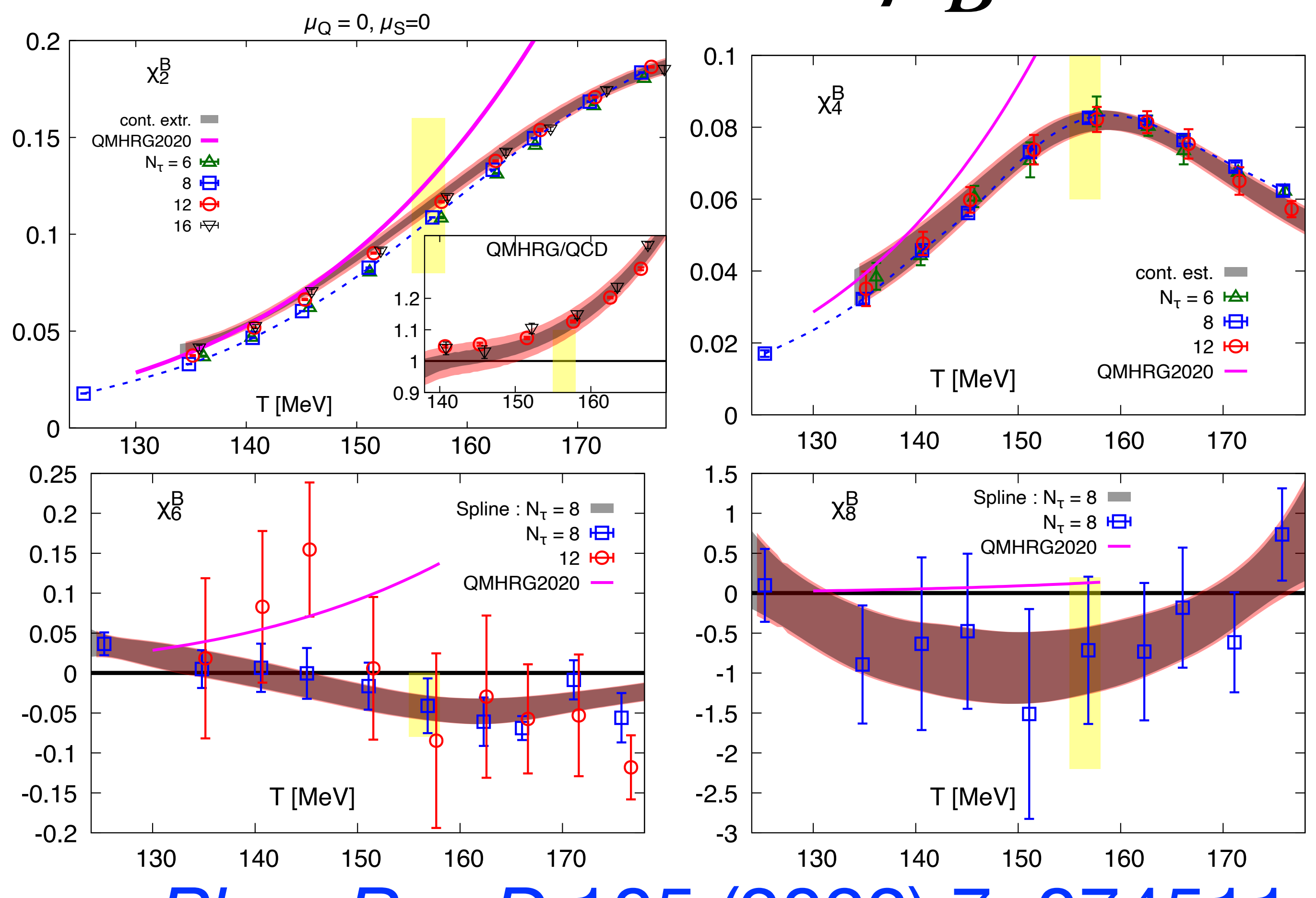
$$\chi_6^B(\mu_B) \simeq \chi_6^B(0) + \frac{\chi_8^B}{2!}\hat{\mu}_B^2.$$



Fu, Luo, Pawlowski, Rennecke, Wen, SY (2022)

Idea of this work

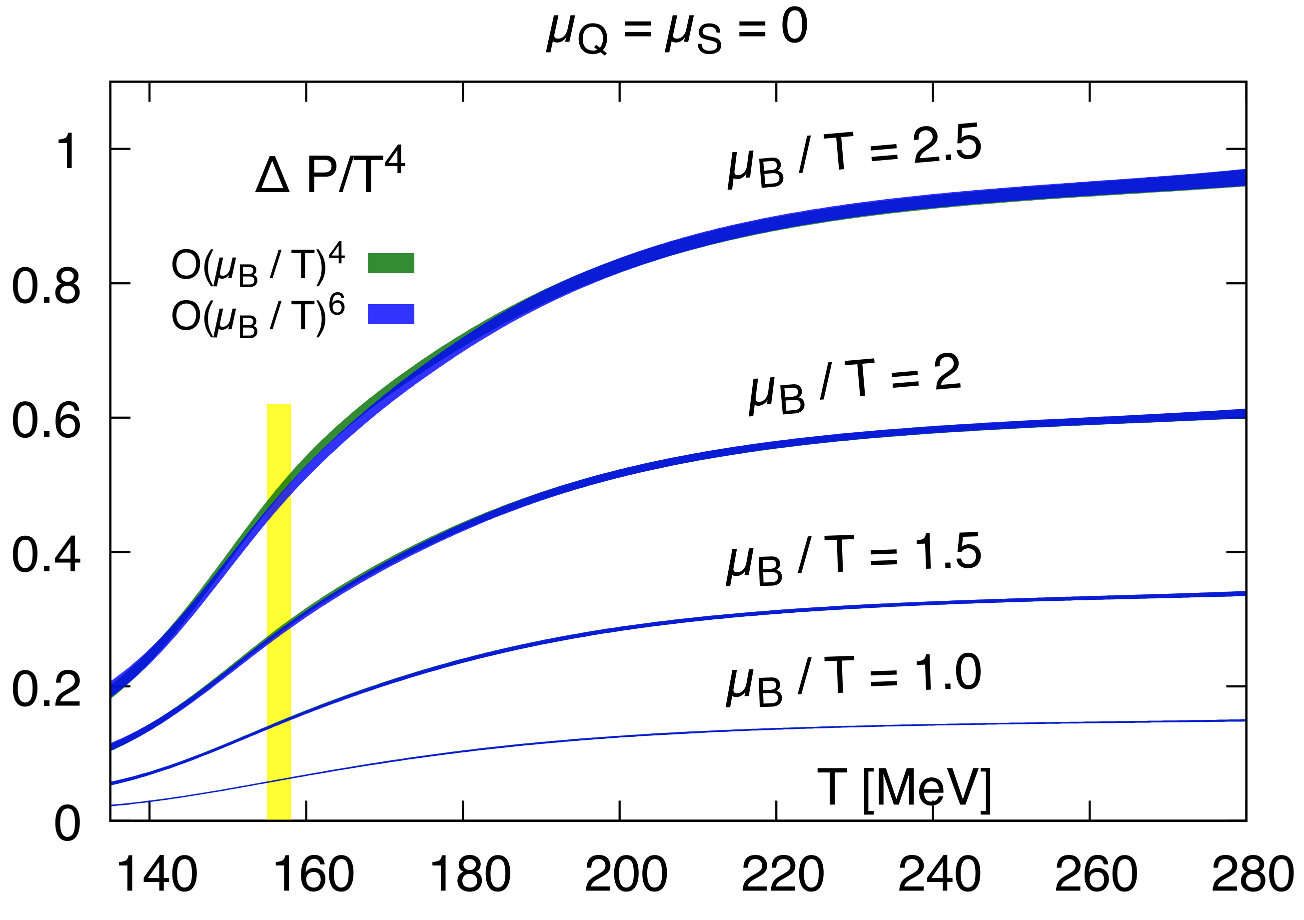
Lattice QCD at $\mu_B = 0$



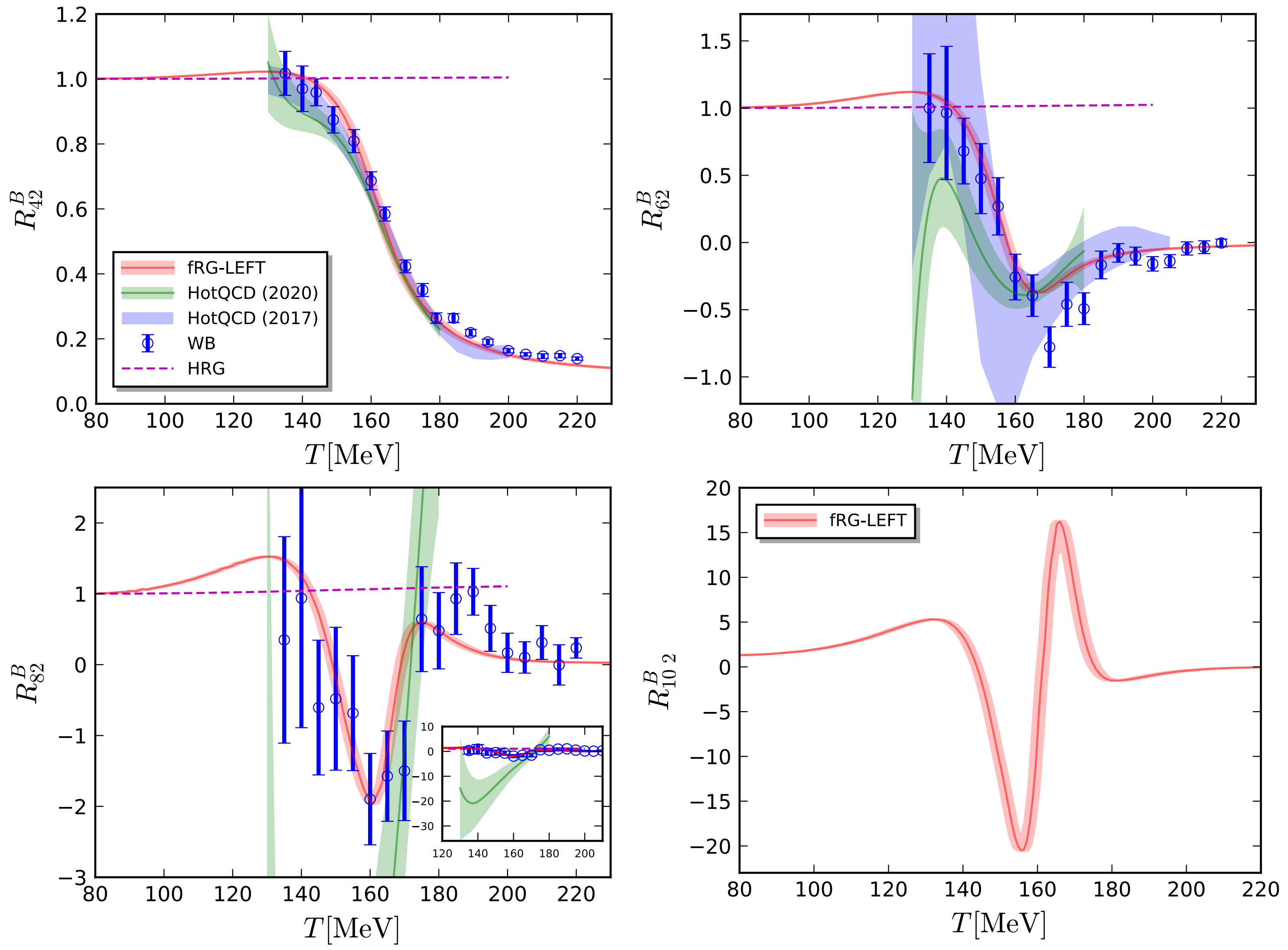
Phys.Rev.D 105 (2022) 7, 074511

Extrapolation

Finite μ_B



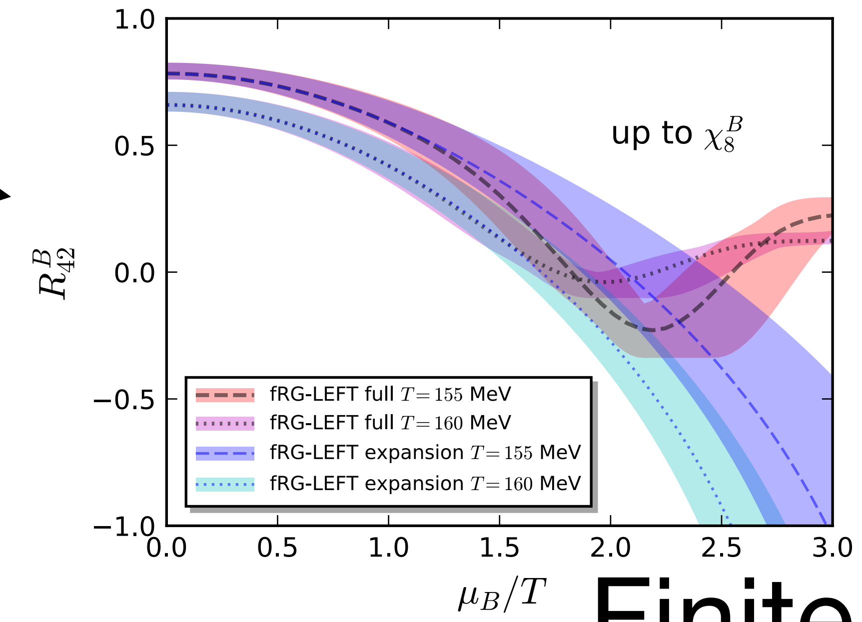
fRG at $\mu_B = 0$



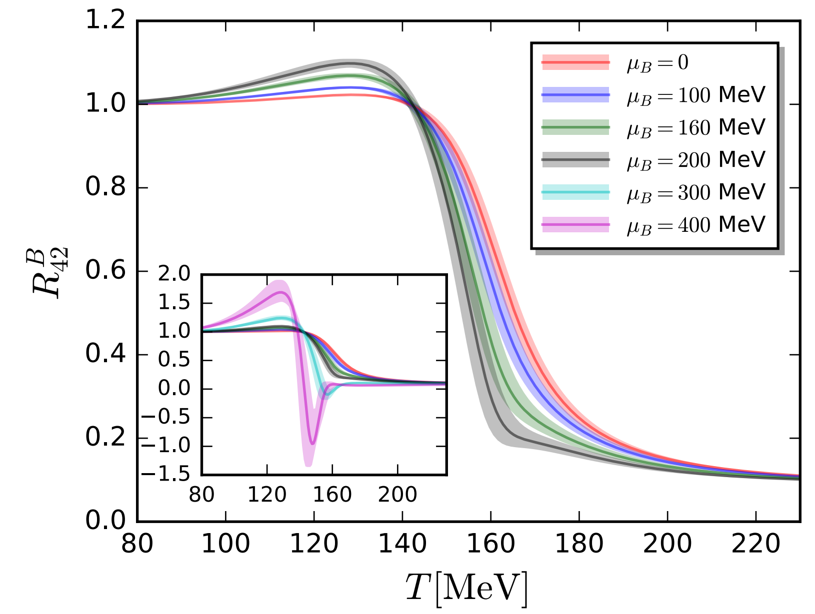
Extrapolation

Compare

Direct calculation



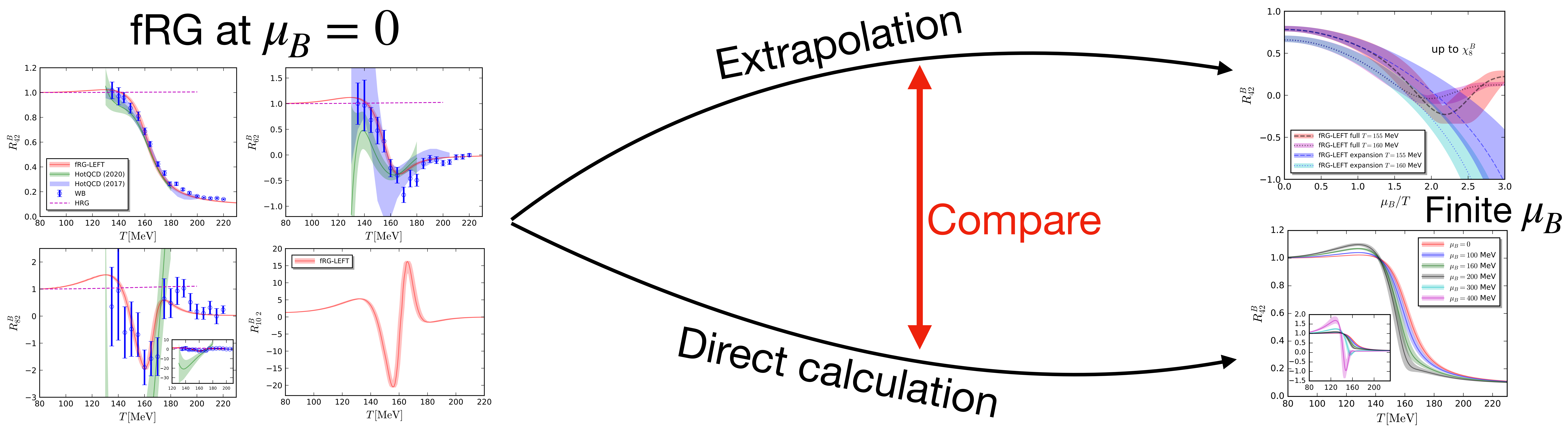
Finite μ_B



Idea of this work

- Match fRG results with lattice QCD results at 0 chemical potential

- Estimate how large is the range that the extrapolation results match with the direct computed results



Polyakov-Quark-Meson (PQM) Model

Effective action:

$$\Gamma_k = \int_x \Big\{ Z_q\, \bar{q}\big[\gamma_\mu \partial_\mu - \gamma_0(\mu + igA_0)\big] q + \frac{1}{2} Z_\phi\, (\partial_\mu \phi)^2$$
$$+ h\, \bar{q}(T^0 \sigma + i\gamma_5 \vec{T} \cdot \vec{\pi}) + V_k(\rho) - c\sigma + V_{\text{glue}}(L, \bar{L}) \Big\}$$

Here we use Local potential approximation (LPA):

$$\partial_t Z_{q/\phi} = 0$$

$$\partial_t h = 0$$

We only consider a simple computation of the effective potential.

Effective potential

Flow equation of effective potential:

$$\partial_t V_k(\rho) = \frac{k^4}{4\pi^2}\left[3\, l_0^{(B)}(m_\pi^2;T) + l_0^{(B)}(m_\sigma^2;T) - 4N_c N_f l_0^{(F)}(m_f^2;\mu,T)\right]$$

The fermion loop for real and imaginary chemical potential

$$\begin{aligned} l_0^{(F)}(m_f^2;\mu,T) &= \frac{k}{3\sqrt{k^2+m_f^2}}\left(1 - n_F(m_f^2;\mu,T;L,\bar{L}) - \bar{n}_F(m_f^2;-\mu,T;L,\bar{L})\right) \\ &= \frac{k}{3\sqrt{k^2+m_f^2}}\left(1 - 2\,\mathrm{Re}\Big(n_F(m_f^2;\mu,T;L,\bar{L})\Big)\right) \end{aligned}$$

Ensure the T and curvature are the same with lattice

$$\begin{aligned} T_{QCD}^{(N_f=2+1)} &= c_T\, T_{PQM}^{(N_f=2)} \\ \mu_{B_{QCD}}^{(N_f=2+1)} &= c_\mu\, \mu_{B_{PQM}}^{(N_f=2)} \end{aligned}$$

Expansion methods

Methods:

A. Taylor Expansion

B. Padé Approximant

C. T’ Expansion

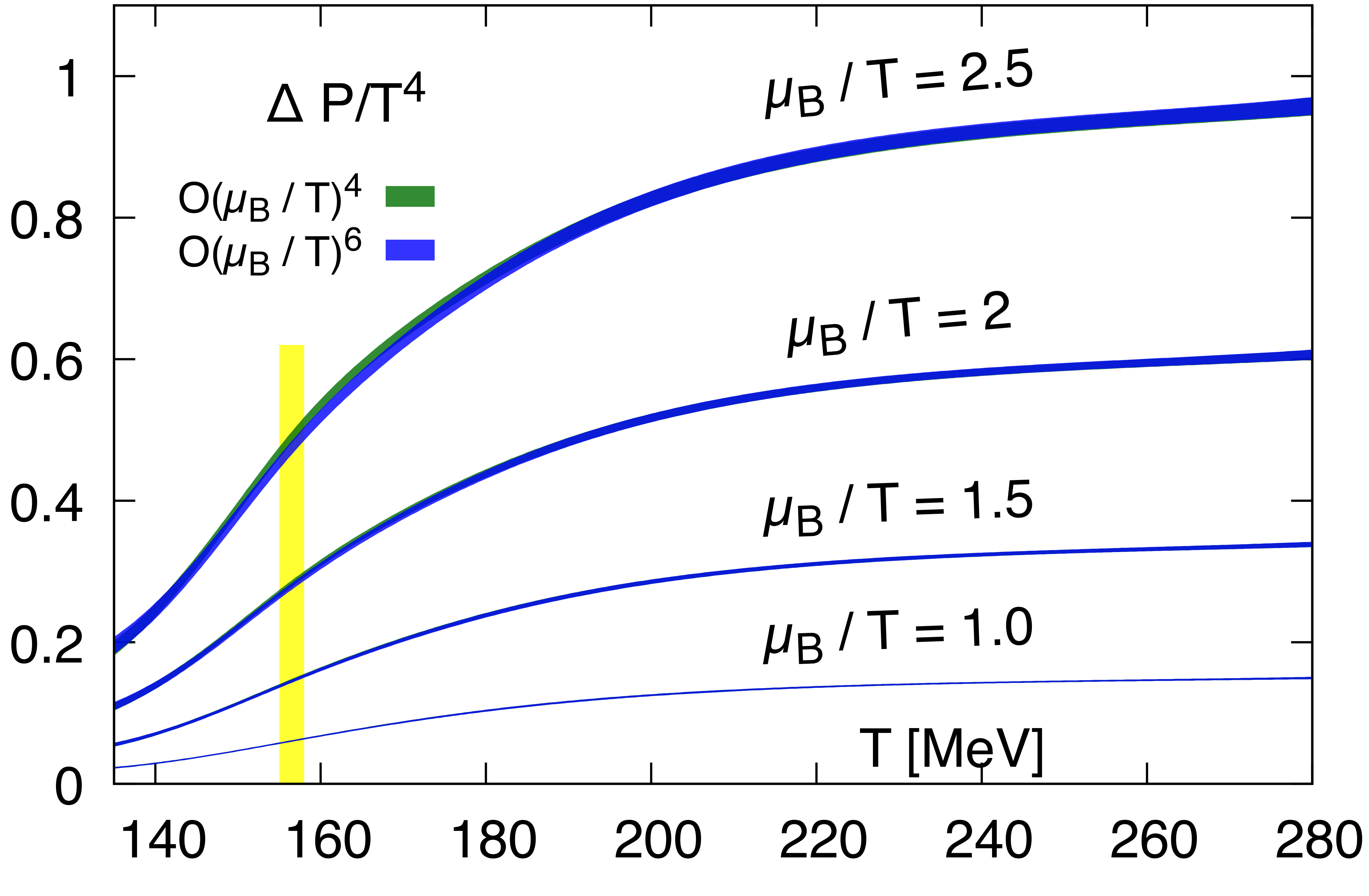
A. Taylor Expansion
B. Padé Approximant
C. T' Expansion

Extrapolation to finite density

A. Taylor Expansion of the pressure:

$$\frac{p(T,\hat{\mu}_B) - p(T,0)}{T^4} = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \chi_{2n}^B(T,0)\, \hat{\mu}_B^{2n}$$

HotQCD: $\mu_Q = \mu_S = 0$



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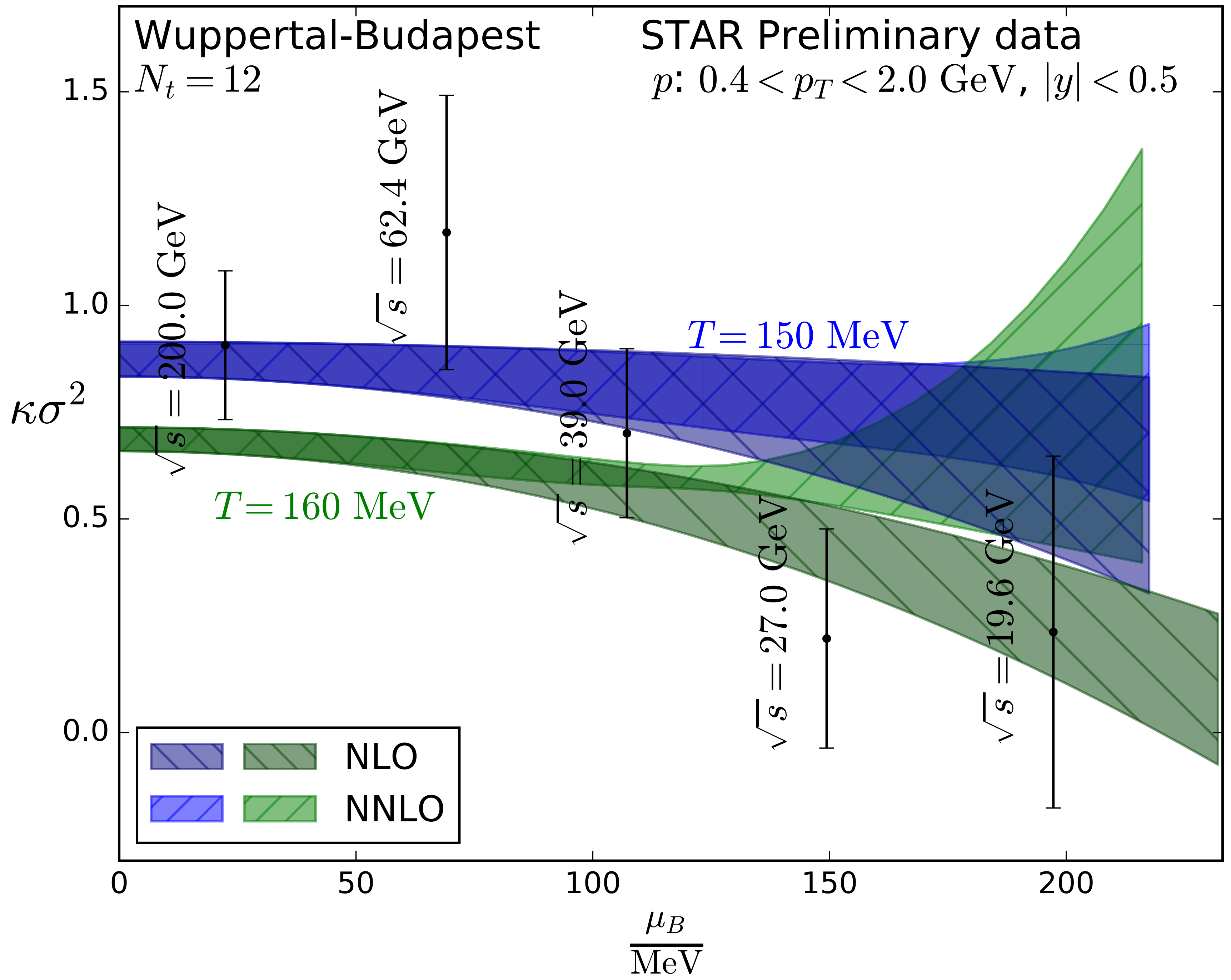
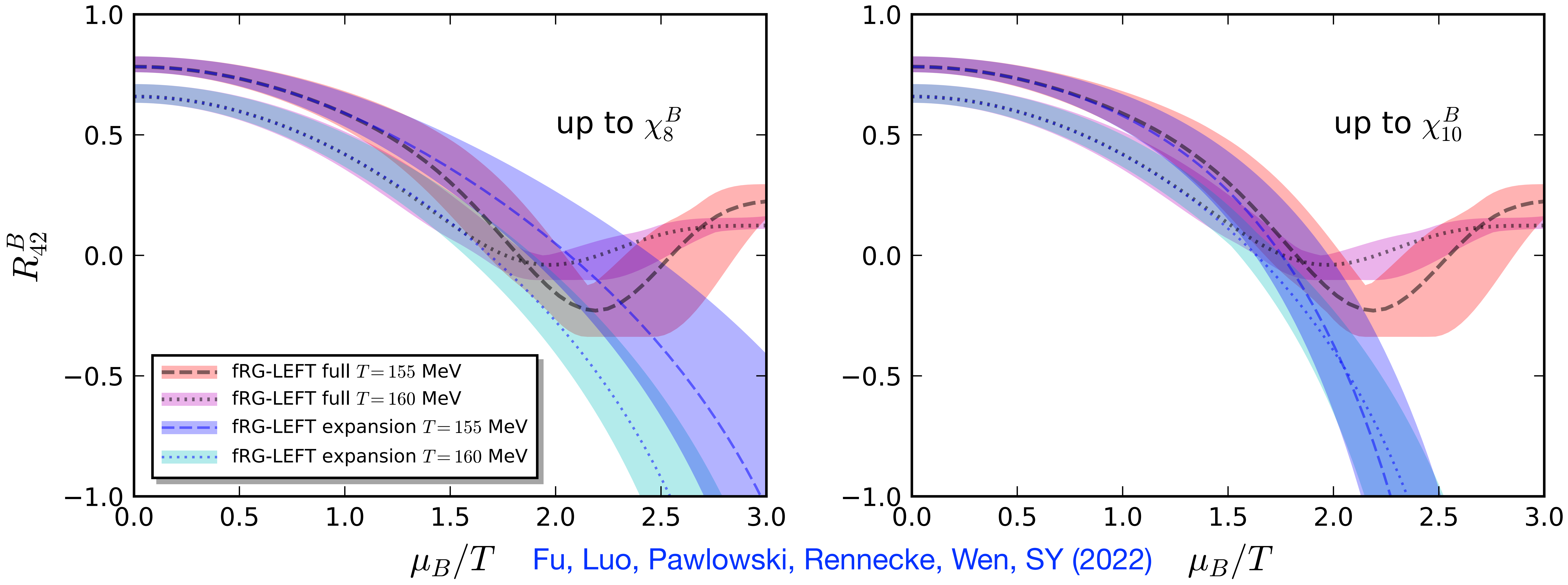
WB:



JHEP 10, 205 (2018)

Example of Taylor Expansion



Fu, Luo, Pawlowski, Rennecke, Wen, SY (2022)

- Direct calculation vs Taylor expansion of R_{42}^B
- The R_{42}^B around $T_{\rm pc}$ exhibits strong fluctuations at high chemical potential, which are difficult to capture with a finite-order Taylor expansion

Extrapolation to finite density

B. Padé approximant:

$$P[m,n] = \frac{p(T,\hat{\mu}_B) - p(T,0)}{T^4} = \frac{\sum_{i=1}^{n/2} a_i \ \hat{\mu}_B^{2i}}{1 + \sum_{j=1}^{m/2} b_i \ \hat{\mu}_B^{2j}}$$

Here the coefficients a_i and b_i are determined by

$$\frac{\partial^i P[m,n]}{\partial \hat{\mu}_B^i} = \chi_i^B$$

- When m=0 the Padé approximant will go back to Taylor expansion
- The poles of Padé approximant can be used to estimate the convergence radius of Taylor expansion

Extrapolation to finite density

Ratio estimator: (the poles of P[2,n])

$$r_{c,2n}^{\text{ratio}} = \left| \frac{(2n+1)(2n+2)\chi_{2n}^{B}}{\chi_{2n+2}^{B}} \right|^{\frac{1}{2}}$$

Mercer-Roberts estimator: (the poles of P[4,n])

$$r_{c,2n}^{\text{MR}} = \left| \left[\frac{\chi_{2n+2}^{B}\chi_{2n-2}^{B}}{(2n+2)!(2n-2)!} - \left(\frac{\chi_{2n}^{B}}{(2n)!} \right)^{2} \right] \right|^{\frac{1}{4}} \left| \left[\frac{\chi_{2n}^{B}\chi_{2n+4}^{B}}{(2n)!(2n+4)!} - \left(\frac{\chi_{2n+2}^{B}}{(2n+2)!} \right)^{2} \right] \right|^{-\frac{1}{4}}$$

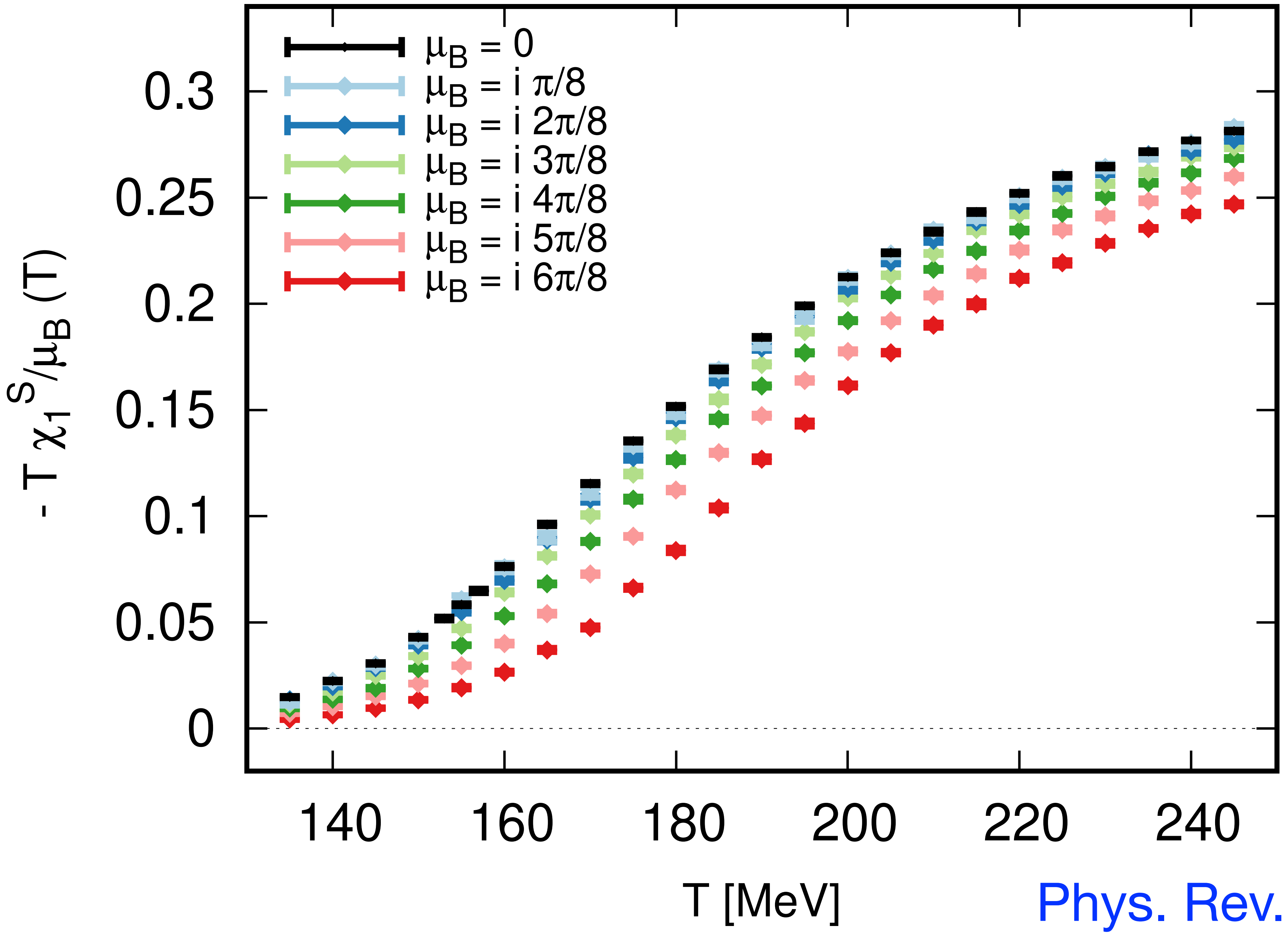
- The estimators are given by the poles of the Padé approximant
- They can give an approximate convergence radius of Taylor expansion

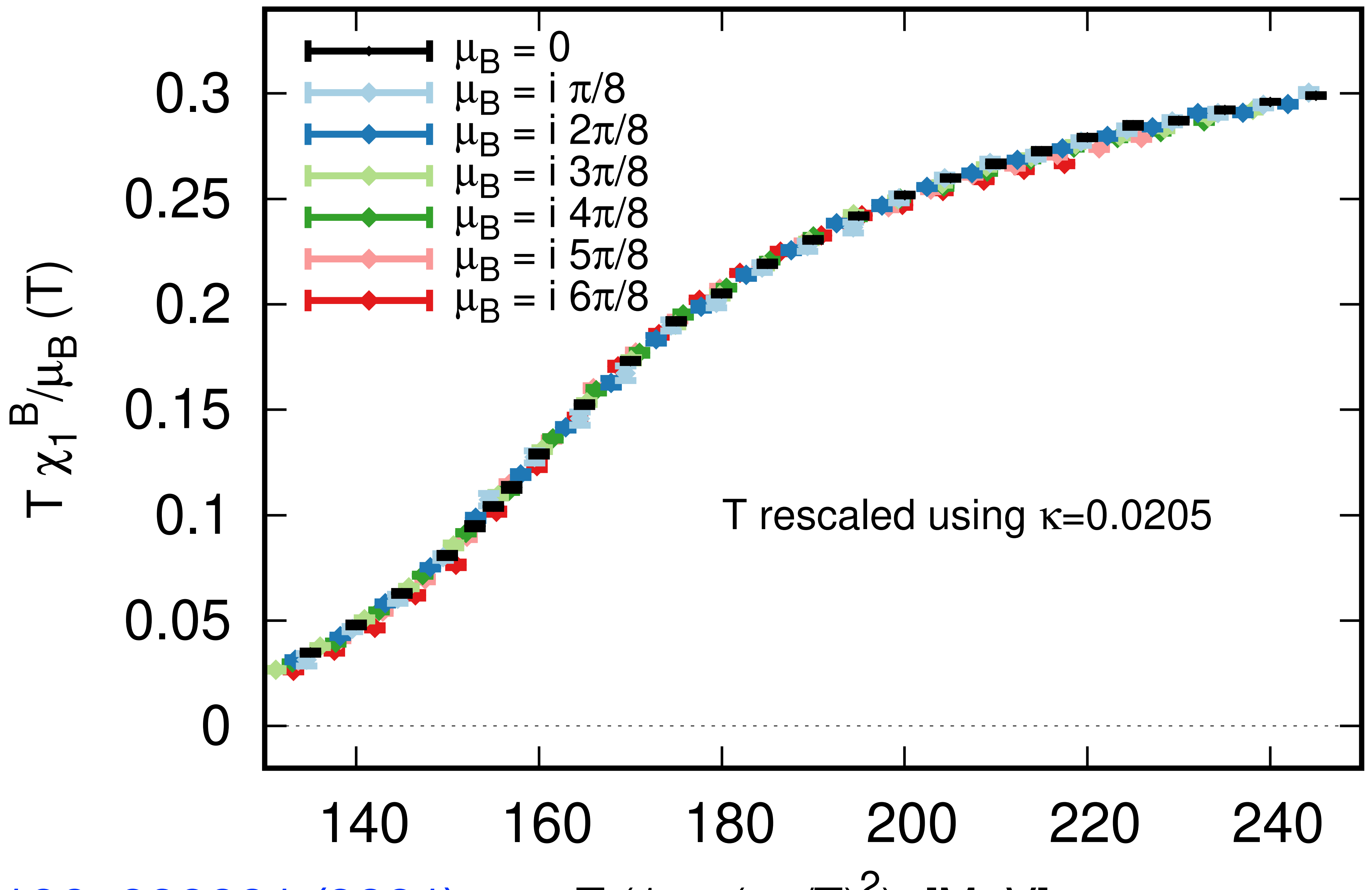
Extrapolation to finite density

C. T' expansion:

$$\frac{\chi_1^B(T,\hat{\mu}_B)}{\hat{\mu}_B} = \chi_2^B(T',0)$$

$$T'(T,\hat{\mu}_B) = T\left(1 + \kappa_2(T)\,\hat{\mu}_B^2 + \kappa_4(T)\,\hat{\mu}_B^4 + \mathcal{O}(\hat{\mu}_B^6)\right)$$

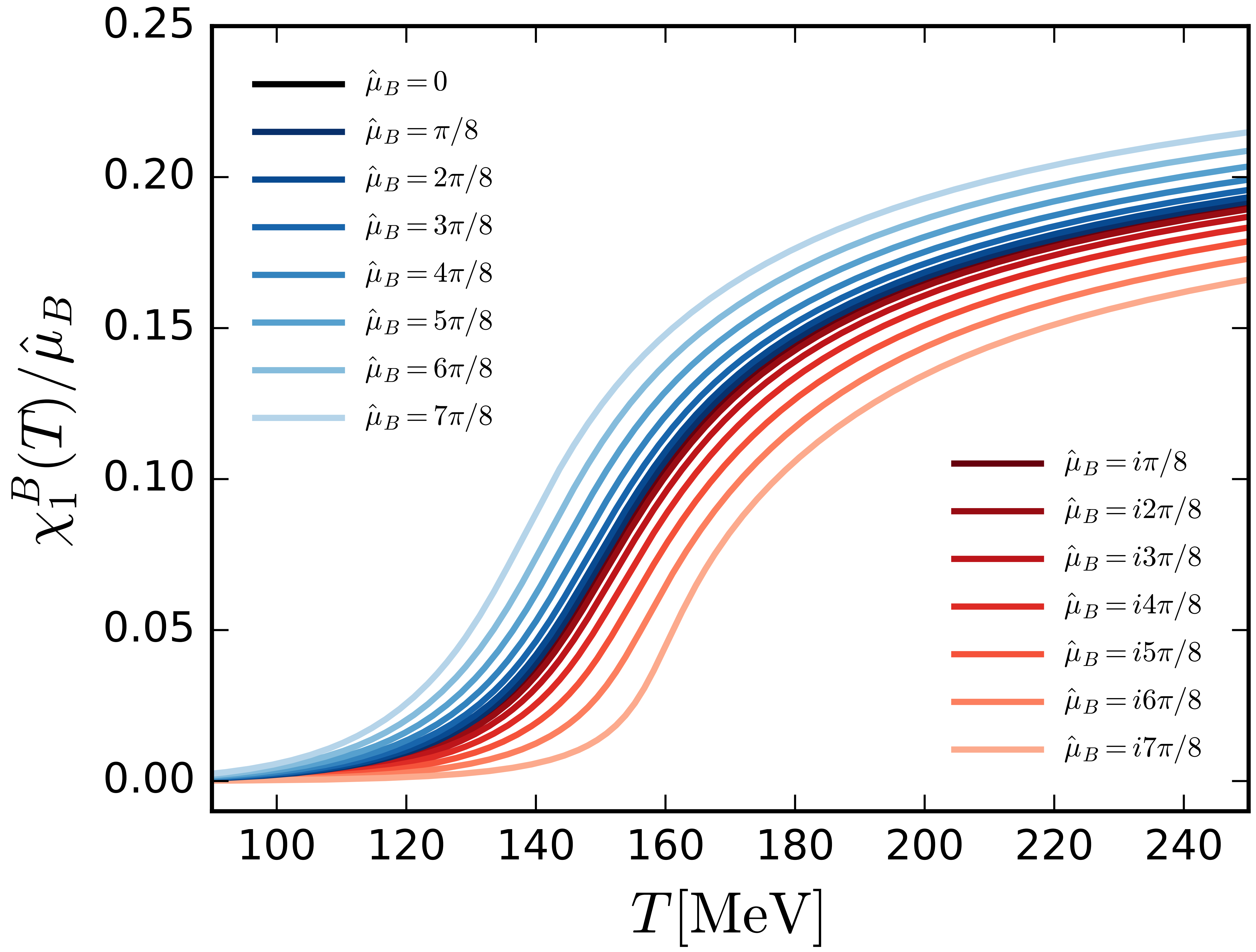




Phys. Rev. Lett. 126, 232001 (2021)

T' Expansion

- Direct compute at both real and imaginary chemical potential
- Apply the T' expansion to perform extrapolation



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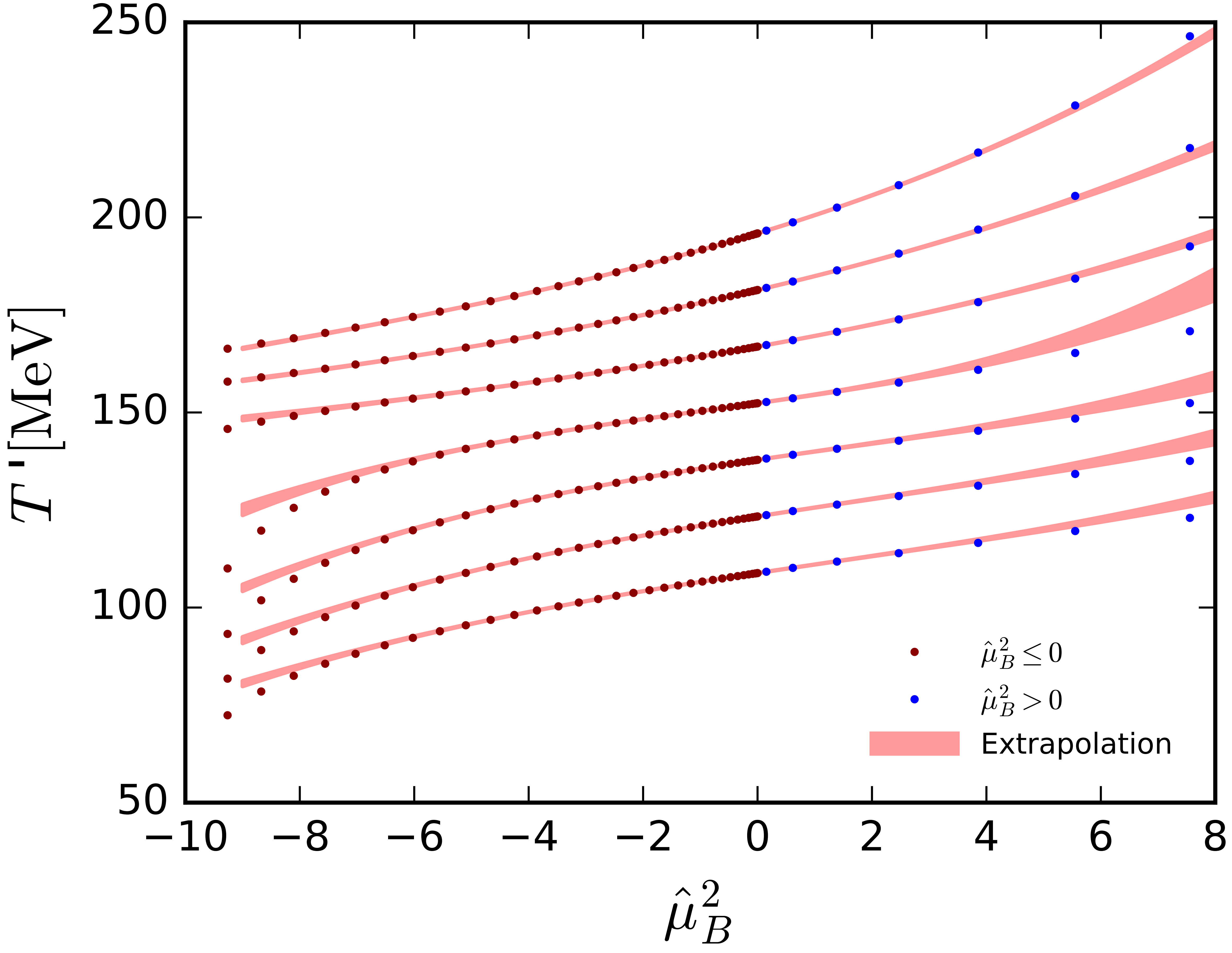
T' Expansion

- Extrapolation by T' expansion

$$T' = T\left(1 + \kappa_2^B(T)\,\hat{\mu}_B^2 + \kappa_4^B(T)\,\hat{\mu}_B^4 + \kappa_6^B(T)\,\hat{\mu}_B^6 + \cdots\right)$$



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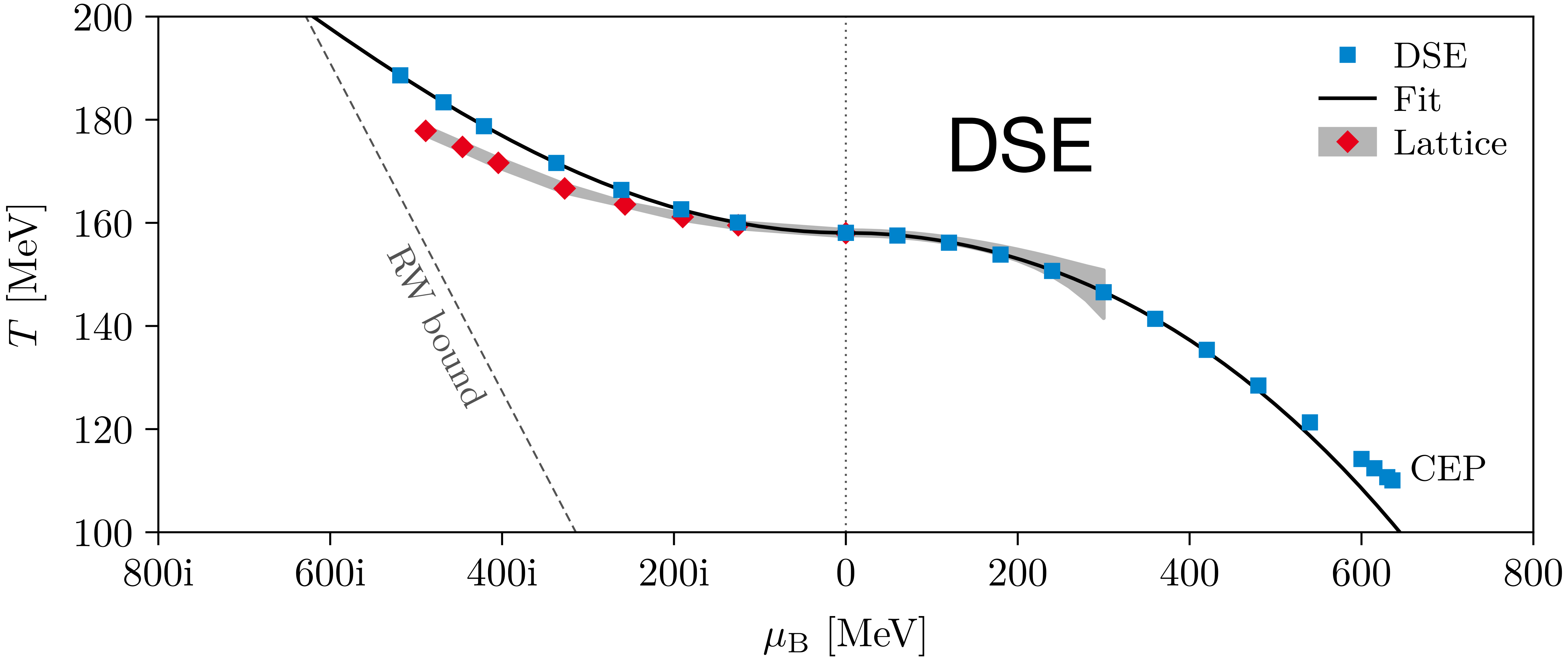
A. Taylor Expansion
B. Padé Approximant
C. T' Expansion

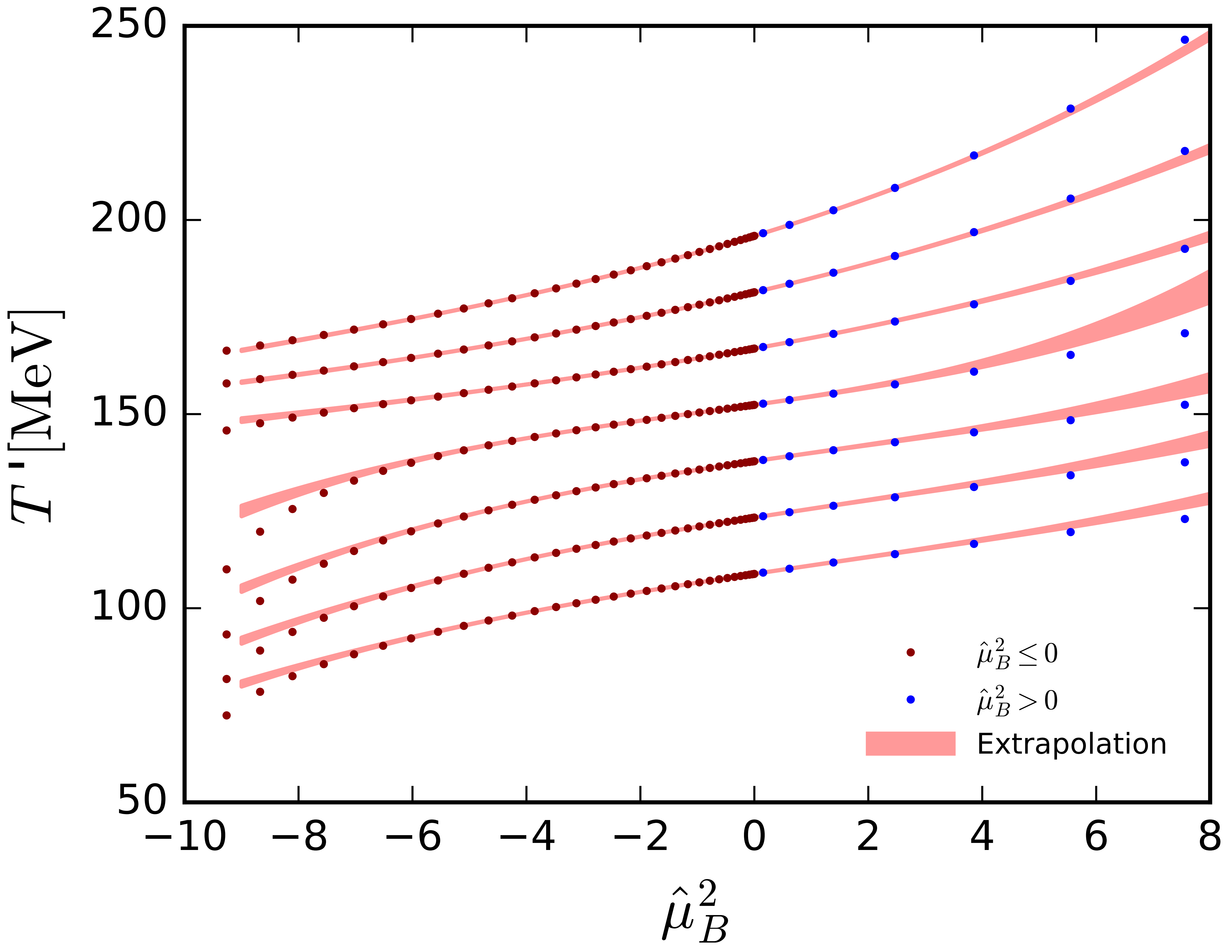
T' Expansion

- Extrapolation by T' expansion

$$T' = T\left(1 + \kappa_2^B(T)\,\hat{\mu}_B^2 + \kappa_4^B(T)\,\hat{\mu}_B^4 + \kappa_6^B(T)\,\hat{\mu}_B^6 + \cdots\right)$$



Bernhardt, Fischer (2023)



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T' Expansion

- Extrapolation by T' expansion

$$T' = T\left(1 + \kappa_2^B(T)\,\hat{\mu}_B^2 + \kappa_4^B(T)\,\hat{\mu}_B^4 + \kappa_6^B(T)\,\hat{\mu}_B^6 + \cdots\right)$$

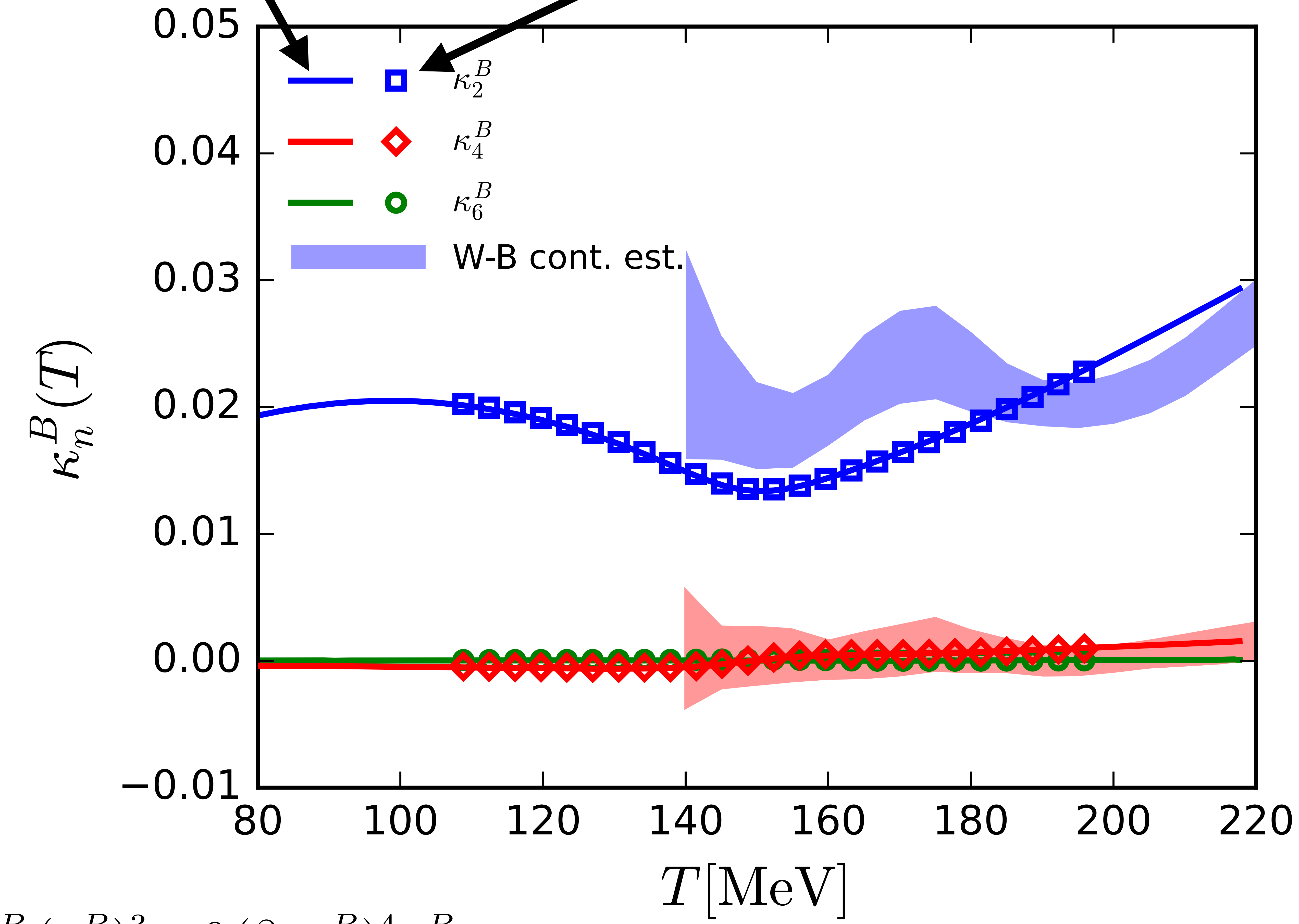
$$\kappa_2^B = \frac{\chi_4^B}{6\,T\,(\partial_T \chi_2^B)}$$

$$\kappa_4^B = \frac{3(\partial_T \chi_2^B)^2\,\chi_6^B - 5\,(\partial_T^2 \chi_2^B)(\chi_4^B)^2}{360\,T\,(\partial_T \chi_2^B)^3}$$

$$\kappa_6^B = \frac{105\,(\partial_T^2\chi_2^B)^2(\chi_4^B)^3 - 63\,\partial_T^2\chi_2^B(\partial_T\chi_2^B)^2\chi_4^B\chi_6^B - 35\,\partial_T^3\chi_2^B\,\partial_T\chi_2^B\,(\chi_4^B)^3 + 9\,(\partial_T\chi_2^B)^4\chi_8^B}{45360\,(\partial_T\chi_2^B)^5\,T}$$

Direct compute

Fit results



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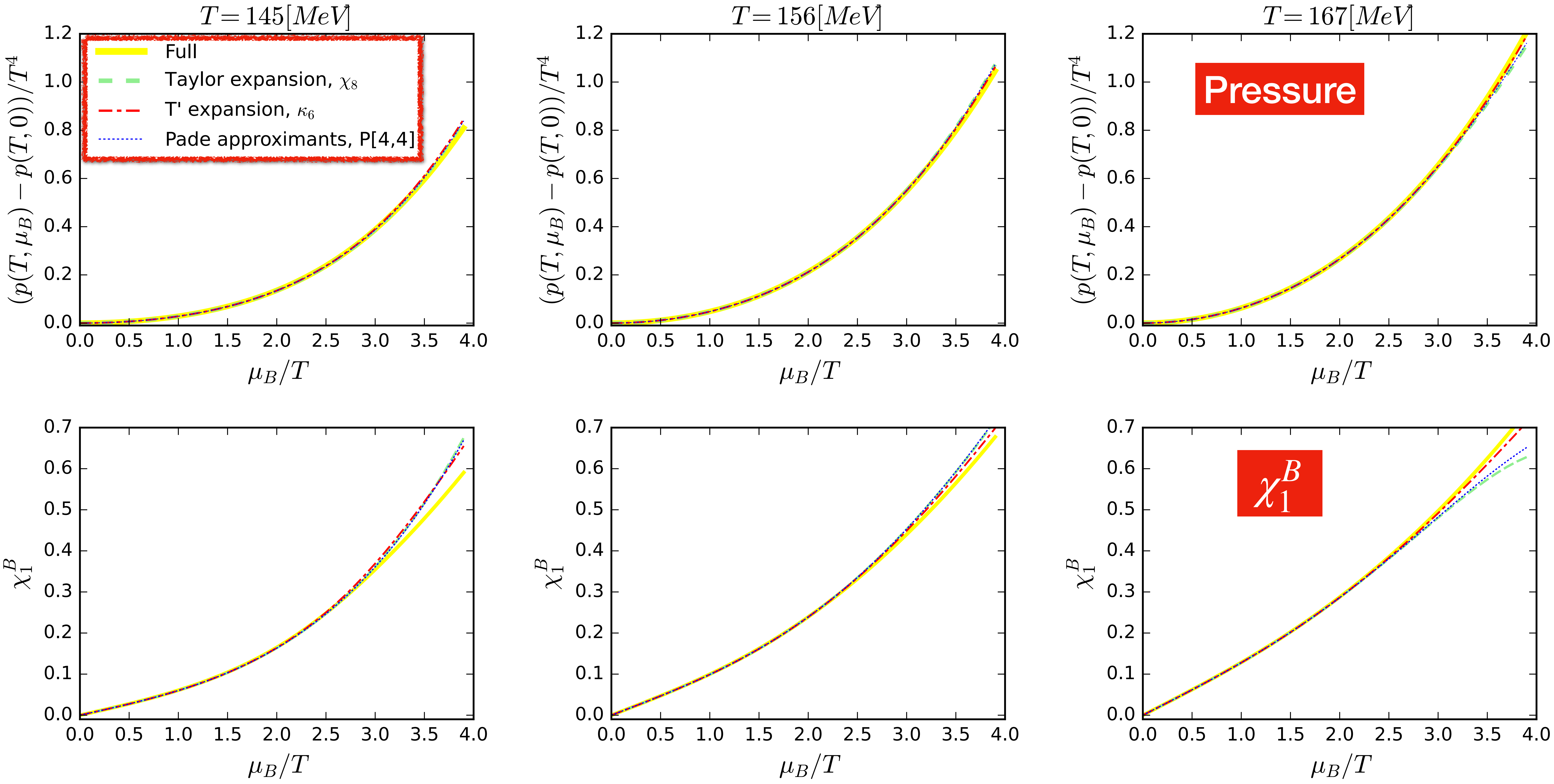
Comparison of different Expansion methods



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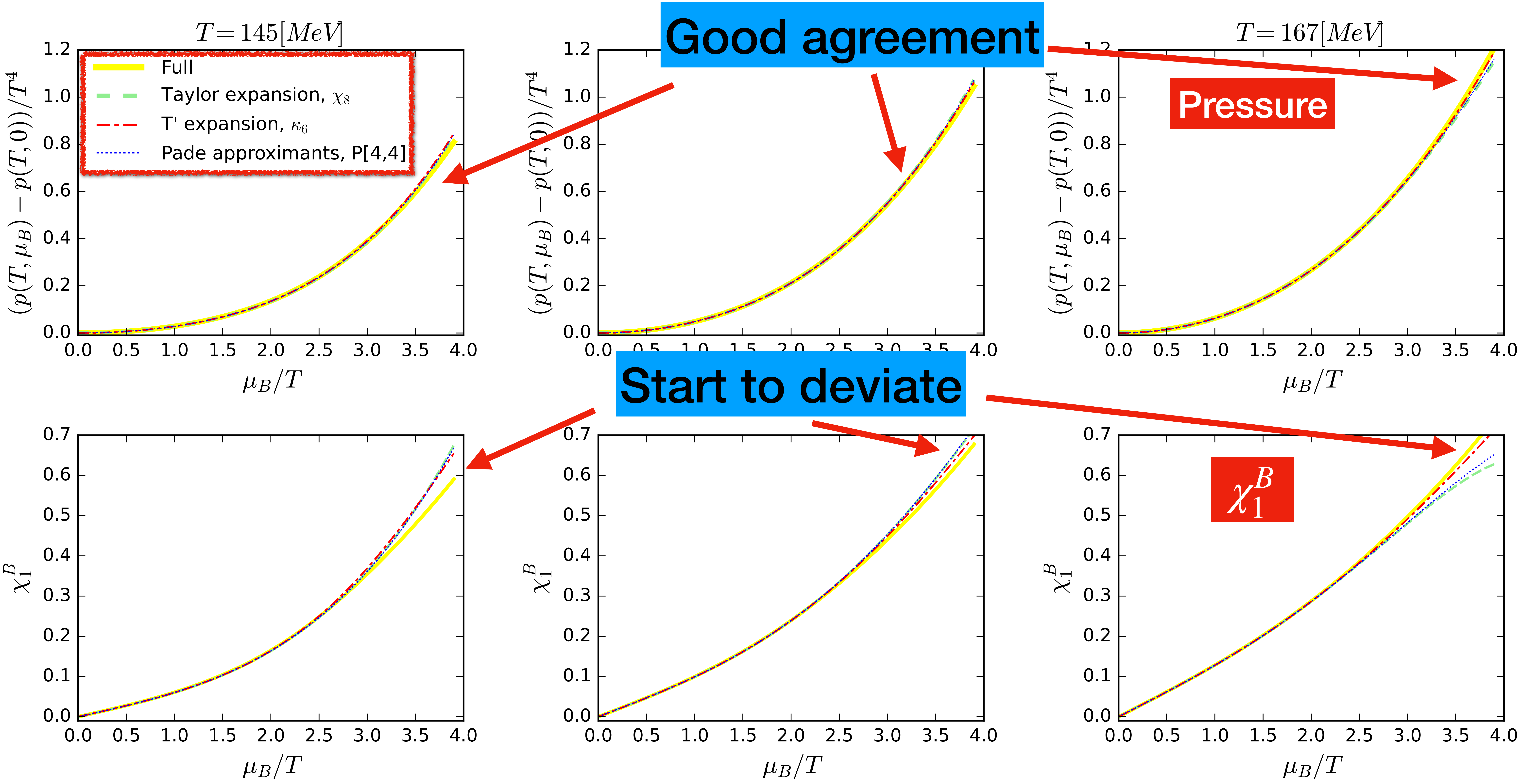

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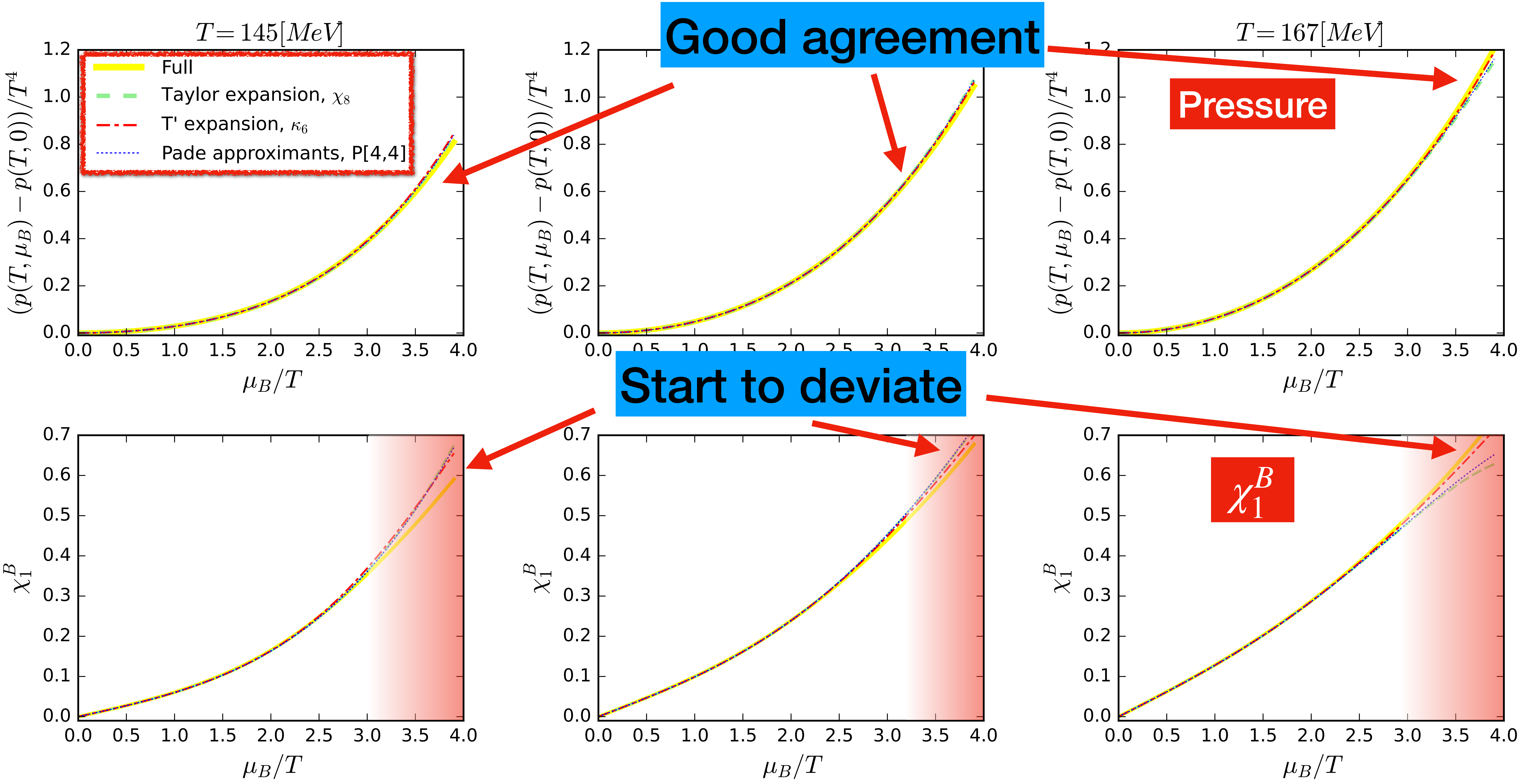

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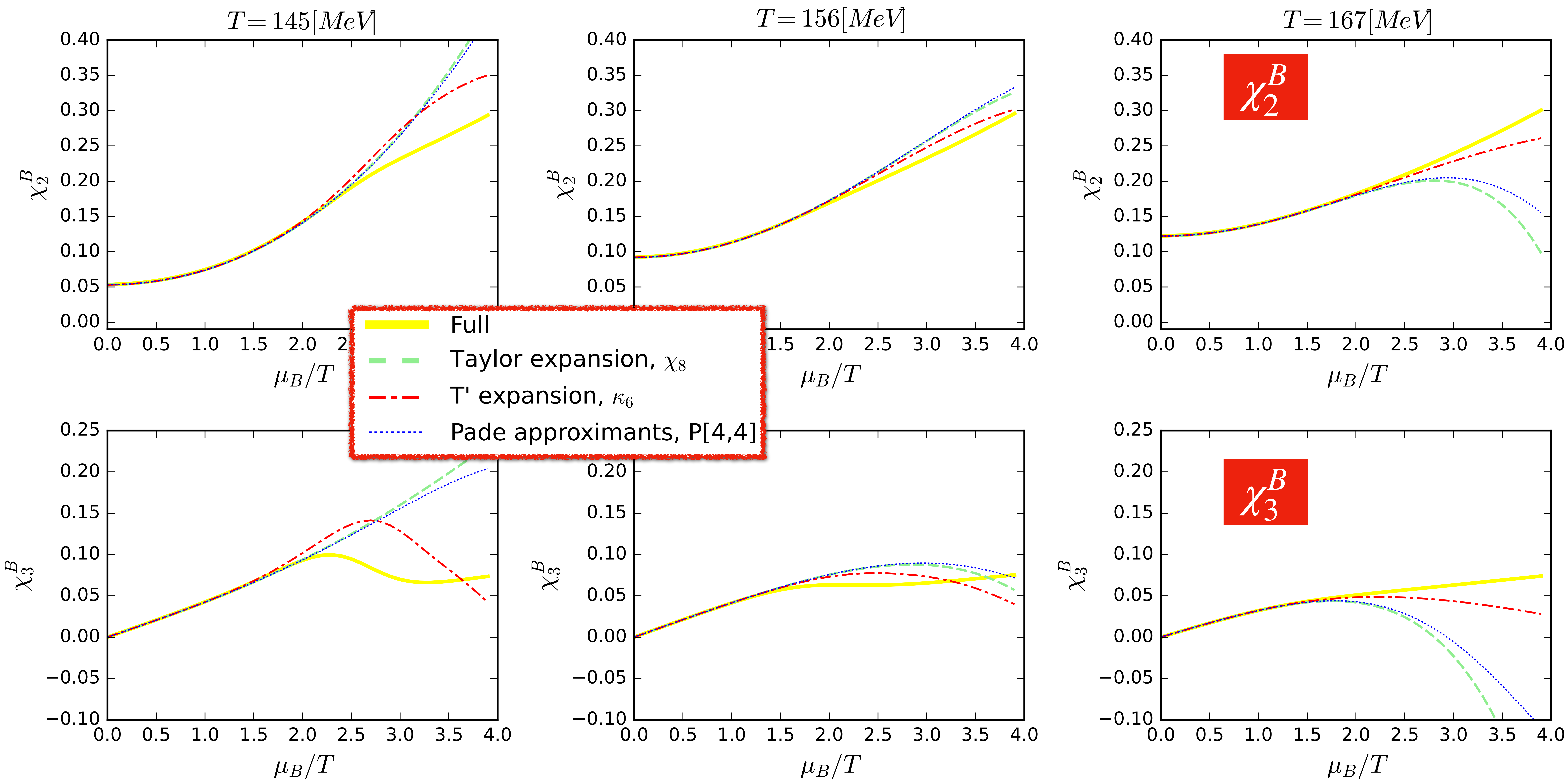
Comparison of different Expansion methods



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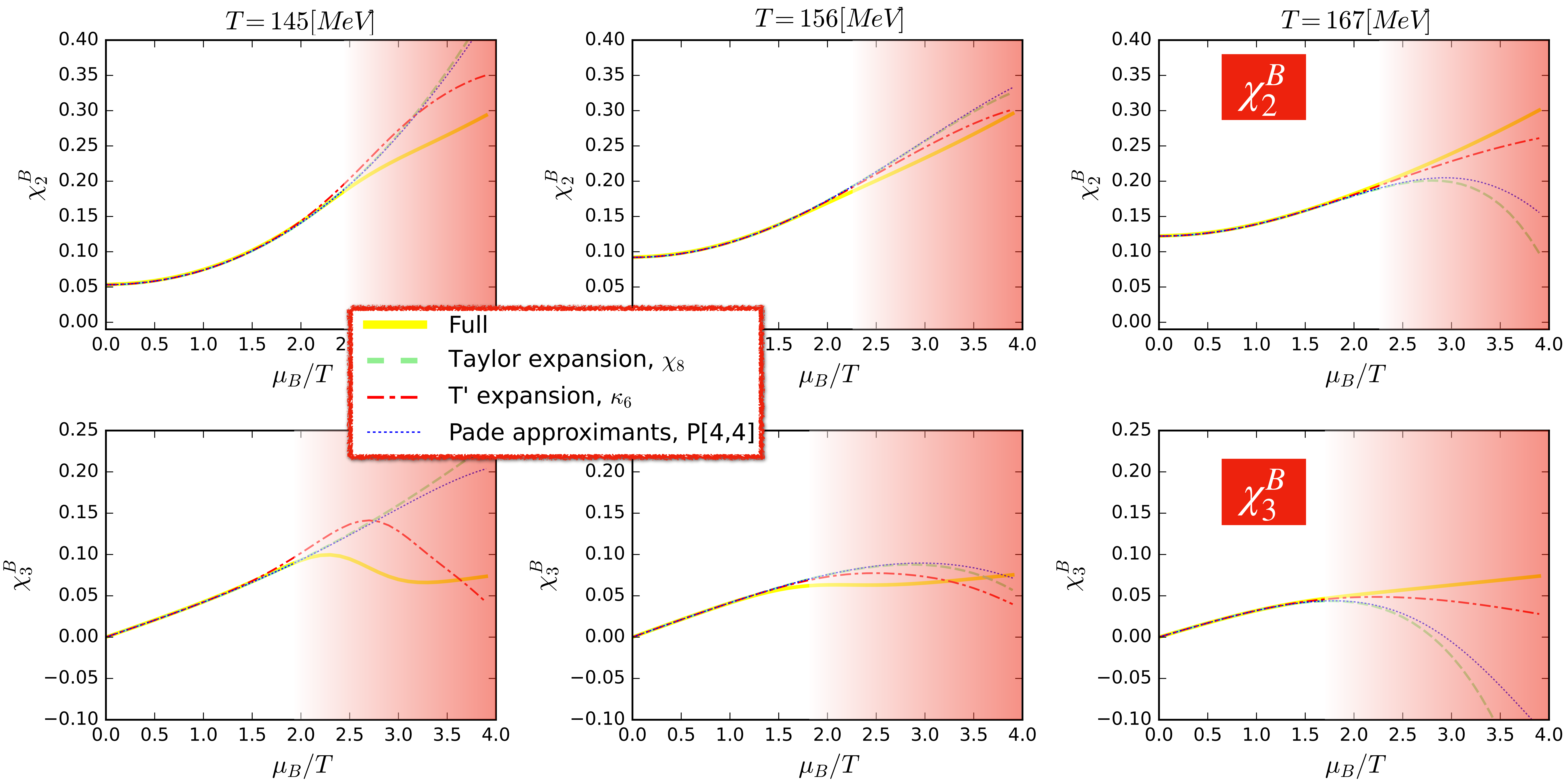
Comparison of different Expansion methods



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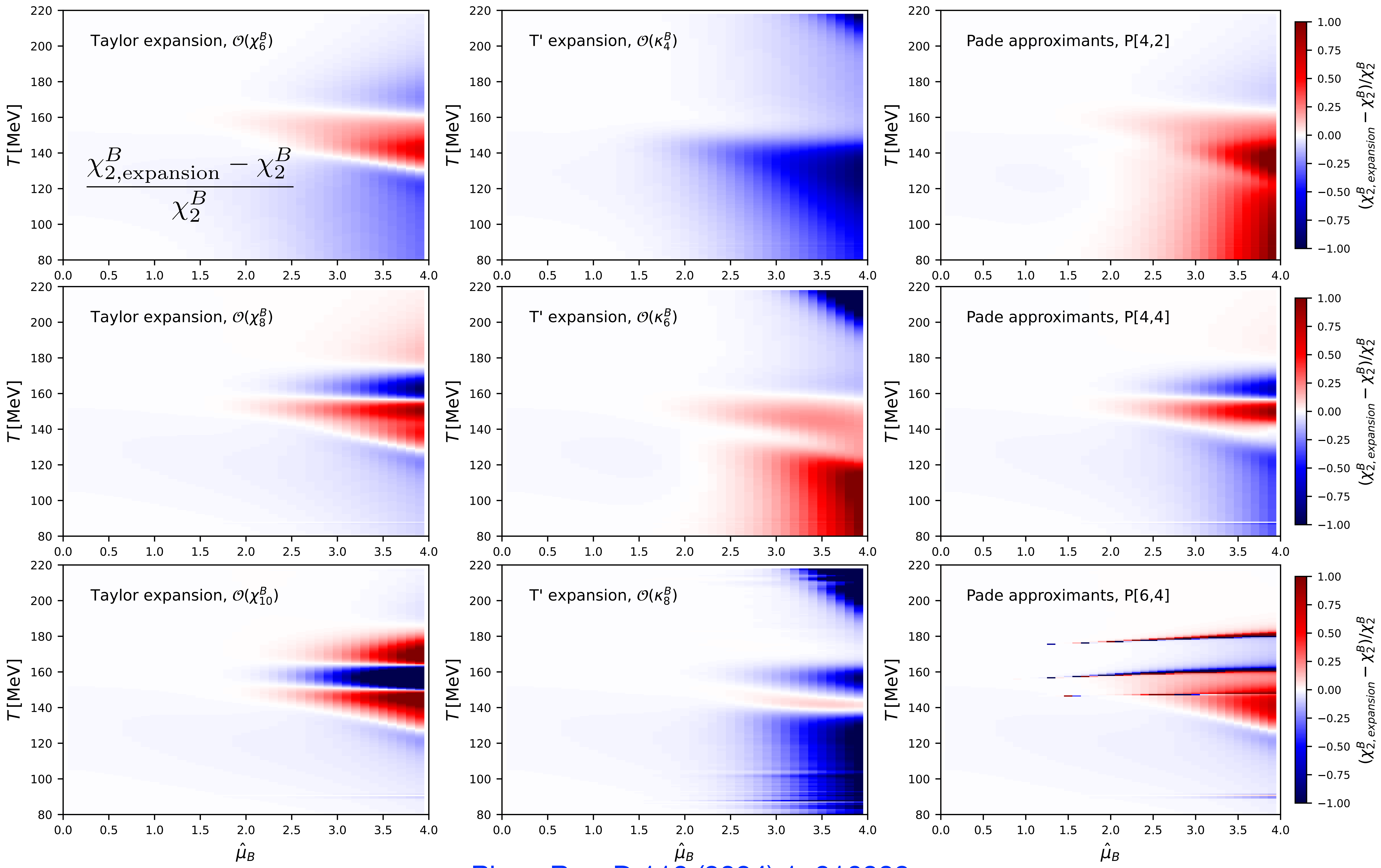

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Comparison of different Expansion methods

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Estimation of Lee-Yang edge

Radius of convergence: Mukherjee, Skokov (2019)

$$R_{\rm conv} = \left| \frac{z_c}{z_0}\left(\frac{m_l^{\rm phys}}{m_s^{\rm phys}}\right)^{\frac{1}{\beta\delta}} - \frac{T-T_c^0}{T_c^0}\right|^{\frac{1}{2}} \frac{1}{\sqrt{\kappa_2}}$$

Parameters:

$$\frac{m_l^{\rm phys}}{m_s^{\rm phys}} = \frac{1}{27}$$

scaling variable: $z_c = |z_c| e^{i\frac{\pi}{2\beta\delta}}$

fRG O(4) $|z_c| = 1.665$

$$z_0 \in [1, 2]$$

Connelly, Johnson, Rennecke, Skokov (2020)

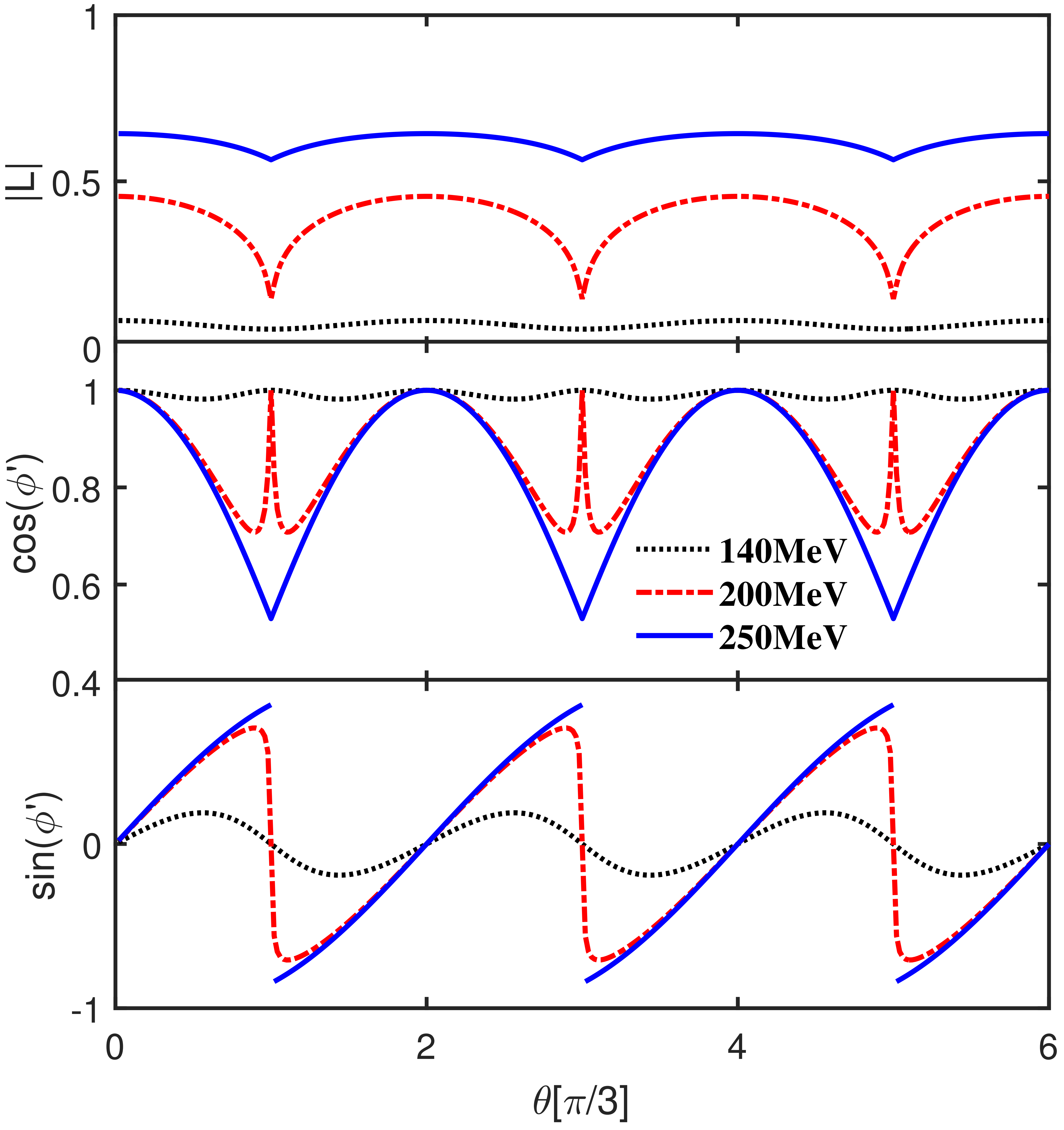
Curvature of phase boundary: $\kappa_2 = 0.0184$

Critical exponents: $\beta = 0.3989$, $\delta = 4.975$

Critical temperature: $T_c^0 = 142.6\,\mathrm{MeV}$

From fRG LPA computation

Roberge-Weiss critical point



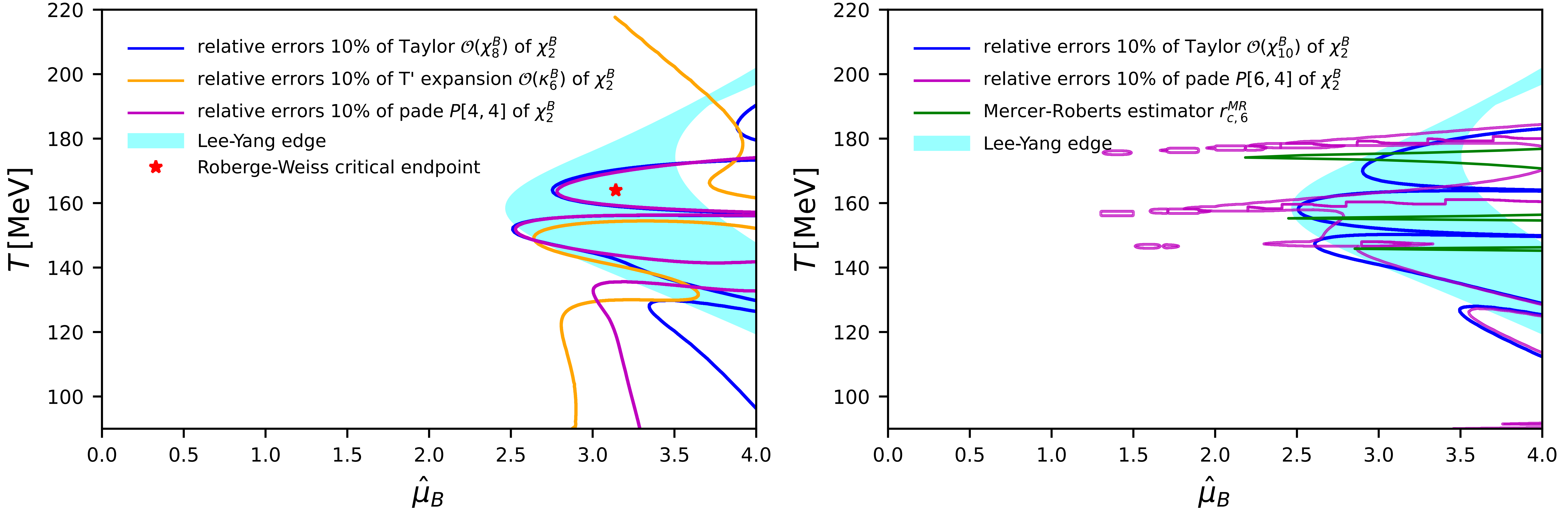
$$\mu^{\mathrm{RW}} = i\frac{\pi}{3}T$$

$$\hat{\mu}_B^{\mathrm{RW}} = i\pi$$

Sun, Wen, Fu (2018)

Estimation of reliable region of expansion

CEP of the model: $(T_{\mathrm{CEP}} = 40\,\mathrm{MeV}, \mu_{B_{\mathrm{CEP}}} = 667\,\mathrm{MeV})$



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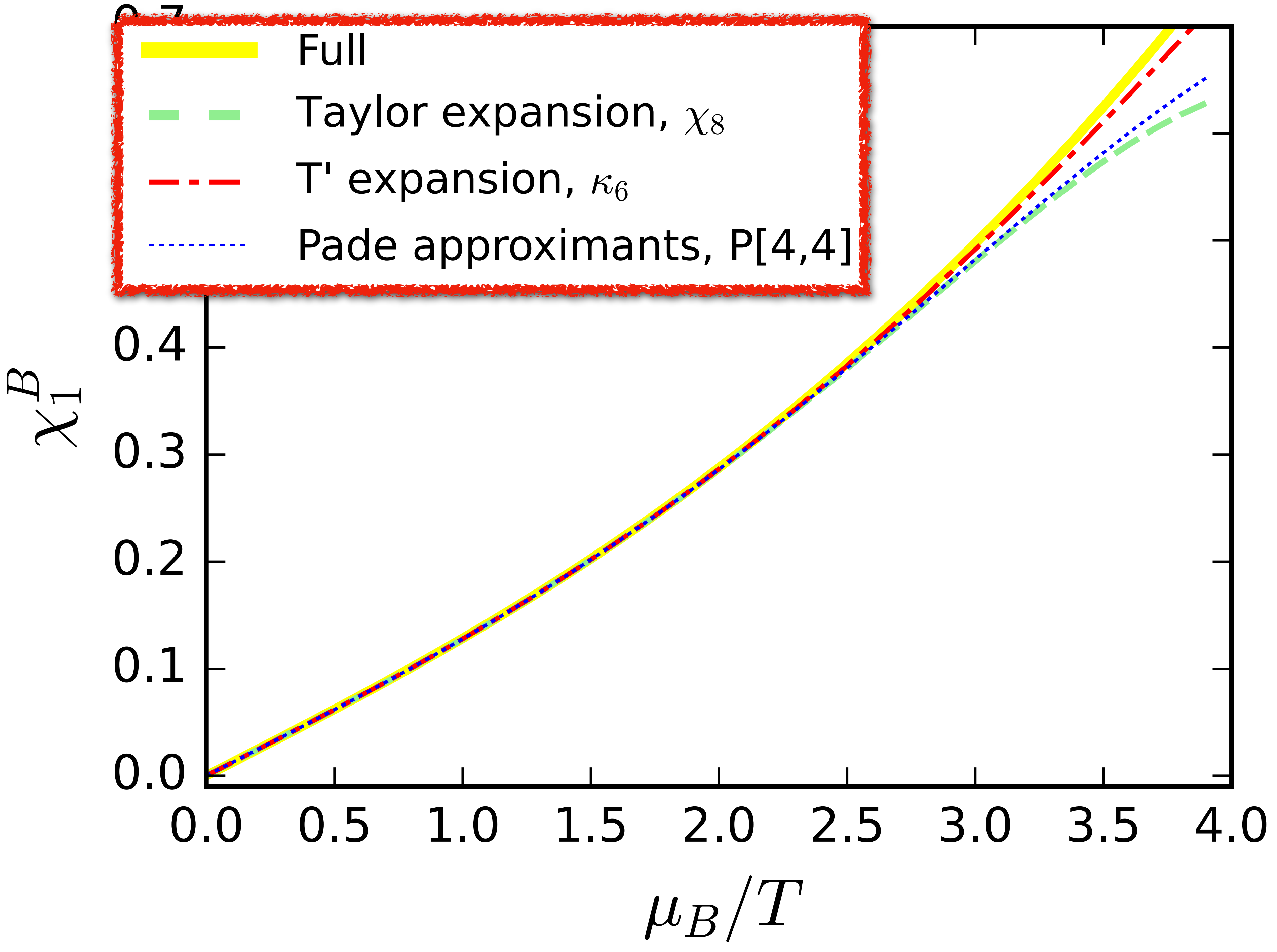
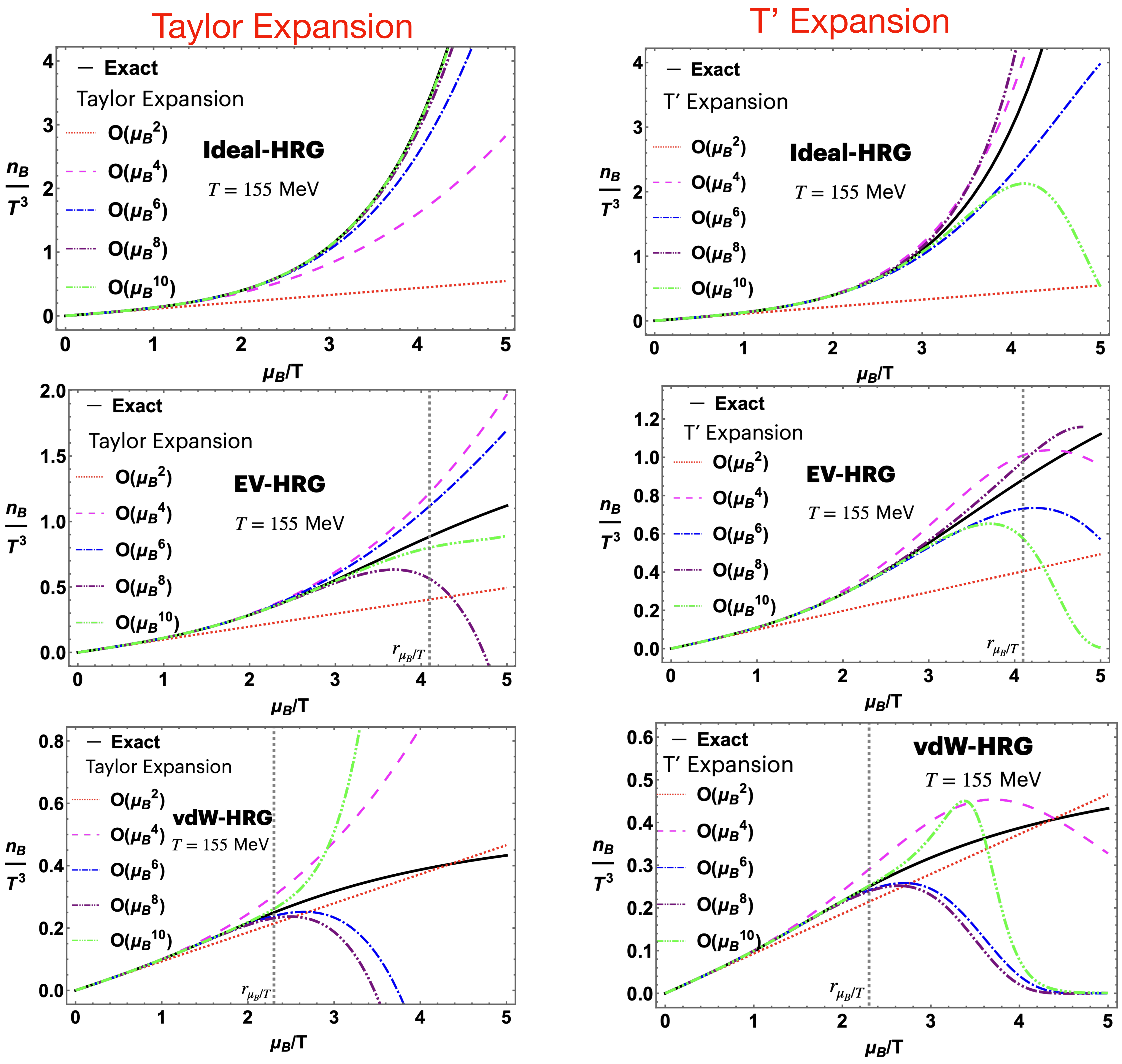
Estimation of reliable region of expansion





Kahangirwe, Ratti, Vovchenko, Gonzalez, Munoz (2025)

Outline

Summary and Conclusion

① Pressure and baryon number fluctuations are computed by fRG at real and imaginary chemical potential within a low energy effective theory

② The convergence of Taylor expansion, Padé approximant and T' expansion are investigated

1. Consistent region: up to $\mathcal{O}(\mu_B^8)$

p/T^4	$\mu_B/T \lesssim 3.5$	around T_{pc}
χ_1^B	$\mu_B/T \lesssim 3.0$	
χ_2^B	$\mu_B/T \lesssim 2.0$	
χ_3^B	$\mu_B/T \lesssim 1.5$	

2. Consistent regions are similar for all these three expansion methods

3. Convergence radius of Lee-Yang edge singularities are in agreement with expansion consistent regions

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Thank you very much!!!