Yang-Lee edge singularities and proton number cumulants

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Analytic Structure of QCD and Yang-Lee Edge Singulary

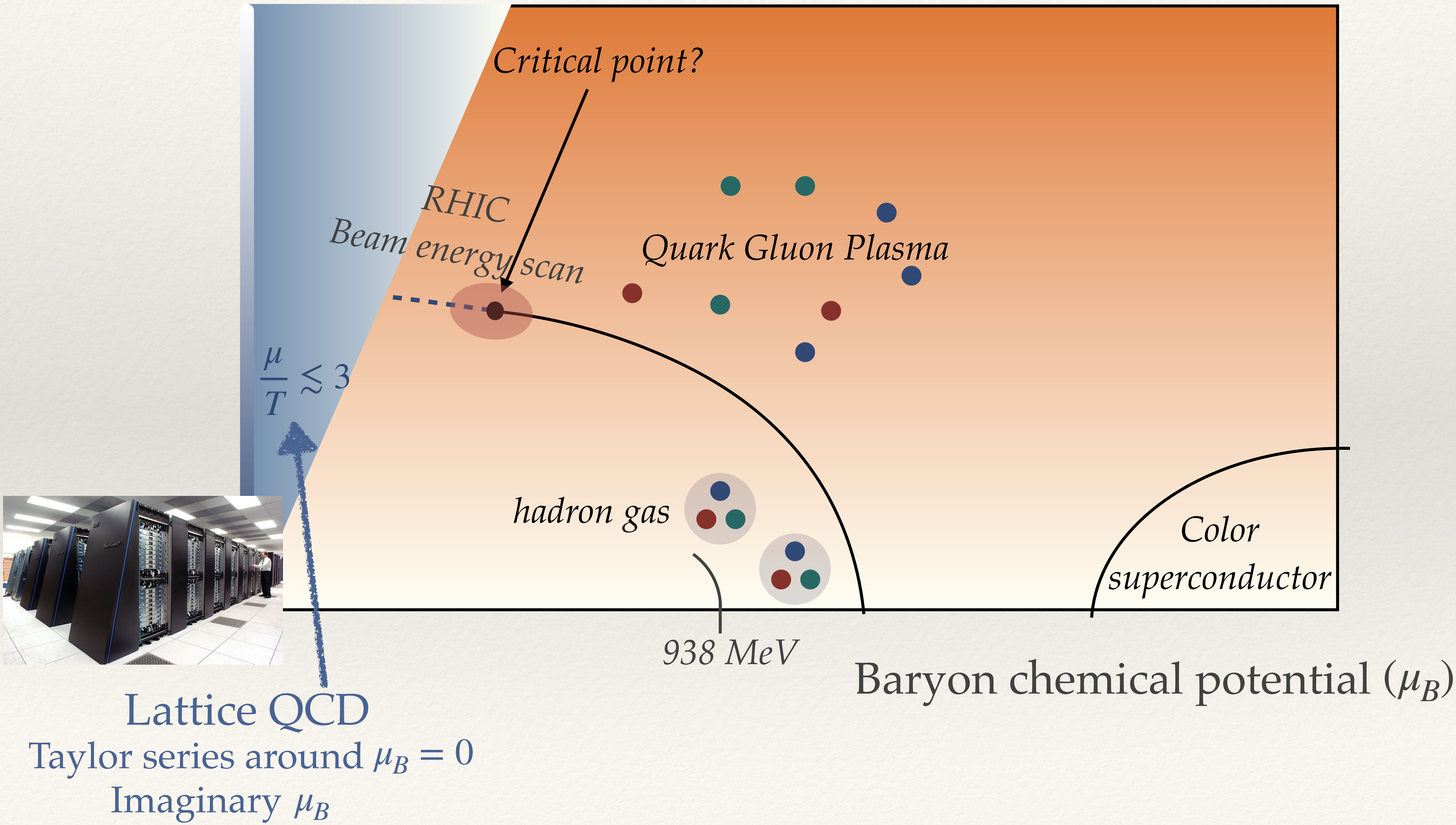
Based on:

GB, *PRL* 127 (2021) 17, 171603; *PRC* 110 (2024) 1, 015203

GB, G. Dunne, Z. Yin *PRD* 105 (2022) 10, 105002

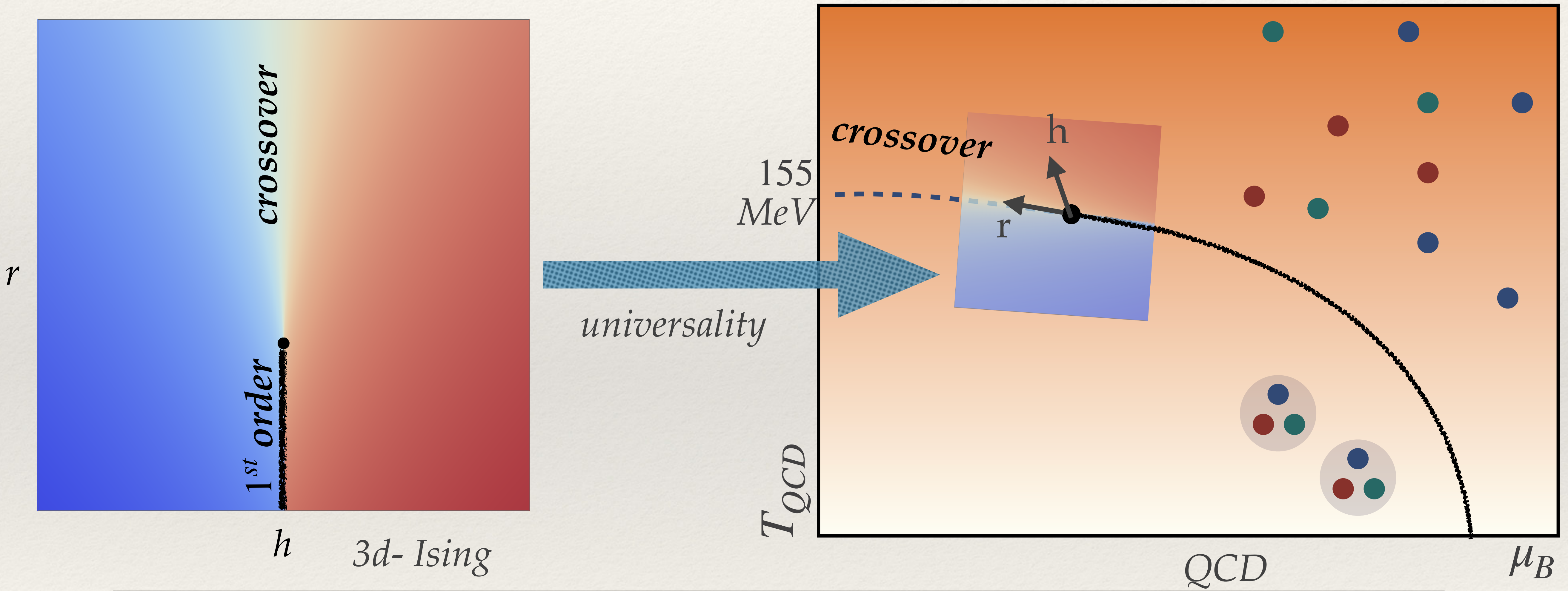
GB, M. Pradeep, M. Stephanov, in progress

Motivations



Motivations

$$\begin{pmatrix} r \\ h \end{pmatrix} = \begin{pmatrix} r_T & r_\mu \\ h_T & h_\mu \end{pmatrix} \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix} = \mathbb{M} \begin{pmatrix} T - T_c \\ \mu - \mu_C \end{pmatrix}$$

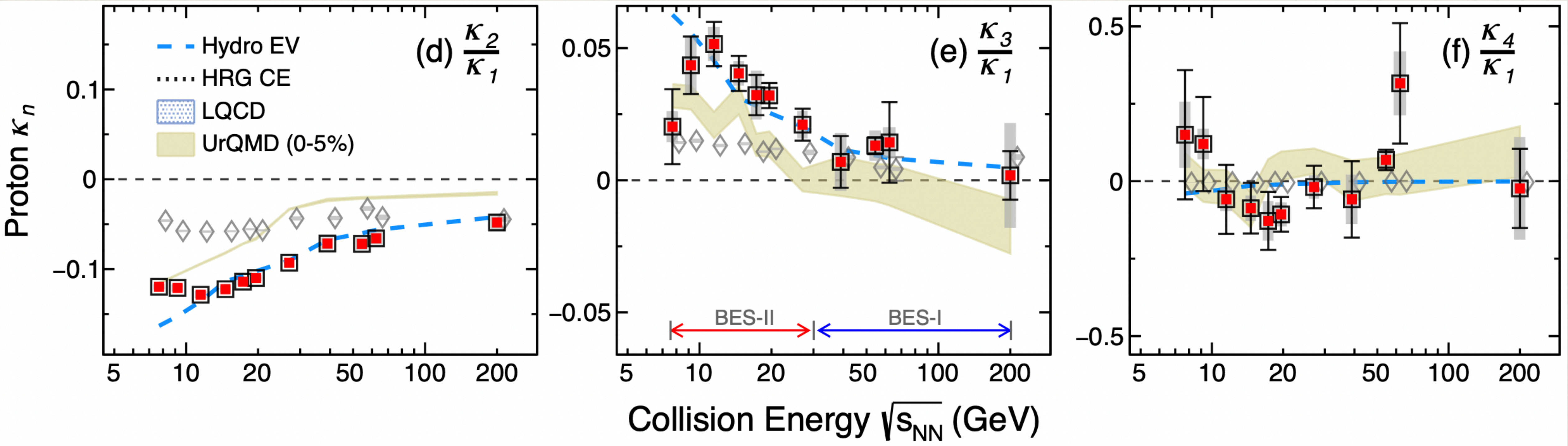


Given the e.o.s. as truncated Taylor series around μ=0, what can we say about *the critical e.o.s* ?

Motivations

Key observable for the search for critical point: *fluctuations*

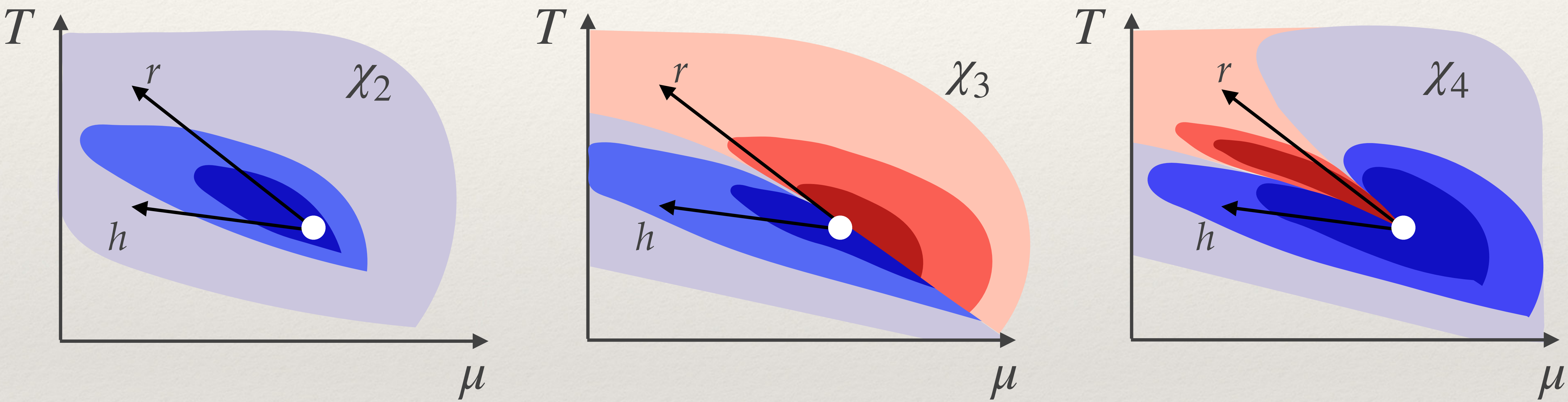
Proton number (factorial) cumulants



[STAR BES II data: 2504.00817]

Motivations

Fluctuations are sensitive to the non-universal mapping parameters



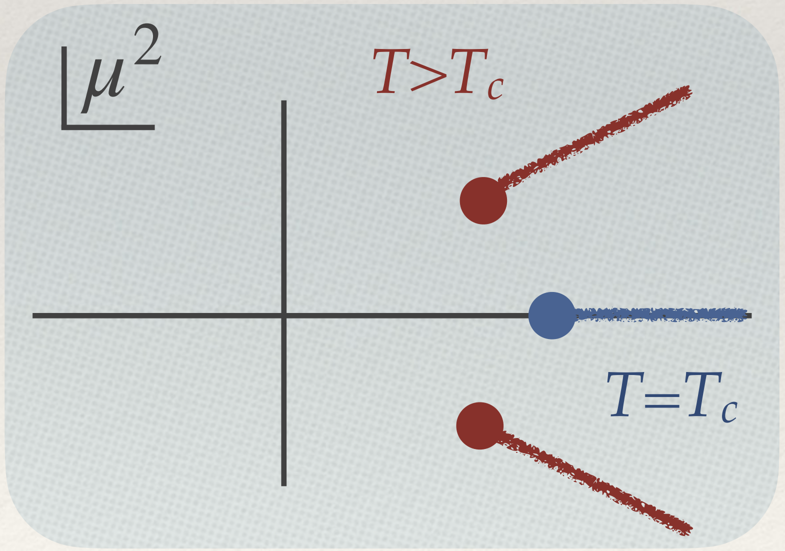
Cartoon images of susceptibilities (red <0, blue >0)

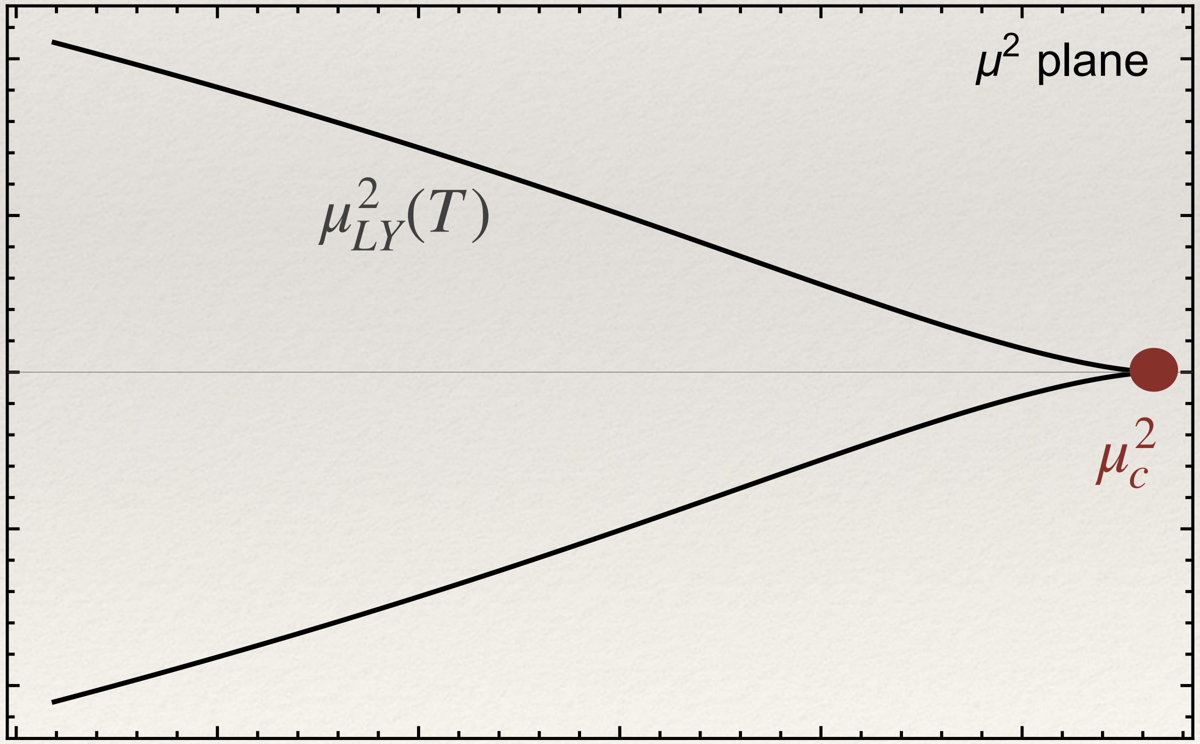
Goal: Extract as much information as possible about the critical point and the mapping parameters from the lattice

Lee-Yang edge singularities

- The equation of state has complex singularities [Lee-Yang, 52′]
- Zeroes of partition function $\mathcal{Z}(\zeta)$ ($\zeta = e^{\mu/T}$: fugacity)
- Coalesce into branch cuts in thermodynamic limit
- Pinch the real axis at a second order transition
- Closest singularity to origin ("extended analyticity conjecture")

[Fonseca, Zamolodchikov '02, An, Mesterházy, Stephanov '17]

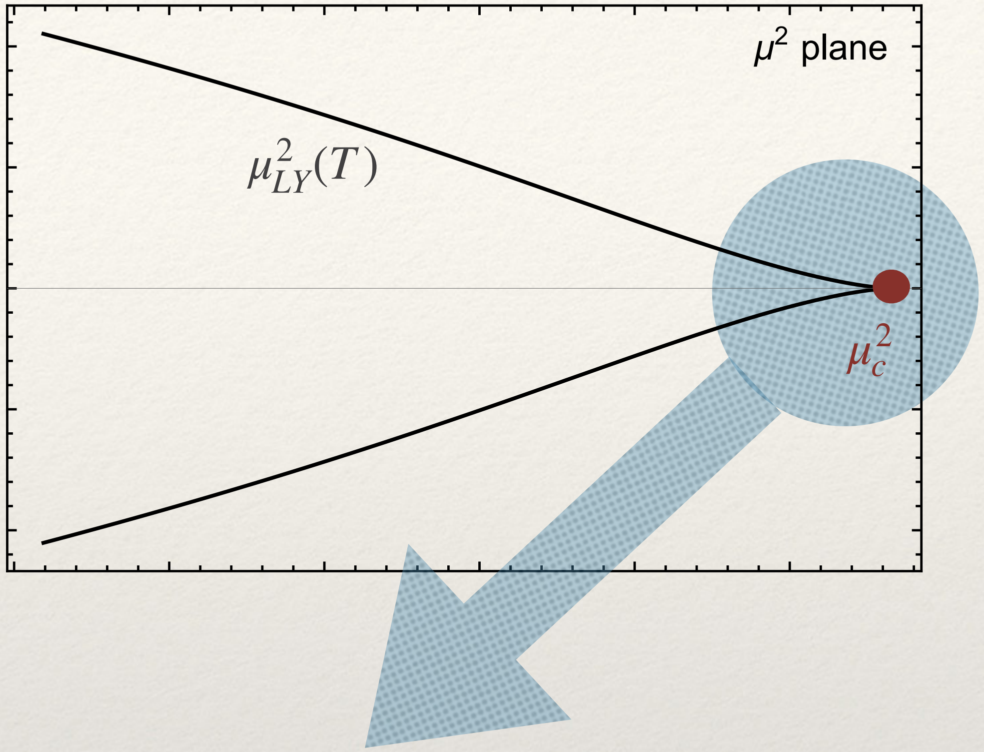




LY singularity near the critical point

Universality:
The scaling e.o.s, $f_s(w)$,
has singularities at

$$w := hr^{-\beta\delta} = \pm i w_{LY}$$



$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) \pm i w_{LY} \frac{\left(\det \mathbb{M}\right)^{\beta\delta}}{h_\mu^{\beta\delta+1}}(T - T_c)^{\beta\delta}$$

[Stephanov '06]

$(\tan\alpha_1)^{-1}$

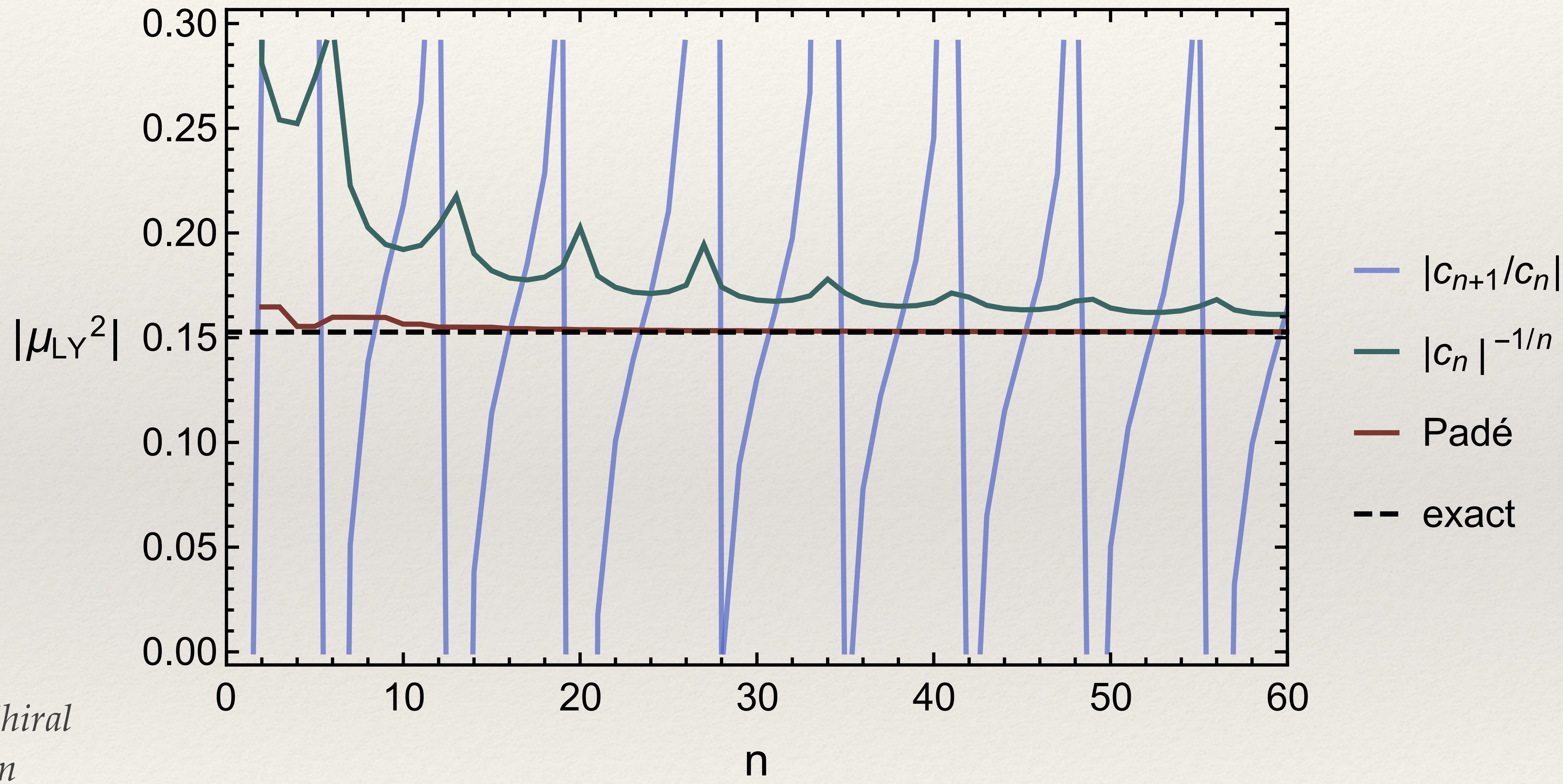
slope of the crossover line

$\det \mathbb{M} \propto (\tan\alpha_2 - \tan\alpha_1)$

relative angle between r, h axes

When life gives you Taylor series...

Darboux theorem: Nearest singularity ⟷ Large order growth



example: Chiral Random Matrix Model
[Halasz et al, 98]

Padé approximants give a much better estimate of the singularity!

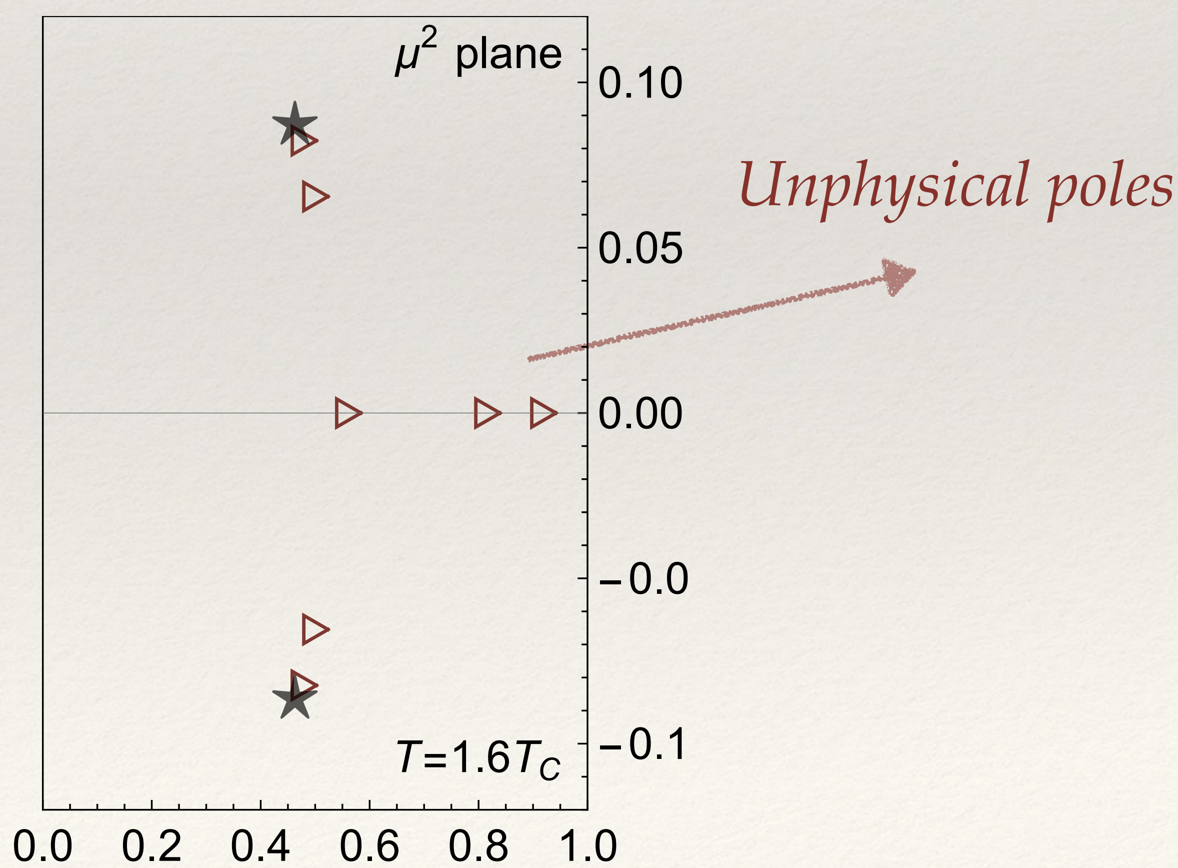
When life gives you Taylor series...

Taylor series: $\chi(\mu^2) = \sum_{n=0}^{N} c_{2n}\mu^{2n}$

Padé approximant (diagonal) $\mathrm{P}_{[N/2,N/2]}f(\mu^2) = \frac{P_{N/2}(\mu^2)}{Q_{N/2}(\mu^2)}$

Singularity of the function ⟷ *poles/zeroes of Padé*

e.g. GN model

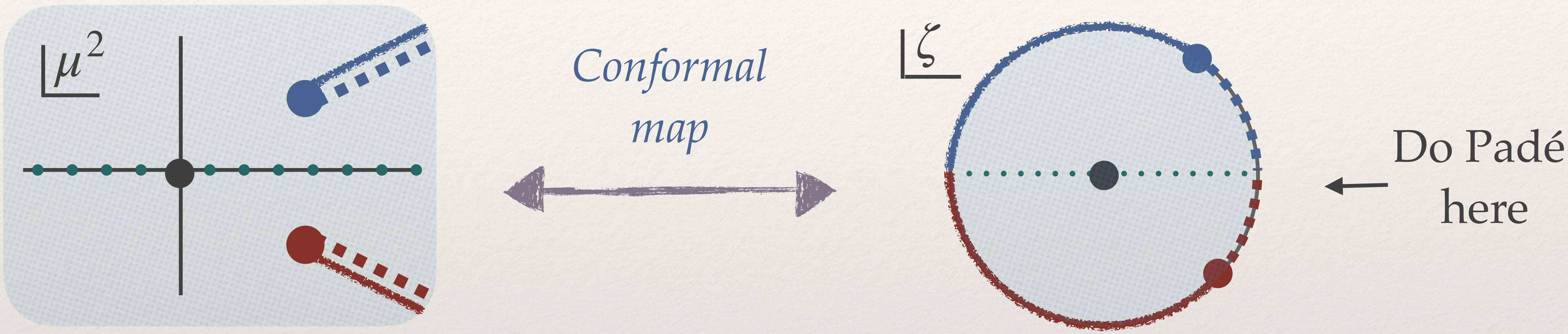


Problem: Padé is fairly good away from the singularity but fails badly near a singularity / branch cut 🤷‍♂️

[Stahl' 97, Costin Dunne '20]

Conformal Maps

Solution: Do Padé after a conformal map



"conformal Padé"

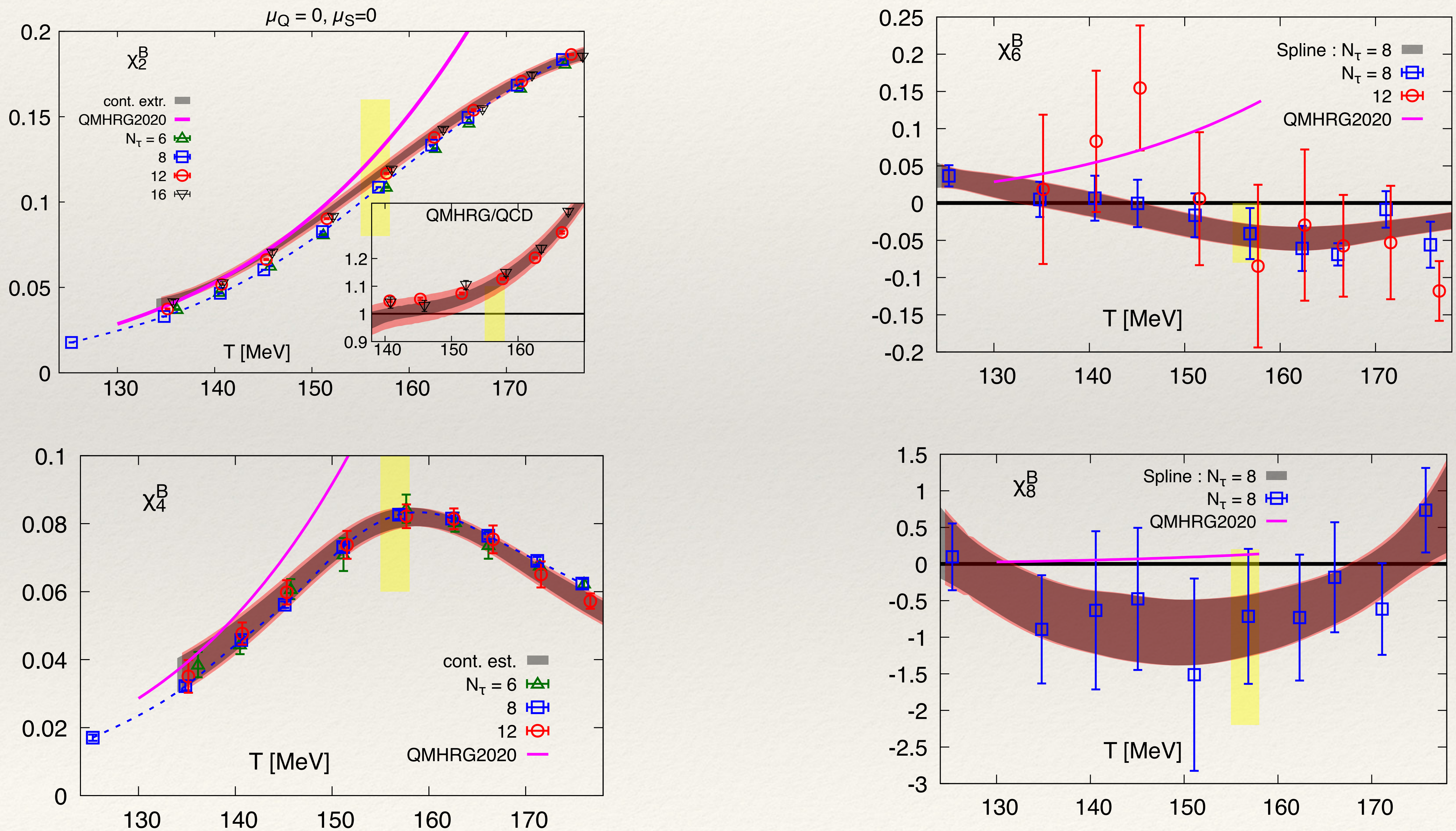
$$\mathrm{P}\chi(T, \phi(\zeta)) = \left.\frac{\tilde{p}_0(T) + \tilde{p}_1(T)\zeta + \ldots + \tilde{p}_{N/2}(T)\zeta^N}{\tilde{q}_0(T) + \tilde{q}_1(T)\zeta + \ldots + \tilde{q}_{N/2}(T)\zeta^N}\right|_{\zeta=\phi^{-1}(\mu^2)}$$

- Captures the singular behavior, no unphysical poles along real axis
- Significantly better approximation than Padé
- Can go beyond the radius of convergence, even to different Riemann sheets!

Taylor coefficients for QCD (HotQCD)

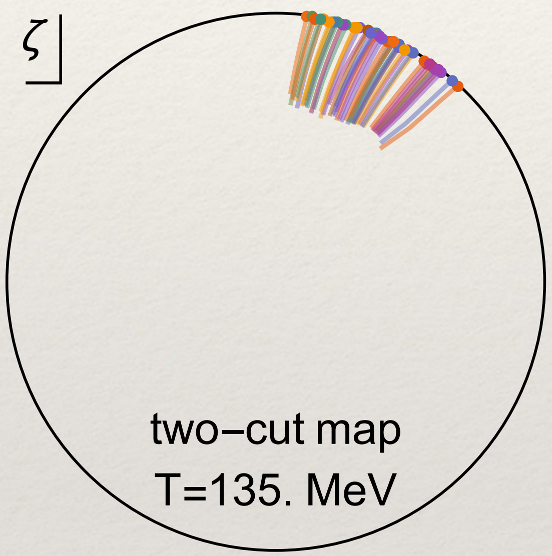
Taylor coefficients from Hot QCD collaboration up to μ_B^8

[Bollweg et al. PRD 105 (2022) 7, 074511]

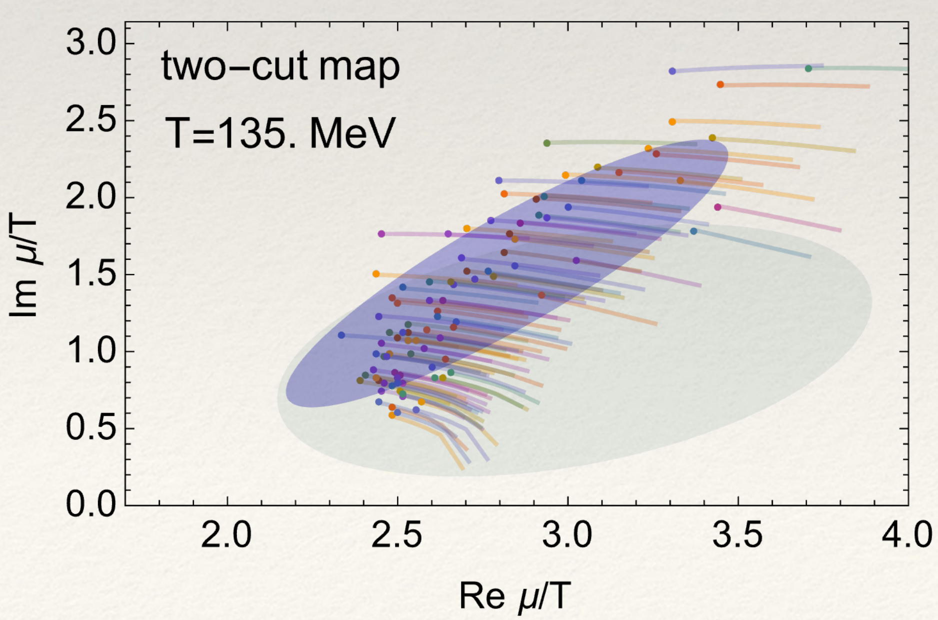


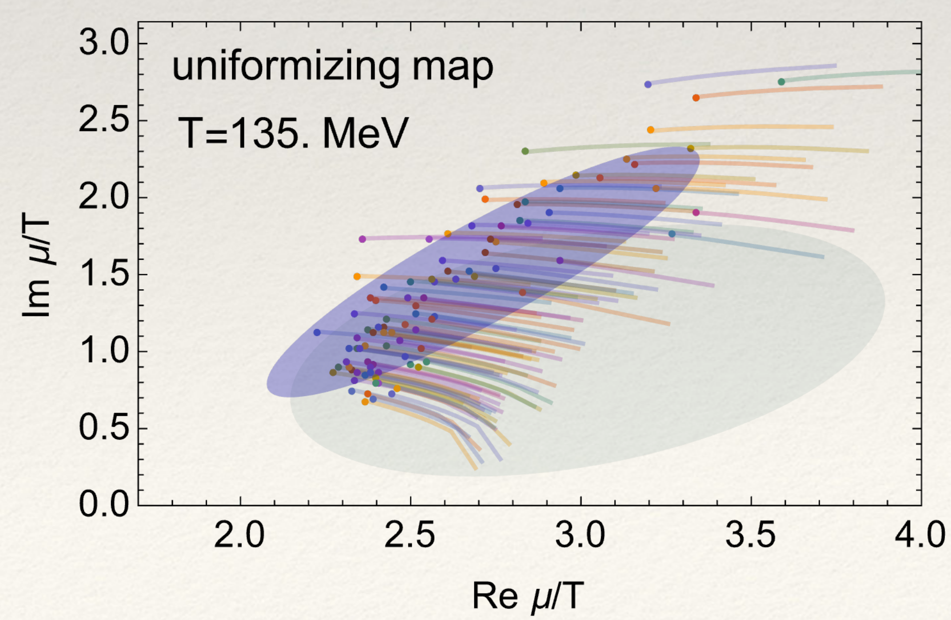
Conformal Pade Algorithm

- Sample Taylor coefficients from a Gaussian ensemble
- Estimate singularity from Pade as an input for conformal map
- Refine the estimate via conformal Pade
- Use the refined value in conformal map
- Repeat



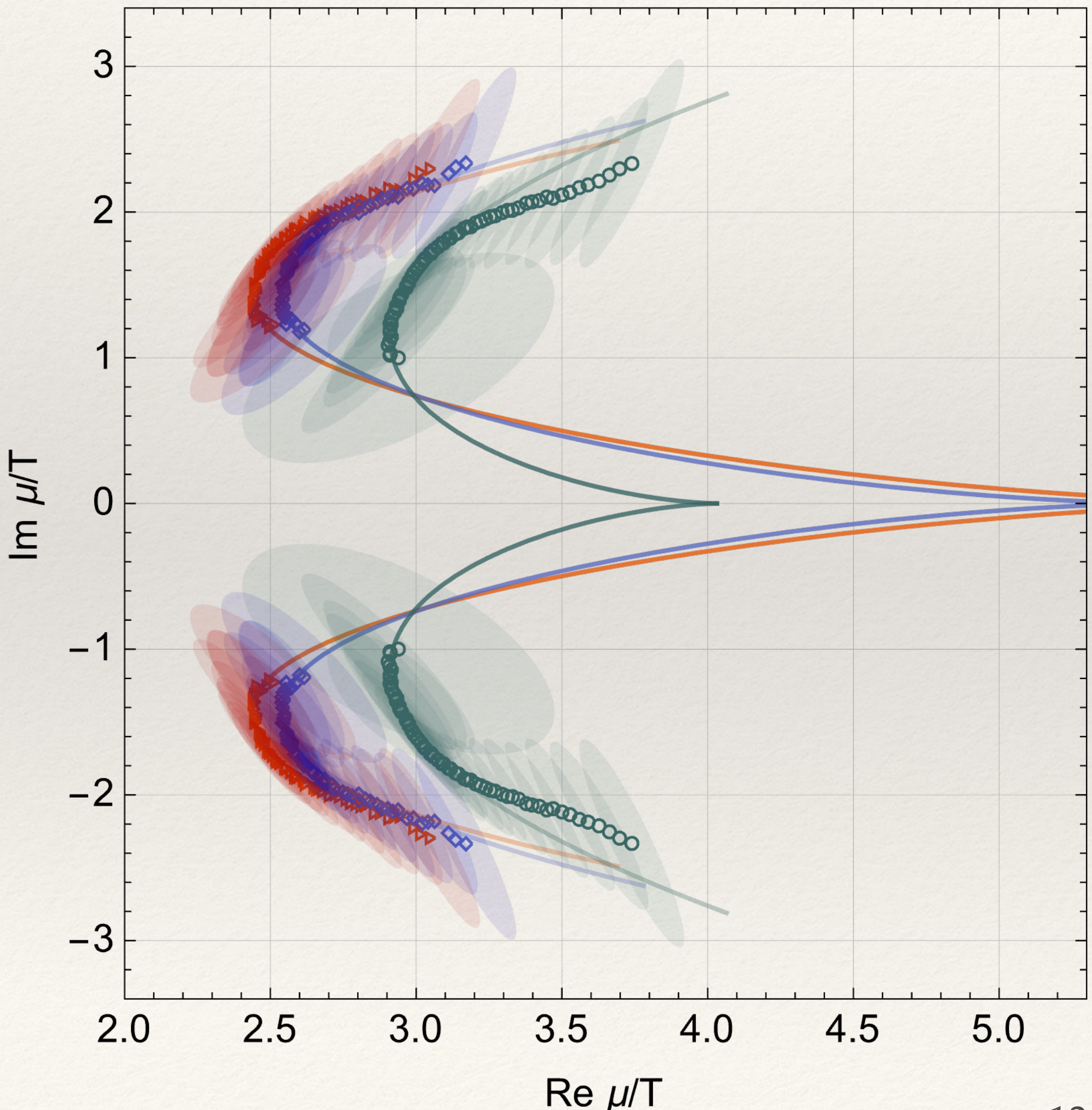
Consistency check:
Estimates of the singularities approach the edge of the unit disk!





Lee Yang Trajectory

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + i w_c \frac{r_\mu^{\beta\delta}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{\beta\delta}(T - T_c)^{\beta\delta}$$



fits:

$$\mathrm{Re}\mu_{LY}(T) = a(T - T_C)^2 + b(T - T_C) + c$$

$$\mathrm{Im}\mu_{LY}(T) = c w_c (T - T_C)^{\beta\delta}$$

$\beta\delta \approx 1.5631$ (3d Ising)

from conformal bootstrap

[Simmons-Duffin, 1502.02033]

$w_c = |z_c|^{-\beta\delta} \approx 0.246$

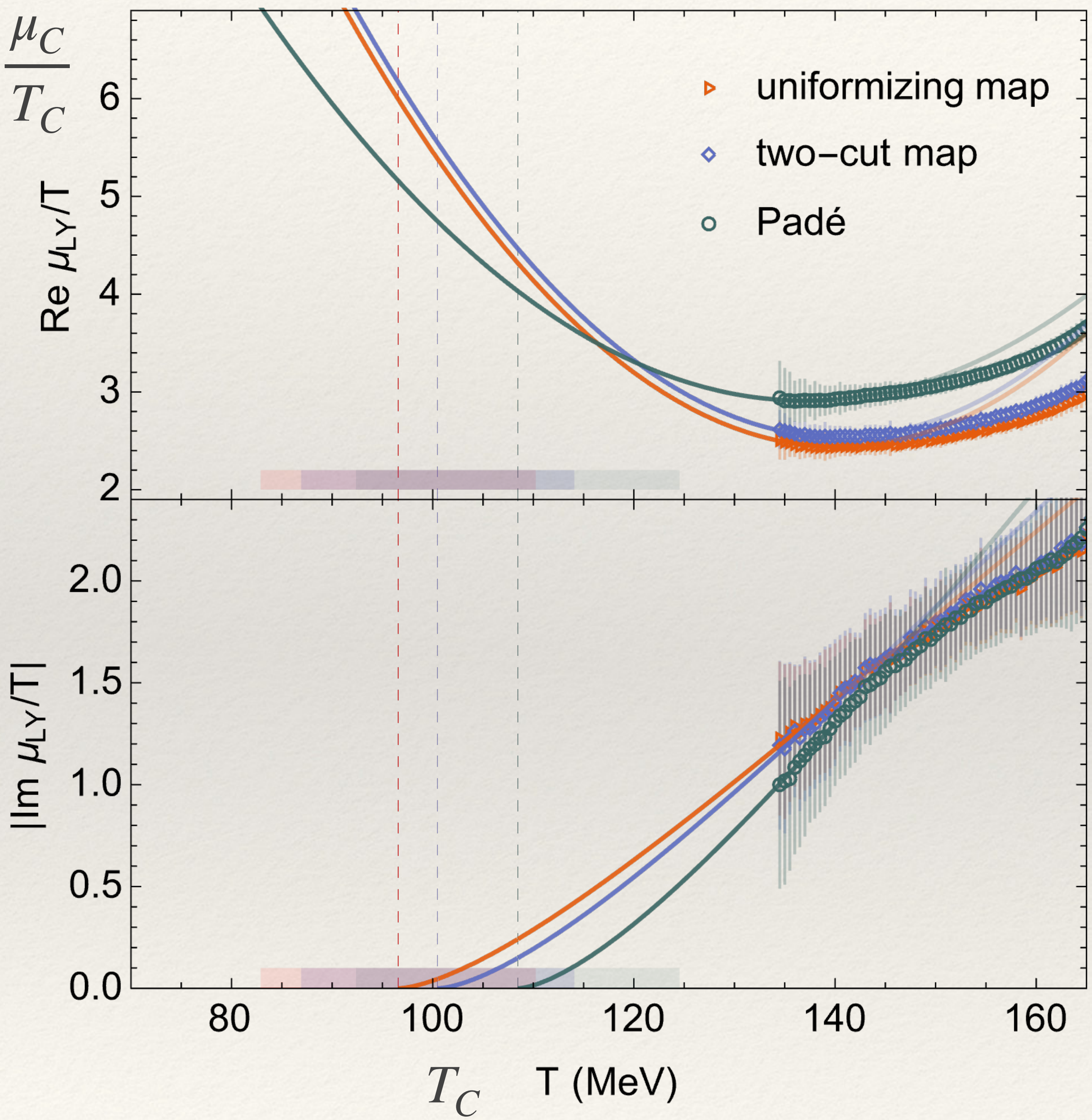
from functional RG

[Connelly et al, 2006.12541]

consistent with the HotQCD results

[Bollweg et al. 2202.09184]

Estimations of QCD critical point



unif. Padé $T_C = 97\,\text{MeV} \quad \mu_C = 579\,\text{MeV}$
$\alpha_1 = 9.40^\circ \quad c = 2.22$

2-cut conf. Padé $T_C = 100\,\text{MeV} \quad \mu_C = 557\,\text{MeV}$
$\alpha_1 = 8.69^\circ \quad c = 2.65$

Padé $T_C = 108\,\text{MeV} \quad \mu_C = 437\,\text{MeV}$
$\alpha_1 = 4.55^\circ \quad c = 3.35$

1 sigma uncertainty:

$T_c :\sim \pm 20\text{MeV}, \quad \mu_c :\sim \pm 200\text{MeV}$

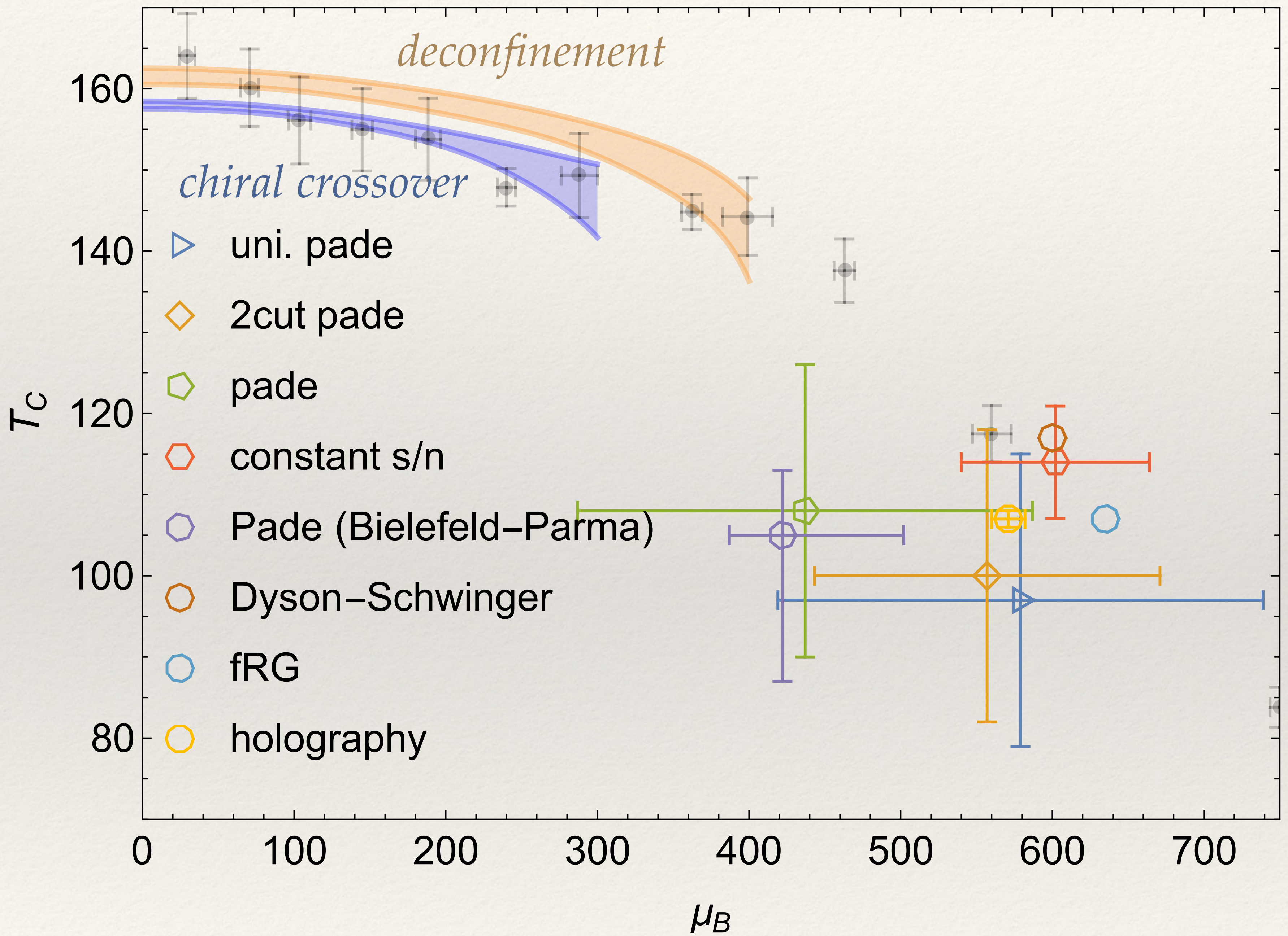
Bielefeld-Parma $T_C \sim 90\,\text{MeV} \quad \mu_C \sim 600\,\text{MeV}$
[Di Renzo, Clarke, Dimopoulos, Goswami, Schmidt '23 Lattice 23]

Functional RG $T_C \sim 107\,\text{MeV} \quad \mu_C \sim 635\,\text{MeV}$
[Fu, Pawlowski, Rennecke '20 PRD 101 054032]

Dyson-Schwinger: $T_C \sim 117\,\text{MeV} \quad \mu_C \sim 600\,\text{MeV}$
[Gunkel, Fischer 21. PRD 104 054022]

Holography: $T_C \sim 104\,\text{MeV} \quad \mu_C \sim 590\,\text{MeV}$
[Hippel et al 2309.00579]

Comparison with other estimates



constant s/n
[Shah et al '24 2410.16206]

Bielefeld-Parma
[Di Renzo et al. Lattice 23]

Dyson-Schwinger
[Gunkel et al '21 PRD 104 054022]

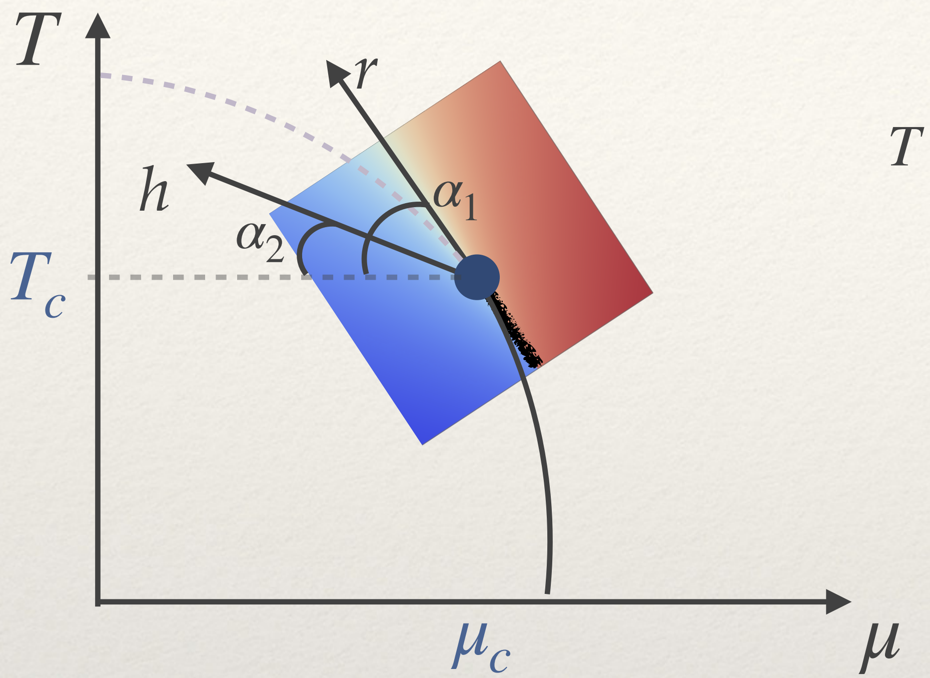
Functional RG
[Fu et al '20 PRD 101 054032]

Holographic model
[Hippel et al '24 PRD 110 094006]

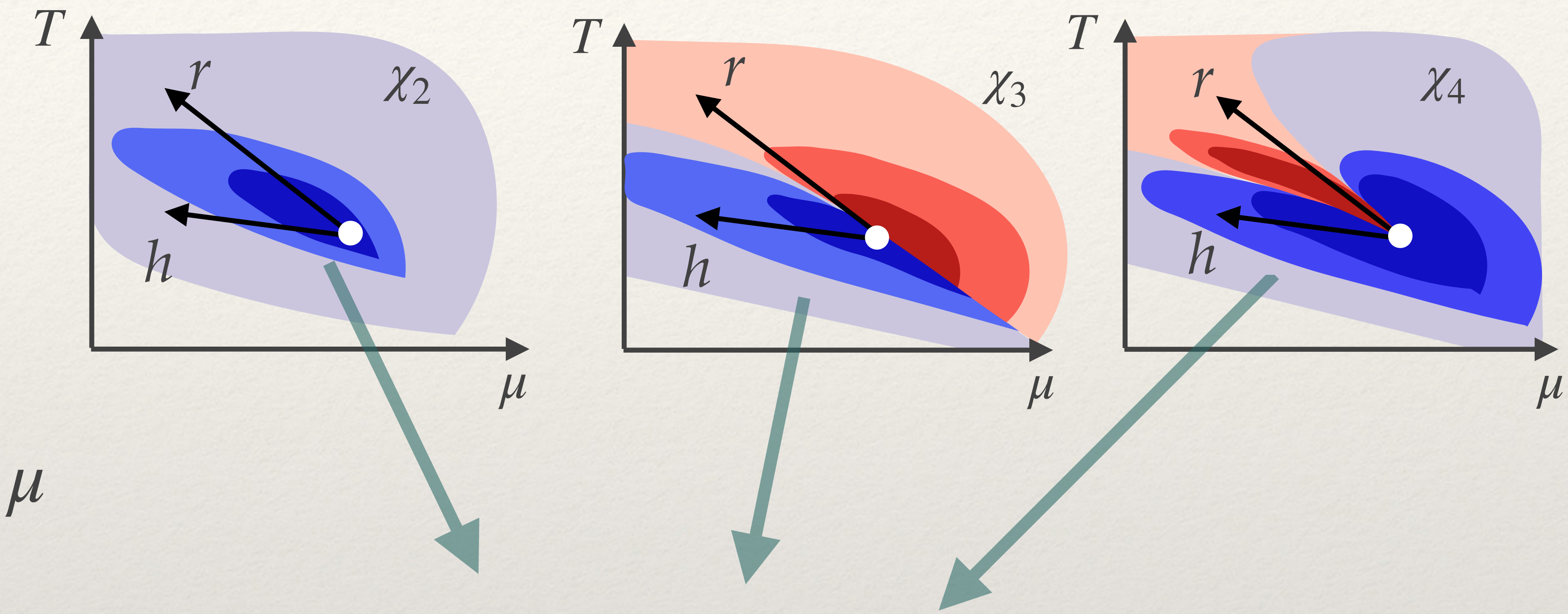
Chiral crossover / deconfinement
[WP data: 2410.06216v1]

Pale black points:
freezout estimates (various)

Constraints on the equation of state



susceptibilities:



shape & size: depend on $\rho, w, \alpha_1, \alpha_2$

$$\mathbb{M} = \frac{1}{w\rho T_c \sin(\alpha_1 - \alpha_2)} \begin{pmatrix} -\rho\cos\alpha_1 & -\rho\sin\alpha_1 \\ \cos\alpha_2 & \sin\alpha_2 \end{pmatrix}$$

[Pradeep, Stephanov, '19]

Note: in this talk everything is in thermal equilibrium, there are no dynamical effects. For *dynamics of fluctuations* see recent review an references therein [GB, 2410.02866]

From Ising to protons

The experiments measure proton number cumulants.

Have to relate thermal fluctuations in QGP to proton number fluctuations
Maximum Entropy Freeze-out [see talk by Stephanov]

Ingredients:

Ising EoS, mapping parameters, Max. Entropy freeze-out, HRG

$T_c, \mu_c, \rho, w, \alpha_1, \alpha_2$

from Padé

No dynamical effects, regular part of EoS (yet…)

work in progress with Pradeep and Stephanov

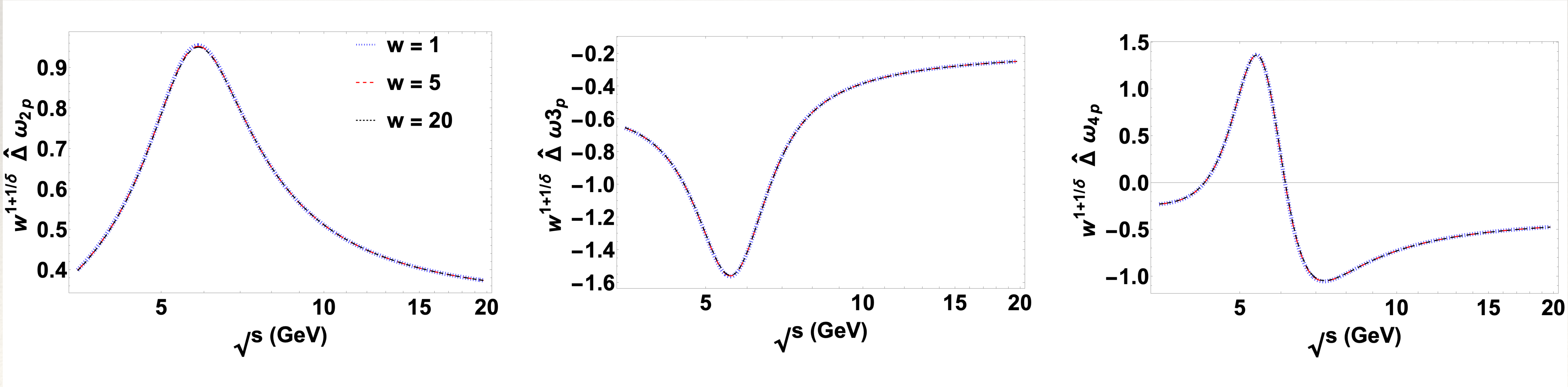
Scaling of the factorial cumulants

$$\kappa_n(\bar{\rho}, w, \ldots) = \left(\frac{w'}{w}\right)^{1+\frac{1}{\delta}} \kappa_n(\bar{\rho}, w', \ldots)$$

$\bar{\rho} = \rho w^{1-\frac{1}{\beta\delta}}$ determines the shape of the cumulant

w determines the overall scale

Example:



Yang-Lee Trajectory and Factorial Cumulants

$$\mu_{LY}(T) \approx \mu_c - c_1(T - T_c) \pm i x_{LY} c_2 (T - T_c)^{\beta\delta}$$

$\mathrm{Re}\mu_{LY}$

Crossover slope

$$c_1 = \tan \alpha_1$$

$\mathrm{Im}\mu_{LY}$

$$\bar{\rho} = \left(\frac{1}{c_2 T_c^{\beta\delta - 1}} \frac{|\sin(\alpha_1 - \alpha_2)|}{|\sin \alpha_1|^{\beta\delta + 1}} \right)^{1/\beta\delta}$$

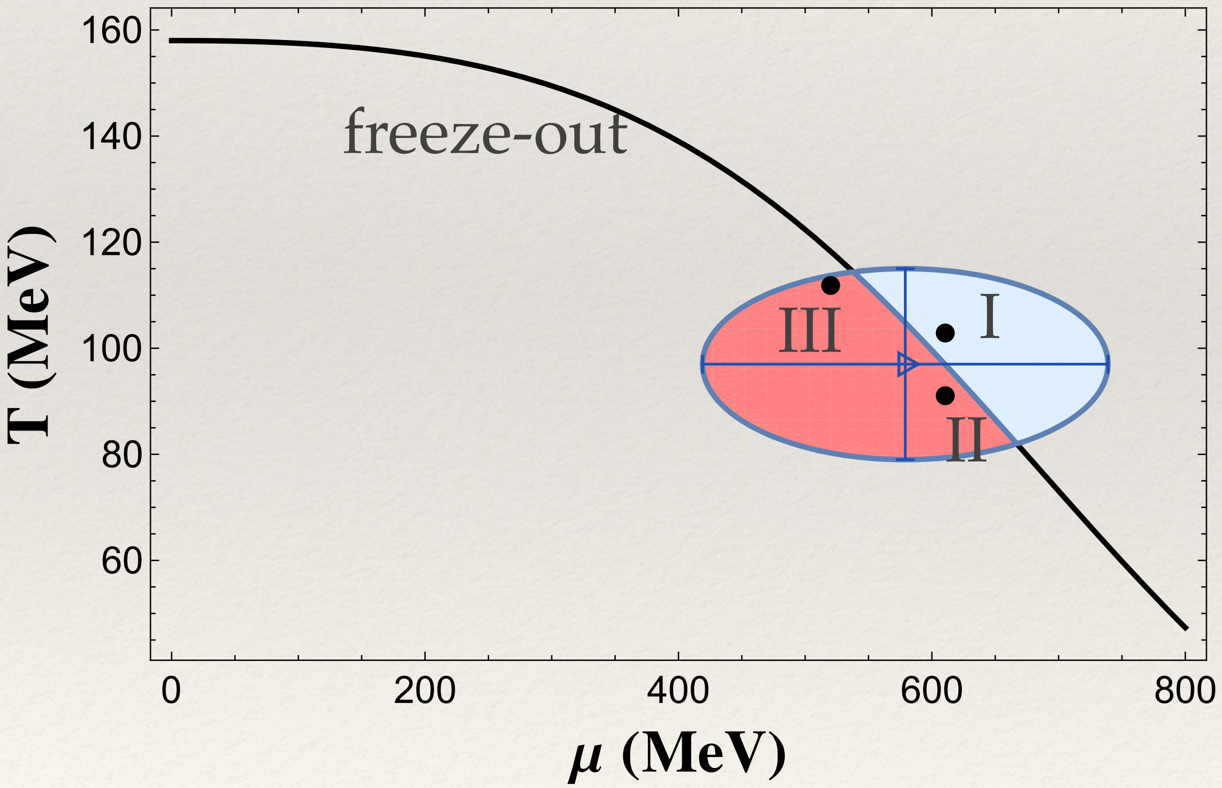
For fixed α_1 and α_2 the imaginary part of the YL trajectory fixes the shape of the factorial cumulants!

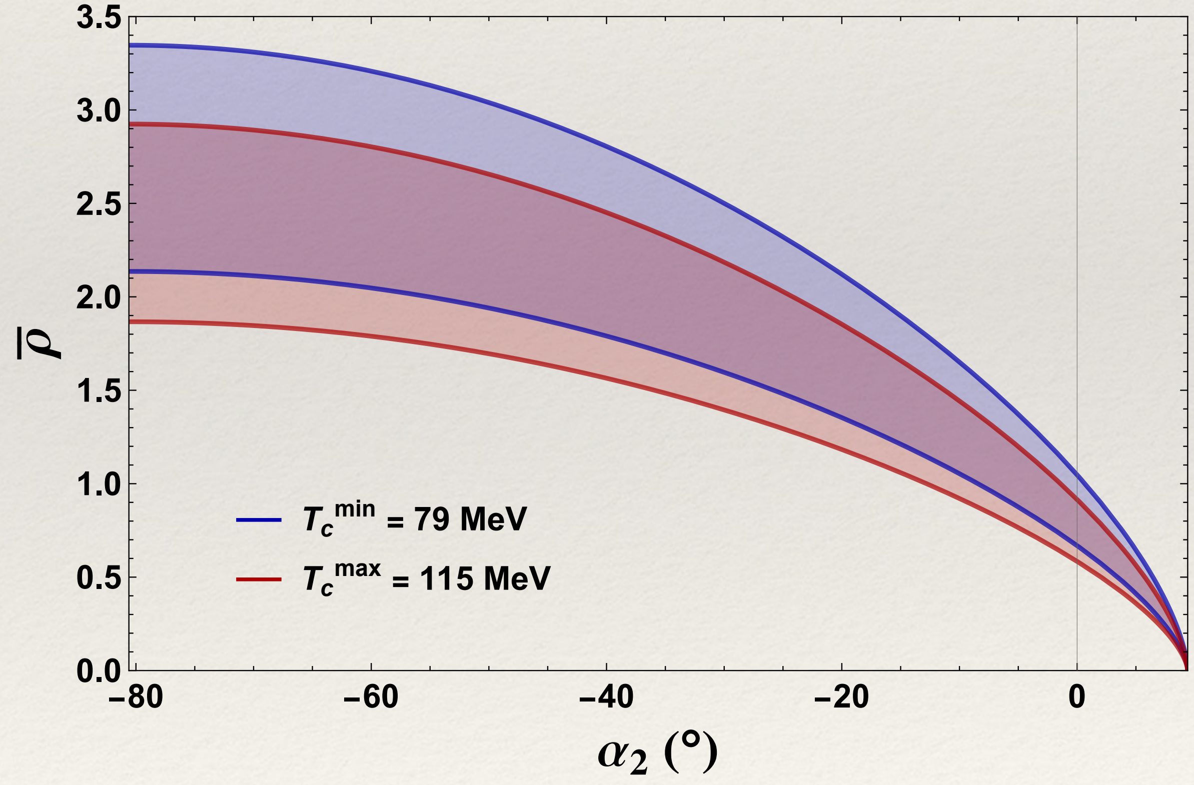
Some examples

We consider three values for the critical point for illustration

Use the values for $\bar{\rho}$ and α_1 from Padè

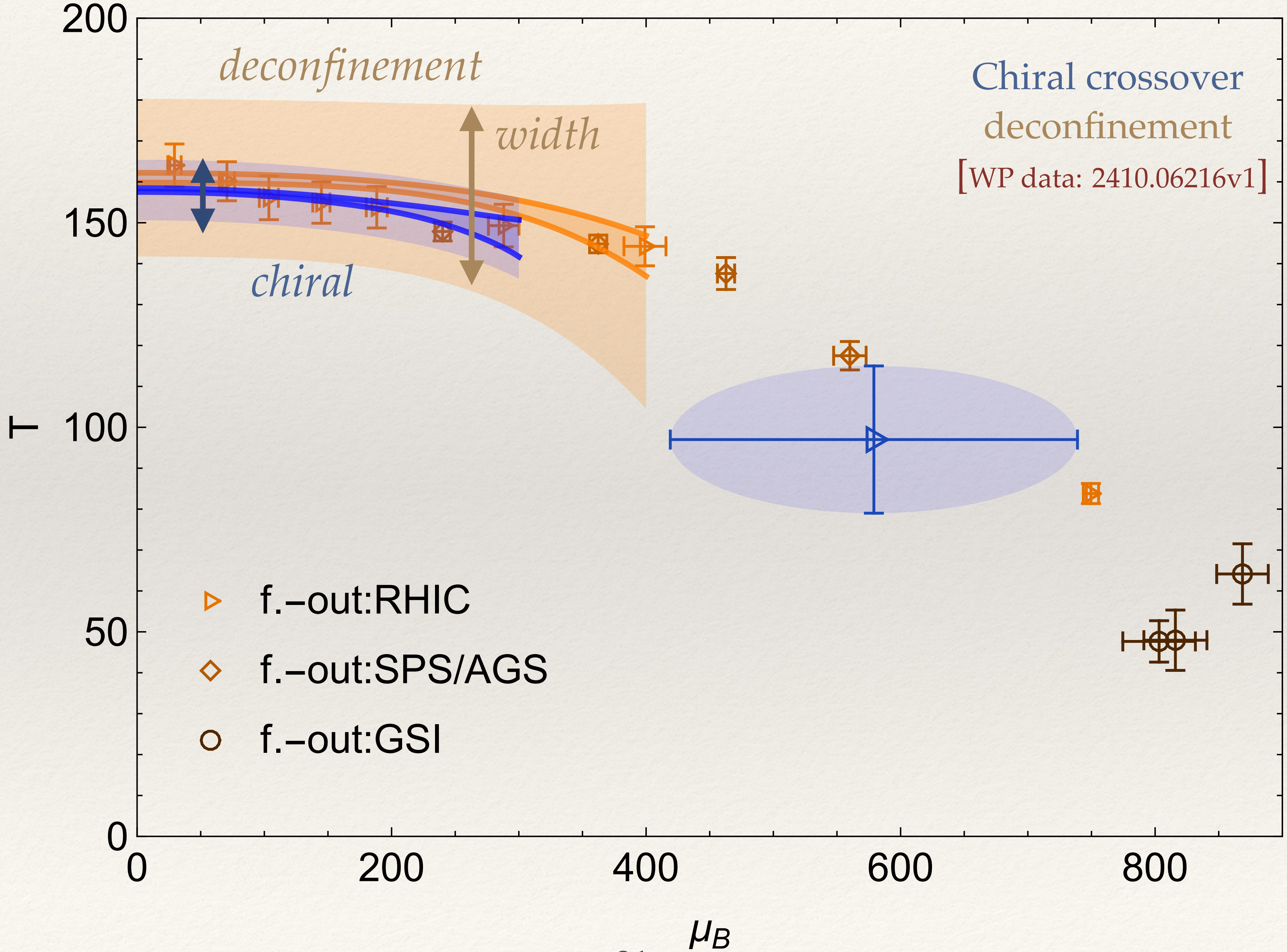
Take a range of α_2 s

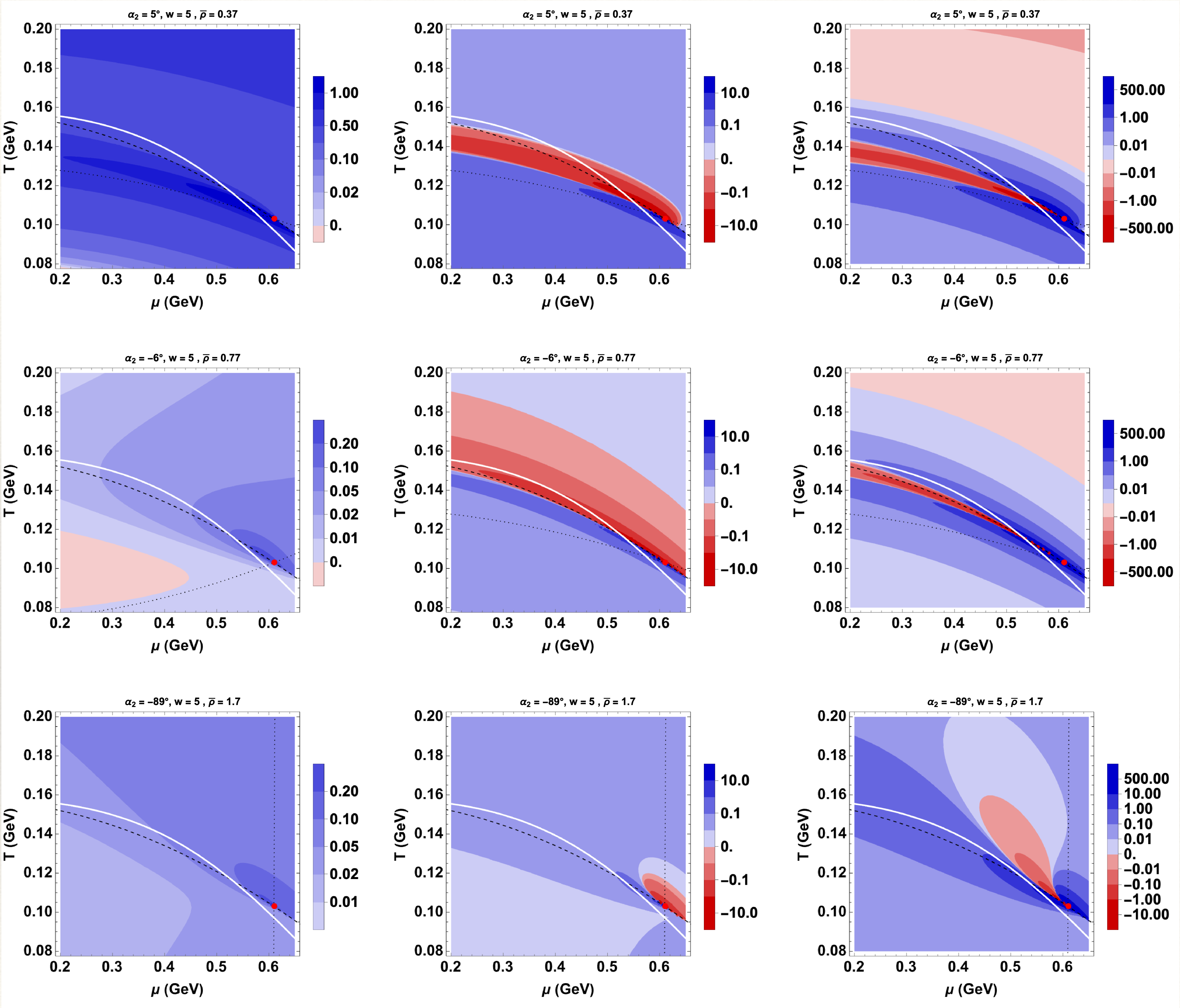




Some examples

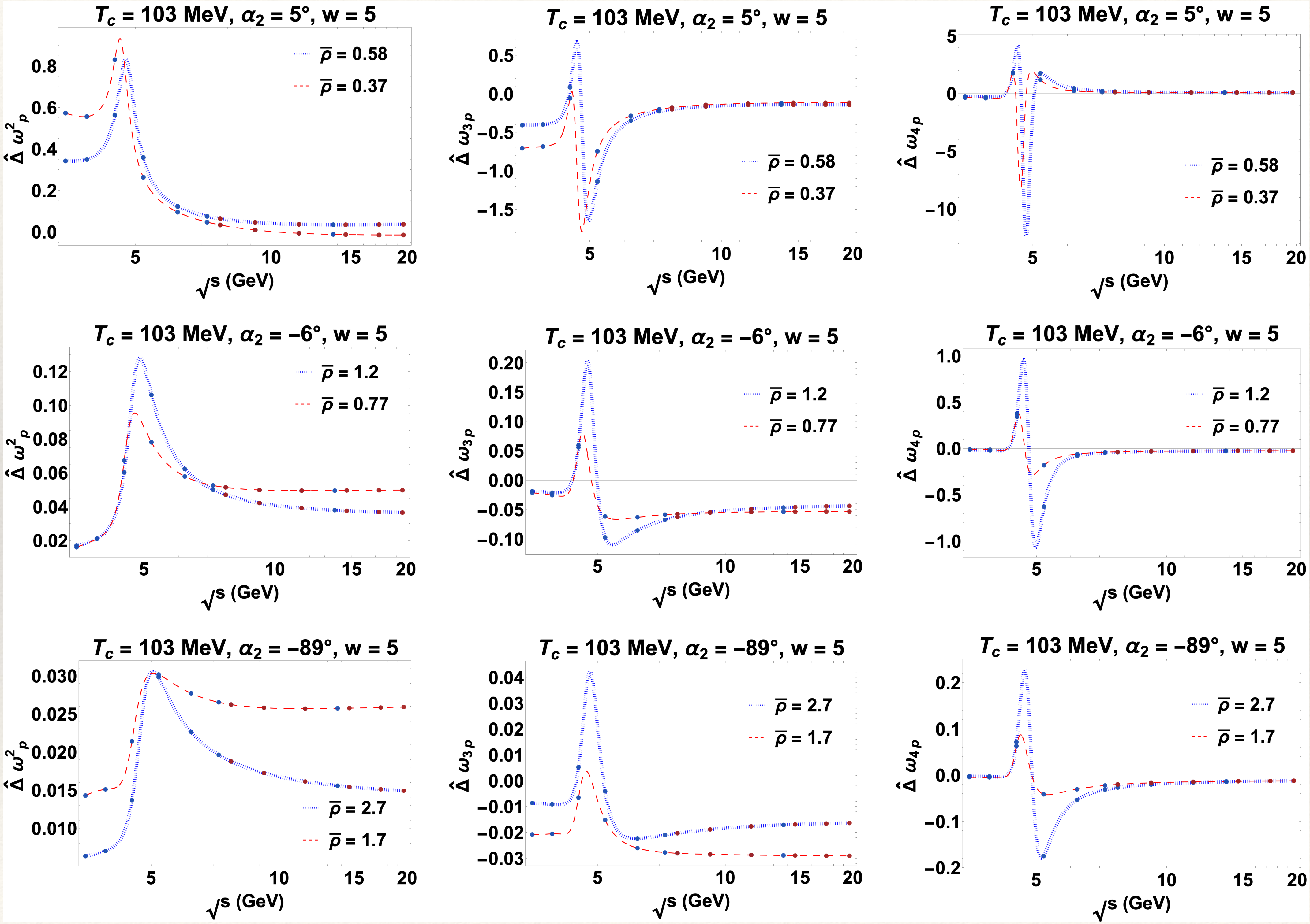
Deconfinement transition is much wider than the chiral crossover



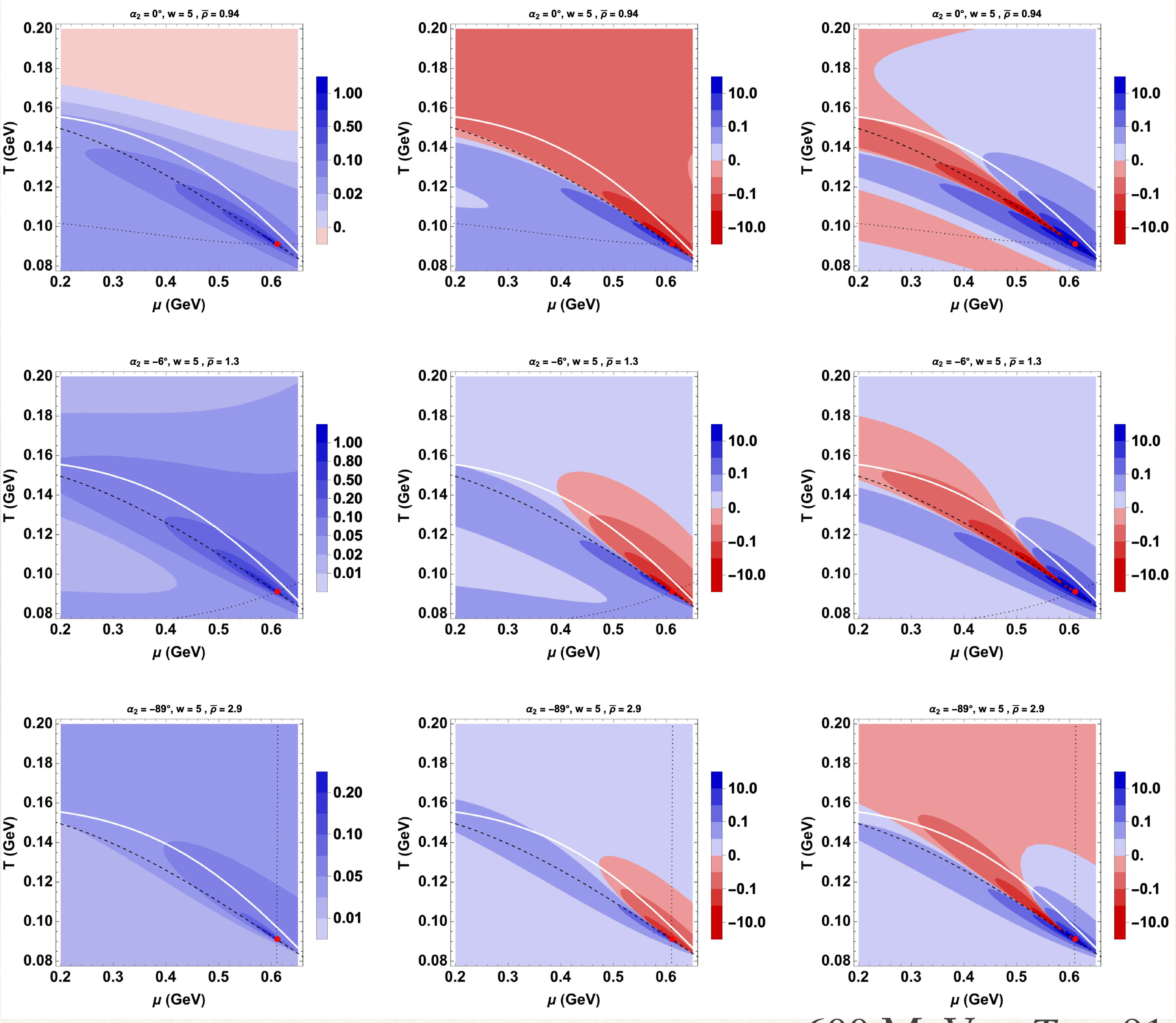


I: $\mu_c = 610$ MeV, $T_c = 103$ MeV

Factorial Cumulants

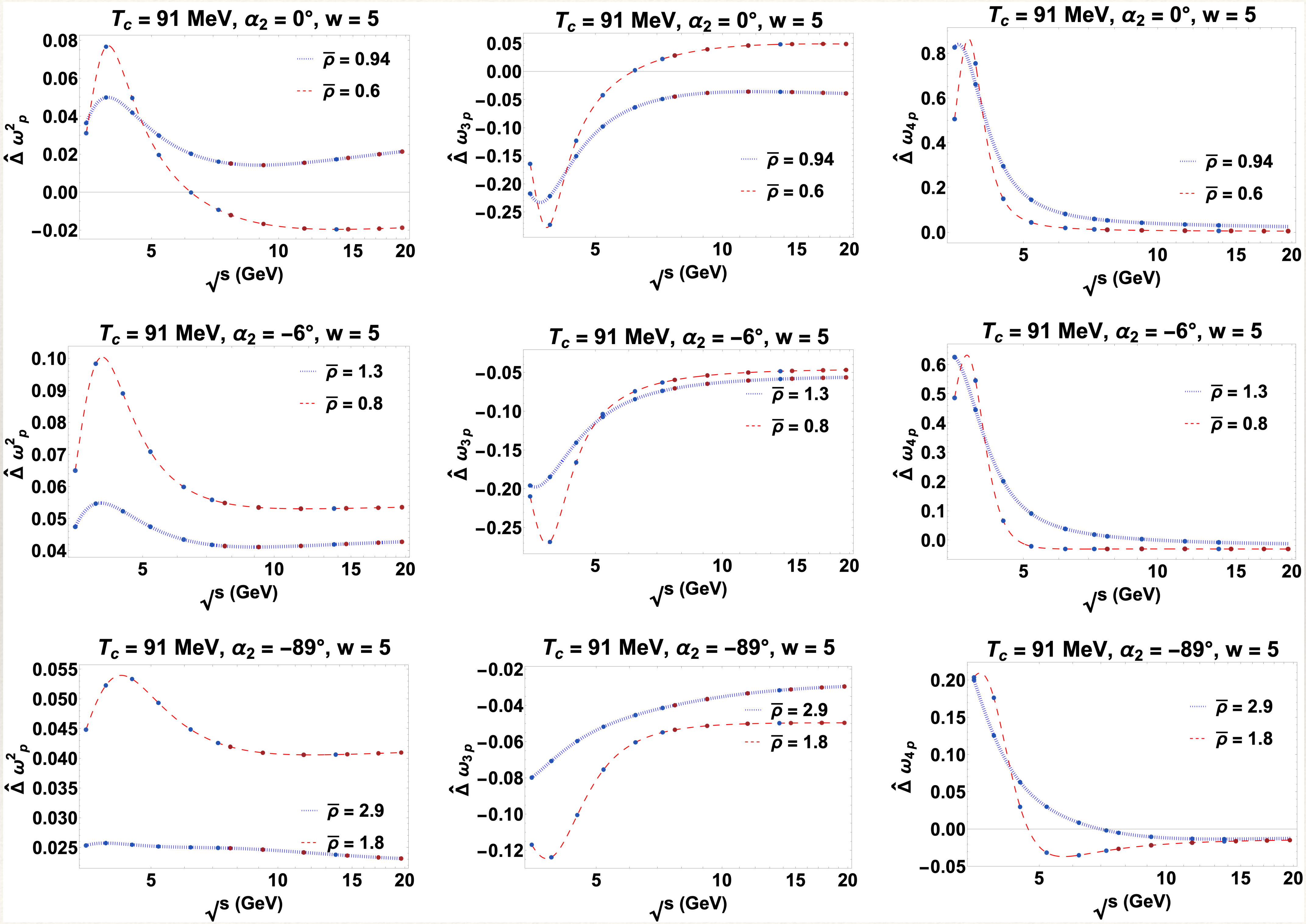


I: $\mu_c = 610$ MeV, $T_c = 103$ MeV

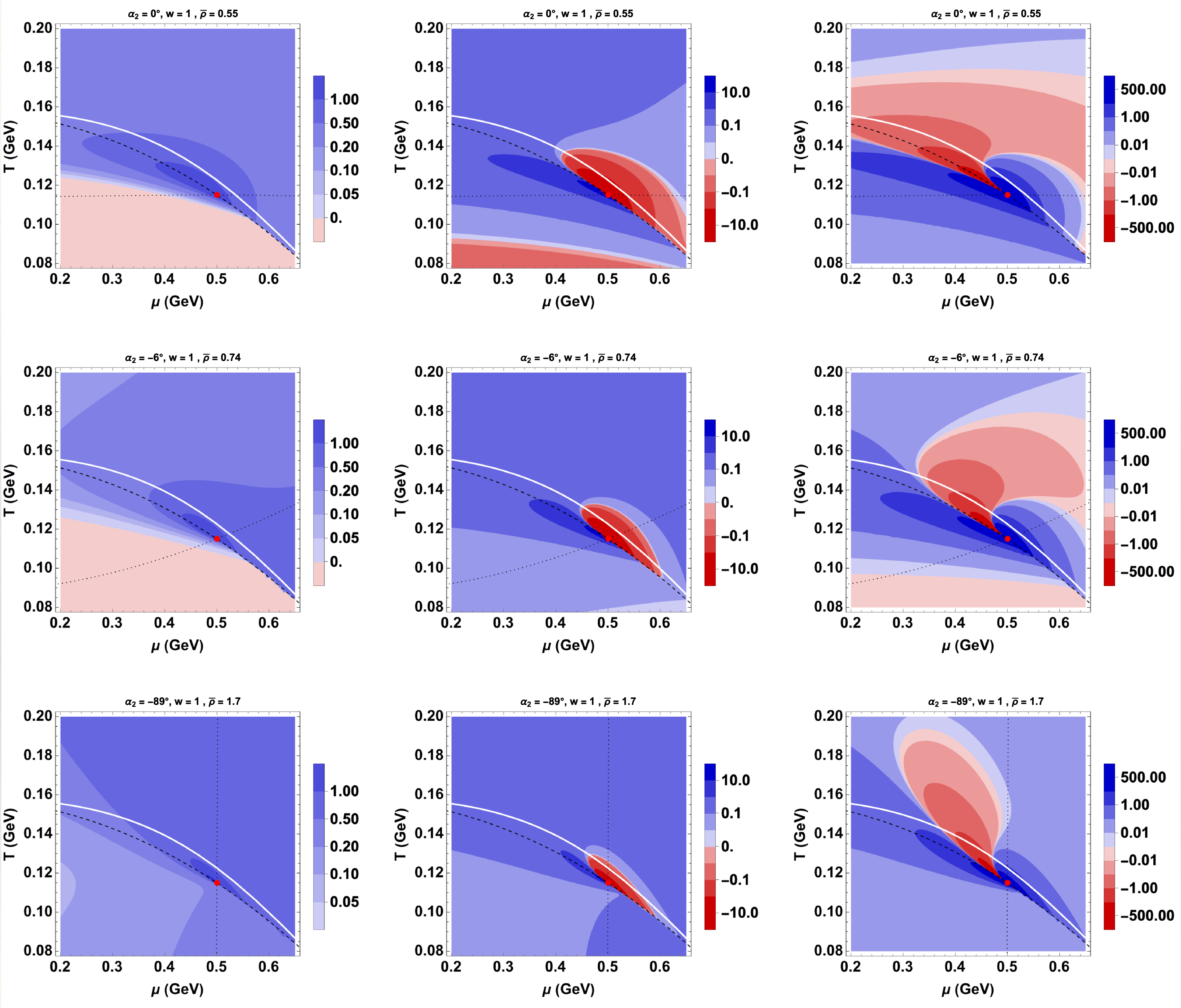


II: $\mu_c = 600$ MeV, $T_c = 91$ MeV

Factorial Cumulants

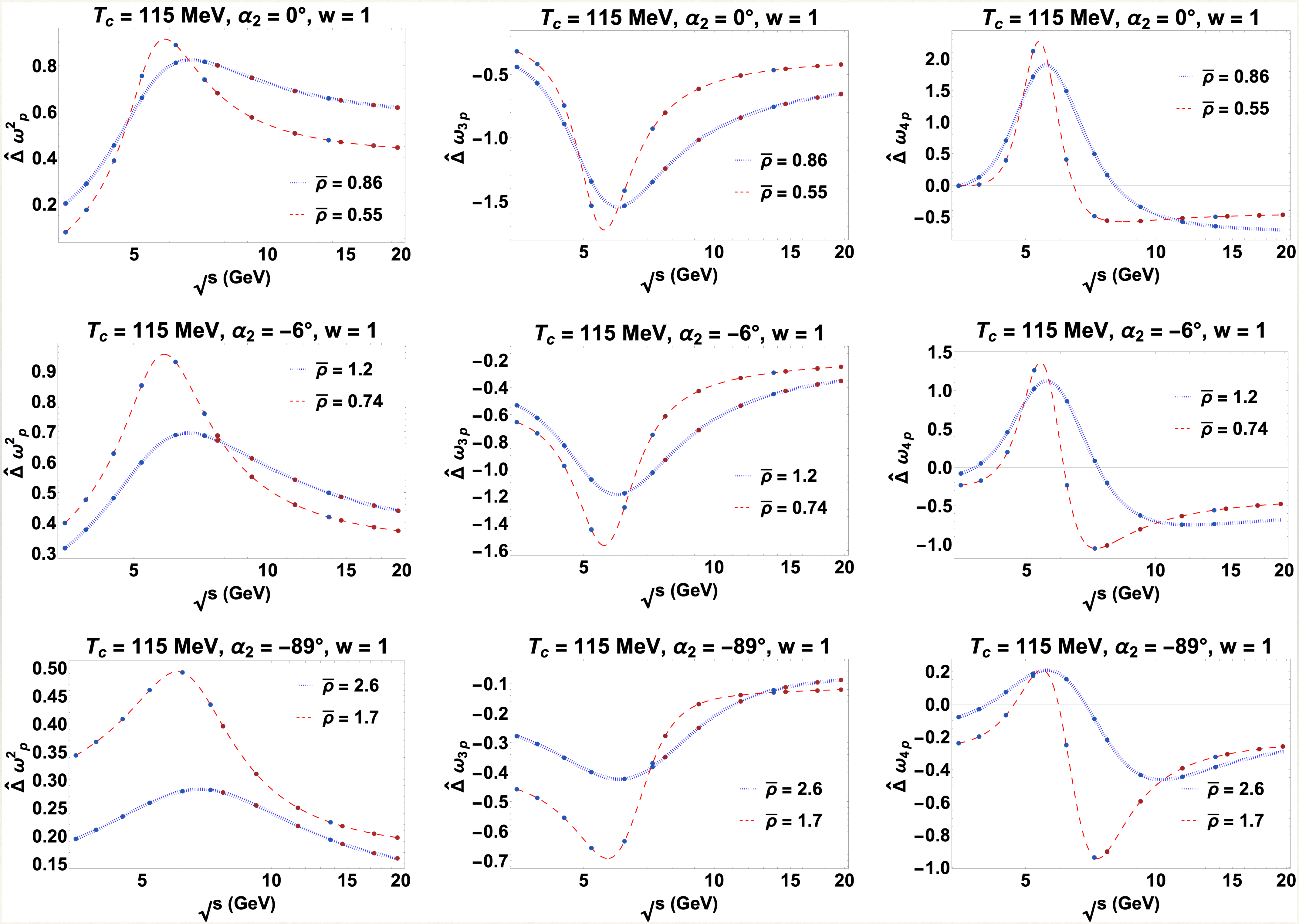


II: $\mu_c = 600$ MeV, $T_c = 91$ MeV



III: $\mu_c = 500$ MeV, $T_c = 115$ MeV

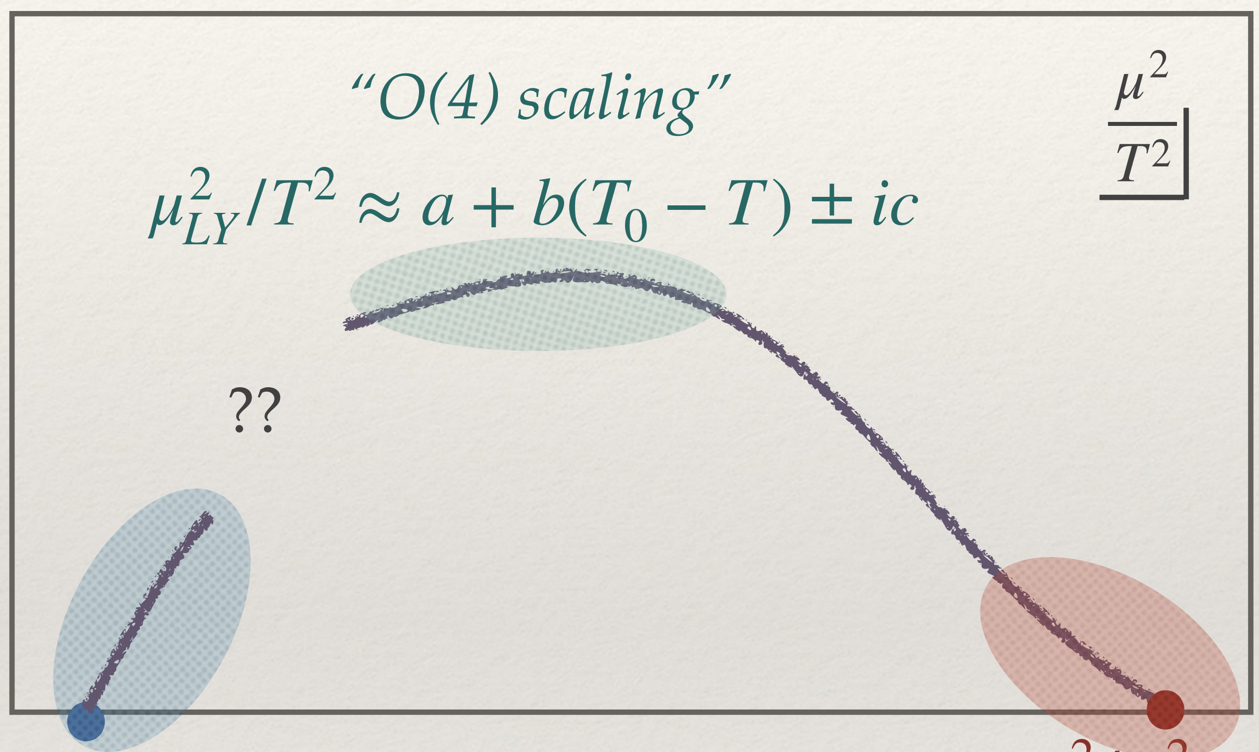
Factorial Cumulants



III: $\mu_c = 500$ MeV, $T_c = 115$ MeV

A parade of singularities

A naive cartoon for Lee-Yang trajectory for QCD …



“Z_2 scaling”

μ_c^2/T_c^2

$-\pi^2$

$\mu_{LY}/T \approx \pm a(T_{RW} - T)^{\beta\delta} \pm i\pi$

$\mu_{LY}/T \approx \mu_c - a(T - T_c) \pm ic(T - T_c)^{\beta\delta}$

“Roberge-Weiss scaling”
(also Z_2)

Conclusions and comments

- From my perspective, with a modest four Taylor coefficients conformal Padé is pretty much the best one can do to extract the LY singularity

- The extrapolation depends on the fit for $\mathrm{R}e\mu_{LY}$ hence the estimate for μ_c has a large uncertainty. Moving forward, lower T data would significantly improve the situation (very difficult, new ideas?)

- $\mathrm{I}m\mu_{LY}$ fixes the shape of the proton number cumulants via $\bar{\rho}$

- The cumulants are sensitive to α_2

EXTRAS

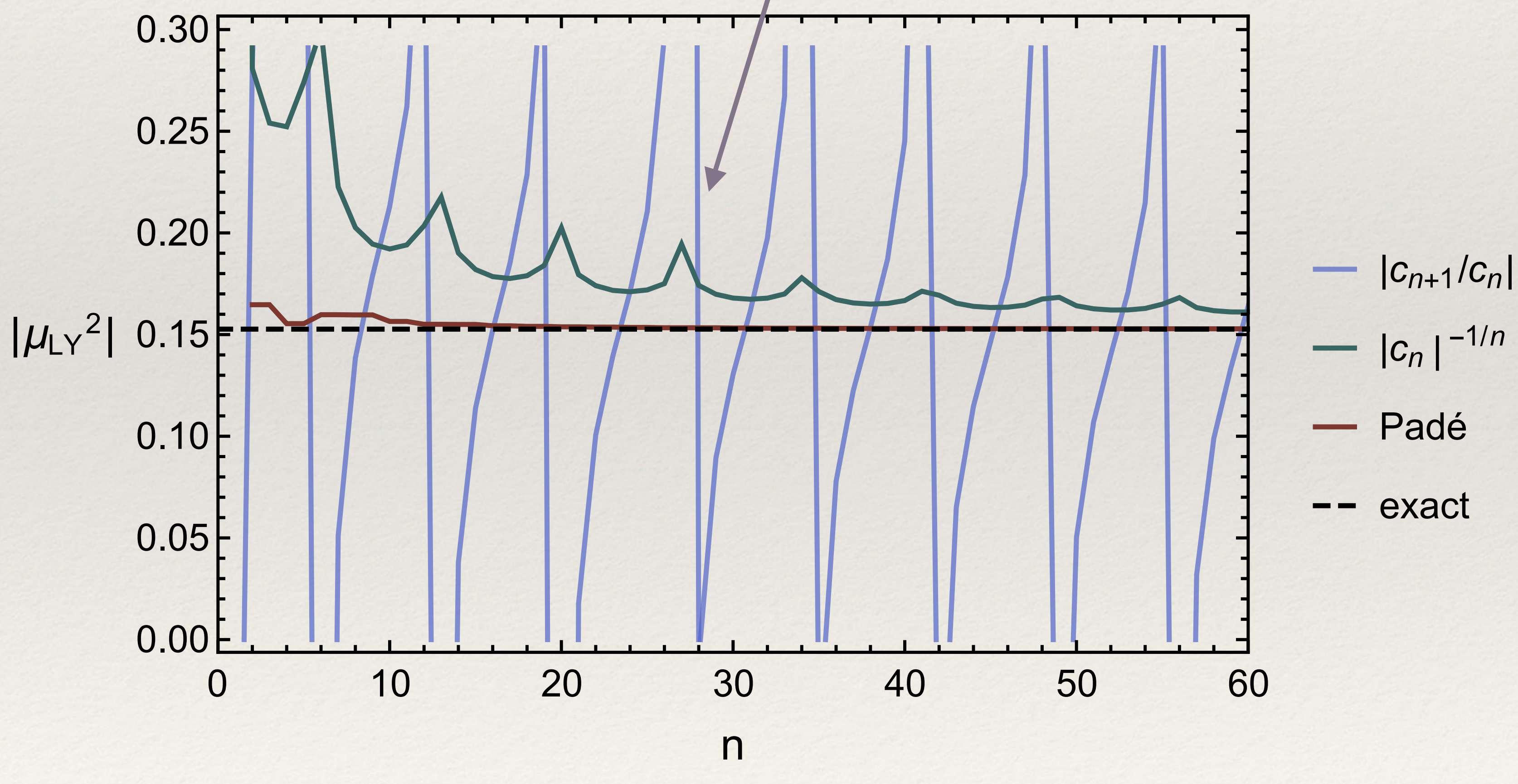
When life gives you Taylor series…

Darboux theorem: Nearest singularity ⟷ Large order growth

$$c_{2n} \sim \frac{\Gamma(\sigma+n)}{\Gamma(1+n)\,|\mu_{LY}^2|^{n+\sigma_{LY}}}\cos(\theta(n+\sigma_{LY})-\pi\sigma), \quad (\theta := \arg \mu_{LY}^2)$$

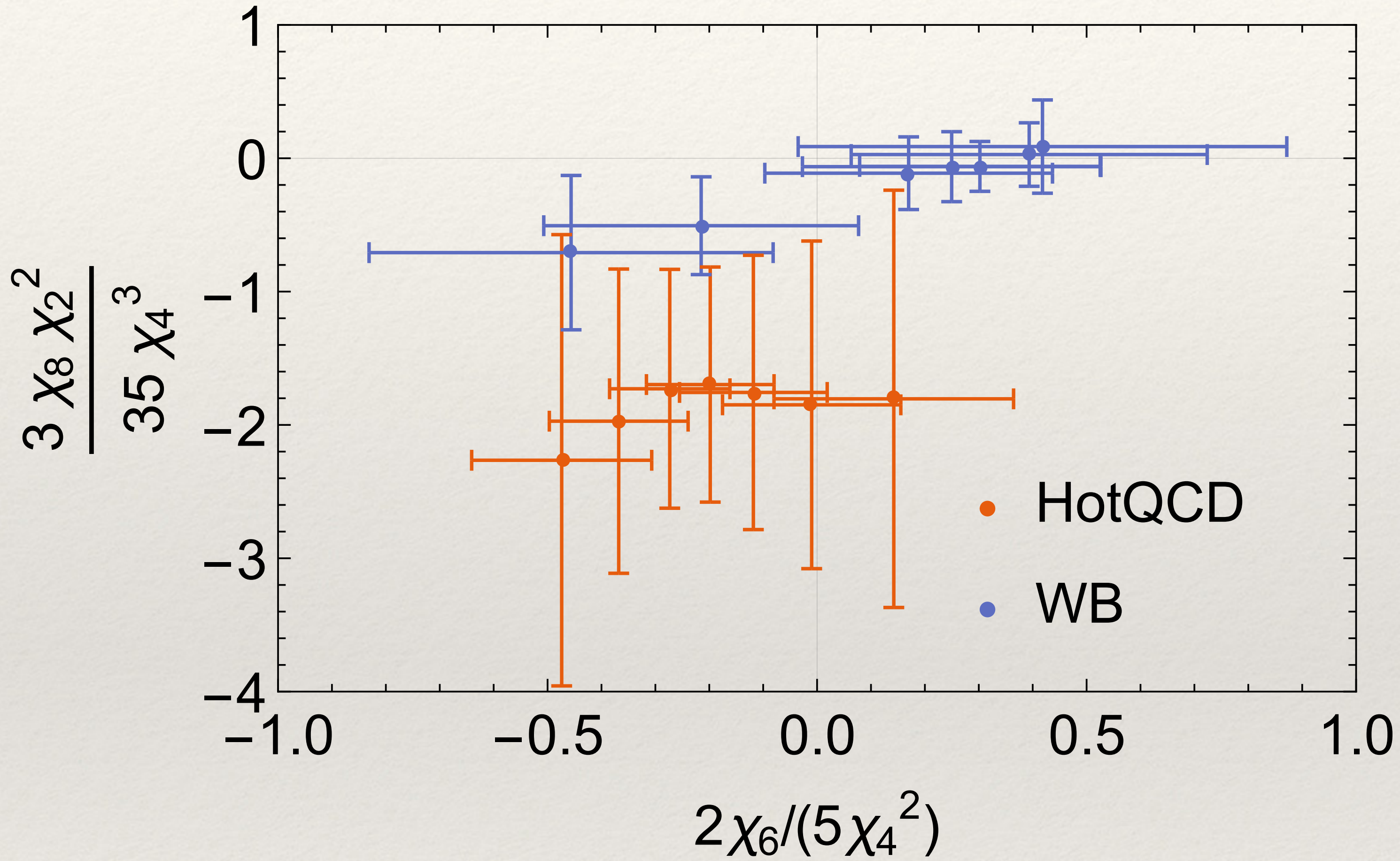
interference…

example: Chiral Random Matrix Model
[Halasz et al, 98]



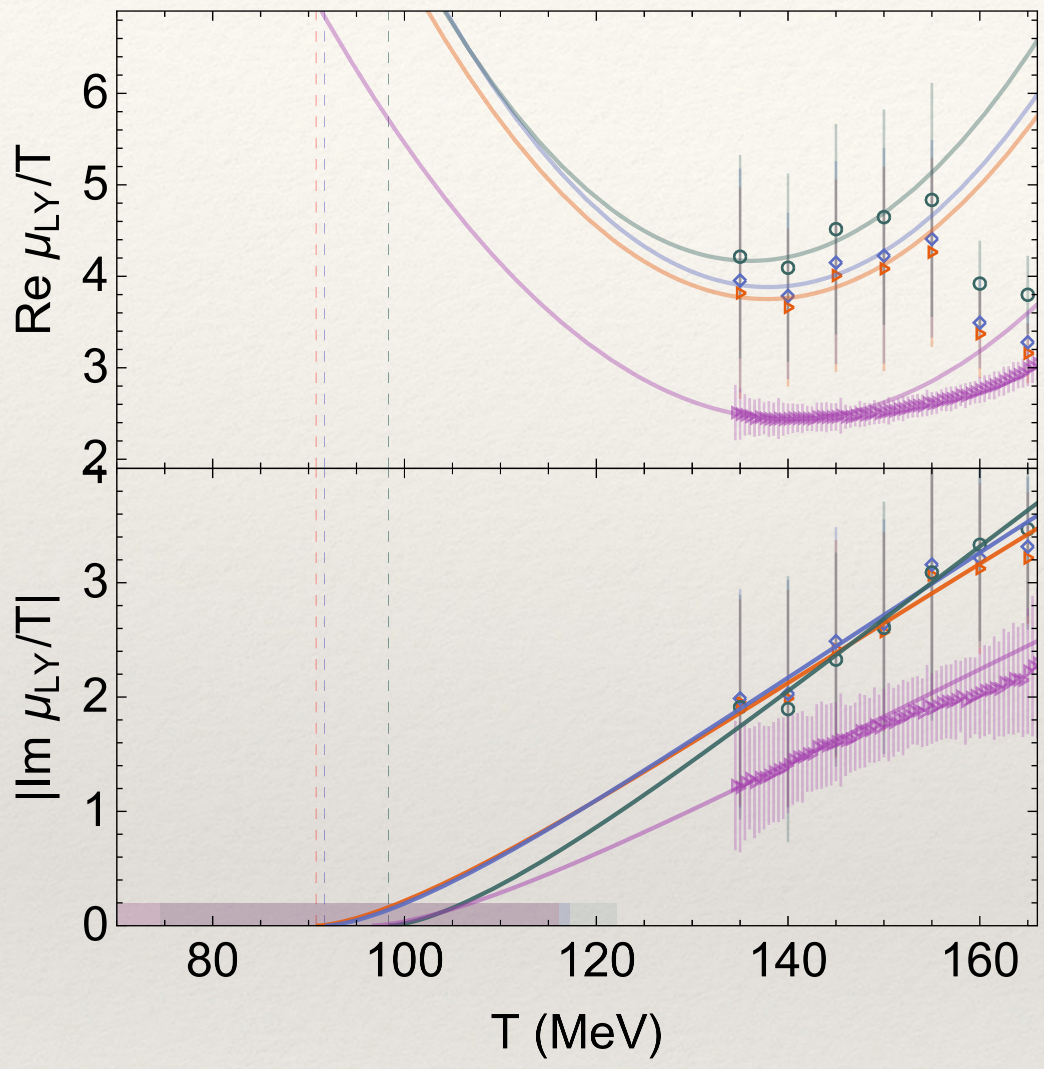
Padé approximants give a much better estimate of the singularity!

Wuppertal-Budapest data



[Wuppertal Budapest: JHEP 10 (2018) 205]

Wuppertal-Budapest data



unif. Padé $T_C = 91 \pm 25\,\mathrm{MeV}$

2-cut conf. Padé $T_C = 92 \pm 25\,\mathrm{MeV}$

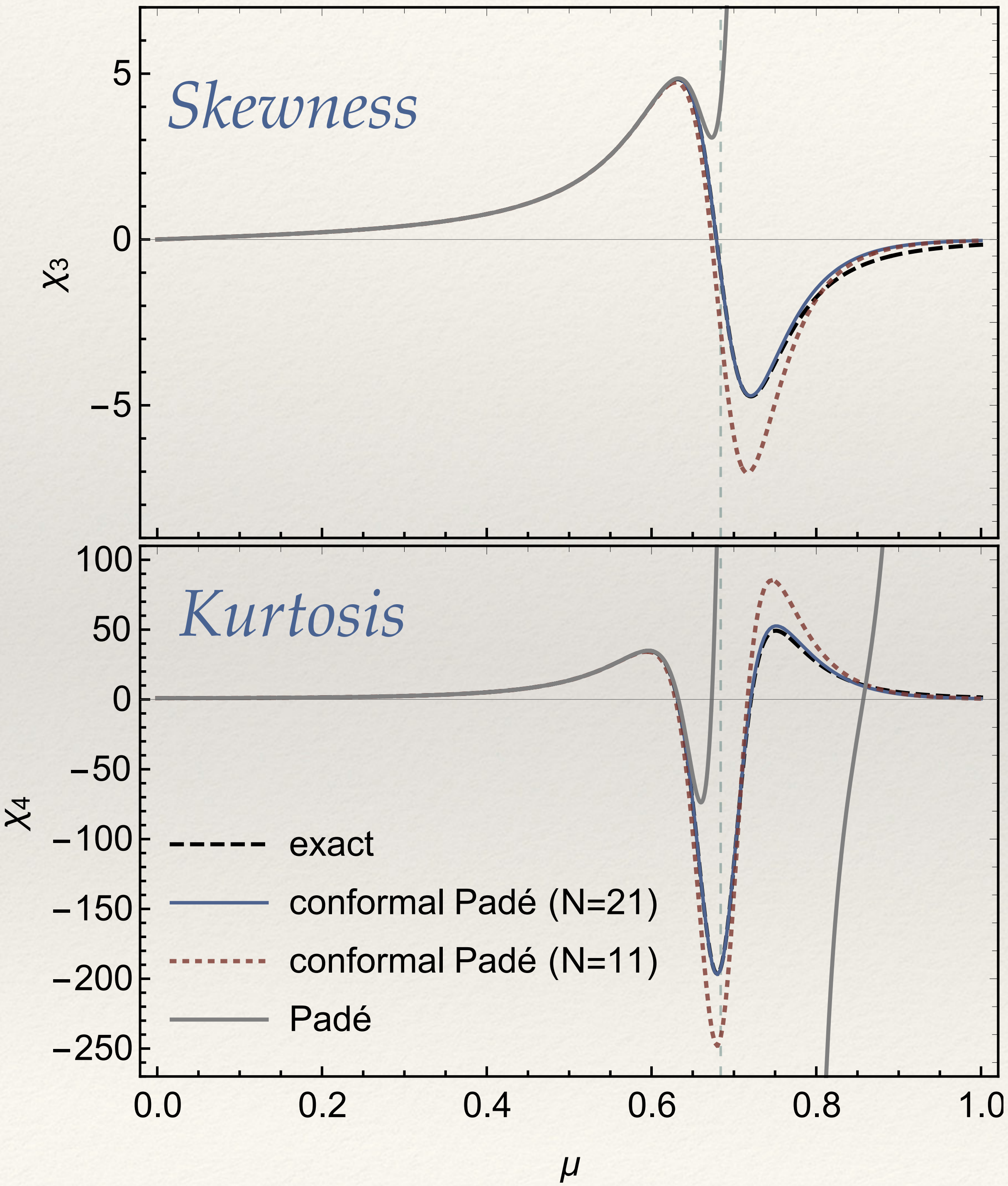
Padé $T_C = 98 \pm 25\,\mathrm{MeV}$

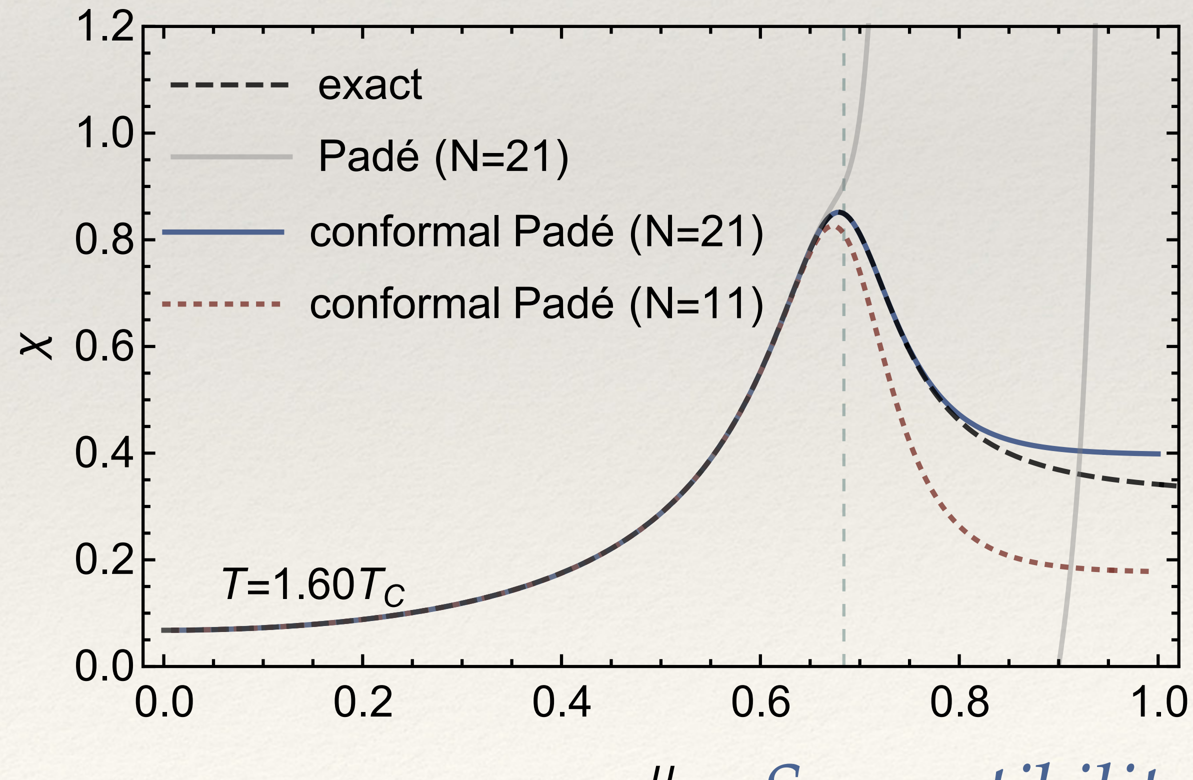
- Higher cumulants differ from HotQCD (systematics?)
- Larger error compared to HotQCD presumably due to uncertainties in χ_8
- Sizable uncertainties, cannot meaningfully extract μ_c
- Low T data is essential

Conformal Maps

Test case: GN model

- conformal Padé does not introduce unphysical poles on the real axis!
- captures the e.o.s. *beyond the radius of convergence*

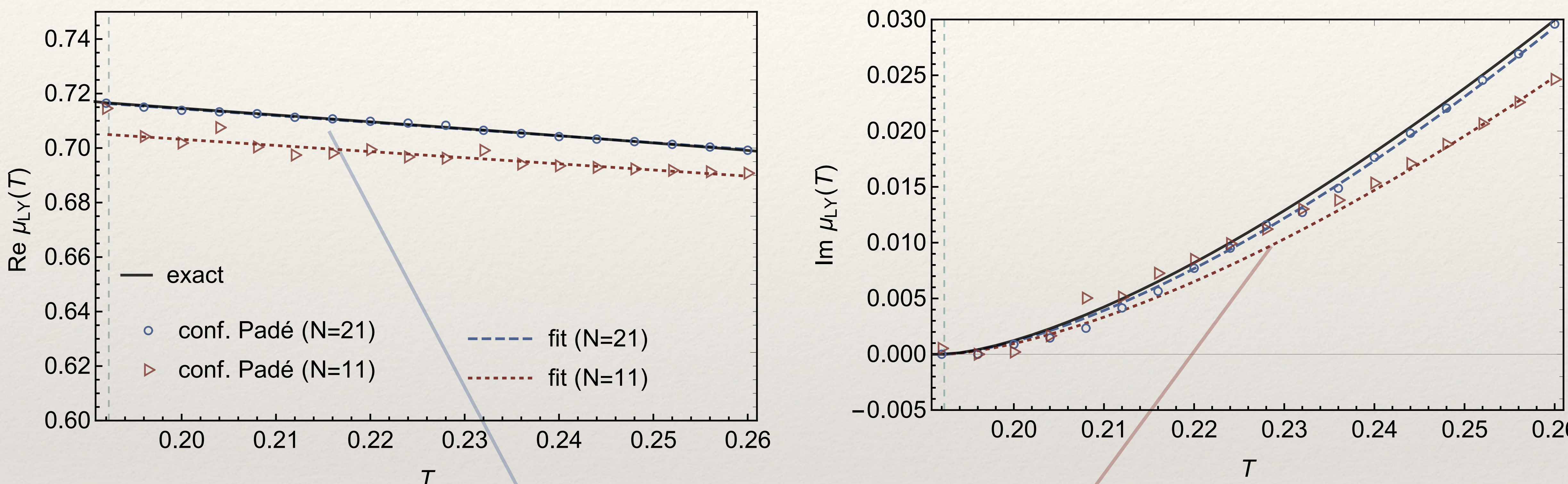




Susceptibility

Lee-Yang trajectory

- Find $\mu_{LY}^2(T)$ from poles of the conformal-Padé (GN model)



fit *fit*

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + i w_{LY} \frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2} (T - T_c)^{3/2} \qquad w_{LY} = \frac{2}{3\sqrt{3}}$$

- Extract μ_c, T_c, crossover slope, $\frac{h_T}{h_\mu}$, and $\frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2}$

Ising parameters

$$\mu_{LY}(T) \approx \mu_c - \frac{h_T}{h_\mu}(T - T_c) + i w_{LY} \frac{r_\mu^{3/2}}{h_\mu}\left(\frac{r_T}{r_\mu} - \frac{h_T}{h_\mu}\right)^{3/2} (T - T_c)^{3/2}$$

$$w_{LY} = \frac{2}{3\sqrt{3}}$$

	T_c	μ_c	h_T/h_μ	c
exact	0.192	0.717	0.249	4.684
conf. Padé (N=21)	0.195	0.716	0.248	4.323
conf. Padé (N=11)	0.185	0.707	0.225	3.666

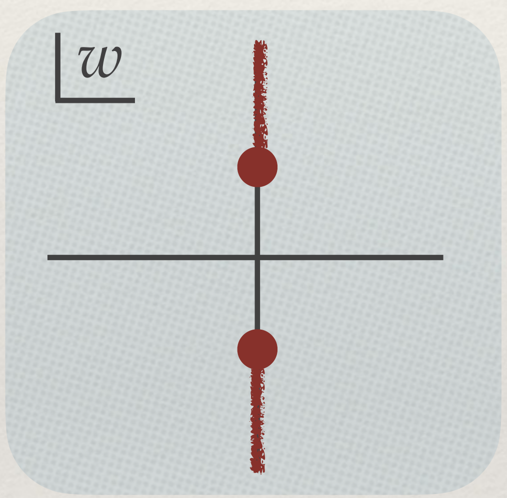
Uniformization Map : crossing the branch cut

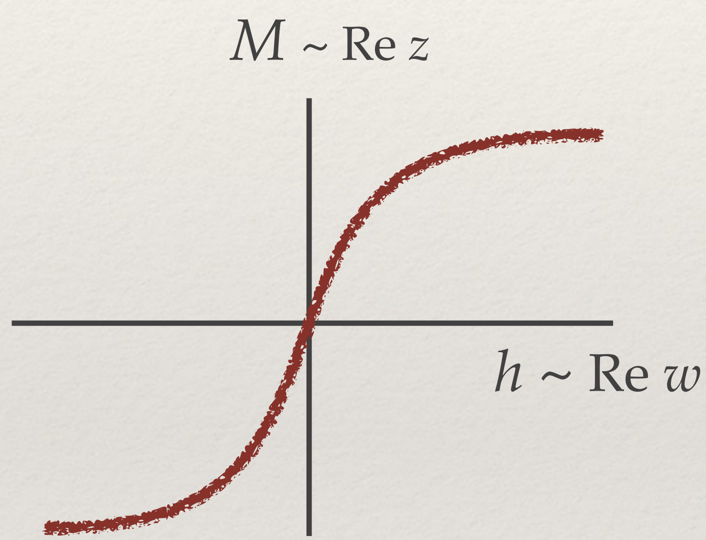
$w = hr^{-\beta\delta}$
$z = Mr^{-\beta}$

Ising model: $w = F(z)$

$F(z) = z + z^3$ (mean field)

High Temperature (T>T_c)





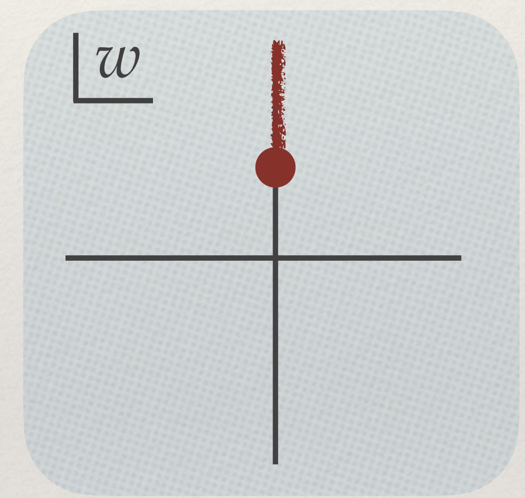
high T sheet
r>0

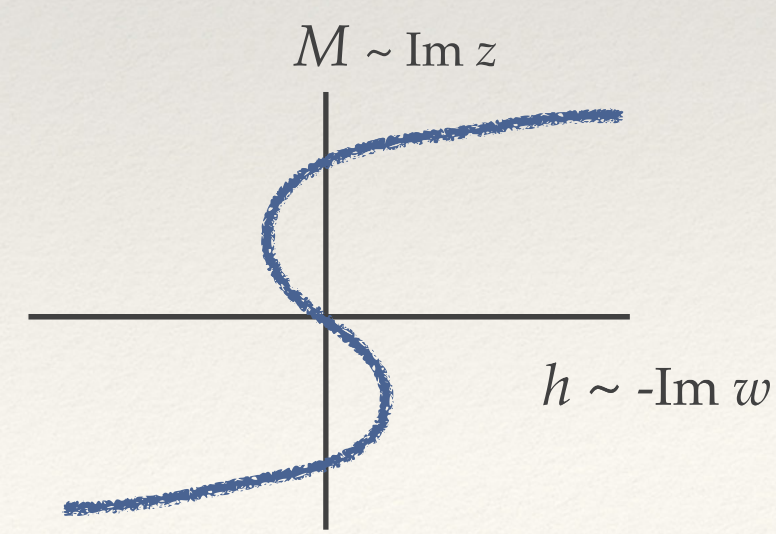
$$z(w) = w - w^3 + 3w^5 - 12w^7 + \ldots$$

high T expansion

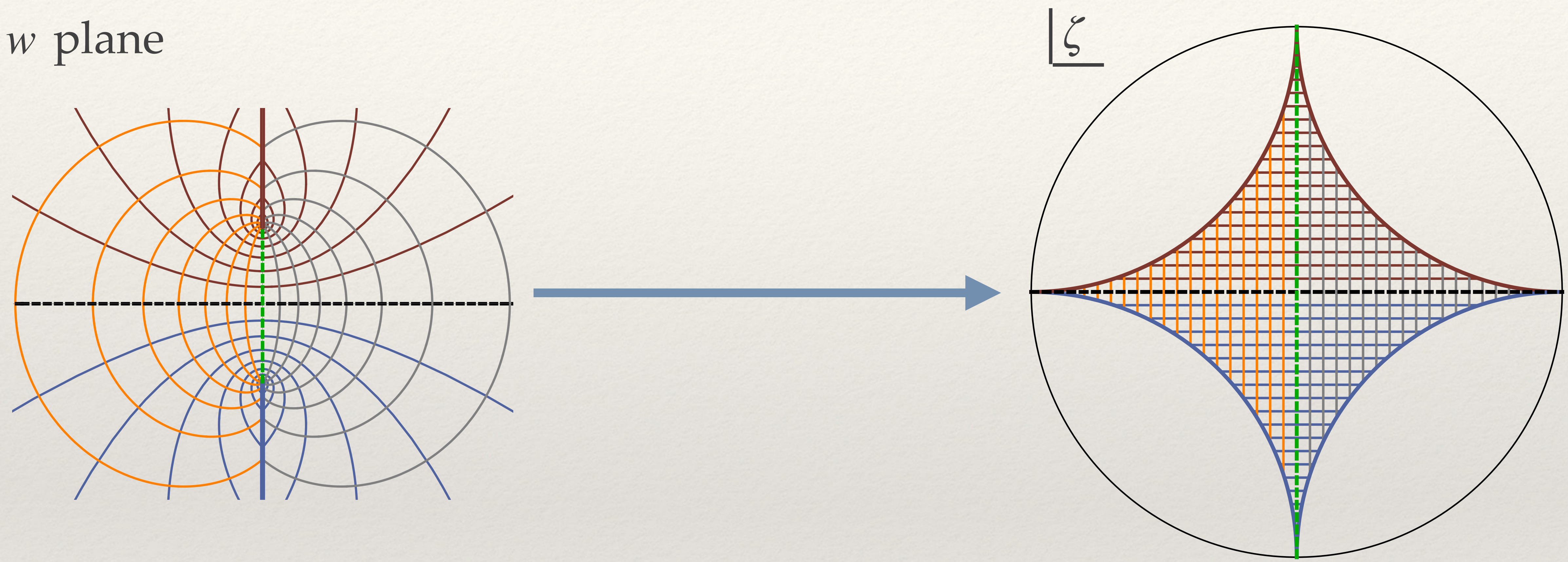
Low Temperature (T<T_c)

low T sheet
r<0, h>0





Uniformization: crossing the branch cut



high T sheet

r>0

$$w \to w(\tau) = i(-1 + 2\lambda(\tau))$$

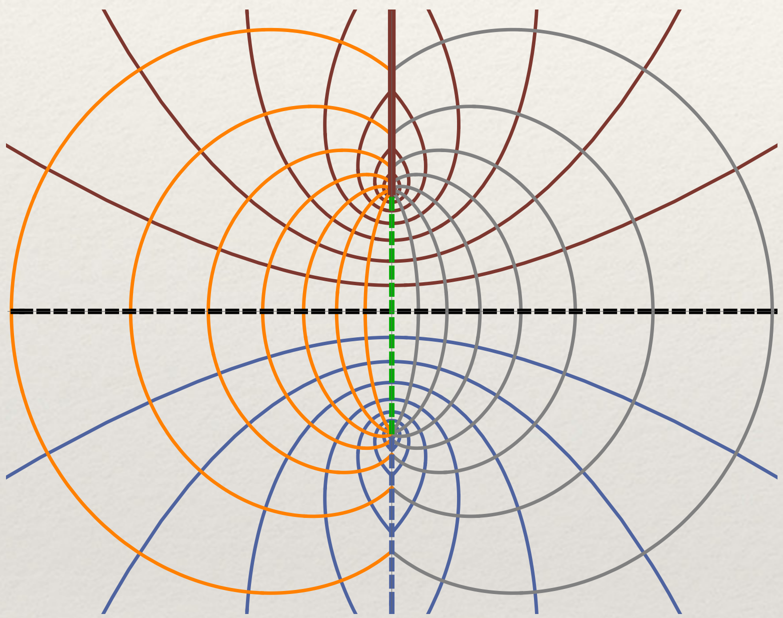
$$\tau(\zeta) = i\left(\frac{1 + i\zeta}{1 - i\zeta}\right)$$

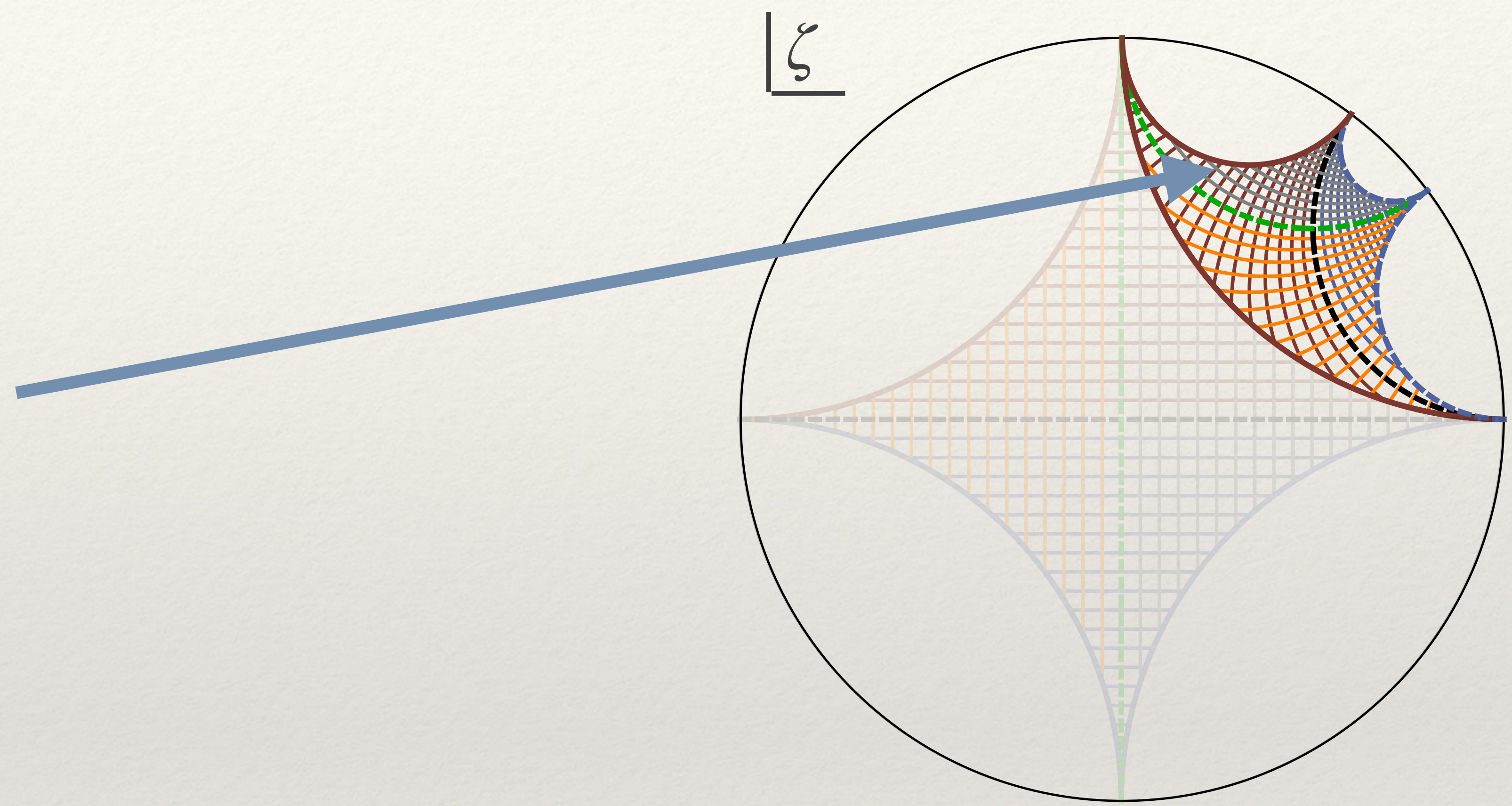
$$\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}$$ (elliptic modular function)

$$\theta_2(\tau) = \sum_{n=1}^{\infty} e^{2\pi i\tau(n+1/2)^2}, \quad \theta_3(\tau) = \sum_{n=1}^{\infty} e^{2\pi i\tau n^2}$$

Uniformization: crossing the branch cut

w plane





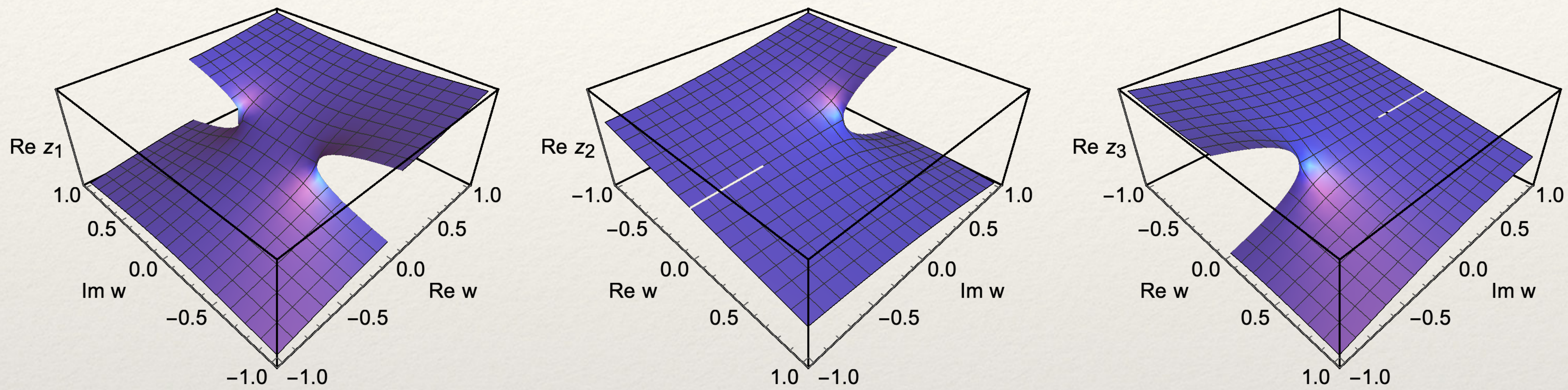
low T sheet

r<0

Low T sheet = Schwartz reflection of the high T sheet
(modular transformation)

Uniformization

$$w = F(z) = z + z^3 \quad \text{(mean field)}$$



$$z_1(w) = -\frac{2i}{\sqrt{3}}\left[{}_2F_1\left(\frac{1}{3},-\frac{1}{3},\frac{1}{2};\frac{1}{2}(1-iw)\right) - \text{c.c.}\right]$$

$$z_2(w) = \frac{2i}{\sqrt{3}}\,{}_2F_1\left(\frac{1}{3},-\frac{1}{3},\frac{1}{2};\frac{1}{2}(1-iw)\right)$$

$$\lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)} \quad \theta_2(\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i\tau(n+1/2)^2}, \quad \theta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i\tau n^2}$$

$$w(\tau) = i(-1 + 2\lambda(\tau))$$

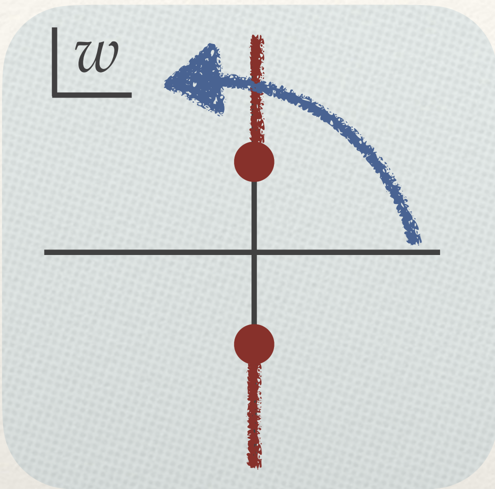
$z(\tau)$: single valued

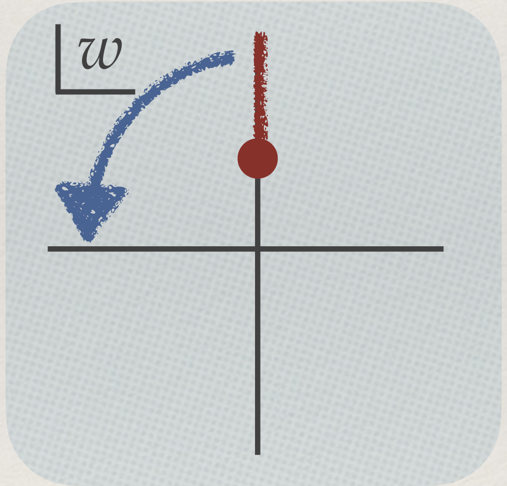
"uniformization"

[Bateman, Higher Transcendal Functions I]

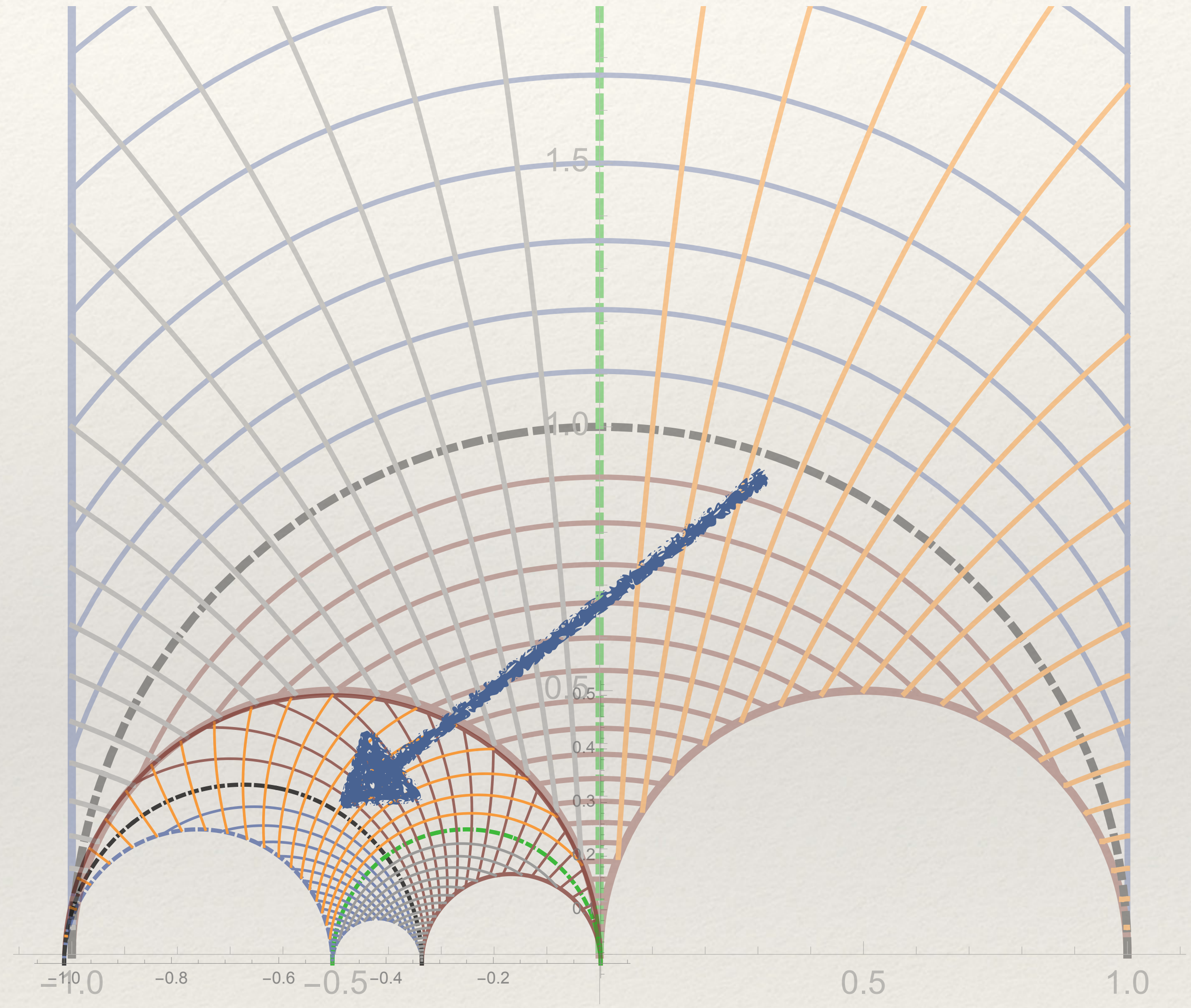
* $w \to 2/(3\sqrt{3})w$

Uniformization





Jumping sheets
in w plane

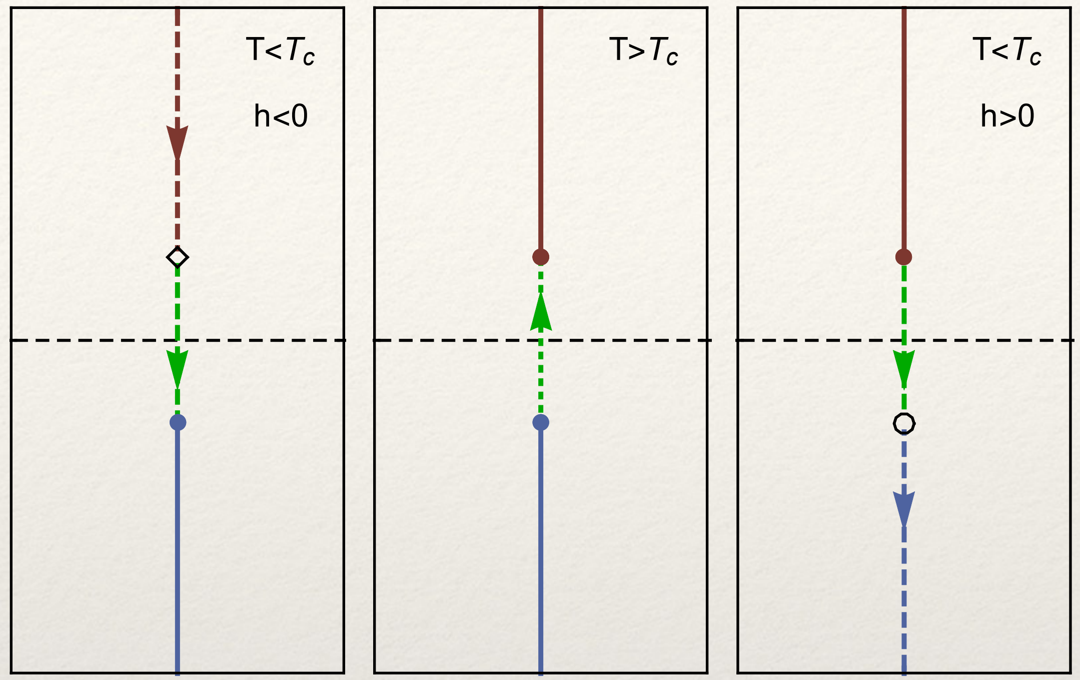


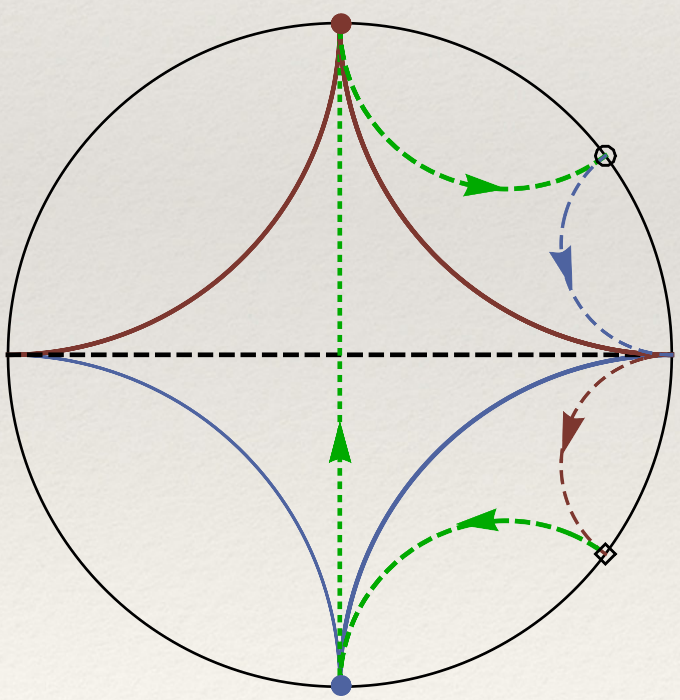
Smooth in
τ plane

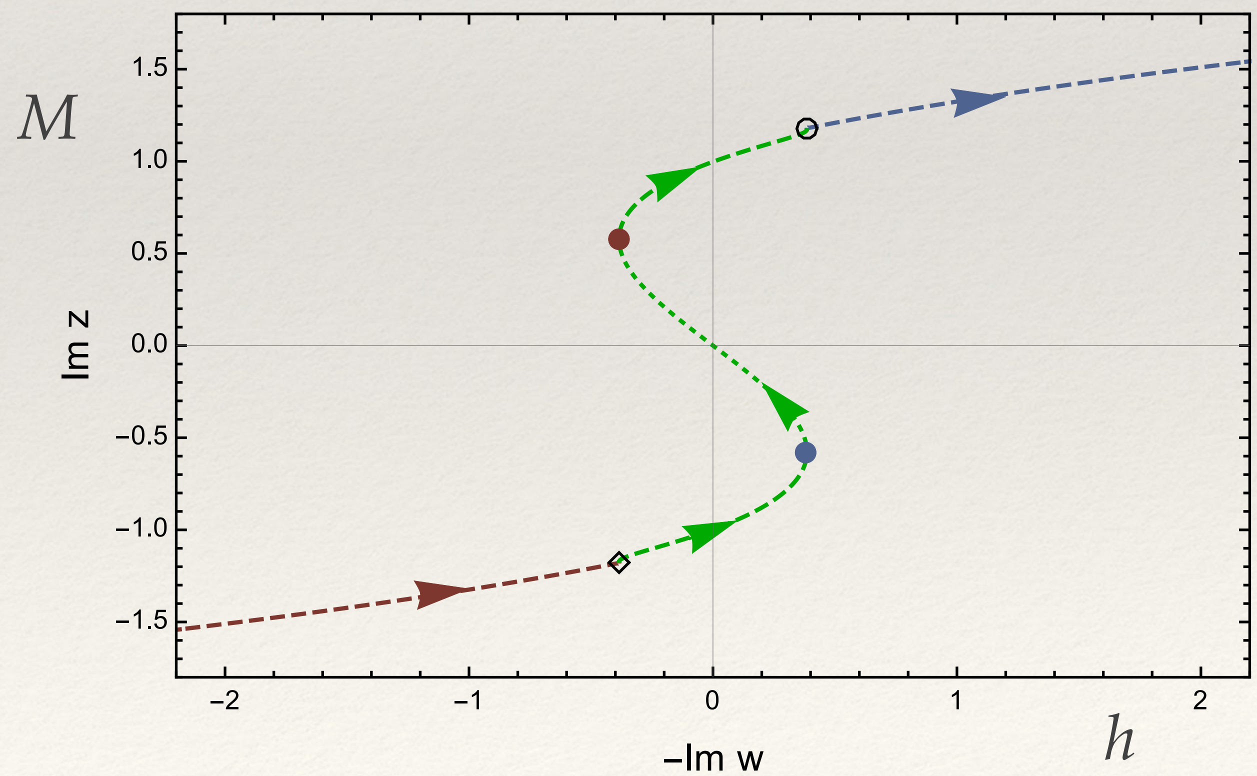
Interactive realization:
https://people.math.osu.edu/costin.9/classes.html

*

Uniformization: higher Riemann sheets







Uniformization: crossing the branch cut

Low T