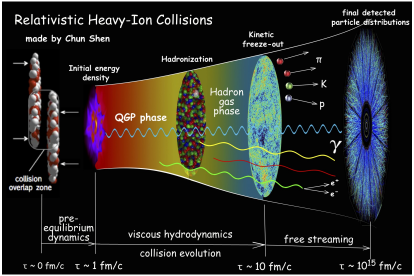
Maximum entropy freezeout of fluctuations

M. Stephanov

with Maneesha Pradeep: PhysRevLett.130.162301 [2211.09142]

Equation of state, heavy-ion collisions and freezeout

- Equation of state is the key ingredient of hydrodynamics.
- Hydrodynamics describes evolution of HIC fireball.
- However, experiments measure particle multiplicities, *not* hydrodynamic variables (densities).
- Freezeout is an essential step connecting theory with experiment.



Standard freezeout (Cooper-Frye)

- In each hydro cell at freezeout local values of $T(x)$, $\mu(x)$, $u(x)$

  are translated into

  local phase-space distribution $\langle f_A(x)\rangle = e^{\hat{\mu}(x)q_A+\beta(x)u(x)\cdot p_A}$.

  where $\hat{\mu} = \mu/T$, $\beta = 1/T$,

  and A is a set of particle quantum numbers

  such as charge (q_A), 4-momentum (p_A),

  i.e., phase-space coordinates.

Standard freezeout (Cooper-Frye)

- In each hydro cell at freezeout local values of $T(x)$, $\mu(x)$, $u(x)$

  are translated into

  local phase-space distribution $\langle f_A(x)\rangle = e^{\hat\mu(x) q_A + \beta(x) u(x)\cdot p_A}$.

  where $\hat\mu = \mu/T$, $\beta = 1/T$,

  and A is a set of particle quantum numbers

  such as charge (q_A), 4-momentum (p_A),

  i.e., phase-space coordinates.

- $\langle f_A\rangle$ gives us single-particle observables.

  For fluctuations we need (at least) $\langle \delta f_A \delta f_B\rangle$.

Fluctuations

- Hydrodynamics is intrinsically stochastic: dissipation means there are fluctuations.
- Fluctuations, especially non-gaussian, are essential for mapping QCD phase diagram and locating QCD critical point.
- Hydrodynamic evolution of fluctuations
  — a lot of recent progress — a subject for a different talk.
- This talk — freezeout of fluctuations.
- Example of application – next talk.

Earlier work, problems and questions

- "Fluctuating Cooper-Frye". Naively: *Kapusta-Muller-MS 2011*

$$\delta f_A = \left(\delta\hat{\mu}\frac{\partial}{\partial\hat{\mu}} + \delta T\frac{\partial}{\partial T} + \delta u\frac{\partial}{\partial u}\right)\langle f_A(\hat{\mu},T,u)\rangle$$

Then, multiplicity fluctuation correlator:

$$\langle\delta f_A\delta f_B\rangle = \underbrace{\langle\delta\hat{\mu}\delta\hat{\mu}\rangle}_{\substack{\text{from hydro}\\ \delta\hat{\mu}=\chi^{-1}\delta n}} \left(\frac{\partial}{\partial\hat{\mu}}\langle f_A\rangle\right)\left(\frac{\partial}{\partial\hat{\mu}}\langle f_B\rangle\right) + \dots \qquad (*)$$

Earlier work, problems and questions

- "Fluctuating Cooper-Frye". Naively: *Kapusta-Muller-MS 2011*

$$\delta f_A = \left(\delta\hat\mu \frac{\partial}{\partial\hat\mu} + \delta T \frac{\partial}{\partial T} + \delta u \frac{\partial}{\partial u}\right)\langle f_A(\hat\mu, T, u)\rangle$$

Then, multiplicity fluctuation correlator:

$$\langle \delta f_A \delta f_B\rangle = \underbrace{\langle\delta\hat\mu\delta\hat\mu\rangle}_{\substack{\text{from hydro} \\ \delta\hat\mu = \chi^{-1}\delta n}} \left(\frac{\partial}{\partial\hat\mu}\langle f_A\rangle\right)\left(\frac{\partial}{\partial\hat\mu}\langle f_B\rangle\right) + \ldots \qquad (*)$$

- Problem:
consider ideal gas, no correlations, i.e. $\langle f_A f_B\rangle = \langle f_A\rangle\delta_{AB}$,

but there are fluctuations of $\delta\hat\mu$, δT, etc. even in ideal gas $\Rightarrow$ equation $(*)$ produces incorrect result: spurious correlations.

Source of the problem and a solution

- Correlated particle pairs erroneously include "pairs" made of the same particle counted twice.

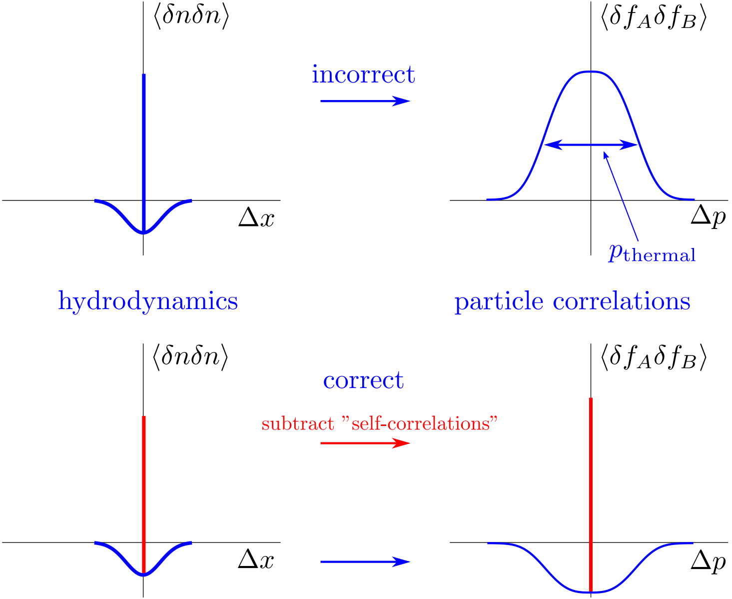
Li-Springer-MS '13, Plumberg-Kapusta '20

- A solution for charge fluctuations – subtract the contribution of ideal gas to $\langle\delta n\delta n\rangle$ in hydrodynamics and apply equation $(*)$ only to the remainder:

$$\langle\delta n\delta n\rangle \equiv \langle\delta n\delta n\rangle_{\mathrm{ideal}} + \Delta\langle\delta n\delta n\rangle, \qquad (\delta n = \chi\delta\hat{\mu})$$

$$\langle\delta f_A\delta f_B\rangle = \langle f_A\rangle\delta_{AB} + \underbrace{\Delta\langle\delta n\delta n\rangle\left(\chi^{-1}\frac{\partial}{\partial\hat{\mu}}\langle f_A\rangle\right)\left(\chi^{-1}\frac{\partial}{\partial n}\langle f_B\rangle\right)}_{\text{balance function}}$$

Thermal smearing and “self-correlations”



Open questions

How to deal with

- Temperature, velocity fluctuations?
- Non-gaussian fluctuations?

Maximum entropy freezeout:

Pradeep-MS PhysRevLett.130.162301 [2211.09142]

Revisit one-point/single-particle observables

- Conservation laws constrain particle phase-space distributions:

  $\langle n(x)\rangle = \sum_A q_A \langle f_A(x)\rangle$ and $\langle \epsilon(x) u^\mu(x)\rangle = \sum_A p^\mu_A \langle f_A(x)\rangle$;

  or, generally, $\langle \Psi_a \rangle = \sum_A P_a^A \langle f_A \rangle$,

  where P_a^A – contribution of particle A to (conserved) quantity Ψ_a.

  Problem: these equations for $\langle f_A \rangle$ have infinitely many solutions.

Revisit one-point/single-particle observables

- Conservation laws constrain particle phase-space distributions:

  $\langle n(x)\rangle = \sum_A q_A \langle f_A(x)\rangle$ and $\langle \epsilon(x) u^\mu(x)\rangle = \sum_A p_A^\mu \langle f_A(x)\rangle$;

  or, generally, $\langle \Psi_a \rangle = \sum_A P_a^A \langle f_A \rangle$,

  where P_a^A – contribution of particle A to (conserved) quantity Ψ_a.

  Problem: these equations for $\langle f_A \rangle$ have infinitely many solutions.

- Which solution maximizes Bolzmann entropy? (minimum bias)

  $S_0 = -\sum_A \langle f_A \rangle \log \langle f_A \rangle$

  Answer: $\langle f_A \rangle = e^{\hat{\mu} q_A + \beta u \cdot p_A}$ — Cooper-Frye.

Revisit one-point/single-particle observables

- Conservation laws constrain particle phase-space distributions:

  $\langle n(x)\rangle = \sum_A q_A \langle f_A(x)\rangle$ and $\langle \epsilon(x) u^\mu(x)\rangle = \sum_A p_A^\mu \langle f_A(x)\rangle$;

  or, generally, $\langle \Psi_a \rangle = \sum_A P_a^A \langle f_A \rangle$,

  where P_a^A – contribution of particle A to (conserved) quantity Ψ_a.

  Problem: these equations for $\langle f_A \rangle$ have infinitely many solutions.

- Which solution maximizes Bolzmann entropy? (minimum bias)

  $S_0 = -\sum_A \langle f_A \rangle \log \langle f_A \rangle$

  Answer: $\langle f_A \rangle = e^{\hat{\mu} q_A + \beta u \cdot p_A}$ — Cooper-Frye.

- Fluctuations?

Pradeep-MS 2022

Maximum entropy freezeout of fluctuations

- We want to match *fluctuations* of Ψ_a at freezeout

  to *fluctuations* of f_A, so that $\Psi_a = \sum_A P_a^A f_A$ event-by-event

  i.e., $G_{AB} \equiv \langle \delta f_A \delta f_B \rangle$ must obey $(P_a^A = \{q_A, p_A, \dots\})$

$$\underbrace{\langle \delta\Psi_a \delta\Psi_b \rangle}_{H_{ab}} = \sum_{AB} P_a^A P_b^B \underbrace{\langle \delta f_A \delta f_B \rangle}_{G_{AB}}$$

Again, for G_{AB}, there are infinitely many solutions.

Maximum entropy freezeout of fluctuations

- We want to match *fluctuations* of Ψ_a at freezeout

  to *fluctuations* of f_A, so that $\Psi_a = \sum_A P_a^A f_A$ event-by-event

  i.e., $G_{AB} \equiv \langle \delta f_A \delta f_B \rangle$ must obey $\qquad (P_a^A = \{q_A, p_A, \dots\})$

$$\underbrace{\langle \delta\Psi_a \delta\Psi_b \rangle}_{H_{ab}} = \sum_{AB} P_a^A P_b^B \underbrace{\langle \delta f_A \delta f_B \rangle}_{G_{AB}}$$

  Again, for G_{AB}, there are infinitely many solutions.

- Entropy?

Entropy of fluctuations

- Entropy is a functional of fluctuations, i.e., of G_{AB}:

$$S_2 = S_0 + \underbrace{\frac{1}{2}\mathrm{Tr}\left[\log G\bar{G} - G\bar{G} + 1\right]}_{\text{relative entropy}}, \text{ where } \bar{G}_{AB} = -\frac{\delta^2 S_0}{\delta\langle f_A\rangle\delta\langle f_B\rangle}.$$

- This follows from $S_{\mathrm{rel}}[\mathcal{P}] = \mathcal{P}\log\bar{\mathcal{P}}/\mathcal{P}$,

  where $\mathcal{P} = \mathcal{P}[f]$ is the probability distribution of f_A, i.e., functional of $\langle f_A\rangle$, $\langle\delta f_A\delta f_B\rangle \equiv G_{AB}$, etc.

  and $\bar{\mathcal{P}}$ is $\mathcal{P}$ for ideal hadron gas (reference distirbution).

  We maximize S with respect to $\mathcal{P}$ subject to conservation laws.

Maximum entropy solution

- Relative entropy is maximized (subject to constraints) by

  $$(G^{-1})^{AB} = (\bar{G}^{-1})^{AB} + (H^{-1} - \bar{H}^{-1})^{ab} P_a^A P_b^B$$

  - Note: when $H = \bar{H} \to G = \bar{G}$ – ideal gas (reference) correlator.
  - Similar expressions for non-gaussian correlators.

- Linearizing in $\Delta H \equiv H - \bar{H}$ we obtain the desired generalization of earlier results:

  $$G_{AB} = \underbrace{\bar{G}_{AB}}_{\text{baseline}} + \underbrace{\Delta H_{ab} P_A^a P_B^b}_{\text{correlations}},$$

  where $P_A^a \equiv (\bar{H}^{-1})^{aa'} P_{a'}^{A'} \bar{G}_{A'A}$.

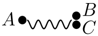
  [cf. $\Delta\langle \delta f_A \delta f_B \rangle = \langle \delta n \delta n \rangle \left(\chi^{-1} q_A \langle f_A \rangle\right) \left(\chi^{-1}) q_B \langle f_B \rangle\right)$]

Non-gaussian correlators ($n \geq 3$ particles)

- Linearied equations are simple and intuitive in terms of

$$\Delta G_{AB} = G_{AB} - \bar{G}_{AB}, \quad \Delta H_{ab} = H_{ab} - \bar{H}_{ab},$$ – "relative correlator"

$$\underbrace{\widehat{\Delta} G_{ABC}}_{\text{irreducible correlation}} = G_{ABC} - \Big[\underbrace{\bar{G}_{ABC}} + \underbrace{3 \Delta G_{AD} \bar{G}^{D}_{BC}} \Big]_{\overline{ABC}} \leftarrow \text{permutation average}$$



$$\widehat{\Delta} H_{abc} = H_{abc} - \Big[\bar{H}_{abc} + 3 \Delta H_{ad} \bar{H}^{d}_{bc} \Big]_{\overline{abc}}$$

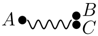
  - $\hat{\Delta} G_{ABC}$ (Irreducibe Relative Correlator)
    — vanishes for an uncorrelated hadron gas,
    as well as for a gas with only 2-particle correlations.
  - $\int_A \int_B \int_C \hat{\Delta} G_{ABC}$ is a factorial cumulant

Non-gaussian correlators ($n \geq 3$ particles)

- Linearied equations are simple and intuitive in terms of

$$\Delta G_{AB} = G_{AB} - \bar{G}_{AB}, \quad \Delta H_{ab} = H_{ab} - \bar{H}_{ab},$$ – "relative correlator"

$$\underbrace{\widehat{\Delta} G_{ABC}}_{\text{irreducible correlation}} = G_{ABC} - \Big[\underbrace{\bar{G}_{ABC}}_{} + \underbrace{3 \Delta G_{AD} \bar{G}^{D}_{BC}}_{} \Big]_{\overline{ABC}}$$ ← permutation average



$$\widehat{\Delta} H_{abc} = H_{abc} - \left[\bar{H}_{abc} + 3 \Delta H_{ad} \bar{H}^{d}_{bc} \right]_{\overline{abc}}$$

  - $\hat{\Delta} G_{ABC}$ (Irreducibe Relative Correlator)
    — vanishes for an uncorrelated hadron gas,
    as well as for a gas with only 2-particle correlations.
  - $\int_A \int_B \int_C \hat{\Delta} G_{ABC}$ is a factorial cumulant

- Maximum entropy method gives:

$$\Delta G_{AB} = \Delta H_{ab} P^{a}_{A} P^{b}_{B}, \quad \hat{\Delta} G_{ABC} = \hat{\Delta} H_{abc} P^{a}_{A} P^{b}_{B} P^{c}_{C}, \quad \text{etc.}$$

Why Factorial Cumulants

Factorial Cumulants (FC) are better measures of particle correlations

Three reasons:

- Maximum Entropy freezeout: *Pradeep, MS 2211.09142*

  FC are integrals of $\hat{\Delta}G_n$ (the irreducible particle correlators), which are directly related to hydrodynamic correlators.

Why Factorial Cumulants

Factorial Cumulants (FC) are better measures of particle correlations

Three reasons:

- Maximum Entropy freezeout: *Pradeep, MS 2211.09142*

  FC are integrals of $\hat{\Delta} G_n$ (the irreducible particle correlators), which are directly related to hydrodynamic correlators.

- Acceptance dependence: *Ling, MS 1512.09125, Bzdak, Koch, Strodthoff 1607.07375*

  FC are monomials of Δy for small Δy; NC are polynomials.

Why Factorial Cumulants

Factorial Cumulants (FC) are better measures of particle correlations

Three reasons:

- Maximum Entropy freezeout: *Pradeep, MS 2211.09142*

  FC are integrals of $\hat{\Delta}G_n$ (the irreducible particle correlators), which are directly related to hydrodynamic correlators.

- Acceptance dependence: *Ling, MS 1512.09125, Bzdak, Koch, Strodthoff 1607.07375*

  FC are monomials of Δy for small Δy; NC are polynomials.

- Normal Cumulants (NC) – quantify non-gaussianity;
  FC – quantify non-poissonianity, i.e., *irreducible* correlations.

  NC are for densities (continuous);
  FC are for multiplicities (discrete).

Comparison to legacy approach for critical fluctuations

- The contribution of critical fluctuations matches the simple model often used in the literature *(MS 2011)*:

  $$\delta f_A^{\rm critical} = \delta\sigma \left(\frac{\partial}{\partial\sigma} \langle f_A \rangle \right)$$

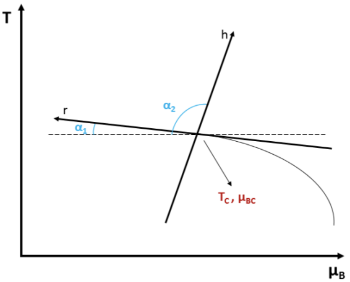
  where critical field σ couples to mass so that $\delta m_A = g_A \delta\sigma$.

  Thus $\langle \delta f_A \delta f_B \rangle = \underbrace{\langle f_A \rangle \delta_{AB}}_{\text{Poisson baseline}} + \underbrace{\langle \delta\sigma\delta\sigma \rangle \left(\frac{\partial}{\partial\sigma} \langle f_A \rangle \right) \left(\frac{\partial}{\partial\sigma} \langle f_B \rangle \right)}_{\text{critical contribution} \sim g_A g_B}$

- Similar. But, within maximum entropy approach, we do not need to know the couplings g_A of the critical mode. All we need is already in the equation of state.

- And ME approach is *not* limited to most singular contributions.

Equation of state near CP



Universality: QCD pressure singularity matches Ising model Gibbs free energy:

$$P_{\text{QCD}}(\mu, T)/T_c^4 = -G_{\text{Ising}}(h, r) + \text{less singular terms}\,,$$

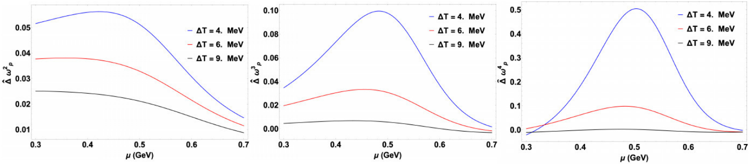
(from Parotto et al.)

$$\begin{aligned} h(\mu, T) &= h_T \Delta T + h_\mu \Delta\mu = -\frac{\cos\alpha_1 \Delta T + \sin\alpha_1 \Delta\mu}{w T_c \sin(\alpha_1 - \alpha_2)}; \\ r(\mu, T) &= r_T \Delta T + r_\mu \Delta\mu = \frac{\cos\alpha_2 \Delta T + \sin\alpha_2 \Delta\mu}{\rho w T_c \sin(\alpha_1 - \alpha_2)}\,, \end{aligned}$$

Matching parameters w, ρ, α_1, α_2 characterize the critical point, in addition to its location T_c, μ_c.

Equilibrium estimates for critical contribution to factorial cumulants of proton multiplicities

Karthein, MP, Rajagopal, Stephanov, Yin (in preparation)
2508.19237

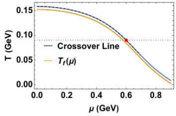


$$T_f(\mu) = T_{\rm co}(\mu) - \Delta T$$

$$\mu_c = 600\,\mathrm{MeV}\,,\ \alpha_2 = 0^\circ\,,\ \rho = 1\,,\ w = 20$$

Example **choice of Mapping Parameters**

$$\hat{\Delta}\omega_p^k = \frac{\hat{\Delta}\left\langle \delta N_p^k \right\rangle}{\langle N_p \rangle}$$



Freeze-out curve

23

Signal depends sensitively on ΔT, w, ρ, e.g.,

the amplitude $\sim w^{-6/5}\Delta T^{k-6/5}$, the width/shift $\sim w^{-2/5}\rho^{-1}\Delta T^{-3/5}$.

Thus, these parameters (T_c, μ_c, w, ρ) could be constrained by data.

Concluding summary

- Maximum entropy approach for single-particle observables $=$ traditional Cooper-Frye freezeout.
- Maximum entropy approach solves the problem of freezing out of hydrodynamic *fluctuations*, respecting conservatoin laws.
- The method is very general and works for gaussian and non-gaussian, for critical and non-critical fluctuations.
- Agrees with existing (*ad hoc*) methods where such are available.
- Allows predicting the magnitude of CP signatures *directly* in terms of the EOS parameters.