Lee-Yang Zero Ratio Method
and
Its Application to the QCD Phase Diagram

Tatsuya Wada（YITP）

Collaborators : Masakiyo Kitazawa (YITP), Kazuyuki Kanaya (Univ. of Tsukuba)

This talk is based on TW, Kitazawa, Kanaya,

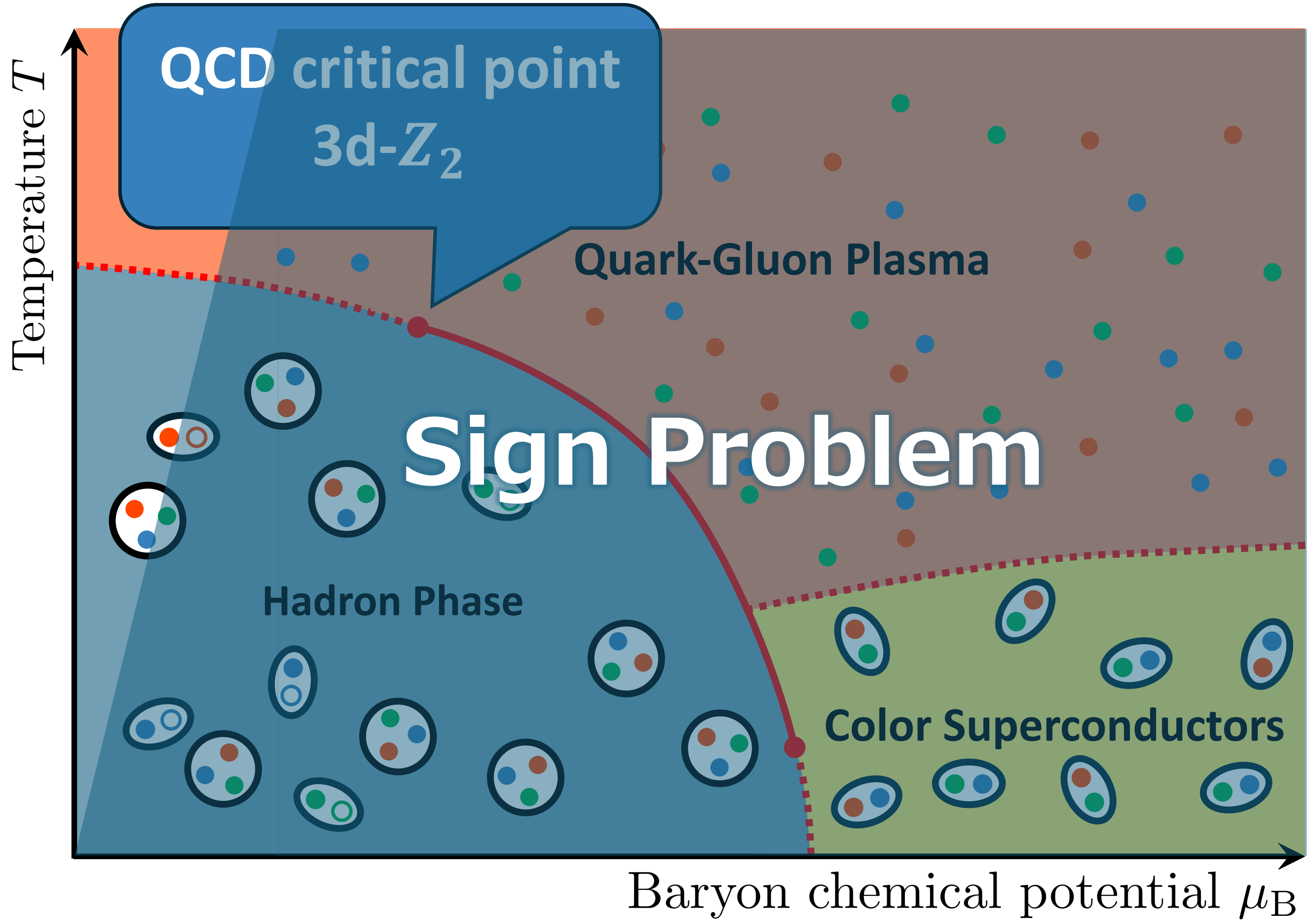
PRL, 134, 162302 and arXiv: 2508.20422

Analytic structure of QCD and Yang-Lee edge singularity @ECT*, Trento Sep 9th

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

QCD Phase Diagram

QCD Phase diagram



Monte Carlo approach

◆ Zero density ($\mu = 0$)
Monte Carlo works well!
The phase changes smoothly.
= Crossover transition

◆ Finite density ($\mu \neq 0$)
The weight e^{-S} is complex.
→ **Sign problem**
No direct Monte Carlo analysis.

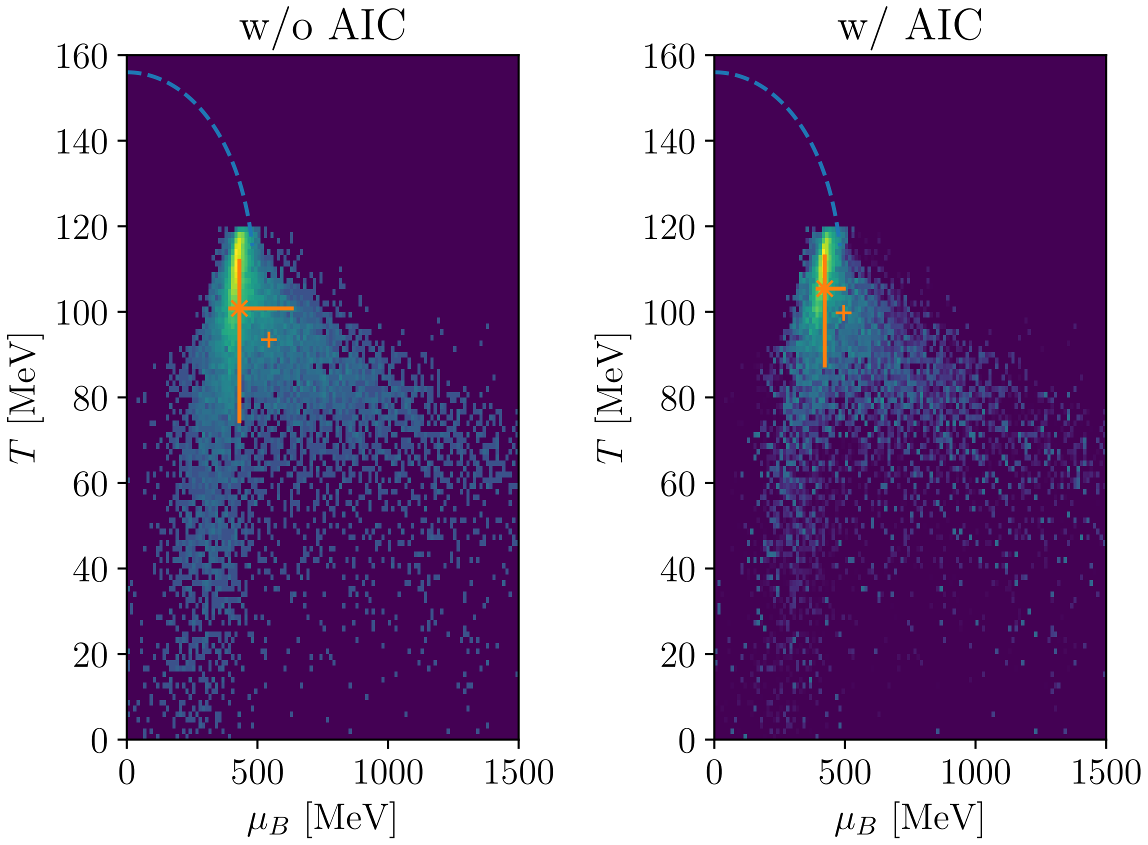
$\mu \neq 0$ region is conjectured by model calculations.

Recent progress in QCD-CP: Lee-Yang zeros

Pure imaginary chemical potential
+ Pade approxmation
+Lee-Yang zeros /edge singularity

- ◆ 2+1 flavor QCD-CP
- ➢ Basar (2024)
- ➢ Clarke, *et al*. (2024)
- ➢ Alexander(2025)

Estimation of location of QCD CP



$$\begin{cases} \mu^{\mathrm{CEP}} = 422^{+80}_{-35}\,\mathrm{MeV} \\ T^{\mathrm{CEP}} = 105^{+8}_{-18}\,\mathrm{MeV} \end{cases}$$

Lee-Yang zero approach is powerful.
But, in practice we always work at finite *V*.
How do we treat finite volume effect?

Review : Lee-Yang Zero

Lee, Yang 1952
Yang, Lee 1952

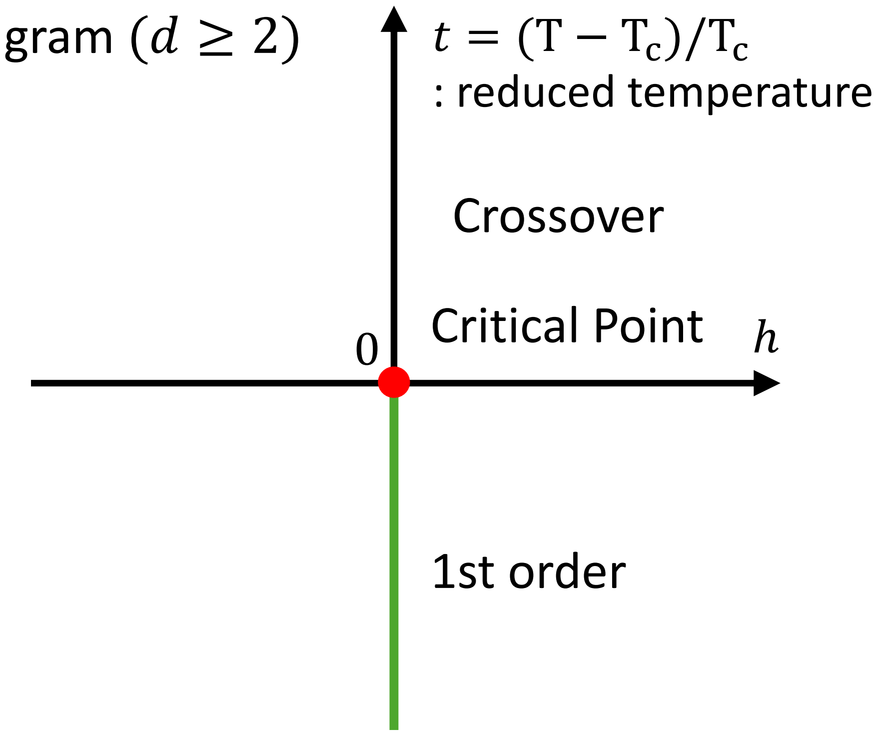
Lee-Yang Zeros
= Zeros of Z in Complex Parameter space

Lee-Yang's circle theorem
LYZs exist only on imaginary axis.

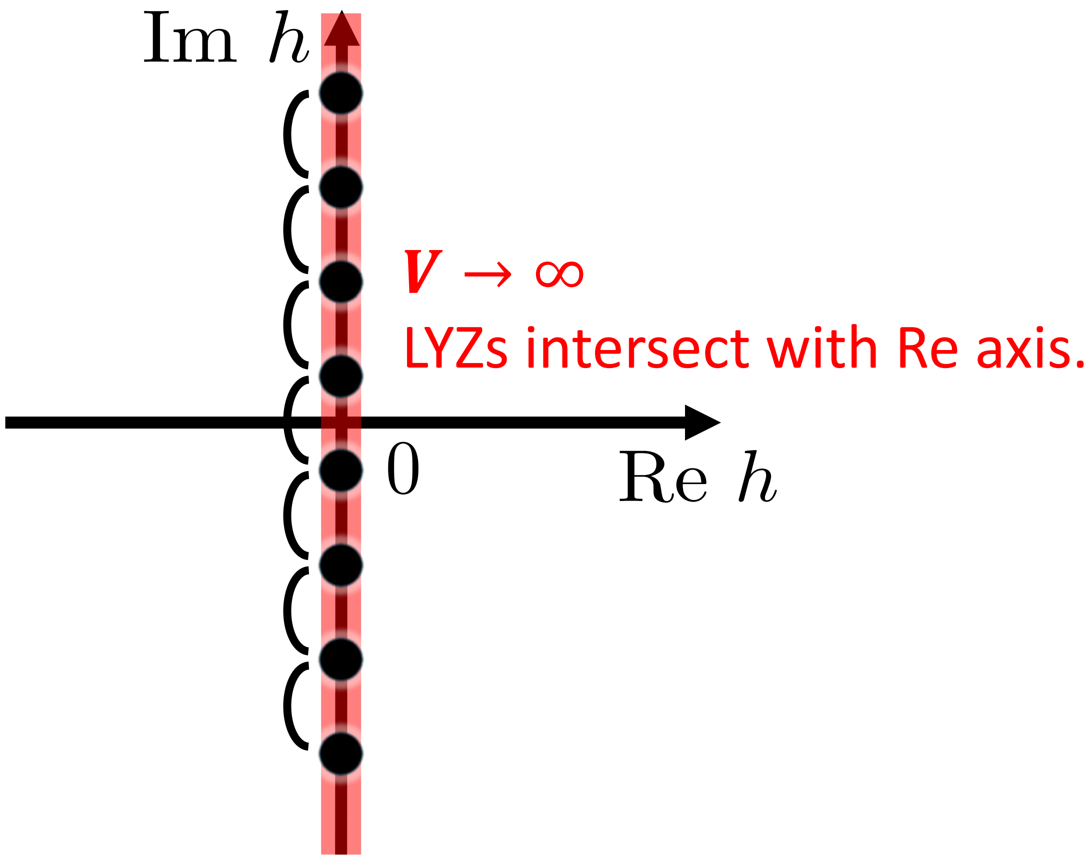
Ex.) d-dimensional Ising model $s_i = \pm 1$

$$Z(t,h,L) = \mathrm{Tr}\exp\left(\frac{1}{T}\sum_{\langle i,j\rangle} s_i s_j + \frac{h}{T}\sum_{i}^{L^d} s_i\right)$$

Phase diagram ($d \geq 2$)



$t < 0$:1st order



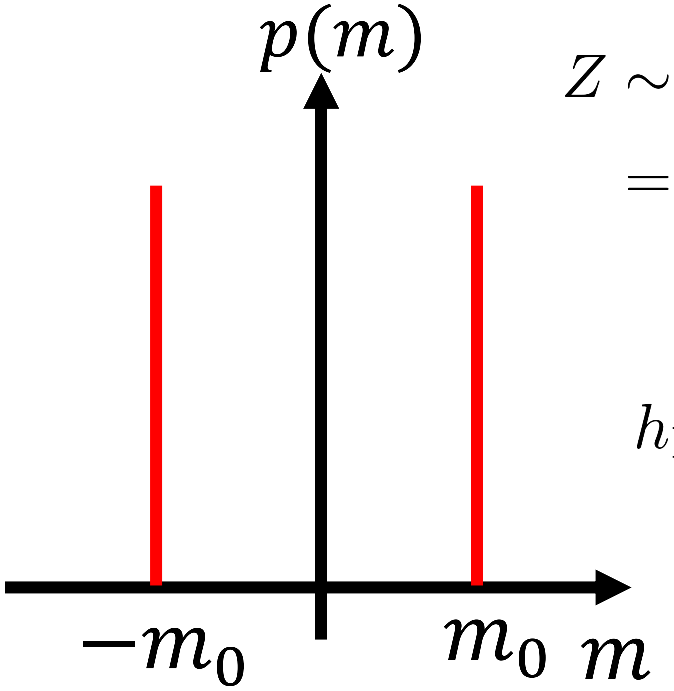
LYZ around 1st order

Lee, Yang 1952
Yang, Lee 1952

Lee-Yang Zeros
= Zeros of Z in Complex Parameter space

Lee-Yang's circle theorem
LYZs exist only on imaginary axis.

$V \to \infty$

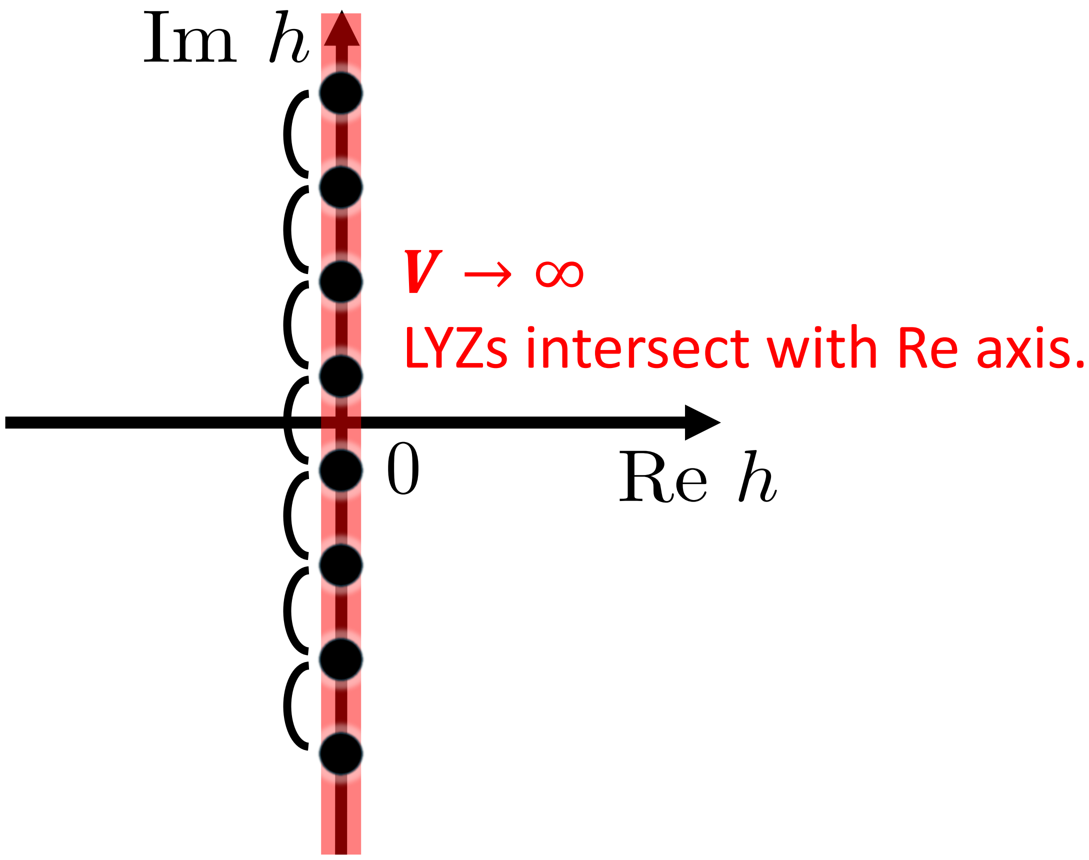


$$Z \sim e^{-V\beta h m_0} + e^{V\beta h m_0}$$
$$= e^{-V\beta h m_0}(1 + e^{2V\beta h m_0})$$

$$h_{\mathrm{LY}}(t) = \frac{(2n+1)\pi i}{2\beta V m_0}$$

The intervals of LYZs are same.

$t < 0$:1st order



LYZ around 1st order

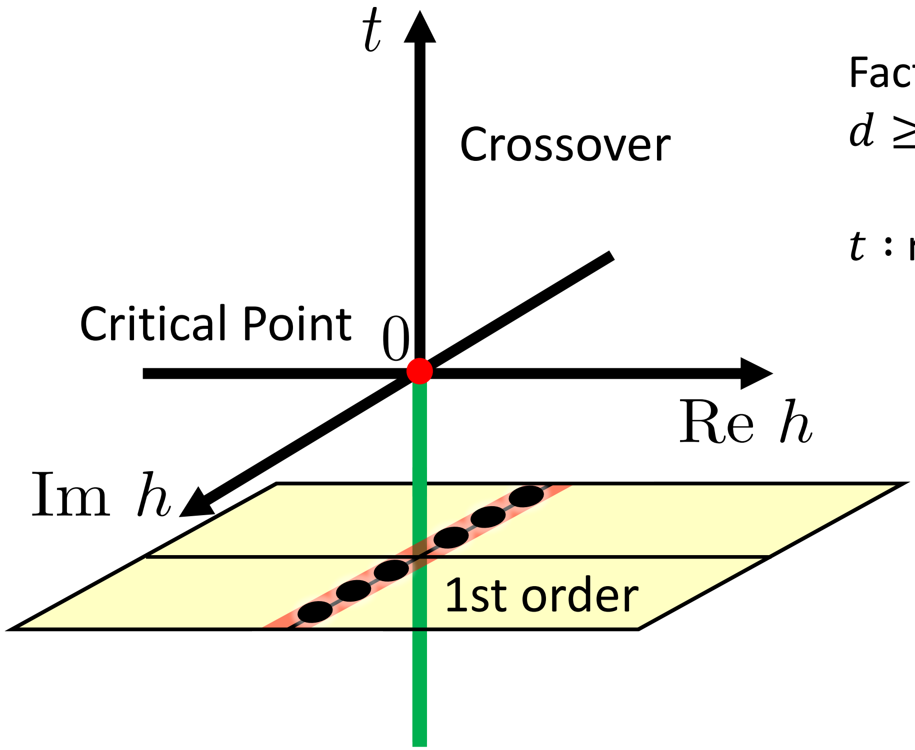
Lee, Yang 1952
Yang, Lee 1952

Lee-Yang Zeros
= Zeros of Z in Complex Parameter space

Lee-Yang's circle theorem
LYZs exist only on imaginary axis.

Ex.) d-dimensional Ising model $s_i = \pm 1$

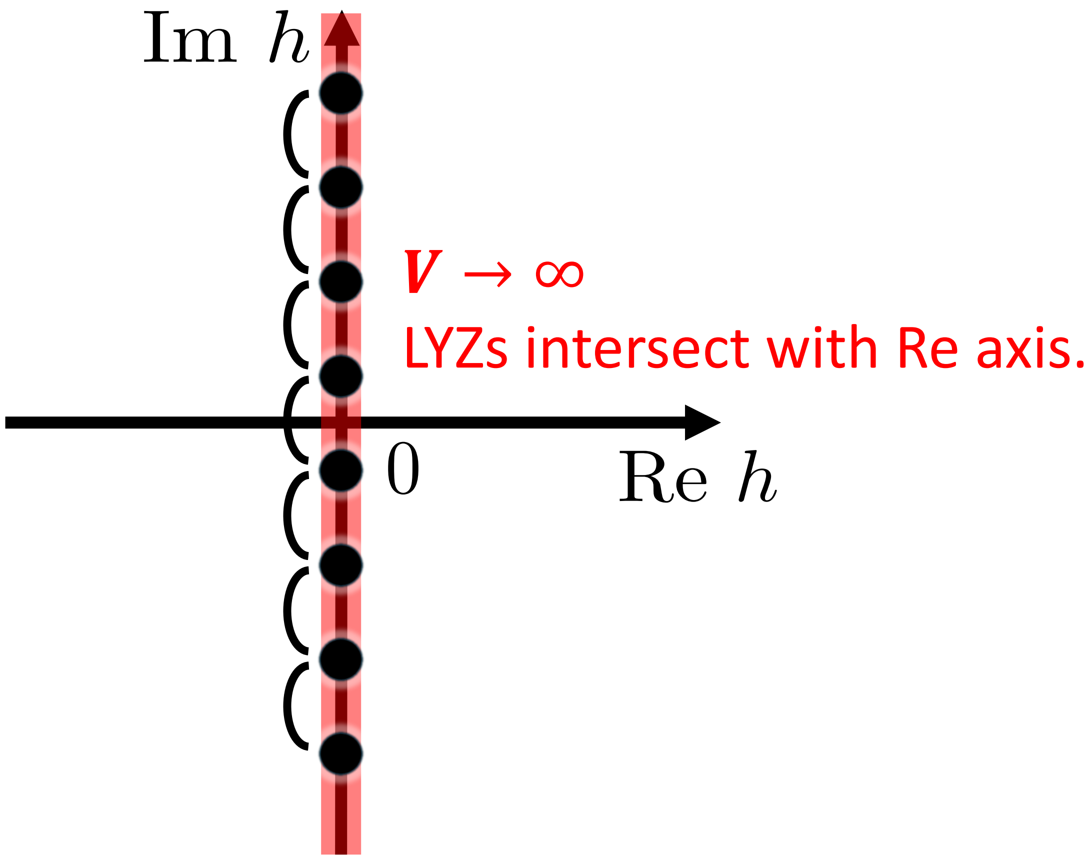
$\boldsymbol{V \to \infty}$
LYZs intersect with Re axis.



Fact:
$d \geq 2$, Ising model has a CP.

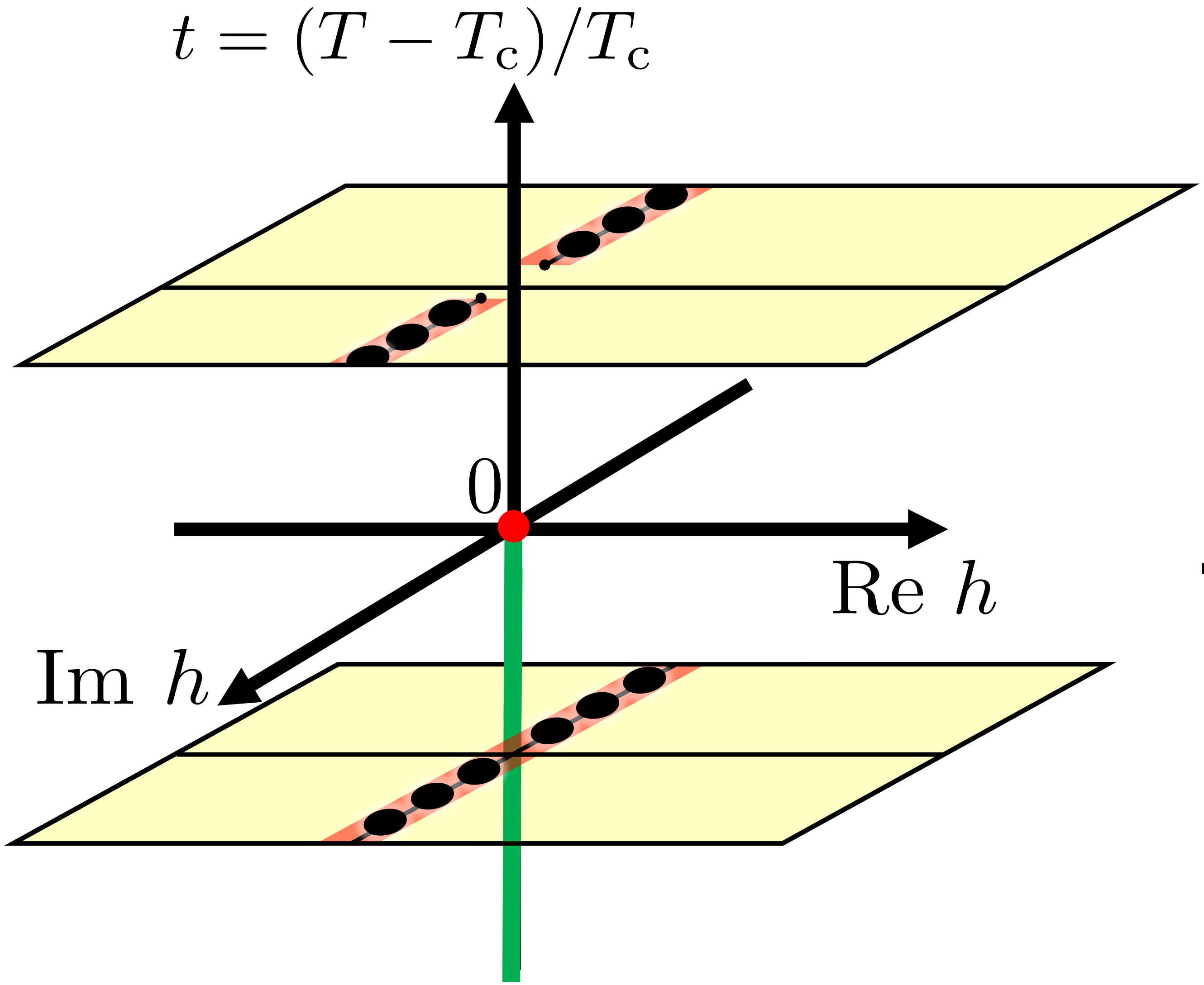
t : reduced temperature

$t < 0$:1st order

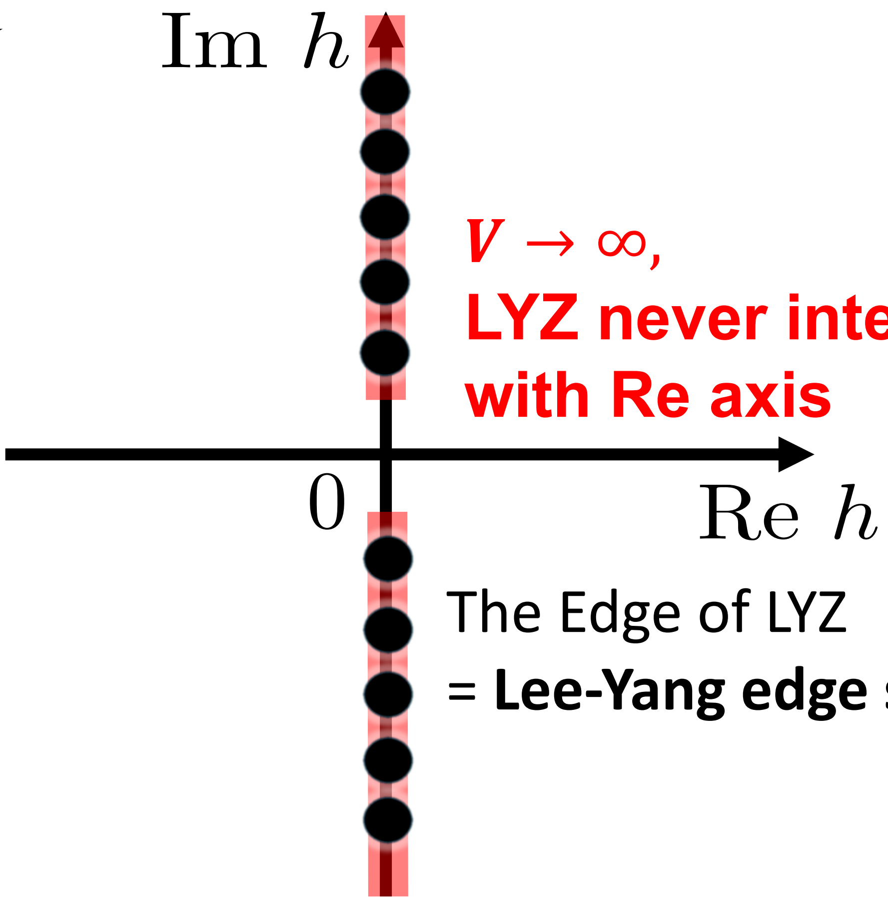


LYZ around Crossover

Lee, Yang 1952
Yang, Lee 1952



$t > 0$: Crossover

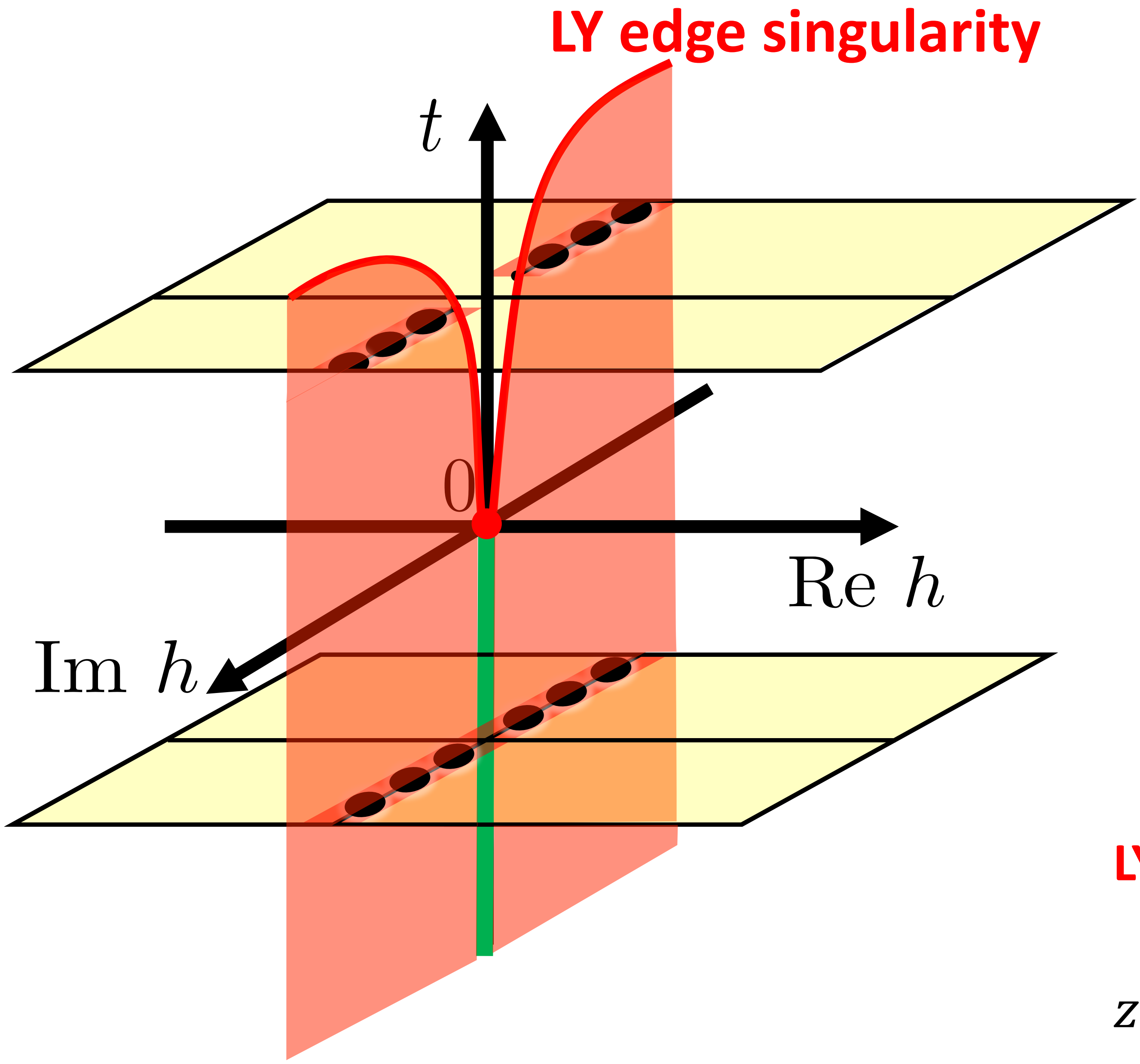


***V* → ∞,**
LYZ never intersect with Re axis

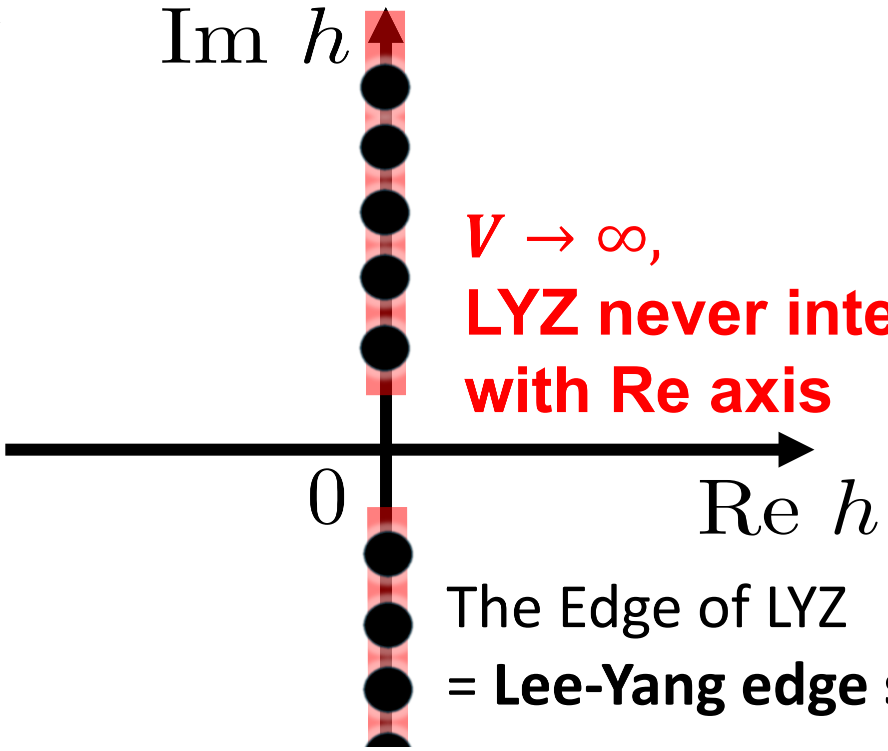
The Edge of LYZ
= **Lee-Yang edge singularity**

Lee-Yang edge singularity

Lee, Yang 1952
Yang, Lee 1952



$t > 0$: Crossover



$V \to \infty$,
LYZ never intersect with Re axis

The Edge of LYZ
= **Lee-Yang edge singularity**

LY edge singularity

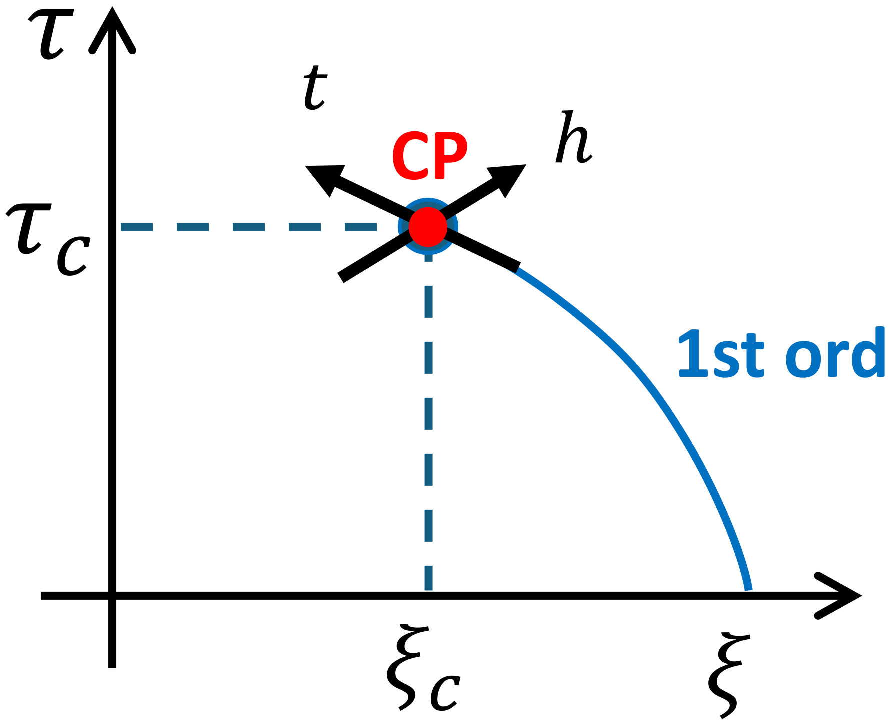
$$\to \quad h_{\mathrm{LYES}}(t) = z_c^{-\beta\delta} t^{\beta\delta}$$

z_c, β, δ : universal constants

Lee-Yang Zero in General model

Lee, Yang 1952
Yang, Lee 1952

General Model w/ Z(2) CP



$\Delta\tau = \tau - \tau_c$
$\Delta\xi = \xi - \xi_c$

$$\begin{pmatrix} t \\ h \end{pmatrix} = A \begin{pmatrix} \Delta\tau \\ \Delta\xi \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \Delta\tau \\ \Delta\xi \end{pmatrix}$$

τ and ξ is written by linear combination t and h around the CP

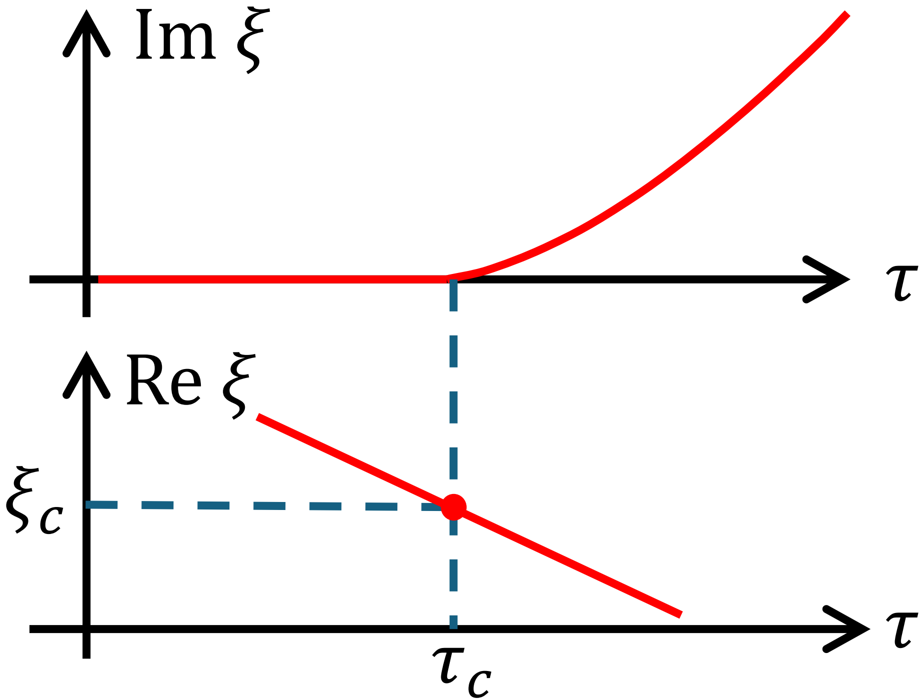
LY edge singularity

$$\rightarrow \quad h_{\mathrm{LYES}}(t) = z_c^{-\beta\delta} t^{\beta\delta}$$

z_c, β, δ : universal constant

$$\begin{cases} \mathrm{Re}\,\xi = \xi_c + c_1 \Delta\tau \\ \mathrm{Im}\,\xi = c_2 \Delta\tau^{\beta\delta} \end{cases}$$

Stephanov (2006)



Recent progress: Lee-Yang zeros

- ◆ 2+1 flavor QCD-CP
- ➢ Clarke, *et al*. (2024)

$$\begin{cases} \mu_c = 422^{+80}_{-35}\mathrm{MeV} \\ T_c = 105^{+8}_{-18}\mathrm{MeV} \end{cases}$$

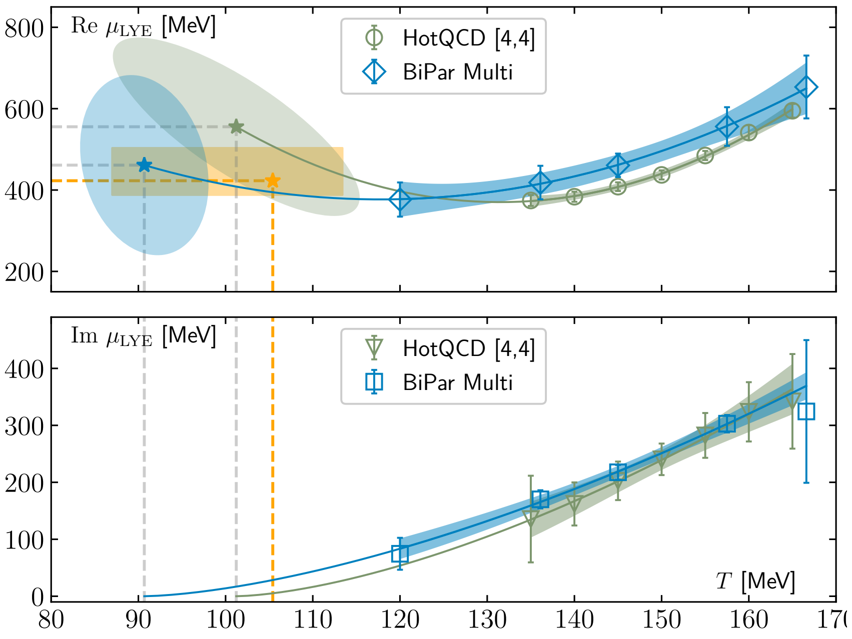
- ➢ Basar (2024)

$$\begin{cases} \mu_c \sim 580\mathrm{MeV} \\ T_c \sim 100\mathrm{MeV} \end{cases}$$

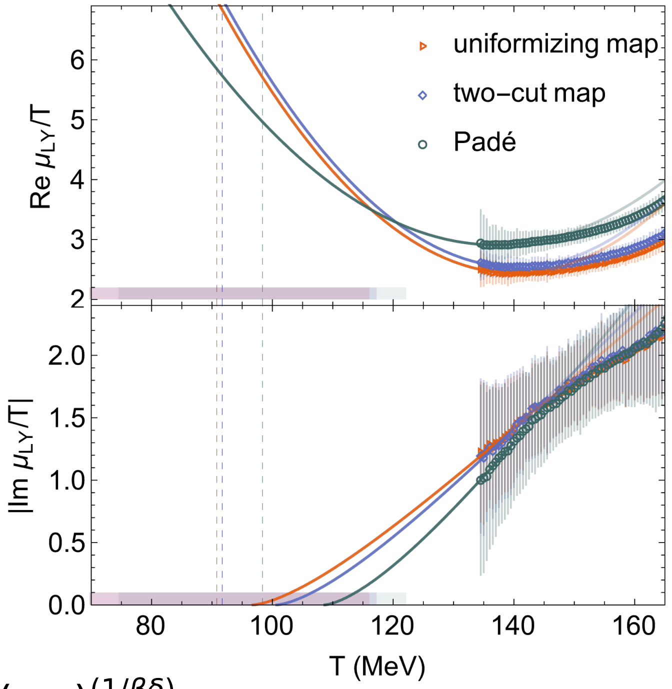
- ➢ Adam, et al.(2025)

$$T_c < 103\mathrm{MeV}$$

or CP does not exist

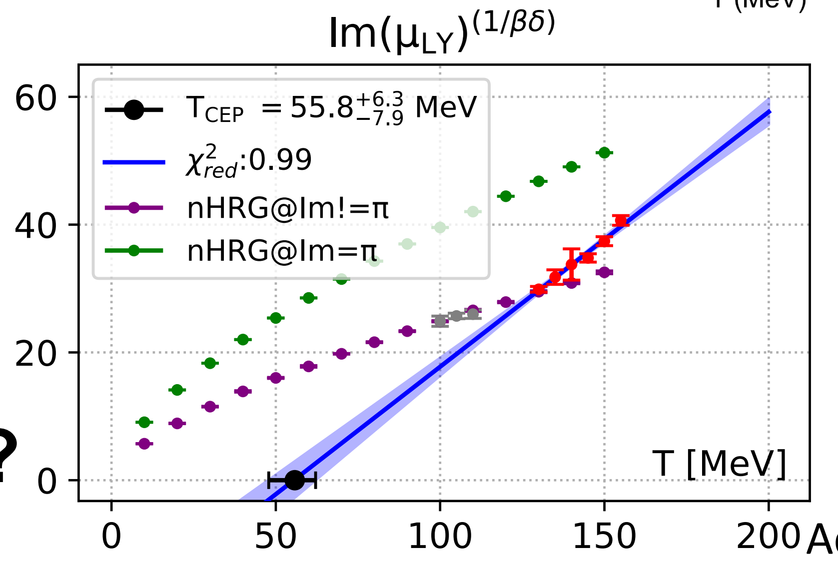


Clarke, *et al*. (2024)



Basar (2024)

How do we treat finite volume effect?



Adam *et al*.(2025)

Problem and Solution

Lee-Yang edge singularity is defined in the **infinite volume limit.**

Lattice simulation is always performed in finite volume.

1st LYZ on finite volume $\stackrel{?}{\simeq}$ **LY edge singularity** $(V \to \infty)$

Solution

Lee-Yang zero ratio method

A new method to locate a CP using **Finite size scaling**

TW, Kitazawa, Kanaya, PRL, 134, 162302 and arXiv: 2508.20422

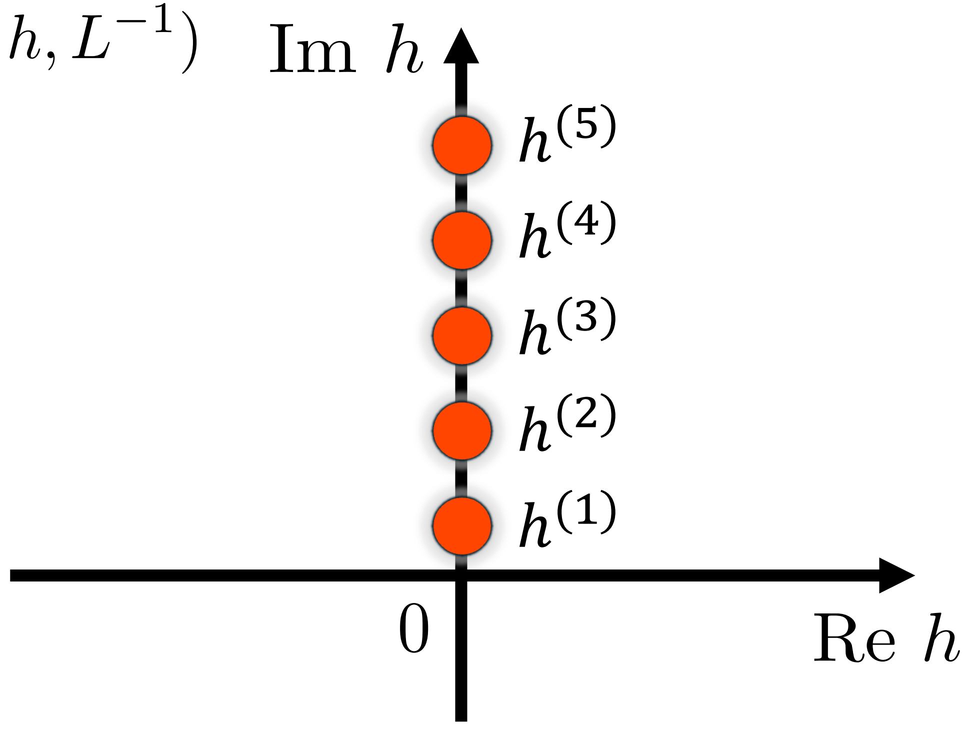
Finite Size Scaling (FSS)

FSS of Free Energy $\beta F_{\rm sing}(t,h,L^{-1}) = -\log Z(t,h,L^{-1})$

$$F_{\rm sing}(t,h,L^{-1}) = \tilde{F}_{\rm sing}(L^{y_t}t, L^{y_h}h)$$

FSS of Partition function

$$Z_{\rm sing}(t,h,L^{-1}) = \tilde{Z}_{\rm sing}(L^{y_t}t, L^{y_h}h)$$



Factorization of Partition function

$$Z_{\rm sing}(t,h,L^{-1}) = \prod_{n=1}^{N}(h - h^{(n)}(t,L^{-1}))$$

LYZ function

$$h = h_{\rm LY}^{(n)}(t,L^{-1})$$

FSS of LYZ

$$L^{y_h}h_{\rm LY}^{(n)}(t,L^{-1}) = \tilde{h}_{\rm LY}^{(n)}(L^{y_t}t)$$

Itzykson, Nucl. Phys. **220** (1983)

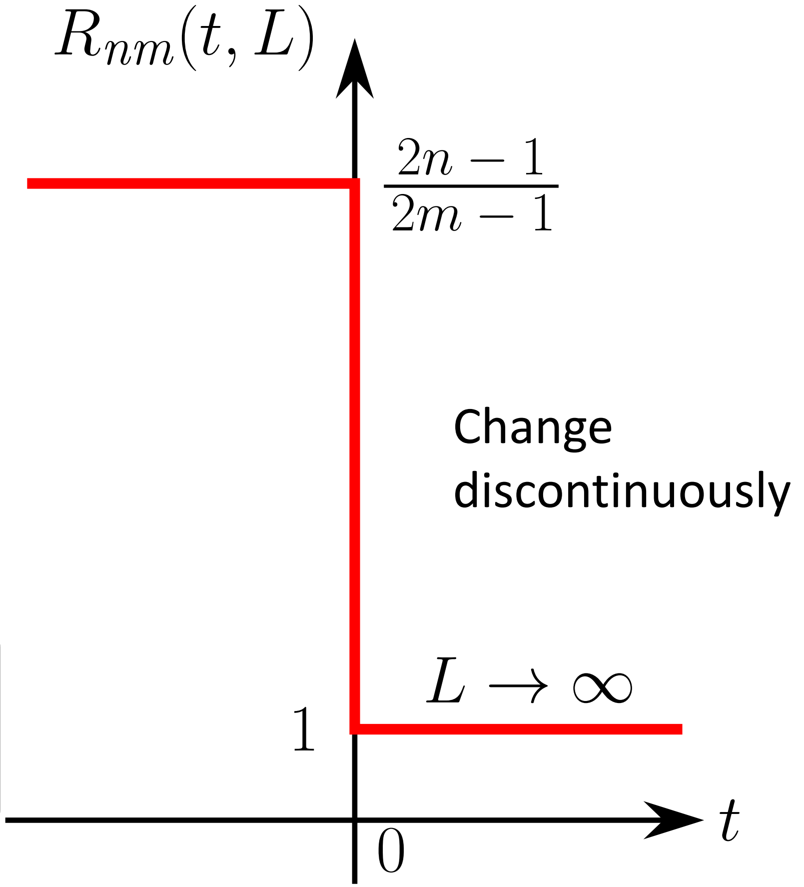
Lee-Yang Zero Ratios

TW, Kitazawa, Kanaya (2025)

LYZ Ratio

$$R_{nm}(t,L) \equiv \frac{h^{(n)}(t,L)}{h^{(m)}(t,L)}$$

$$R_{nm}(t,L) \xrightarrow[L\to\infty]{} \begin{cases} \dfrac{2n-1}{2m-1} & (t<0) \\ 1 & (t>0) \end{cases}$$



◆ $t > 0$: Crossover

All LYZ accumulate at LYES in $V \to \infty$

→ Ratio = 1

◆ $t < 0$: 1st order phase transition

The distance between LYZ and Re-axis is 1:3:5… in $V \to \infty$

→ Ratio = $(2n-1)/(2m-1)$

How is the behavior in finite volume?

LYZR near CP

TW, Kitazawa, Kanaya (2025)

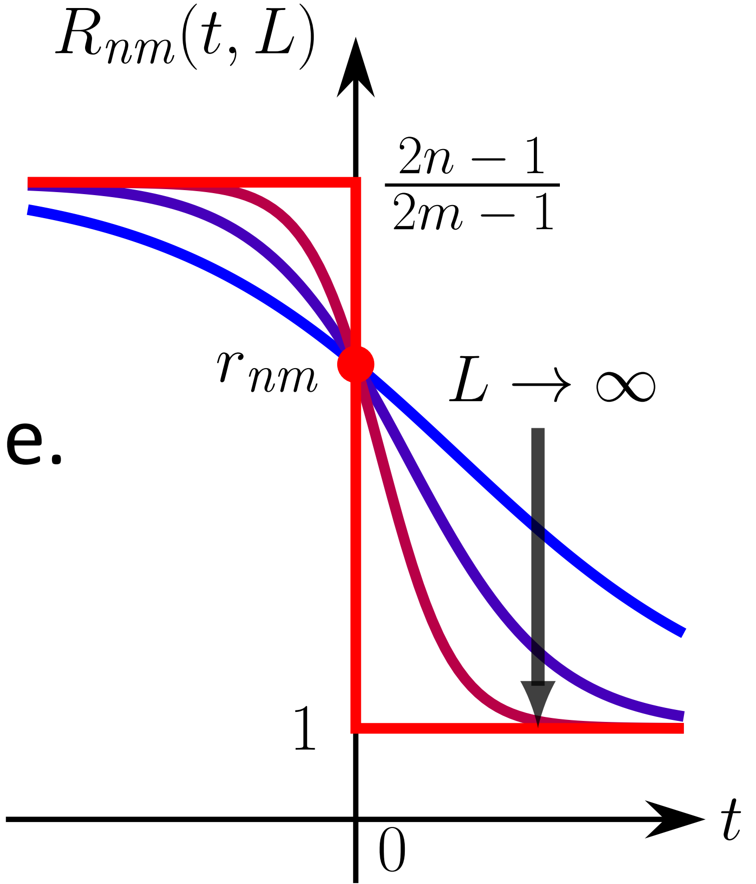
FSS & Taylor Expansion near the CP

$$L^{y_h} h_{\rm LY}^{(n)}(t, L^{-1}) = X_n + Y_n L^{y_t} t + \mathcal{O}(t^2)$$

$$r_{nm} = \frac{X_n}{X_m},\ c_{nm} = r_{nm}\left(\frac{Y_n}{X_n} - \frac{Y_m}{X_m}\right)$$

$$R_{nm}(t, L) \equiv \frac{h^{(n)}(t, L)}{h^{(m)}(t, L)} = \frac{X_n + Y_n L^{y_t} t + \mathcal{O}(t^2)}{X_m + Y_m L^{y_t} t + \mathcal{O}(t^2)}$$

$$R_{nm}(t, L) = \left(r_{nm} + c_{nm} L^{y_t} t + \mathcal{O}(t^2)\right)$$



$\boldsymbol{R_{nm}(t=0)}$ does not depend on the volume.

→ Useful to locate CP

Cf. Similar to Binder Cumulant $B_4(t = 0) = b_4$

Crossing Point = Critical Point

3-dimensional Ising model

TW, Kitazawa, Kanaya, 2508.20422

Numerical Setting

- Algorithm：Wolff Algorithm
- Determine the 1st~4th LYZ using reweighting method
- Volume $L = 16, \cdots, 96, 128, 192, 256$
- # of measurement = 2×10^6

Previous result: Ferrenberg et. al. (2018)

$L = 16, \cdots, 1024$

$\#\text{measurement} \sim 5 \times 10^6 \times 10^4$

- χ^2 fitting

  Non-linear fitting to $(L^{y_t} t)^2$

$$R_{nm}(t, L) = r_{nm} + c_{nm} L^{y_t} t + d_{nm} (L^{y_t} t)^2$$

Aim : To evaluate r_{nm} and to compare with Binder cumulant.

How to determine the LYZ

We cannot calculate partition function in Monte Carlo method.
However, we can detect LYZ using **reweighting method**.

$$\langle \hat{O} \rangle_0 = \frac{1}{Z_0} \sum_{\sigma_i} \hat{O}(t_0, h_0) e^{-H(t_0,h_0)} \qquad Z_0 = \sum_{\sigma_i} e^{-H(t_0,h_0)}$$

Normalized Partition function

$$\frac{Z(t, h_R + ih_I)}{Z(t, h_R)} = \frac{Z_0^{-1} \sum_{\sigma_i} e^{-(H(t,h_R+ih_I)-H(t_0,h_0))} e^{-H(t_0,h_0)}}{Z_0^{-1} \sum_{\sigma_i} e^{-(H(t,h_R)-H(t_0,h_0))} e^{-H(t_0,h_0)}}$$

$$= \frac{\langle e^{-(H(t,h_R+ih_I)-H(t_0,h_0))} \rangle_0}{\langle e^{-(H(t,h)-H(t_0,h_0))} \rangle_0}$$

The zeros of this function is Lee-Yang zeros because the denominator is always positive.
To avoid the sign problem (Overlap problem), we need to generate a lot of config..

FSS of LYZ

TW, Kitazawa, Kanaya, 2508.20422

LYZ function

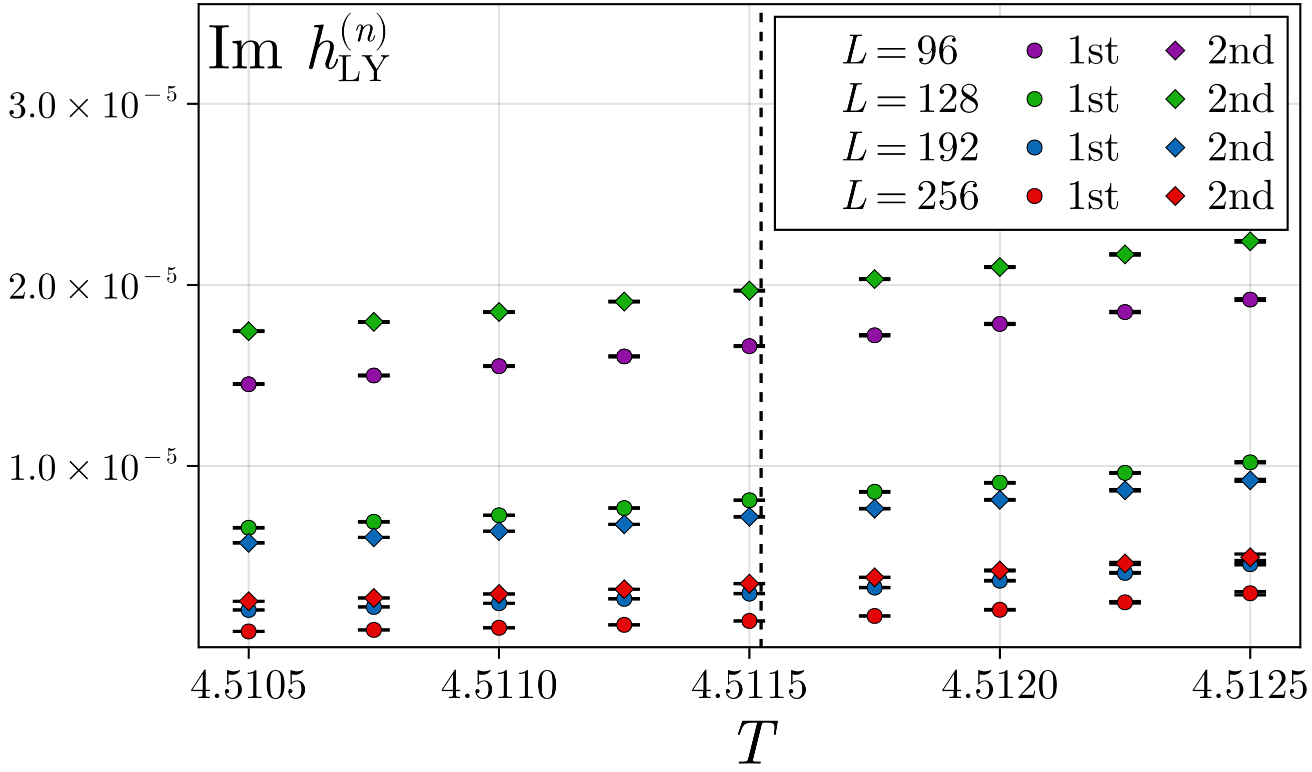
$$h = h_{\mathrm{LY}}^{(n)}(t, L^{-1})$$

FSS of LYZ in Ising model

$$L^{y_h} h_{\mathrm{LY}}^{(n)}(t, L^{-1}) = \tilde{h}_{\mathrm{LY}}^{(n)}(L^{y_t} t)$$

[Itzykson, Nucl. Phys. **220** (1983)]

LYZ w/ error bars



Dashed line : T_c (Ferrenberg 2018)

Below $T = T_c$, LYZ on finite volume never intersect with real axis.

FSS of LYZ

TW, Kitazawa, Kanaya, 2508.20422

LYZ function

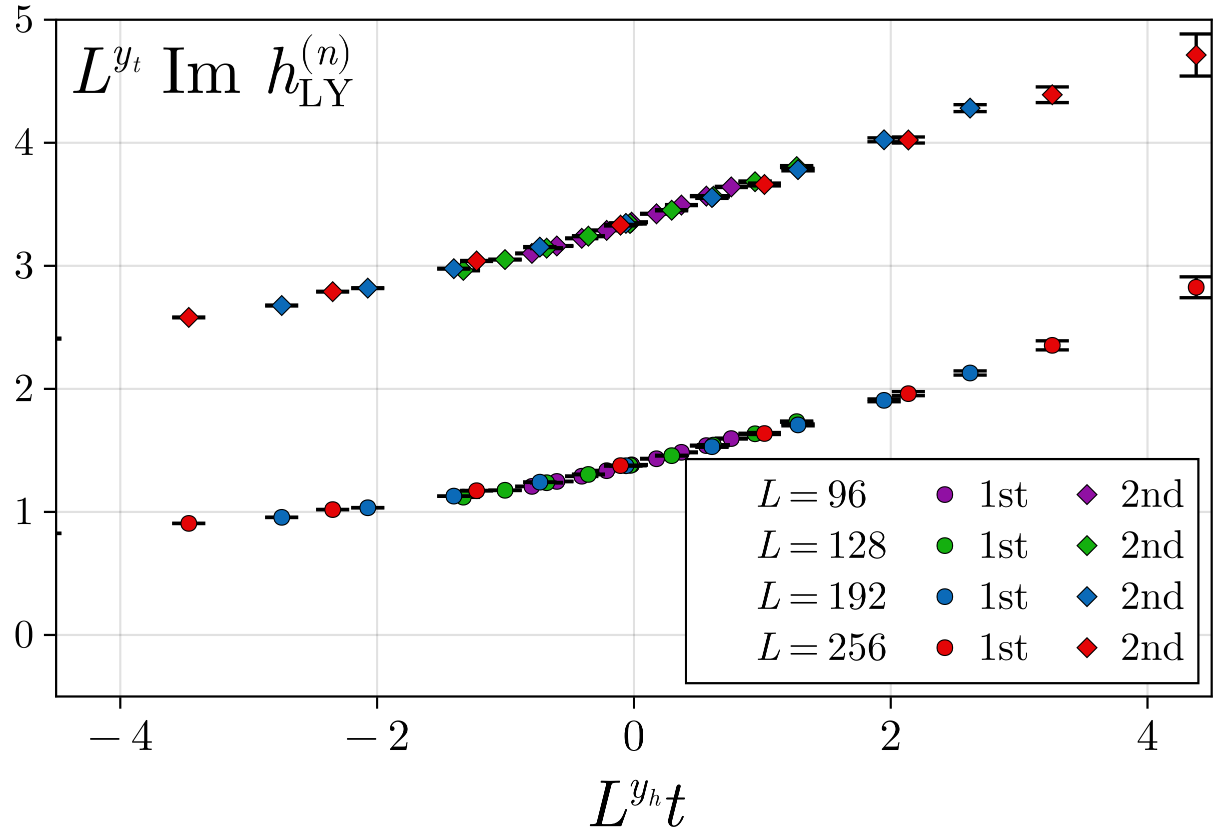
$$h = h_{\mathrm{LY}}^{(n)}(t, L^{-1})$$

FSS of LYZ in Ising model

$$L^{y_h} h_{\mathrm{LY}}^{(n)}(t, L^{-1}) = \tilde{h}_{\mathrm{LY}}^{(n)}(L^{y_t} t)$$

[Itzykson, Nucl. Phys. **220** (1983)]

LYZ w/ error bars



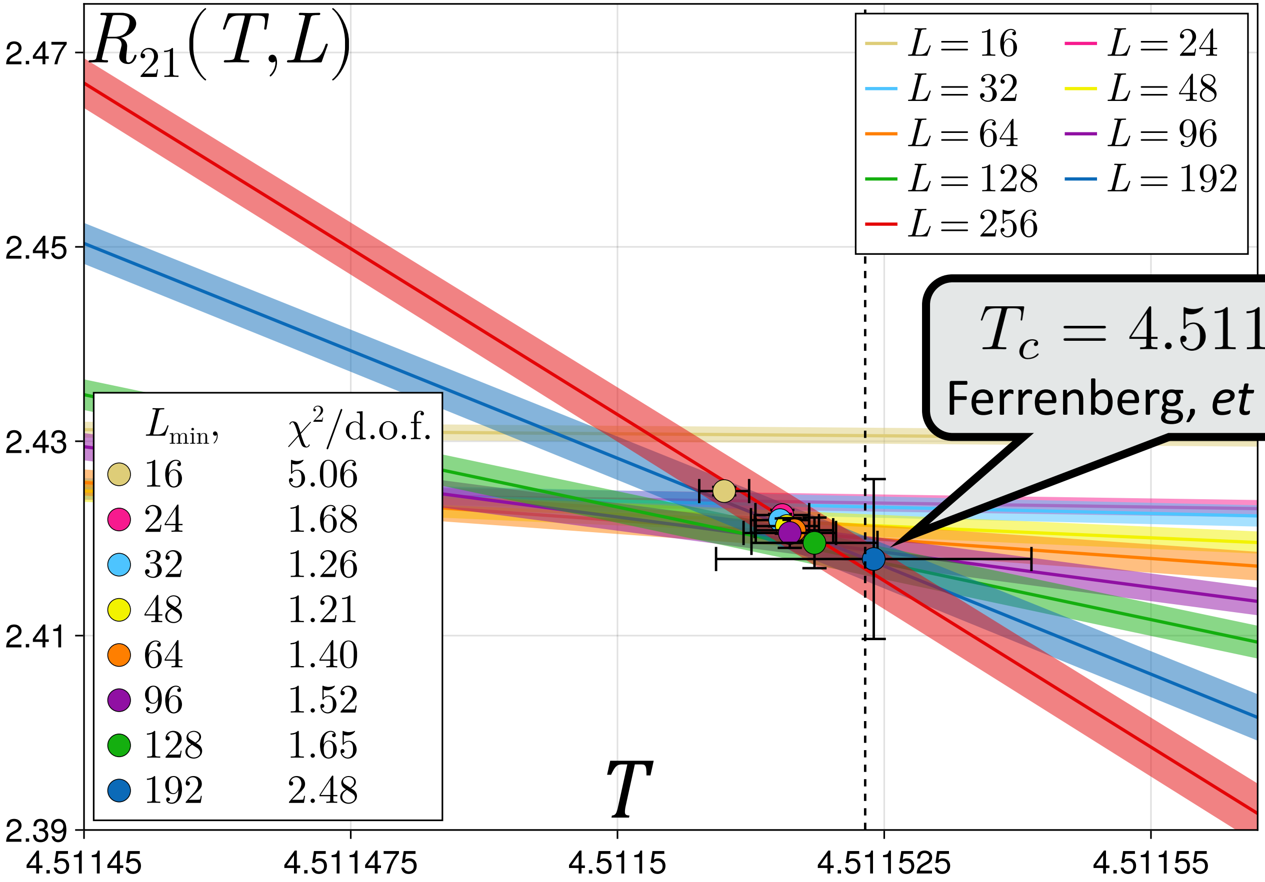
LYZ accumulates a single line.

→ LYZ obey Finite Size Scaling law.

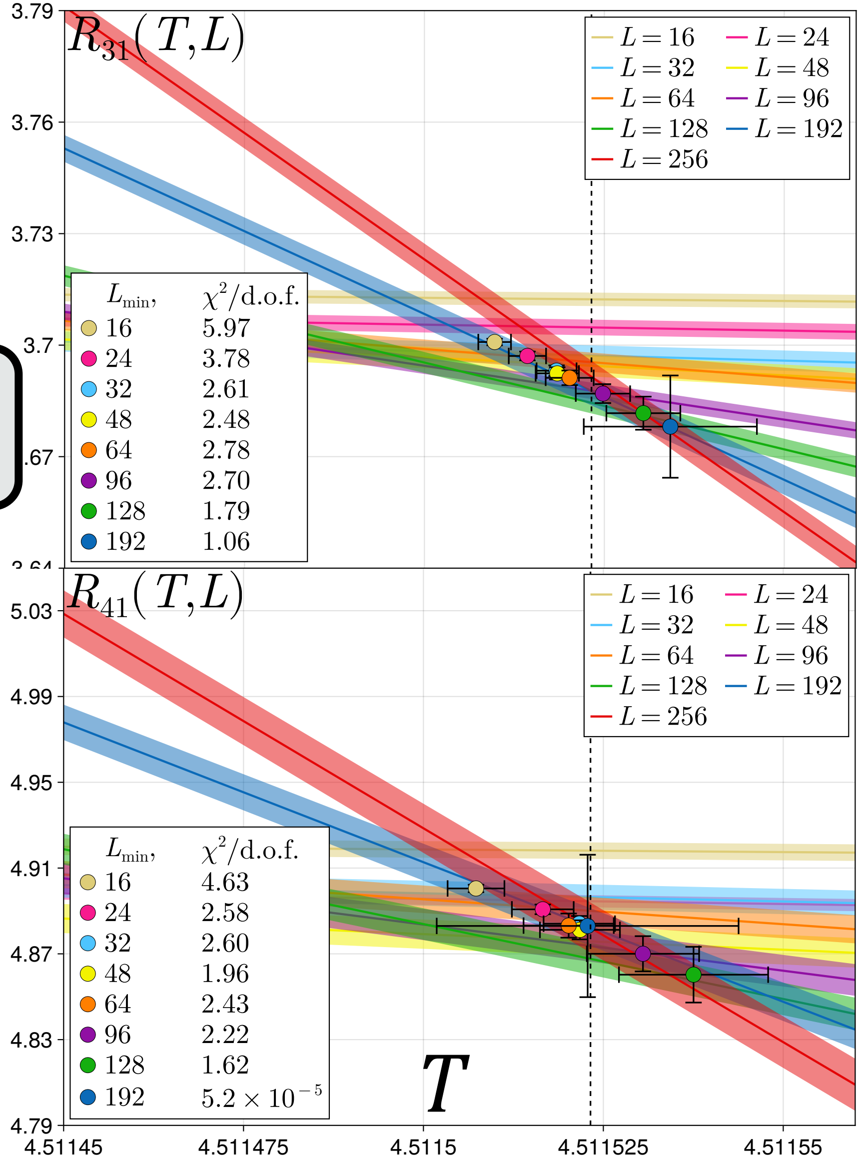
LYZ Ratio

TW, Kitazawa, Kanaya, 2508.20422

$2^{nd}/1^{st}$ LYZ Ratio



	$\mathbf{T_c}$	$\boldsymbol{y_t}$
R_{21} Non-linear	4.5115185(59)	1.587(12)
B_4 (Ferrenberg 2018)	4.5115232(1)	1.58723(22)

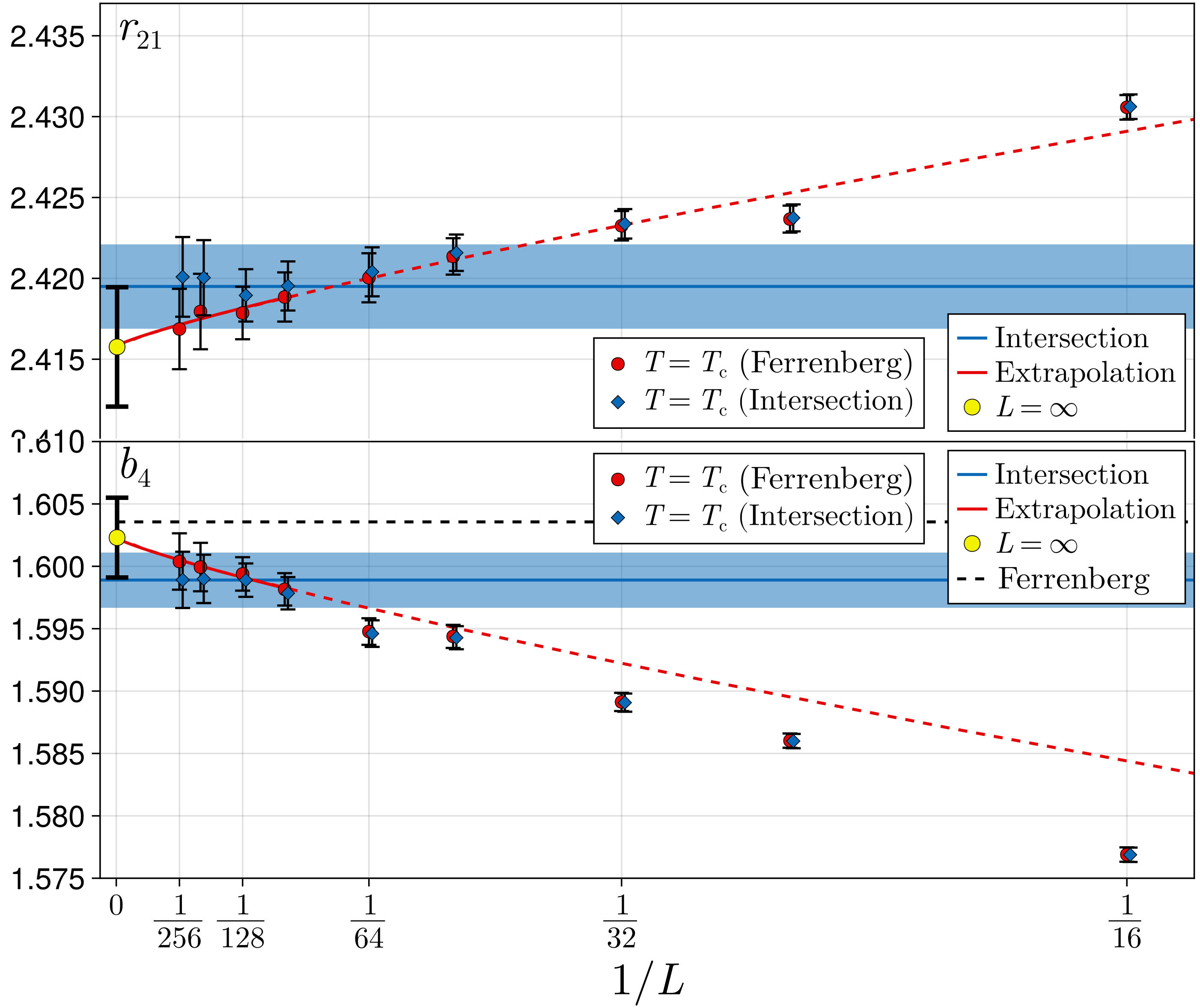


18/30

Consistent with previous result within error

Violation of FSS

TW, Kitazawa, Kanaya, 2508.20422



FSS violation on leading order

$$r_{nm}(L) = r(1 + bL^{-\omega_1})$$

$\omega_1 \simeq 0.83$

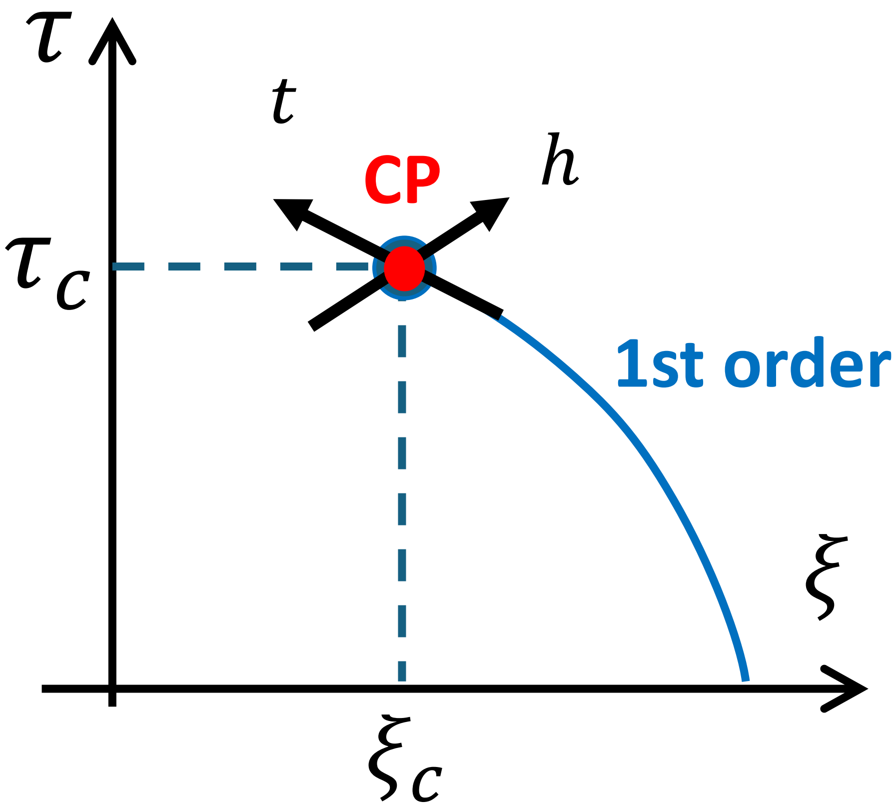
r and b are fitting parameter

Fitting in $L = 256 \sim 96$

Ferrenberg 2018

$$T_c = 4.5115232(1)$$

General System w/ Z_2-CP



Linear Approximation

$$\begin{pmatrix} t \\ h \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix}$$

$$h^{(n)} L^{y_h} = \tilde{h}^{(n)}_{\mathrm{LY}}(tL^{y_t}) \simeq i(X_n + Y_n t L^{y_t})$$

For fixed $\tau \in \mathbb{R}$,
Search LYZ in Complex ξ-plane

$$\bar{y} = y_t - y_h < 0$$

$$\begin{cases} \xi_R^{(n)} = & \xi_c - \dfrac{a_{21}}{a_{22}}\delta\tau + \mathcal{O}(L^{2\bar{y}}) \\ \xi_I^{(n)} = & \dfrac{X_n}{a_{22}} L^{-y_h} + \dfrac{\det A \;\; Y_n}{a_{22}^2}\delta\tau L^{\bar{y}} + \mathcal{O}(L^{2\bar{y}}) \end{cases}$$

$$L \to \infty$$

Consistent with general LYES

$$\begin{cases} \xi_R^{(n)} = & \xi_c - \dfrac{a_{21}}{a_{22}}\delta\tau \\ \xi_I^{(n)} = & \text{const.} \times \delta\tau^{\beta\delta} \end{cases}$$

Stephanov (2006)

General System w/ Z_2-CP

$$\xi_I^{(n)} L^{y_h} = \frac{X_n}{a_{22}} + \frac{\det A \; Y_n}{a_{22}^2} L^{y_t} \delta\tau + \mathcal{O}\left(L^{2\bar{y}}\right)$$

$$y_t - y_h = -0.894 \quad \text{for 3d-Z(2)}$$

$$C = \frac{\det A}{a_{22}} \left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1} \right), \quad D = \frac{a_{12}^2}{a_{22}^2} (Y_1^2 - Y_2^2)$$

$$\mathcal{R}_{nm}(\tau, L) = \frac{\xi_I^{(n)}(\tau, L)}{\xi_I^{(m)}(\tau, L)} = \left(r_{nm} + C_{nm}(L^{y_t} \delta\tau) + \mathcal{O}(\tau^2)\right)\left(1 + D_{nm} L^{2\bar{y}} + \mathcal{O}\left(L^{4\bar{y}}\right)\right)$$

Ising term

Mixed term from $a_{12} \neq 0$

Ising Model $\quad R_{nm}(t, L) = \left(r_{nm} + c_{nm} L^{y_t} t + \mathcal{O}(t^2)\right)$

- In the $L \to \infty$ limit, **Crossing Point = Critical Point !**
- r_{nm} is dependent on only universality class

» Guideline for critical point search
Determination of universality class

Comparison to Binder cumulant method

$$\bar{y} = y_t - y_h = -0.894 \quad \text{for 3d-Z(2)}$$

Binder Cumulant

$$\mathcal{B}_4(t, h, L^{-1}) = \left(b_4 + c_4 t L^{y_t} + \mathcal{O}(t^2)\right) \times \left(1 + d_4 L^{\bar{y}} + \mathcal{O}(L^{2\bar{y}})\right)$$

Jin, *et al*. PR**D96** (2017)

LYZ Ratio

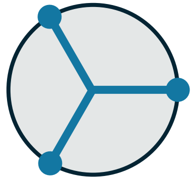
$$\mathcal{R}_{nm}(\tau, L) = \left(r_{nm} + C_{nm}(\tau L^{y_t}) + \mathcal{O}(\tau^2)\right) \times \left(1 + D_{nm} L^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$$

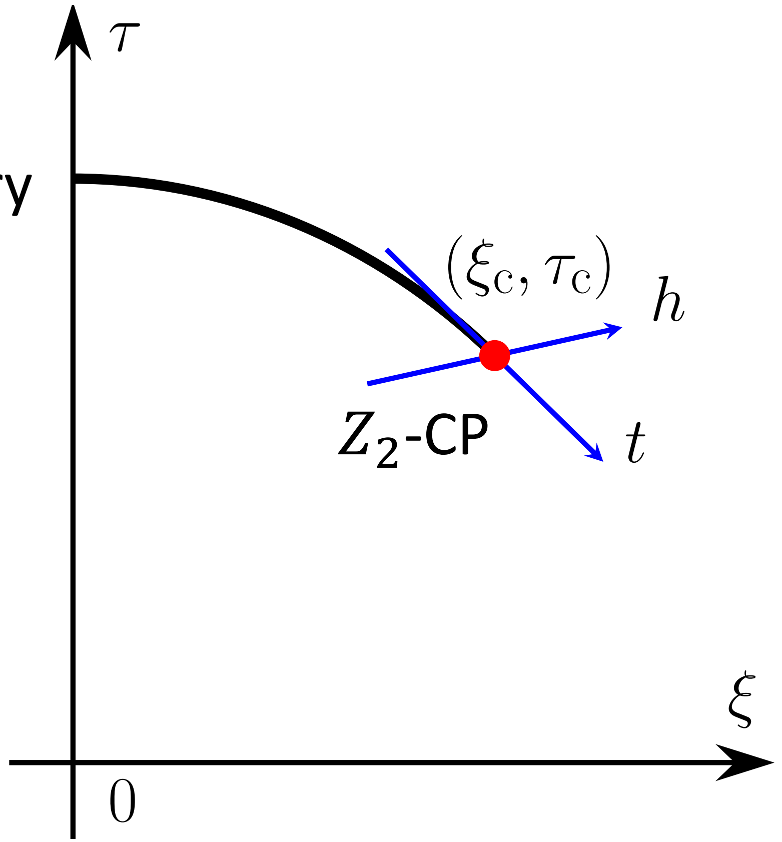
Purely magnetic

Parameter mixing ($a_{21} \neq 0$)

◆ **The suppression of finite volume effect in LYZR is larger than Binder cumulant.**

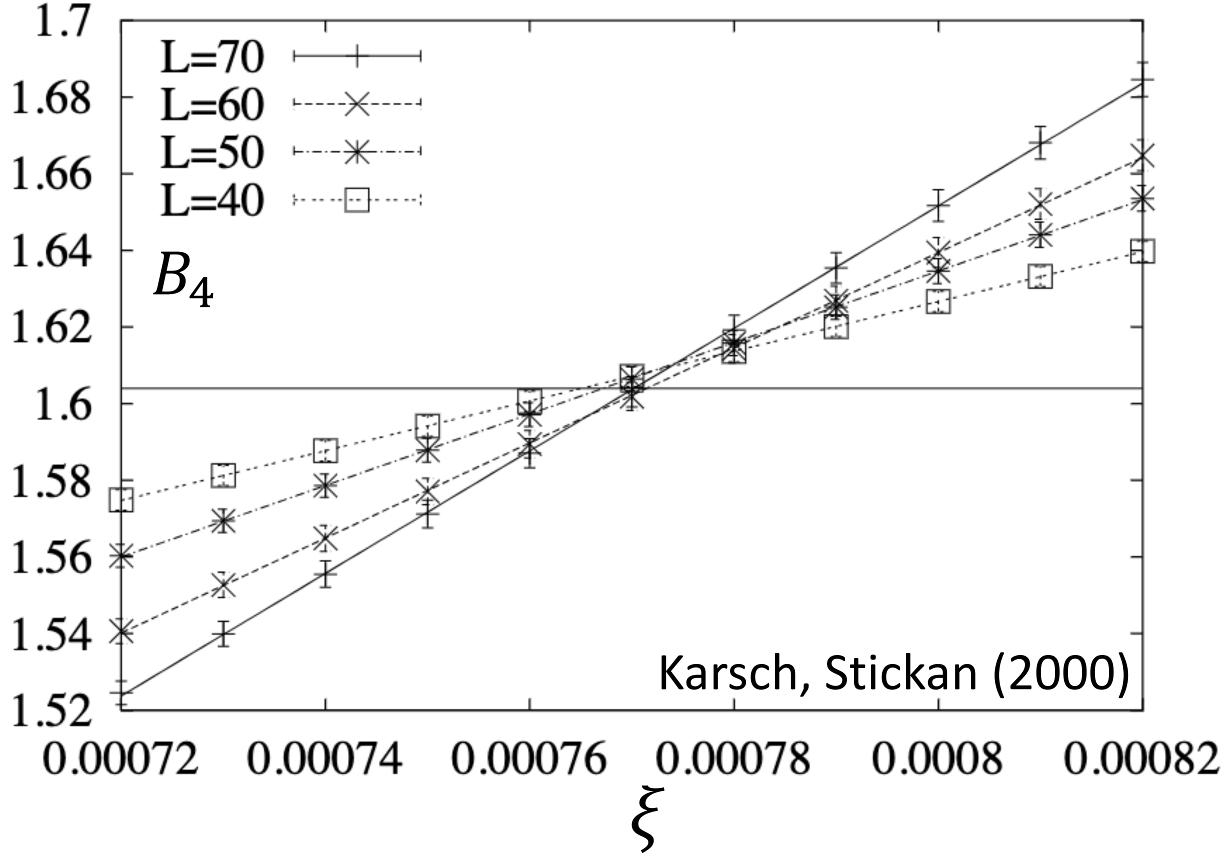
3d-3state Potts model

$$H = -\tau \sum_{i,j} \delta_{\sigma_i,\sigma_j} - \xi \sum_i \delta_{\sigma_i,1} \quad (\sigma_i = 1,2,3 =$$  $)$



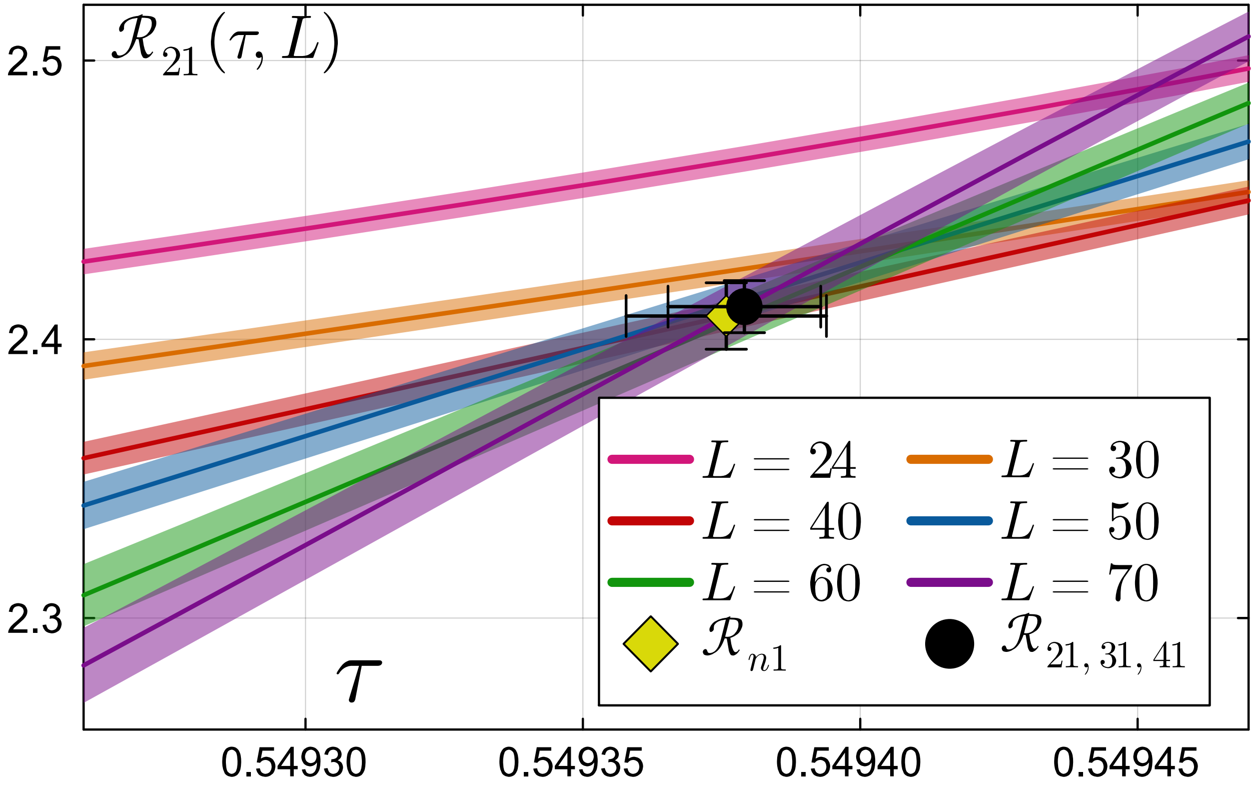
# of measurement=10^6
Heatbath method

Binder cumulant

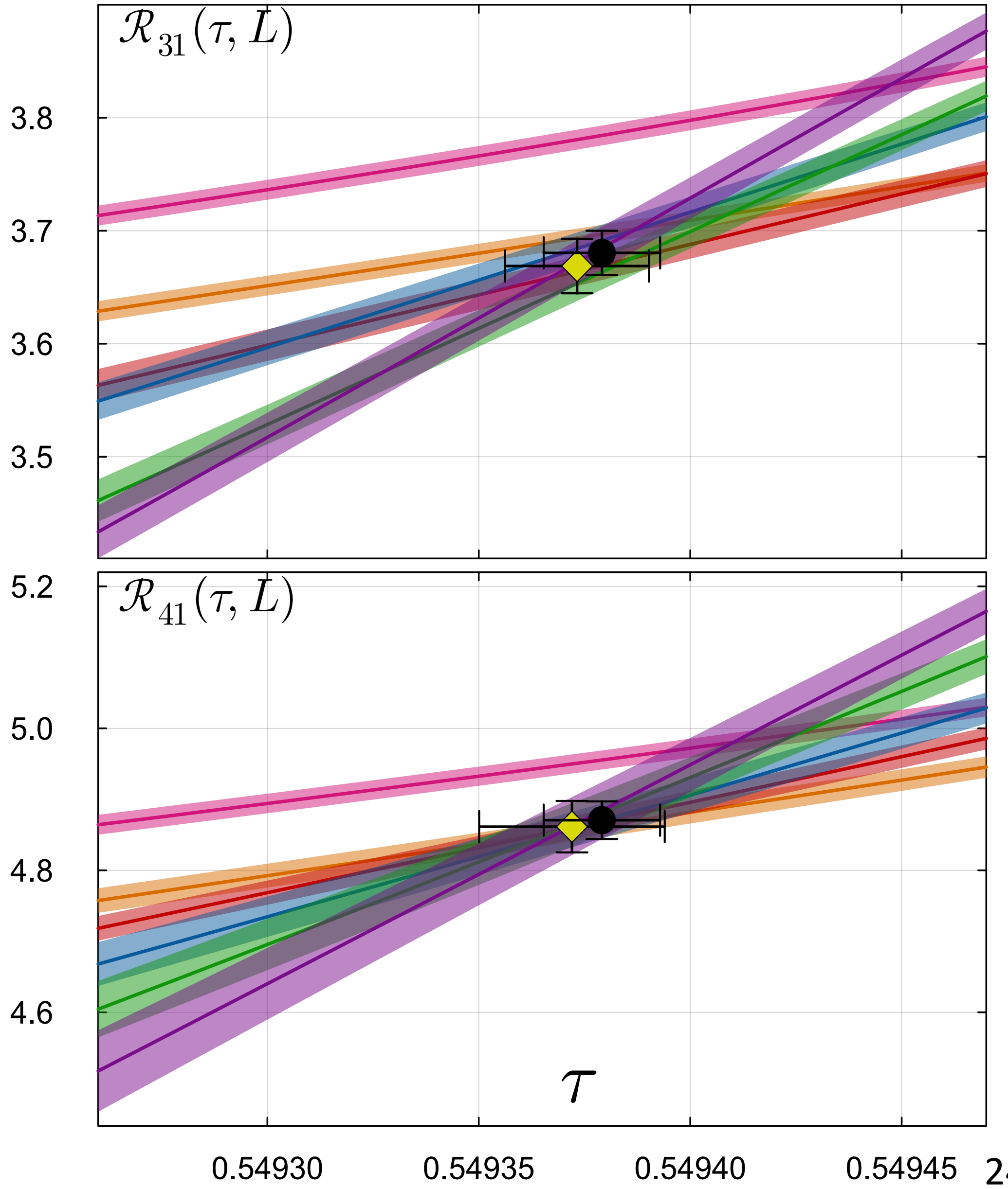


3d-3state Potts model

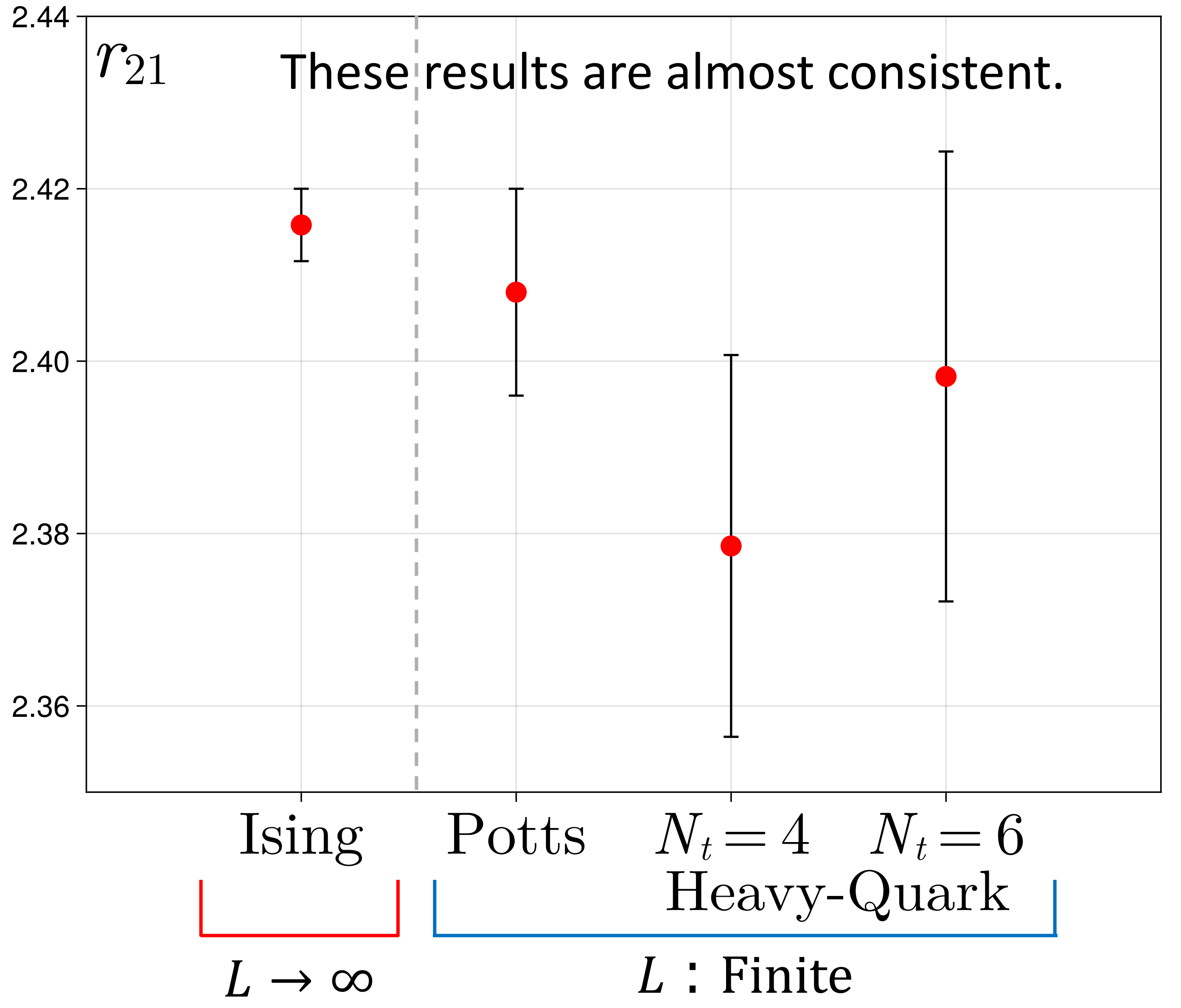
2^{nd} /1^{st} LYZR in Potts model



	τ_c
$\mathcal{R}_{21}$	0.549375(18)
B_4(Karsch 2000)	0.54938(2)



Comparing the r_{21}

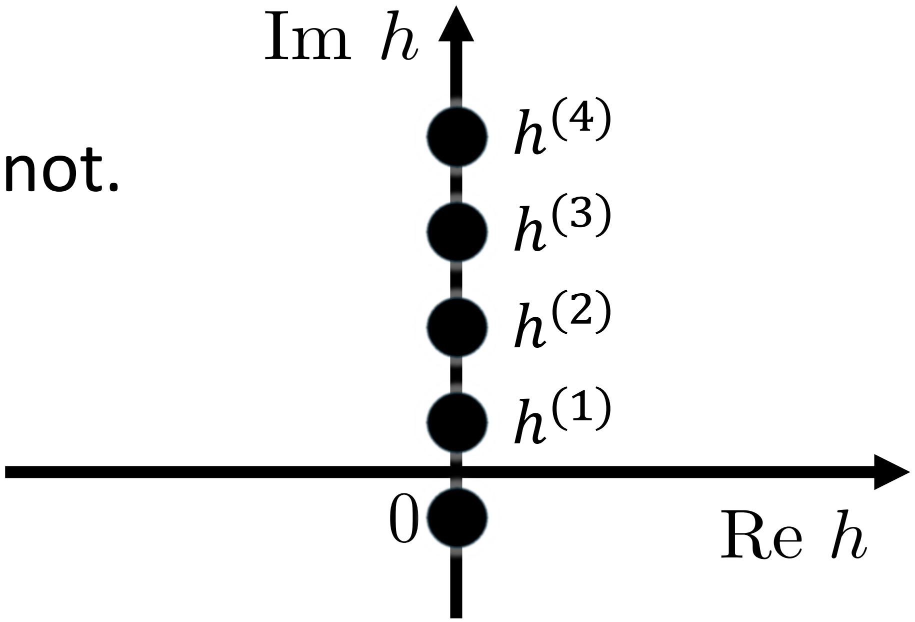


Single LYZ method

TW, Kitazawa, Kanaya, 2508.20422

$N_f = 2 + 1$ QCD

1st LYZ is investigated, but more than 2nd is not.



LYZ function

$$h = h_{\mathrm{LY}}^{(n)}(t, L^{-1})$$

FSS of LYZ in Ising model

$$L^{y_h} h_{\mathrm{LY}}^{(n)}(t, L^{-1}) = \tilde{h}_{\mathrm{LY}}^{(n)}(L^{y_t} t)$$

[Itzykson, Nucl. Phys. **220** (1983)]

Expansion

$$\tilde{h}_{\mathrm{LY}}^{(n)}(L^{y_t} t) = X_n + Y_n L^{y_t} t + \mathcal{O}(t^2)$$

This value is independent to V at t = 0

Crossing point = Critical Point

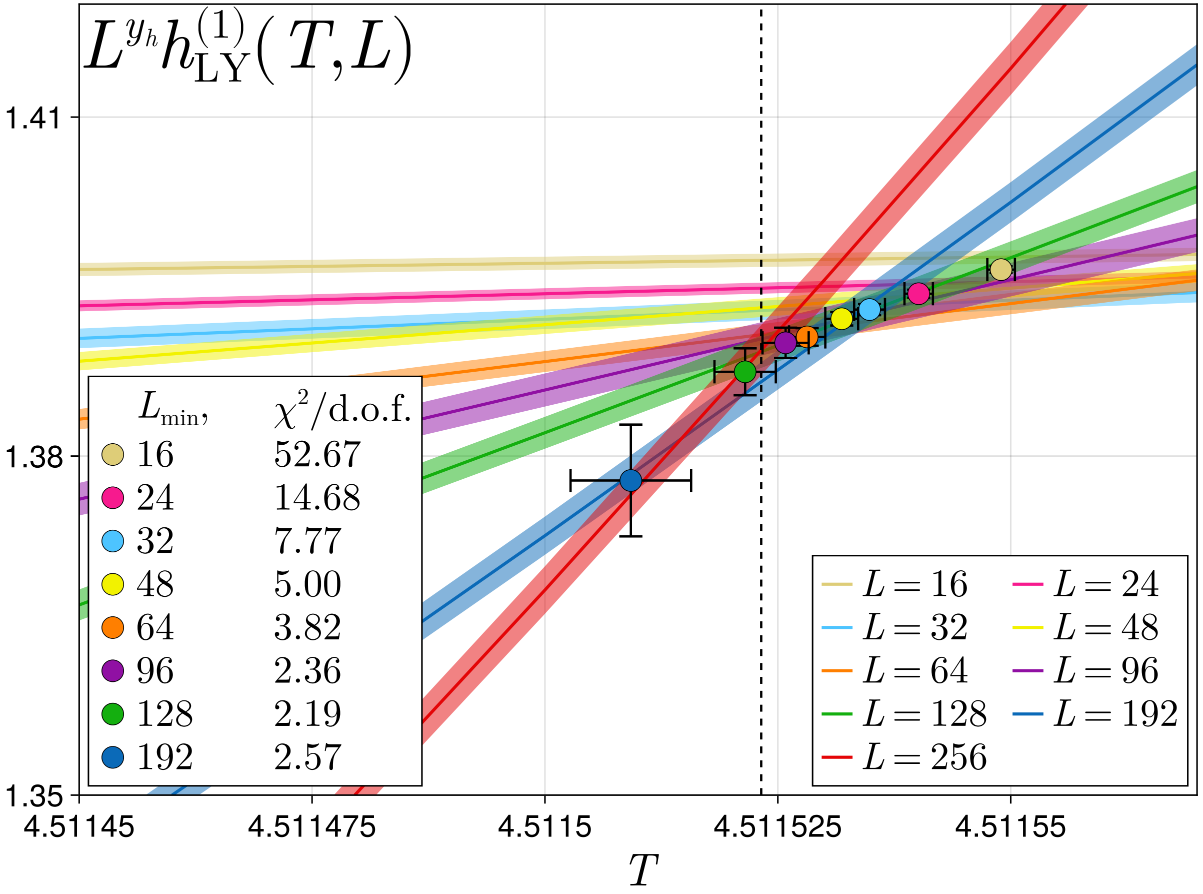
※We must assume critical exponent.

Single LYZ method in Ising model

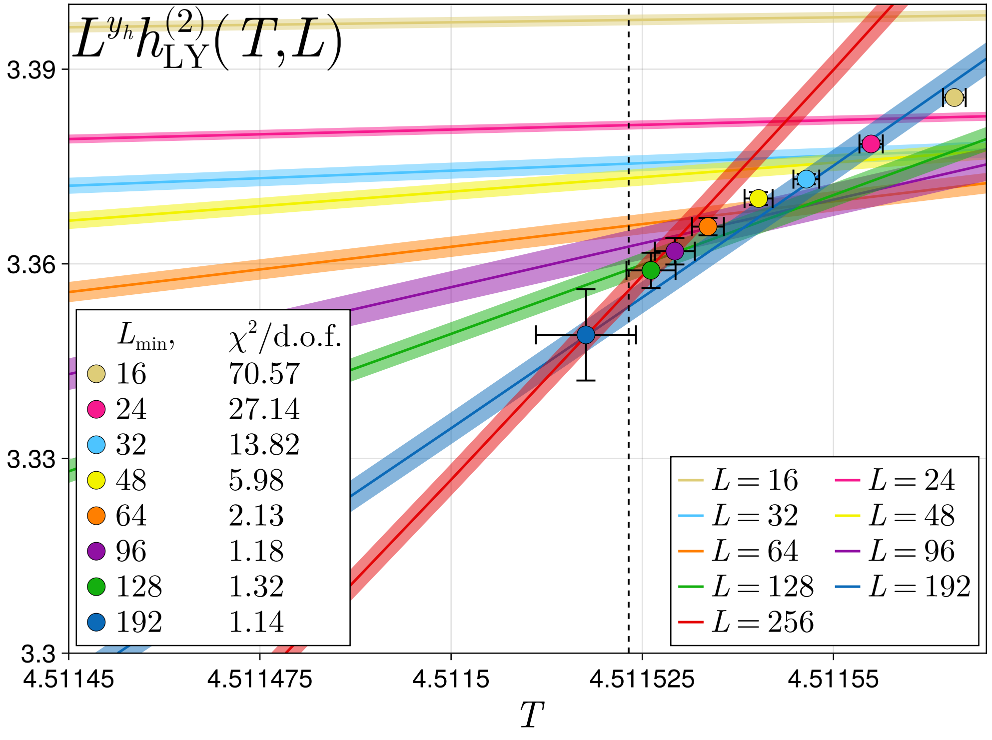
TW, Kitazawa, Kanaya, 2508.20422

$$L^{y_h} h_{\mathrm{LY}}^{(n)}(L^{y_t} t) = X_n + Y_n L^{y_t} t + \mathcal{O}(t^2)$$

1st LYZ in Ising model



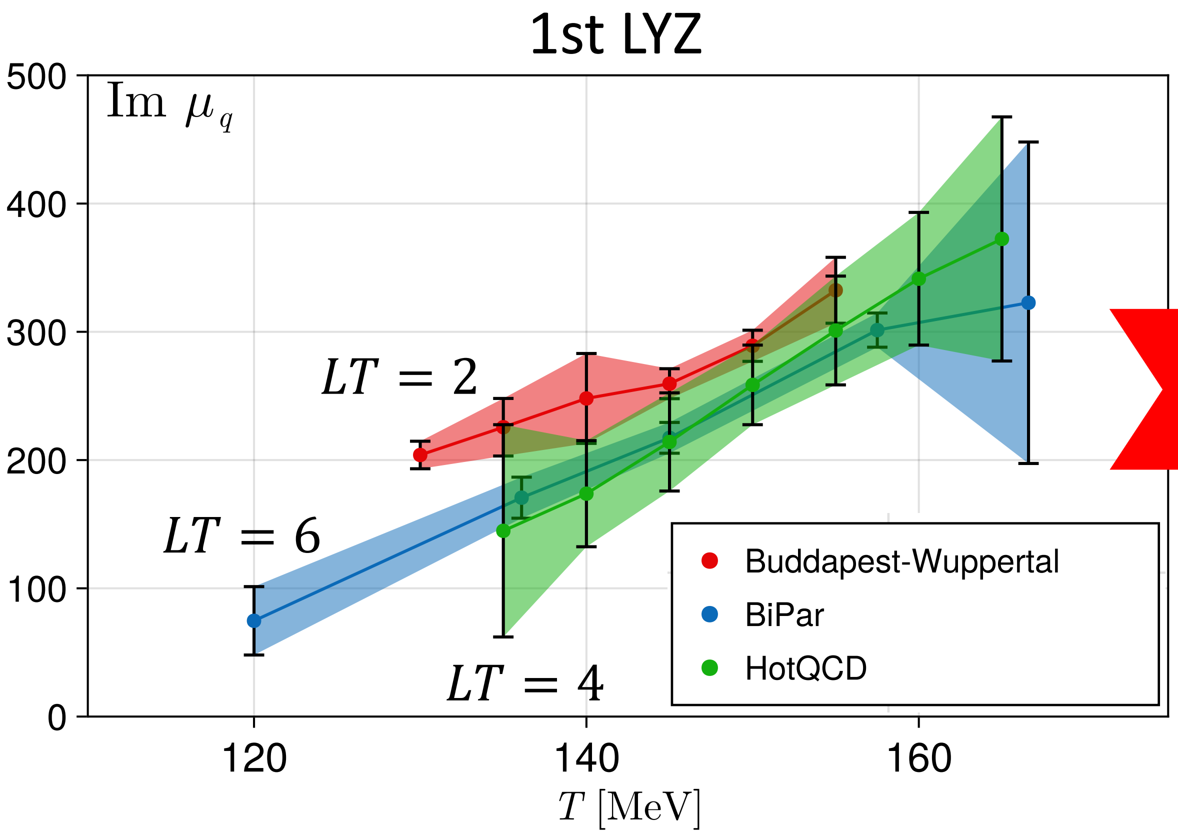
2nd LYZ in Ising model

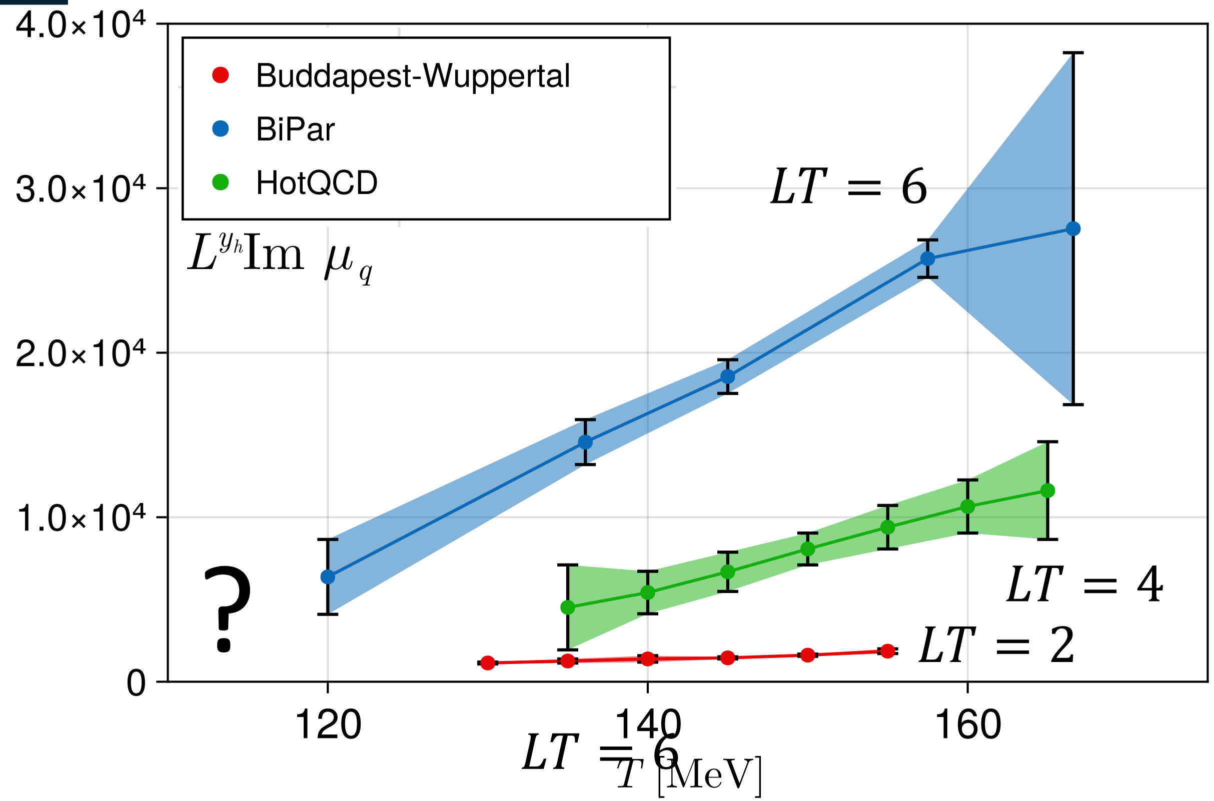


Finite volume effect is large.
However, this value is a good indicator of CP.
(If we know the universality class.)

Single LYZ method in N_f=2+1 QCD

$$L^{y_h} h^{(n)}_{\mathrm{LY}}(L^{y_t} t) = X_n + Y_n L^{y_t} t + \mathcal{O}(t^2)$$





Pure imaginary chemical potential
+ Pade approximation

Lattice result is still high temperature.
$T_c = 90 - 100$ MeV?

Summary

- ✓ We propose a novel and general method to locate the critical point using the Lee-Yang zeros with the finite volume effect. - **Lee-Yang Zeros Ratio method**
- ✓ We verify the LYZR method numerically.
  - ➢ 3d-Ising model
  - ➢ 3d 3state Potts model
  - ➢ Heavy-Quark QCD (Kitazawa's talk on Wendnday)

Future Works

- ☐ Application to 2+1 flavor QCD
  →Roberge-Weiss phase transition,
  QCD critical point (We need 2nd LYZ!)
- ☐ Application to analytic system (Mean field, Large-N etc.)
- ☐ Application to other phase transition (Quantum PT, KT)

Application of LYZR method

Lee-Yang zeros Ratio method

- Ising Model $R_{nm}(t,L) = \left(r_{nm} + c_{nm}L^{y_t}t + \mathcal{O}(t^2)\right)$
- General system $\mathcal{R}_{nm}(\tau,L) = \left(r_{nm} + C_{nm}(L^{y_t}\delta\tau) + \mathcal{O}(\tau^2)\right)\left(1 + D_{nm}L^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$

Crossing Point = Critical Point！

Numerically Application

- 3d-Ising model
- 3d 3-states Pott model
- Heavy-Quark QCD (Kitazawa’s talk on Wendnday)