Finite-size scaling of Lee-Yang zeros in heavy-quark QCD

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In collab. with Tatsuya Wada, Kazuyuki Kanaya

Special thanks to S. Ejiri

Wada, MK, Kanaya, PRL**134**, 162302 ('25), arXiv:2508. 20422 [cond-mat.stat-mech]

Contents

Finite-Size Scaling (FSS)

Scaling Hypothesis

$$F_{\text{sing}}(t, h, L^{-1}) = \tilde{F}_{\text{sing}}(L^{y_t} t, L^{y_h} h)$$

$$Z_{\text{sing}}(t, h, L^{-1}) = \tilde{Z}_{\text{sing}}(L^{y_t} t, L^{y_h} h)$$

$$F = F_{\text{sing}} + F_{\text{reg}}$$

$$Z = Z_{\text{sing}} \times Z_{\text{reg}}$$

LYZ in the scaling region on finite volume

$$Z(t, h, L^{-1}) \sim \tilde{Z}_{\text{sing}}(L^{y_t} t, L^{y_h} h) = 0$$

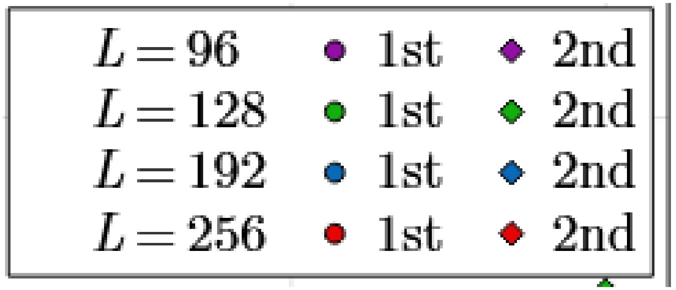
$$L^{y_h} h^{(n)} = \tilde{h}^{(n)}_{\text{LY}}(L^{y_t} t)$$

Itzykson+, ('83)

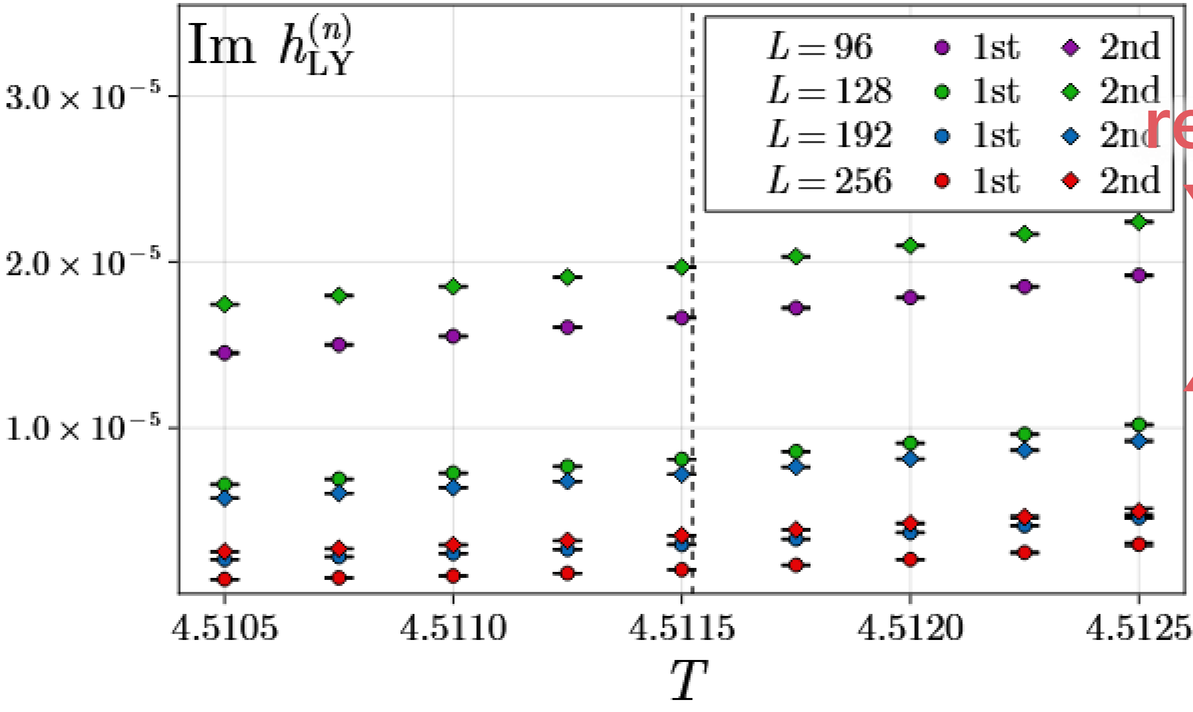
LYZ in 3d Ising Model

Wada, MK, Kanaya, arXiv:2508. 20422

$$L^{y_h} h^{(n)} = \tilde{h}^{(n)}_{\rm LY}(L^{y_t} t)$$

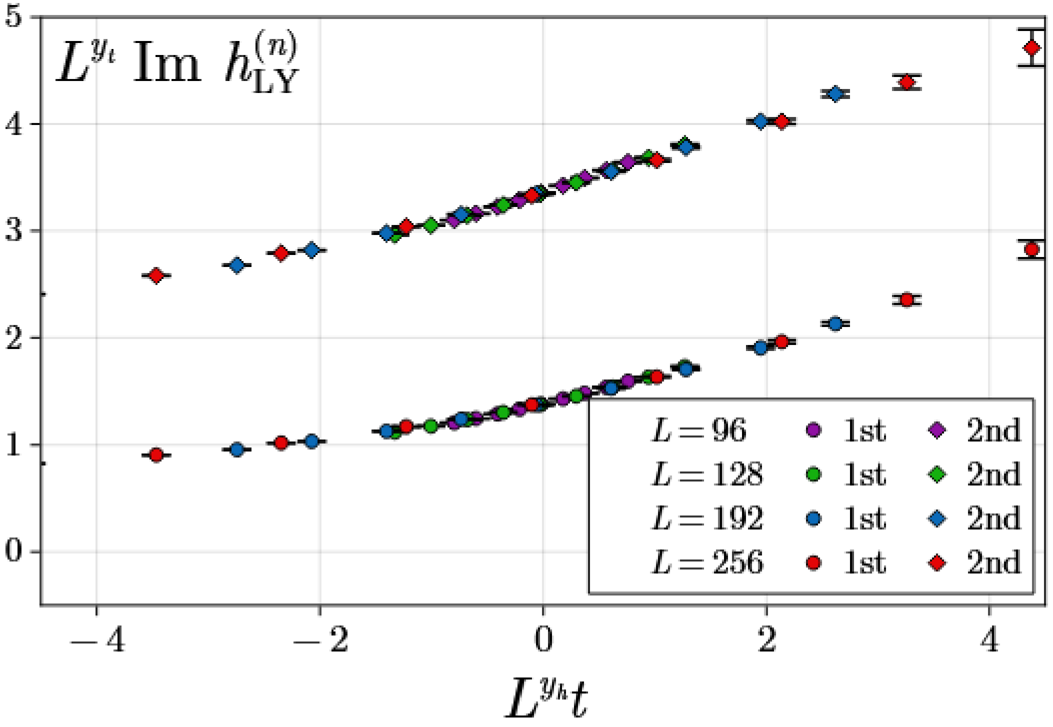


1st & 2nd LYZ



rescale

Rescaled

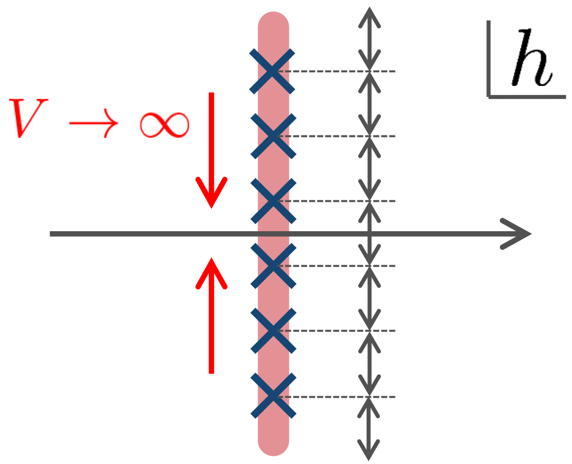


Wada, MK, Kanaya, PRL ('25)

Lee-Yang Zero Ratios

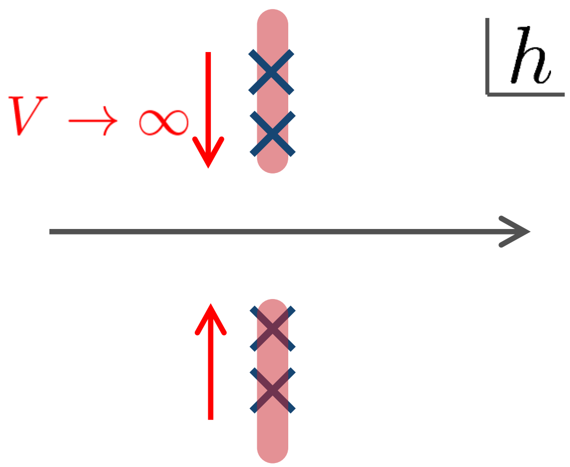
$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$$

First-Order Side $(t < 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} \frac{2n-1}{2m-1}$$

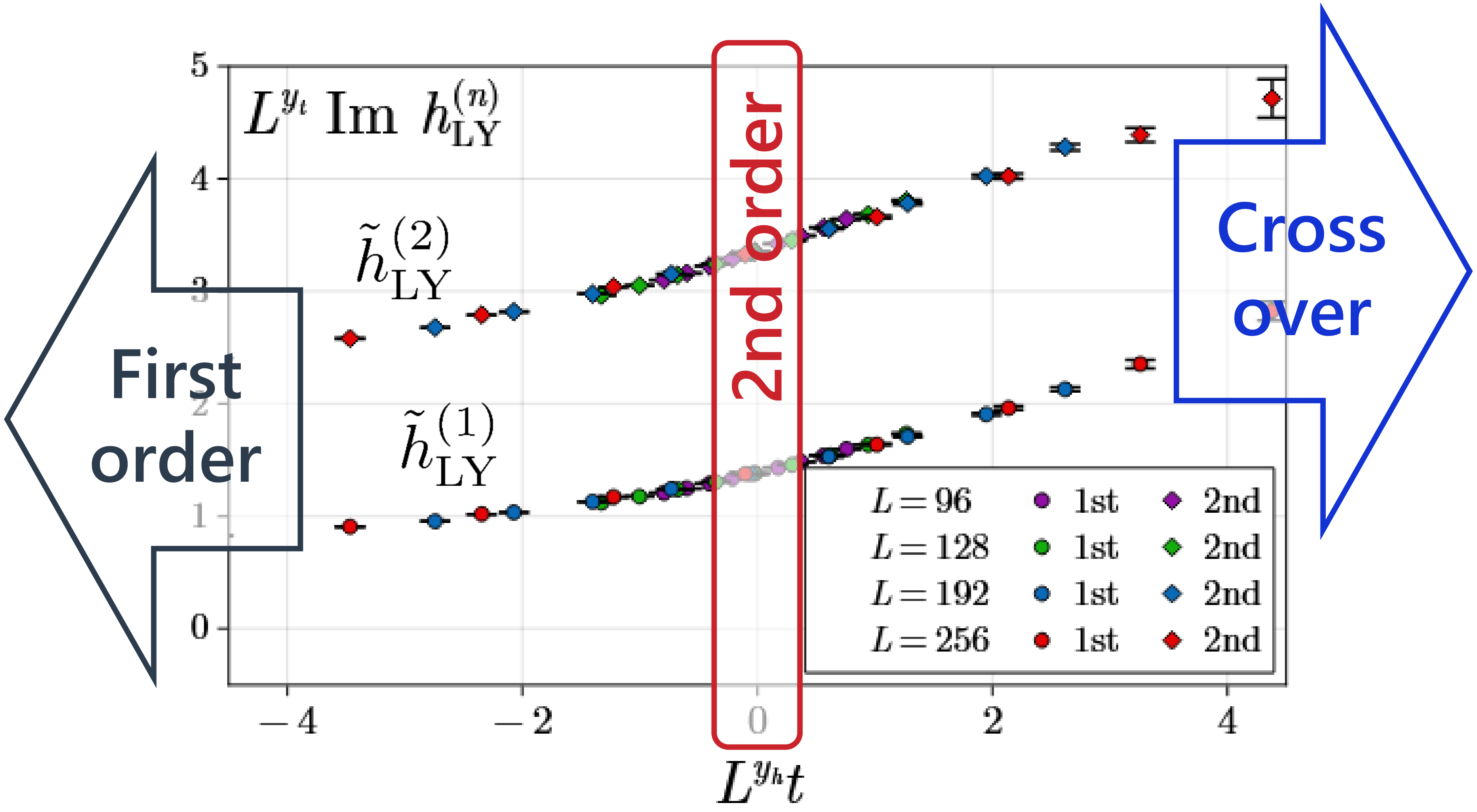
Crossover Side $(t > 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} 1$$

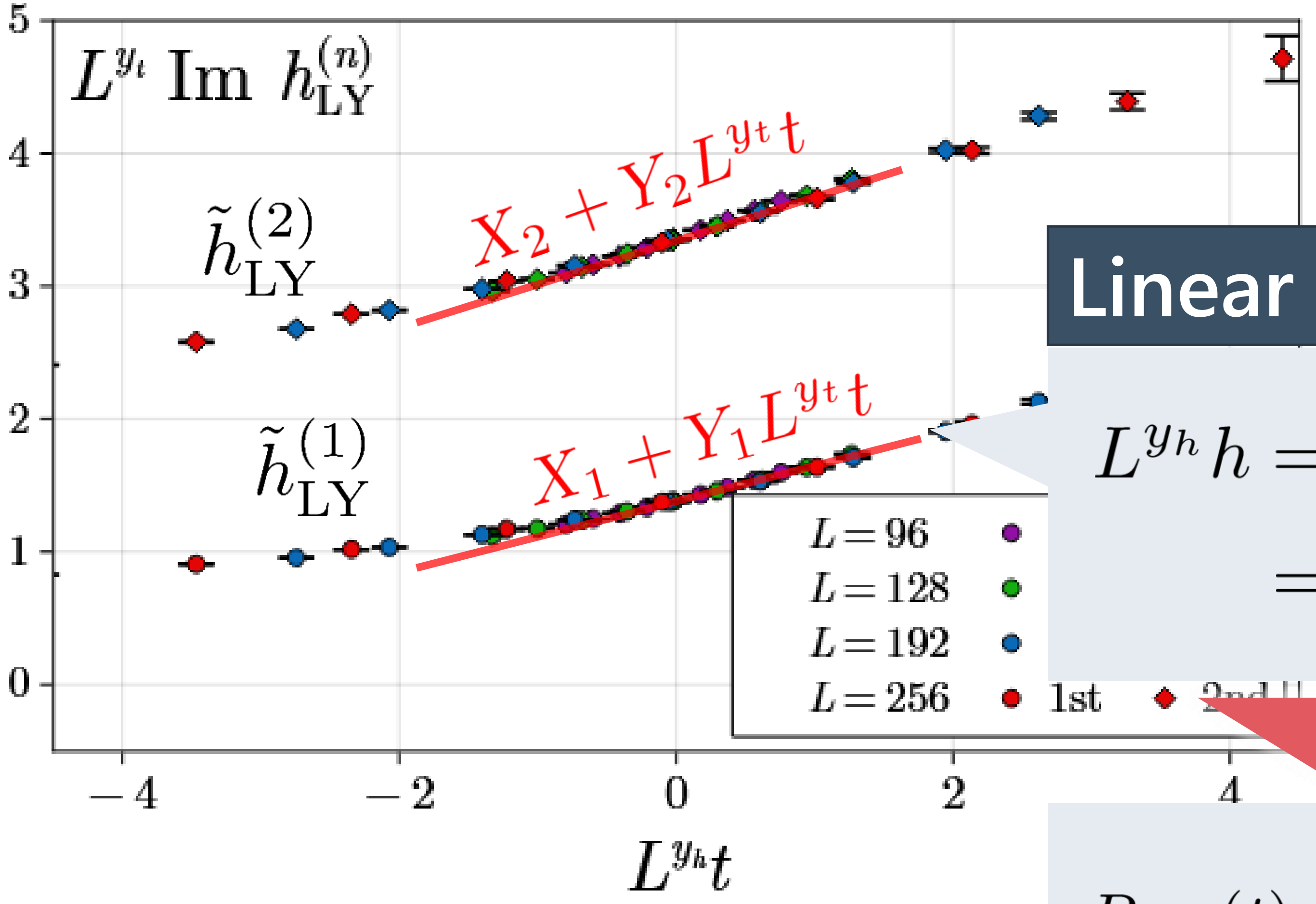
LYZ near $t = 0$

Wada, MK, Kanaya, PRL ('25)



LYZ near $t = 0$

Wada, MK, Kanaya, PRL ('25)



Linear Approx. at t = 0

$$L^{y_h} h = \tilde{h}_{\rm LY}^{(i)}(L^{y_t} t)$$

$$= X_i + Y_i L^{y_t} t + \mathcal{O}(t^2)$$

$$R_{nm}(t) = \frac{X_n}{X_m}\left(1 + C_{nm} t L^{y_t} + \mathcal{O}(t^2)\right)$$

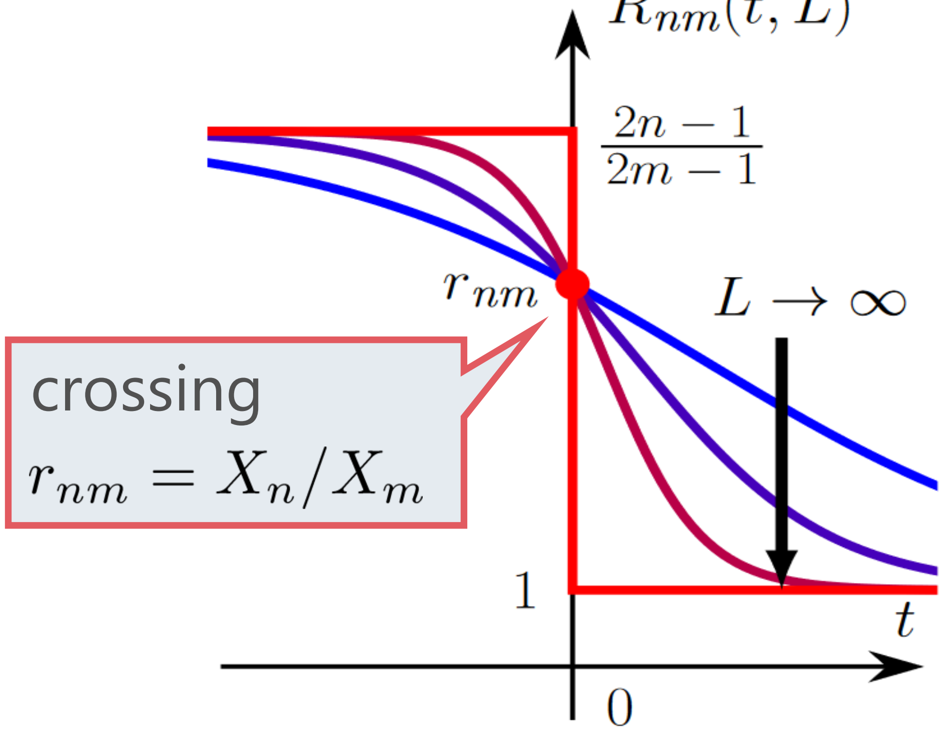
LYZ Ratios near $t = 0$

$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$$

$$R_{n1}(t) \xrightarrow[V\to\infty]{} \begin{cases} 2n-1 & t<0 \ \text{(1st order)} \\ 1 & t>0 \ \text{(crossover)} \end{cases}$$

$$R_{n1}(t) = r_{nm} + c_{nm} L^{y_t} t + \mathcal{O}(t^2)$$

near $t = 0$



— $r_{nm} = R_{nm}(0)$ is L independent, the universal value.

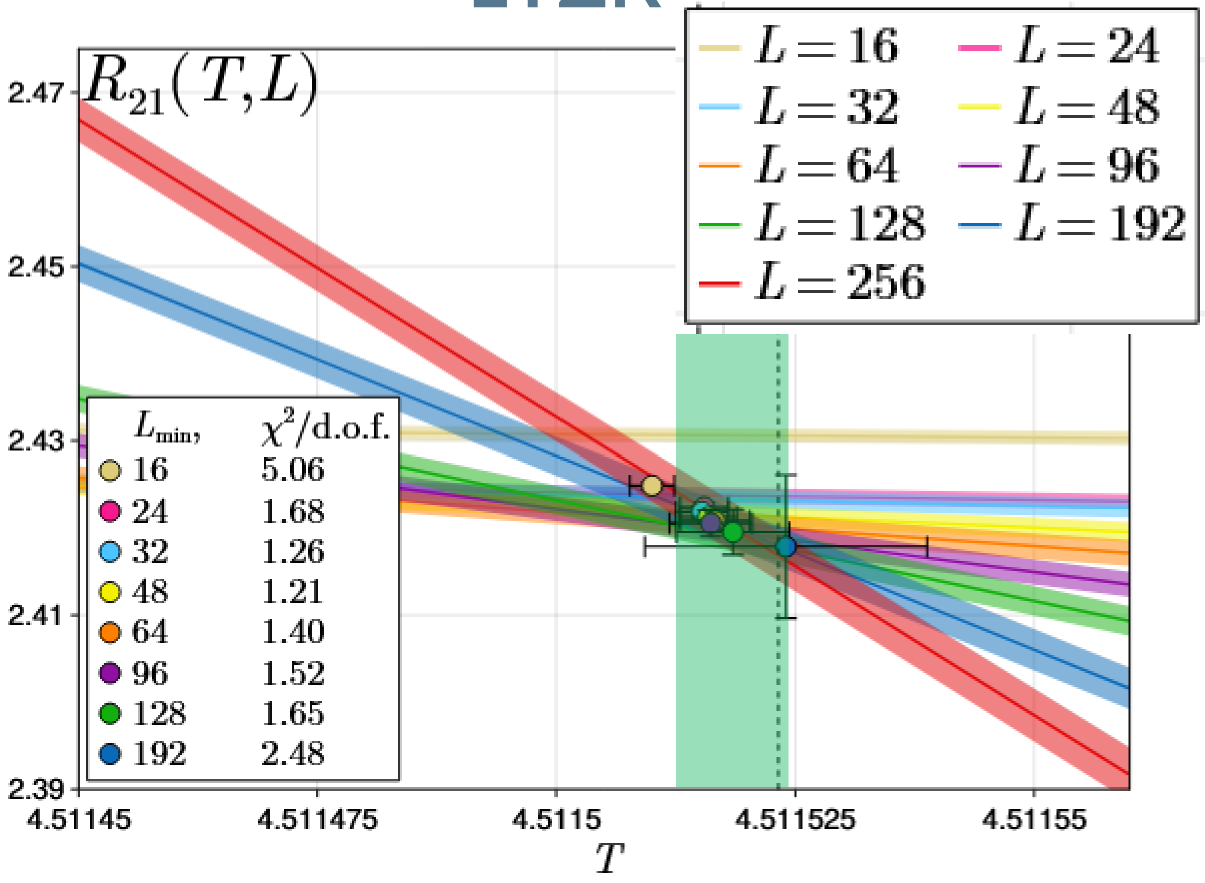
—Crossing point of various L gives the CP.

—Reminiscent of Binder-cumulant analysis

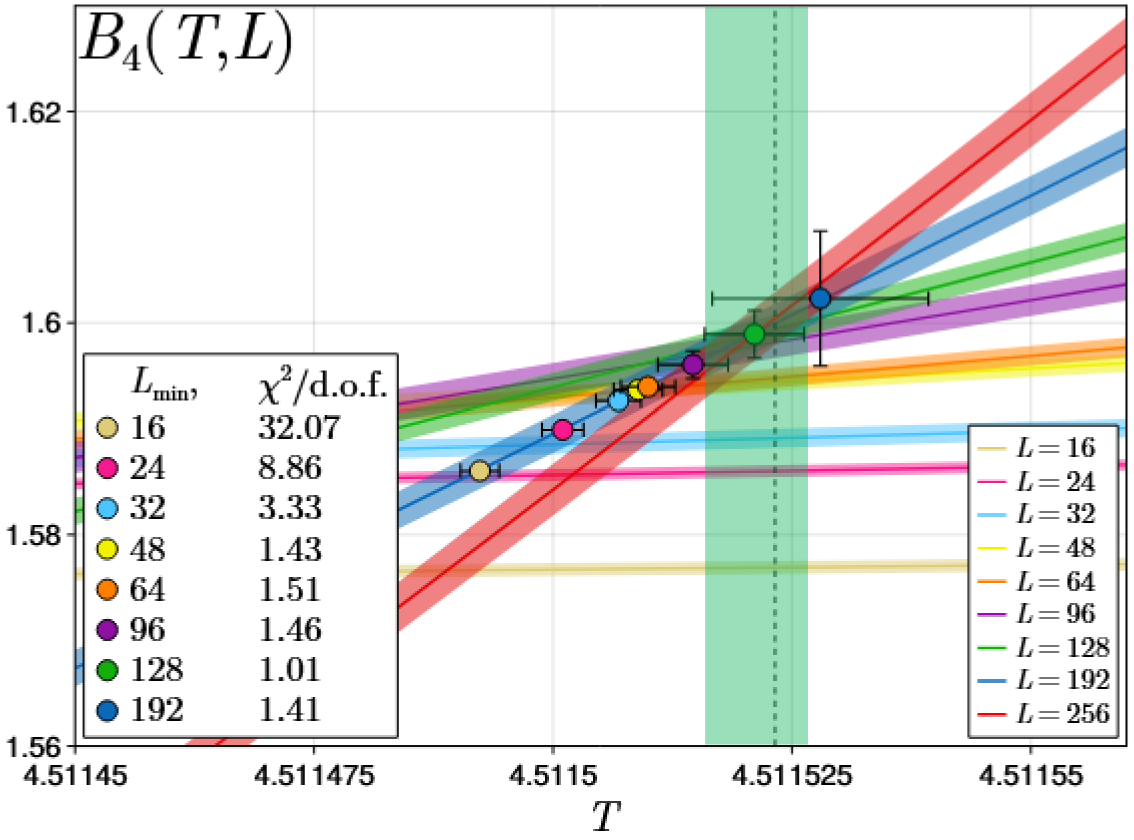
Wada, MK, Kanaya, PRL ('25)

LYZR vs Binder Cumulant in 3d-Ising

LYZR



Binder cumulant

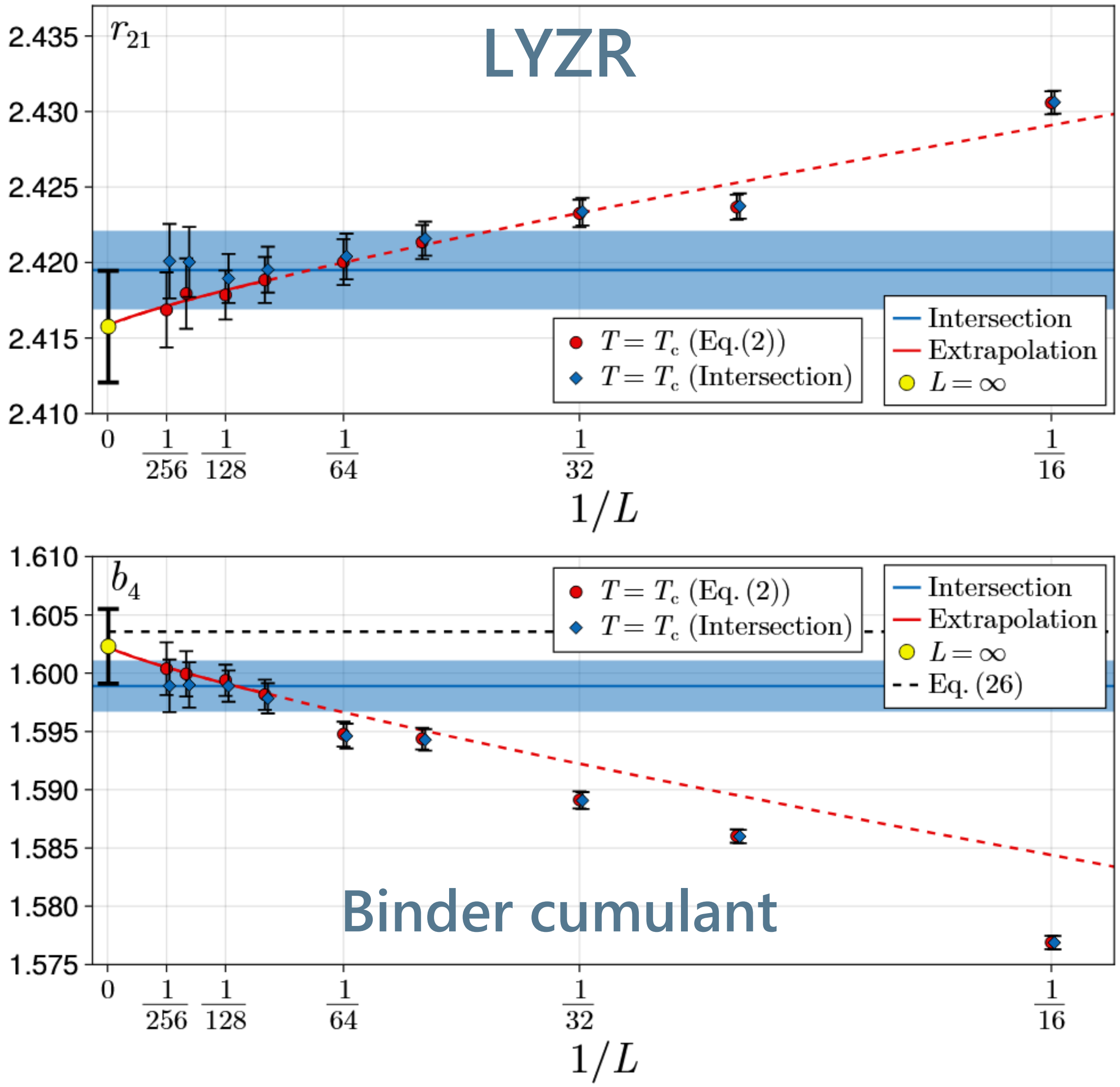


Faster convergence of the violation of FSS in LYZ?

Wada, MK, Kanaya, arXiv:2508. 20422

Convergence at $T = T_c$

Wada, MK, Kanaya, arXiv:2508. 20422



Red: T_c of Ferrenberg ('18)
Blue: T_c of intersection point

In 3d-Ising,
violation of FSS is more quickly suppressed in the LYZR than the Binder cumulant as $L \to \infty$.

$$Z = Z_{\text{sing}} \times Z_{\text{reg}}$$

Curvature

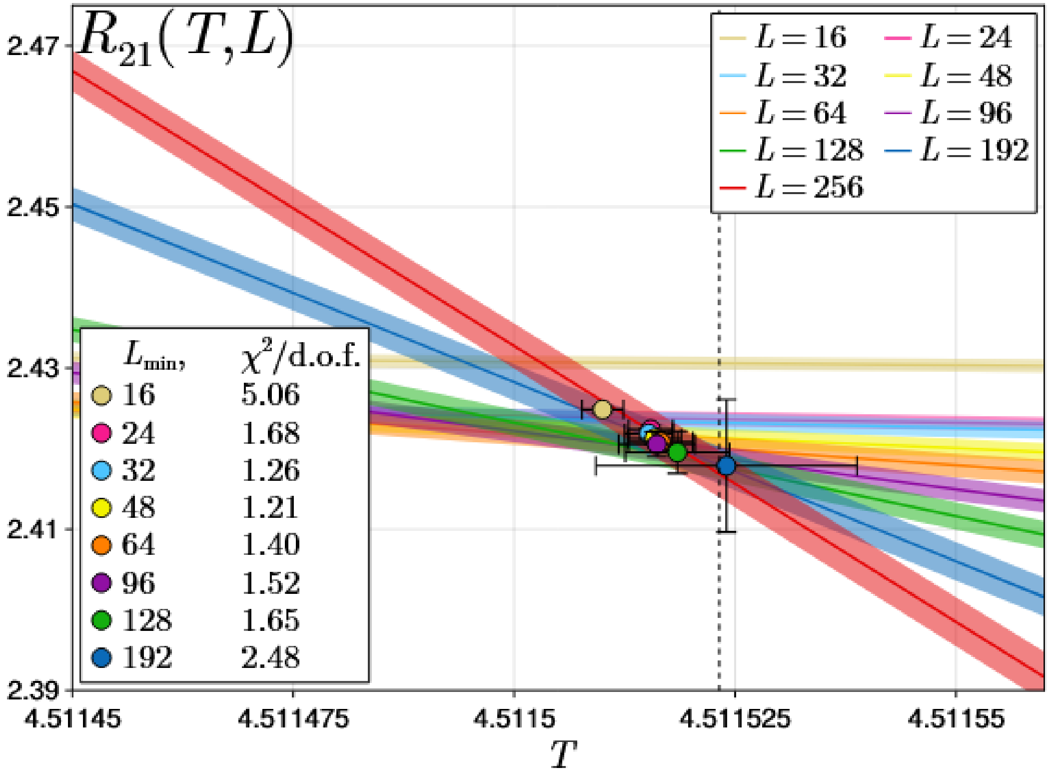
Wada, MK, Kanaya, arXiv:2508. 20422

Our fits to $R_{nm}(t, L), B_4(t, L)$ need a non-linear term.

$$R_{n1}(T, L) = r_{n1} + c_{n1}\, L^{y_t} \left(\frac{T - T_c}{T_c}\right) + d_{n1}\, L^{2y_t} \left(\frac{T - T_c}{T_c}\right)^2$$

Measure of non-linearity

$$C_f = L^{-y_t} \frac{\partial^2 f / \partial T^2}{\partial f / \partial T}\bigg|_{T=T_c}$$



Curvature

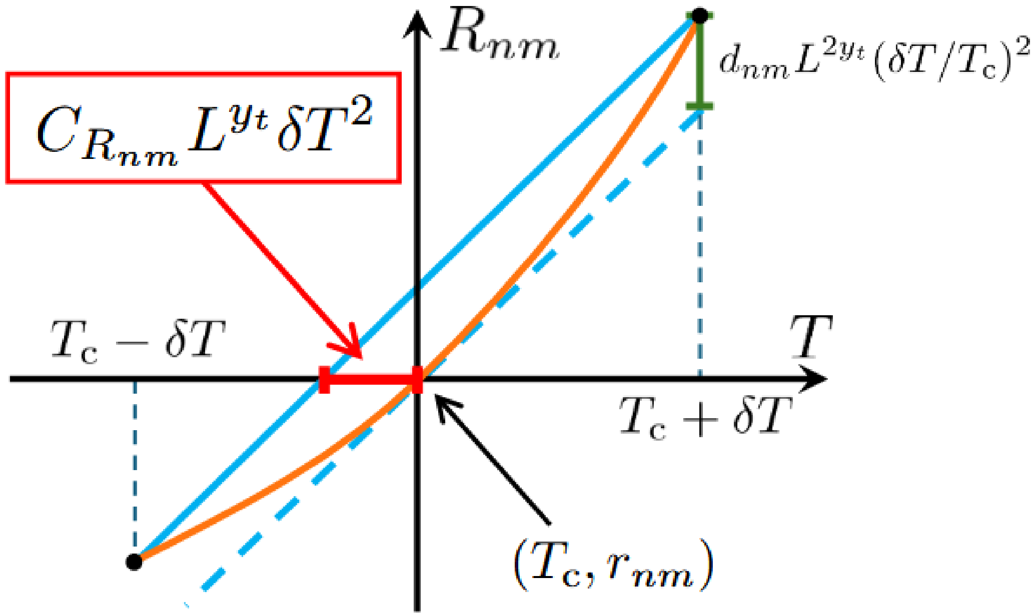
Wada, MK, Kanaya, arXiv:2508. 20422

Measure of non-linearity

$$C_f = L^{-y_t} \frac{\partial^2 f/\partial T^2}{\partial f/\partial T}\bigg|_{T=T_c}$$

f	R_{21}	R_{31}	R_{41}	B_4
C_f	0.0093(17)	0.0053(17)	-0.0010(34)	0.0364(37)

f	$h_{\rm LY}^{(1)}$	$h_{\rm LY}^{(2)}$	$\langle M^2\rangle_c$
C_f	0.07149(59)	0.04230(101)	-0.07209(44)



C_f is invariant under

$$\begin{pmatrix} t' \\ h' \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} t \\ h \end{pmatrix}$$

CP in a General System

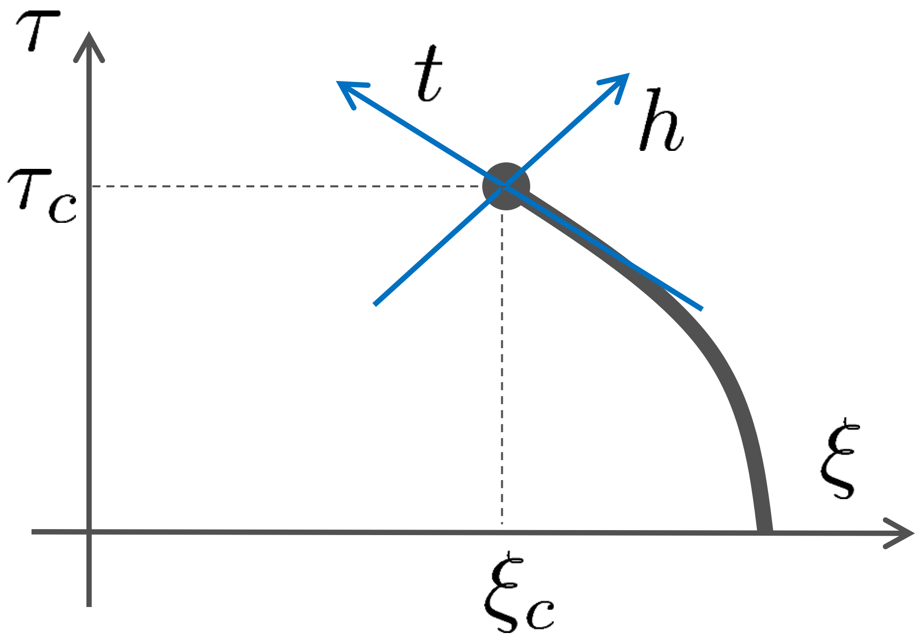
Wada, MK, Kanaya, PRL ('25)

$\bar{y} = y_t - y_h = -0.894$

- CP on a $\tau - \xi$ plane
- **LYZ on the complex ξ plane**

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)} \simeq X_i + Y_i L^{y_t} t$$



Wada, MK, Kanaya, PRL ('25)

CP in a General System

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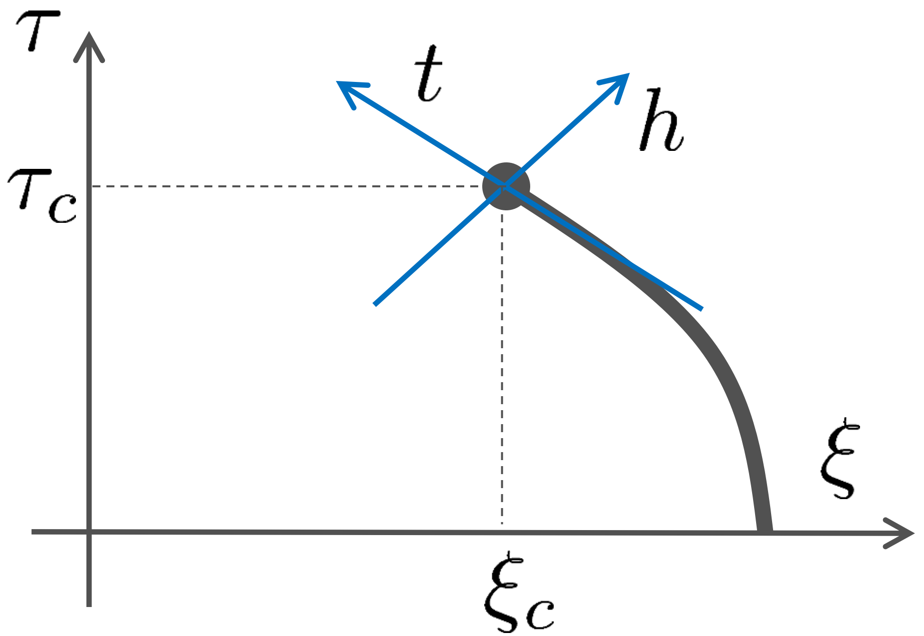


$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)} \simeq X_i + Y_i L^{y_t} t$$

$$\begin{cases} \xi_{\mathrm{R}}^{(n)} = \xi_c - \dfrac{a_{21}}{a_{22}} \delta\tau + \mathcal{O}(L^{2\bar{y}}) \\ \xi_{\mathrm{I}}^{(n)} = \dfrac{X_n}{a_{22}} L^{-y_h} + \dfrac{\det AY_n}{a_{22}^2} \delta\tau L^{\bar{y}} + \mathcal{O}(L^{2\bar{y}}) \end{cases}$$

$L \to \infty$

generalization to finite V

LY Edge Singularity

$$\begin{cases} \mathrm{Re}\xi_{\mathrm{LYES}} \simeq c_1 \tau \\ \mathrm{Im}\xi_{\mathrm{LYES}} \simeq c_2 \tau^{\beta\delta} \end{cases}$$

Stephanov, 2006

LYZ Ratios for General CP

Wada, MK, Kanaya, PRL ('25)

$$\bar{y} = y_t - y_h = -0.894$$

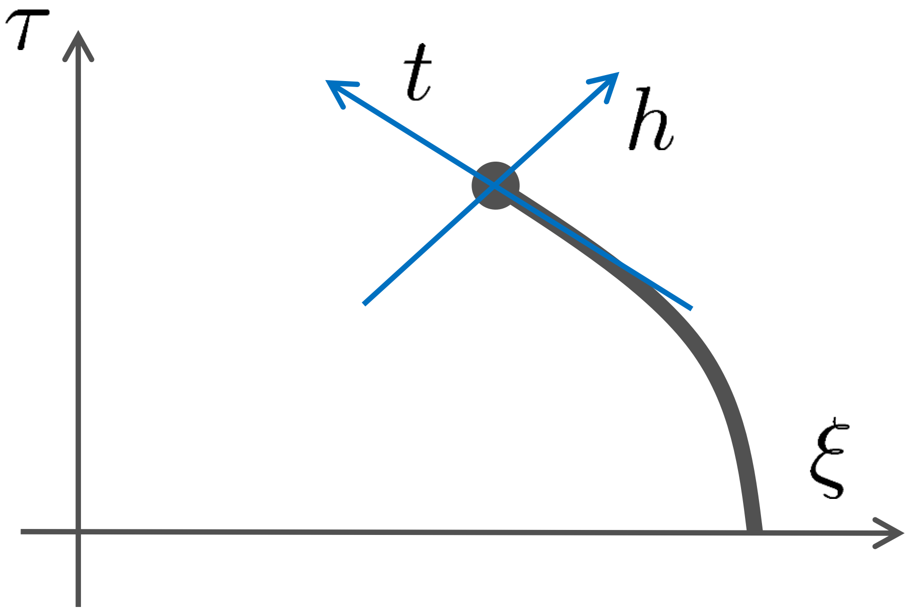
LYZ Ratio

$$R_{nm}(t) = \frac{\xi_{\mathrm{I}}^{(n)}(\tau)}{\xi_{\mathrm{I}}^{(m)}(\tau)} = \frac{X_n}{X_m}\left(1 + C\tau L^{y_t} + \mathcal{O}(\tau^2)\right)\left(1 + DL^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

$$C = \frac{\det A}{a_{22}}\left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1}\right), \quad D = \frac{a_{12}^2}{a_{22}^2}(Y_1^2 - Y_2^2)$$

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A\begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$



LYZ Ratios vs Binder Cumulant

Wada, MK, Kanaya, PRL ('25)

$$\bar{y} = y_t - y_h = -0.894$$

LYZ Ratio

$$R_{nm}(t) = \frac{\xi_{\mathrm{I}}^{(n)}(\tau)}{\xi_{\mathrm{I}}^{(m)}(\tau)} = \frac{X_n}{X_m}\left(1 + C\tau L^{y_t} + \mathcal{O}(\tau^2)\right)\left(1 + DL^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$$

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Binder cumulant

Jin+, PRD86, 2017

$$B_4(t) = b_4\left(1 + c\tau L^{y_t} + \mathcal{O}(t^2)\right)\left(1 + dL^{\bar{y}} + \mathcal{O}(L^{2\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

Deviation at $t = 0$ due to $a_{12} \neq 0$ converges faster in LYZ ratio.

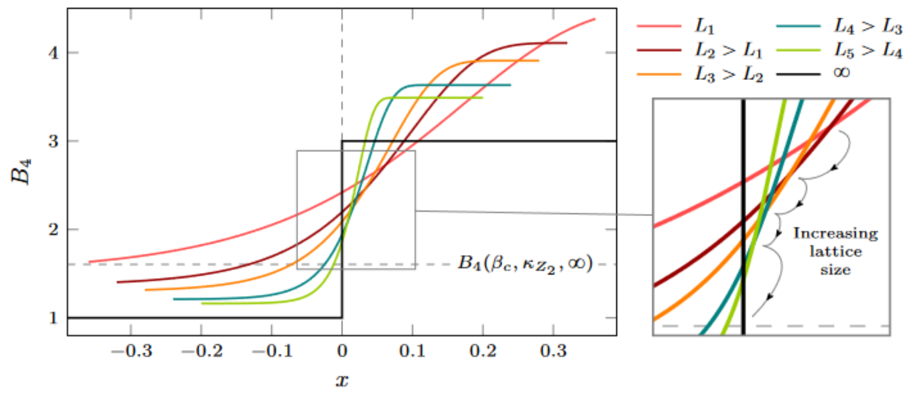
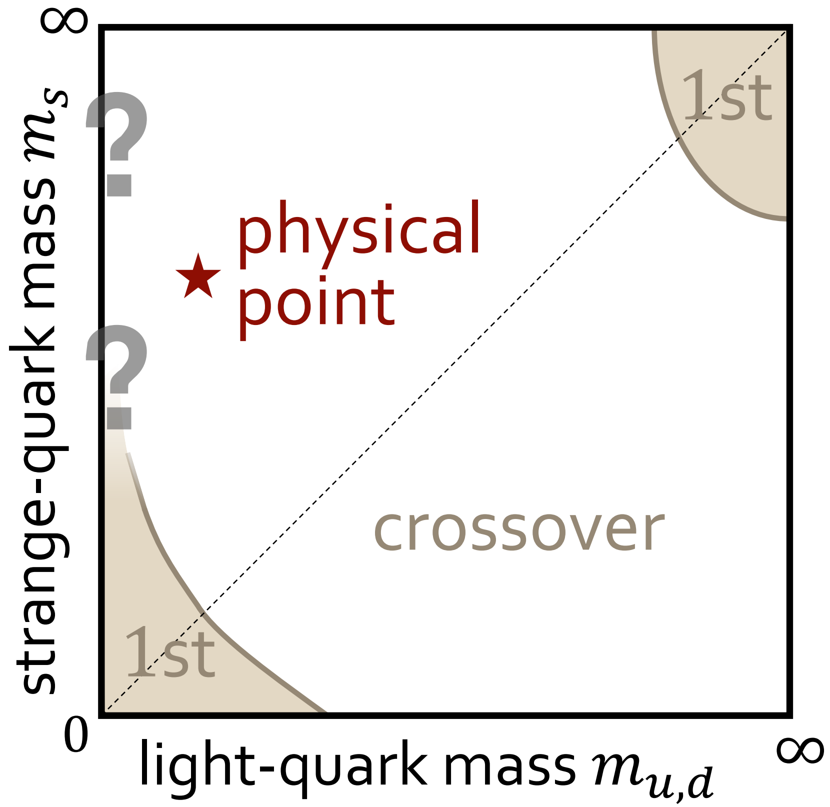


fig from Cuteri+, PRD ('21)

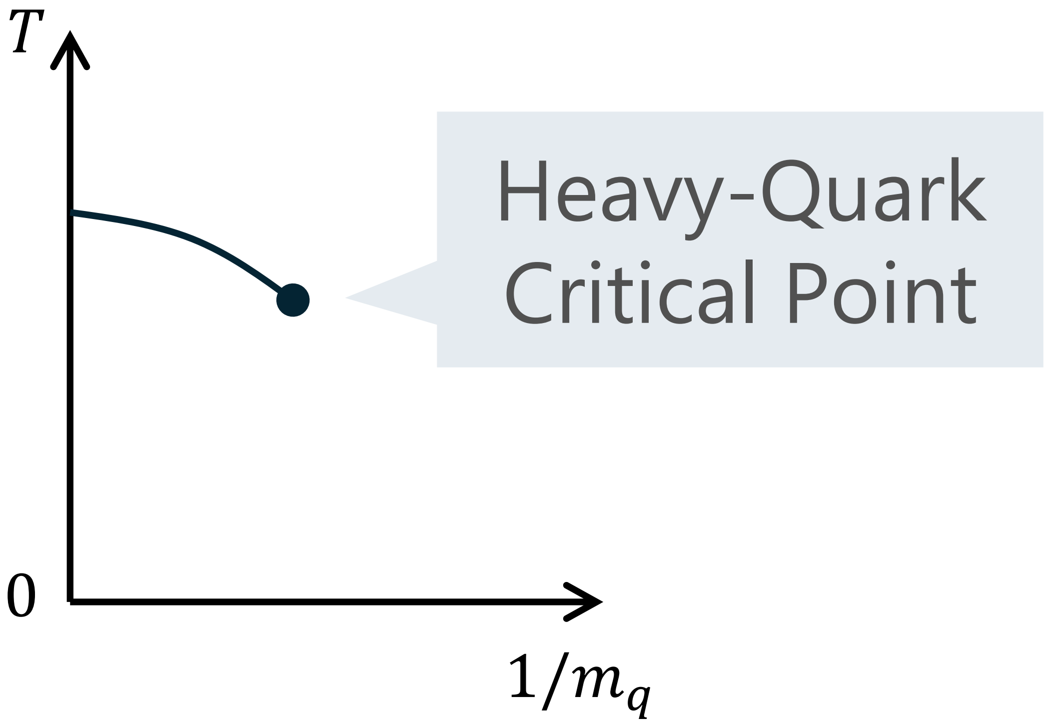
Contents

Motivations

Columbia Plot



Phase Diagram



Can we determine the location of the HQ-CP?

Our Strategy

Kiyohara+, PRD104 ('21)
Ashikawa+, PRD110 ('24)

We want to control **finite-volume effects** → Large volume simulations up to $\boldsymbol{LT = N_x/N_t = 18}$

→ —Simulations on **coarse lattices** $(N_t = 1/aT = 4, 6, 8)$
—Employ **hopping-parameter expansion (HPE)**

Hopping-Parameter Expansion: $\ln \det M(\kappa) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}[B^n]\kappa^n$

Hopping-Parameter Expansion

Wilson Fermion

$$S_q = \sum_{x,y} \bar{\psi}_x M_{xy} \psi_y$$

$$M_{xy} = \delta_{xy} - \kappa B_{xy} \qquad \kappa \sim \frac{1}{2m_q a}$$

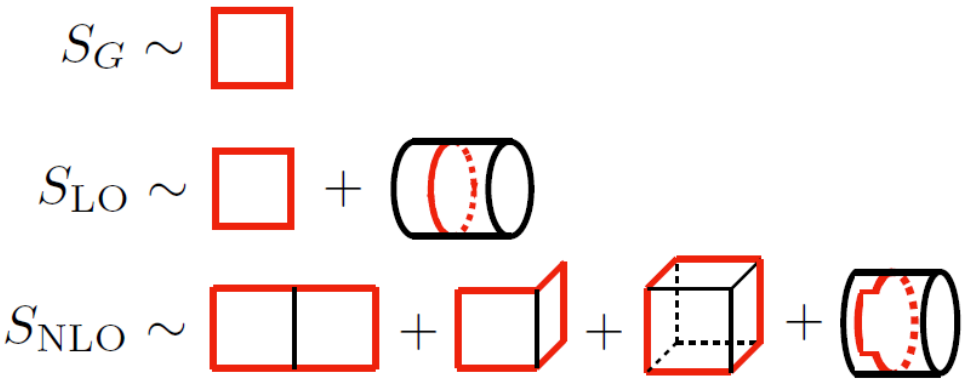
$$B_{xy} = \sum_{\mu=1}^{4} \left[(1-\gamma_\mu) U_{x,\mu} \delta_{y,x+\hat{\mu}} + (1+\gamma_\mu) U^\dagger_{y,\mu} \delta_{y,x-\hat{\mu}} \right]$$

nonzero only for neighboring sites

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}U \mathcal{O} e^{-S_g + \ln \det M(\kappa)}$$

$$\ln \det M(\kappa) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}[B^n] \kappa^n$$

$\mathrm{tr}[B_n]$ are represented by closed trajectories of length n.

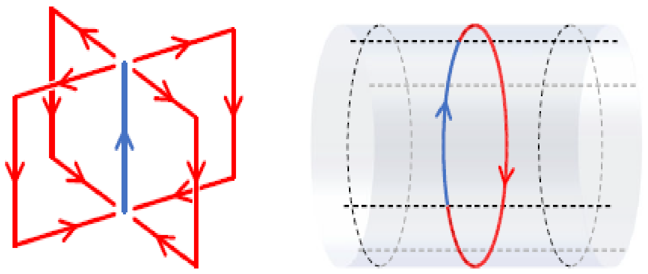


Simulation Method 1

Monte Carlo Updates at the LO

Heat bath & over relaxation with modified staple

➤ Almost the same numerical cost as the pure YM!



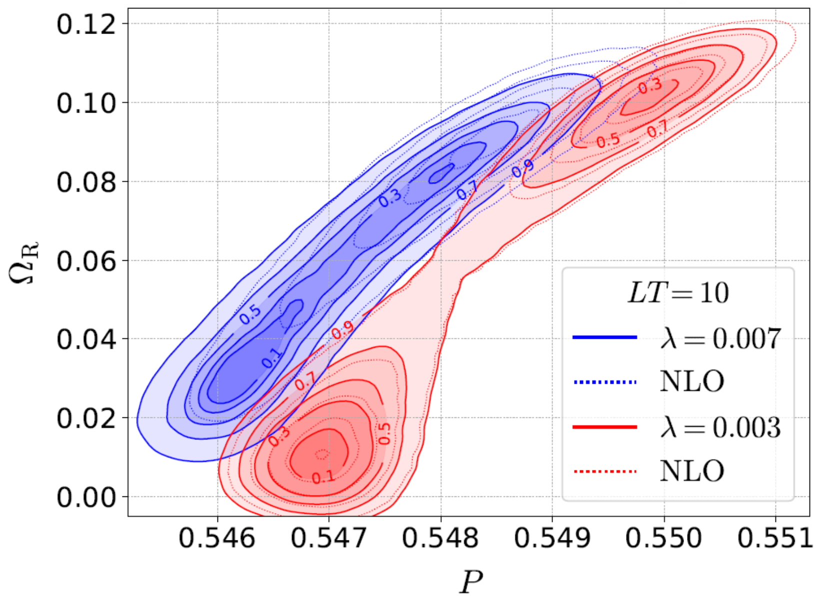
NLO terms by Reweighting

$$\langle \mathcal{O} \rangle_{\mathrm{NLO}} = \frac{\langle \hat{O} e^{-S_{\mathrm{NLO}}} \rangle_{\mathrm{LO}}}{\langle e^{-S_{\mathrm{NLO}}} \rangle_{\mathrm{LO}}}$$

Significant suppression of overlapping problem

➤ Realize high-statistical analysis!

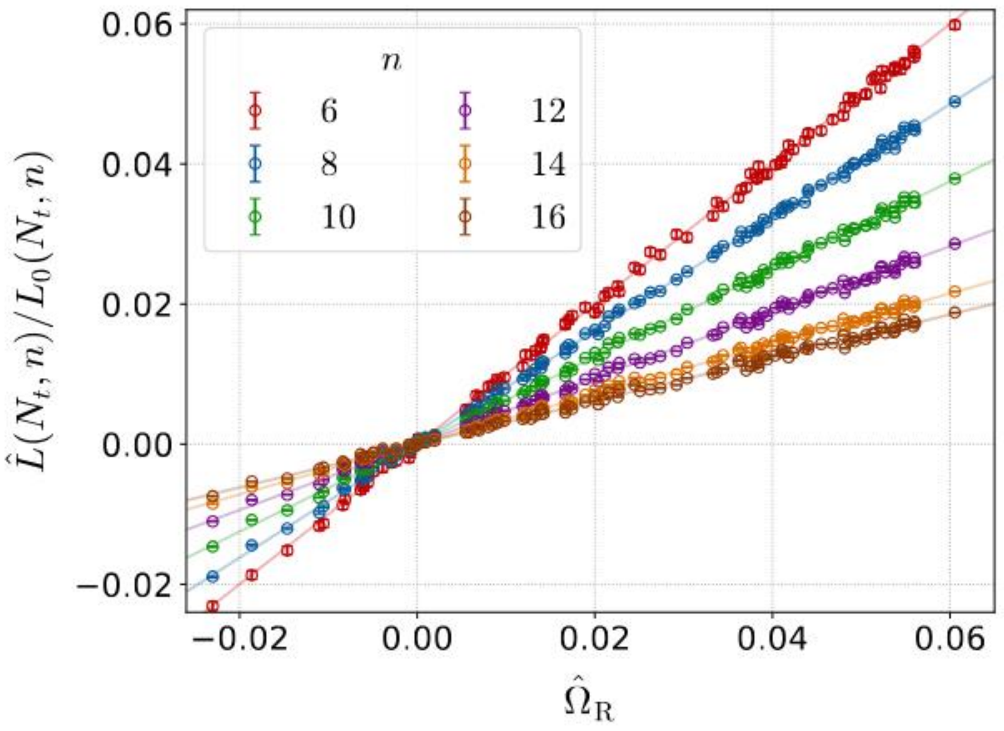
$$\lambda = 2^{N_t+2} N_c N_f \kappa^{N_t}$$



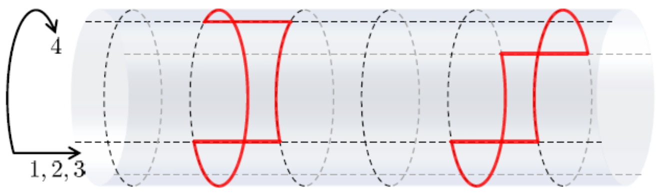
Simulation Method 2

Wakabayashi+ ('22)
Ashikawa+ ('24)

Effective incorporation of yet higher-order terms



$L(N_t, n)$: Winding loops of length n



Strong correlations between different n

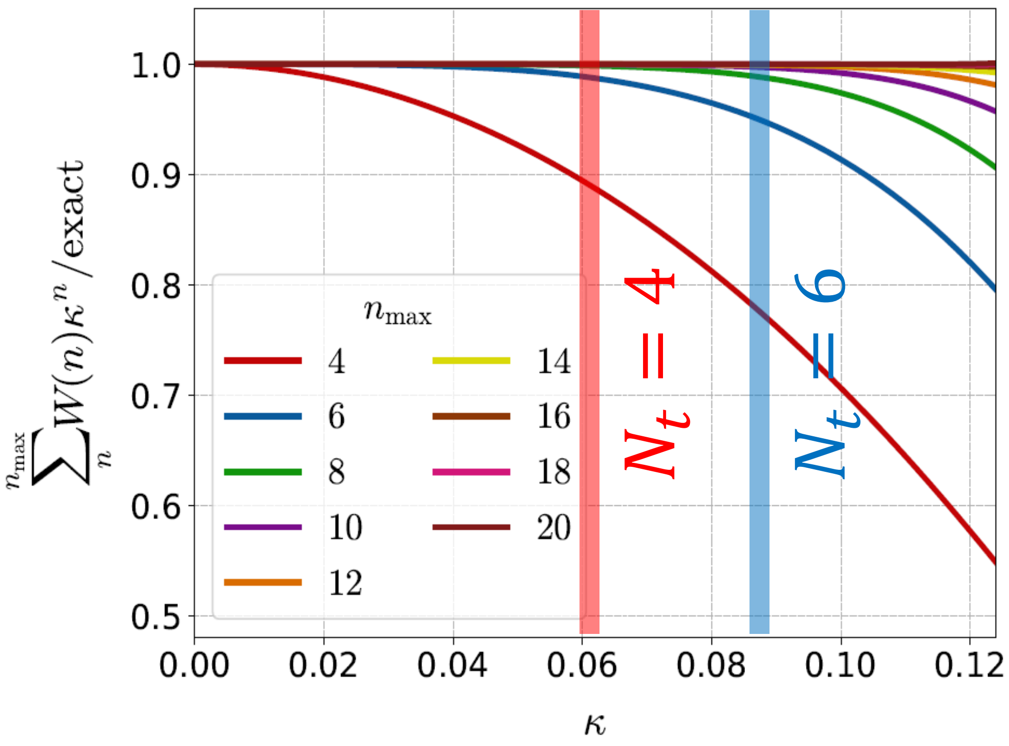
Higher-order terms in the HPE are approximately proportional to the NLO term

$$\frac{\hat{L}(N_t, n)}{L_0(N_t, n)} \simeq \tilde{c}_n \frac{\hat{L}(N_t, N_t + 2)}{L_0(N_t, N_t + 2)} = \tilde{c}_n \mathrm{Re}\hat{\Omega}_{N_t+2}$$

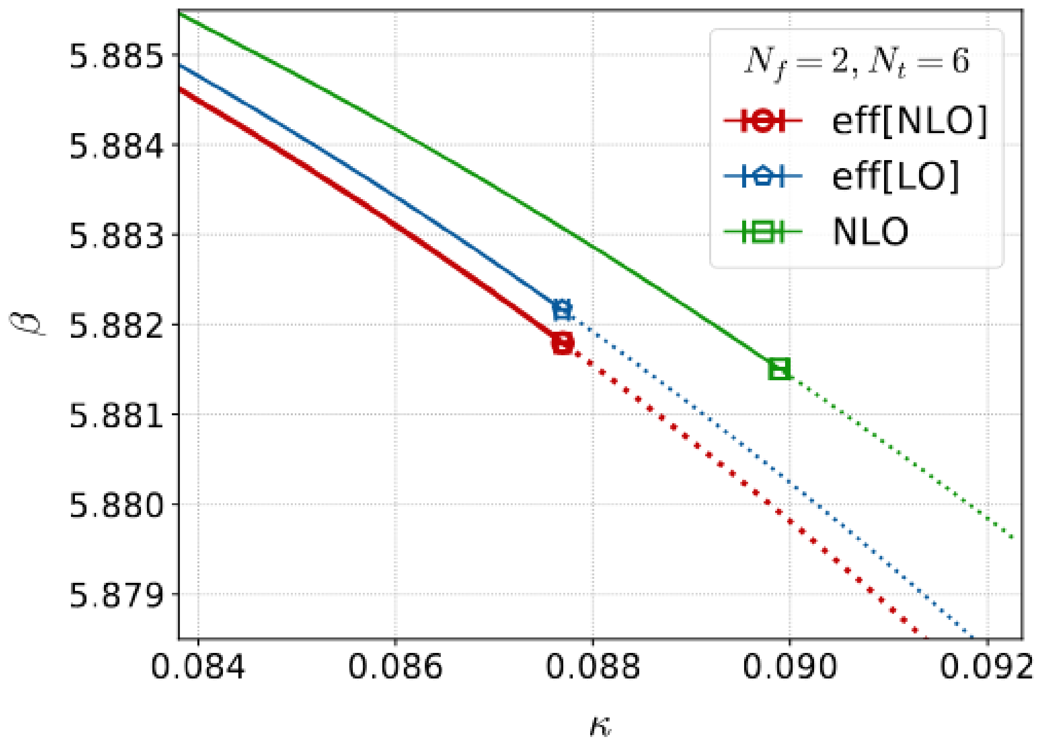
Convergence of HPE

Convergence of Free Fermions

Wilson-loop-type



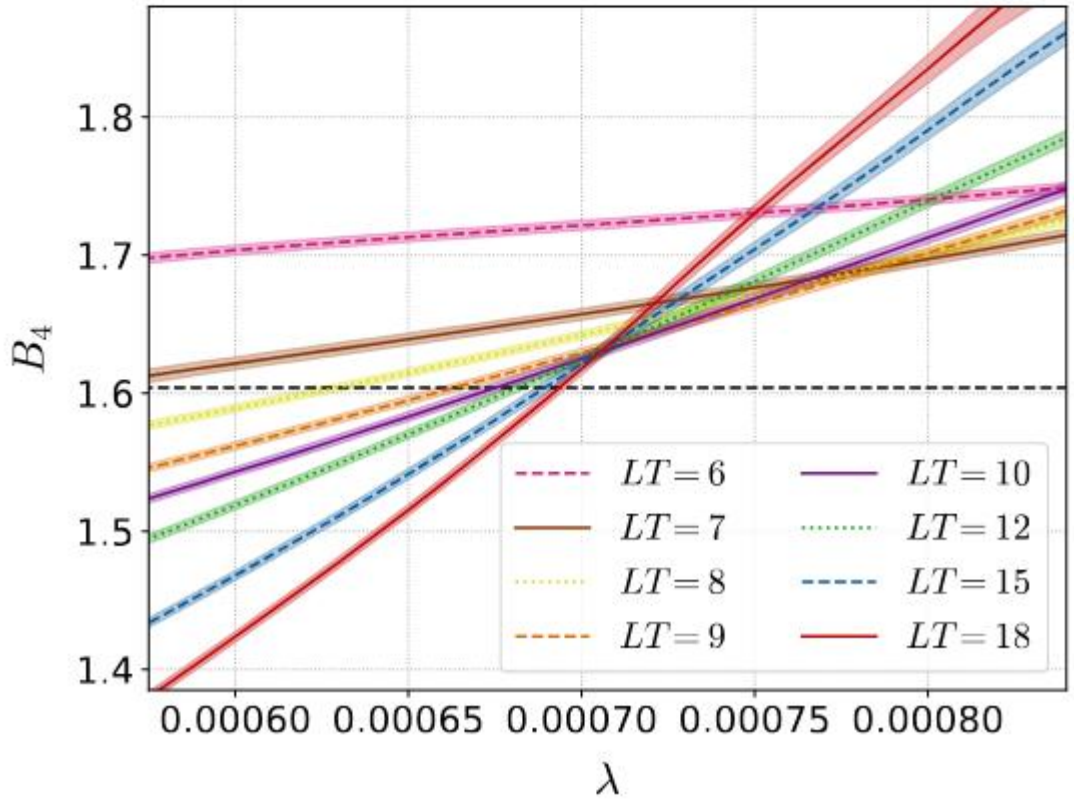
Phase Diagram

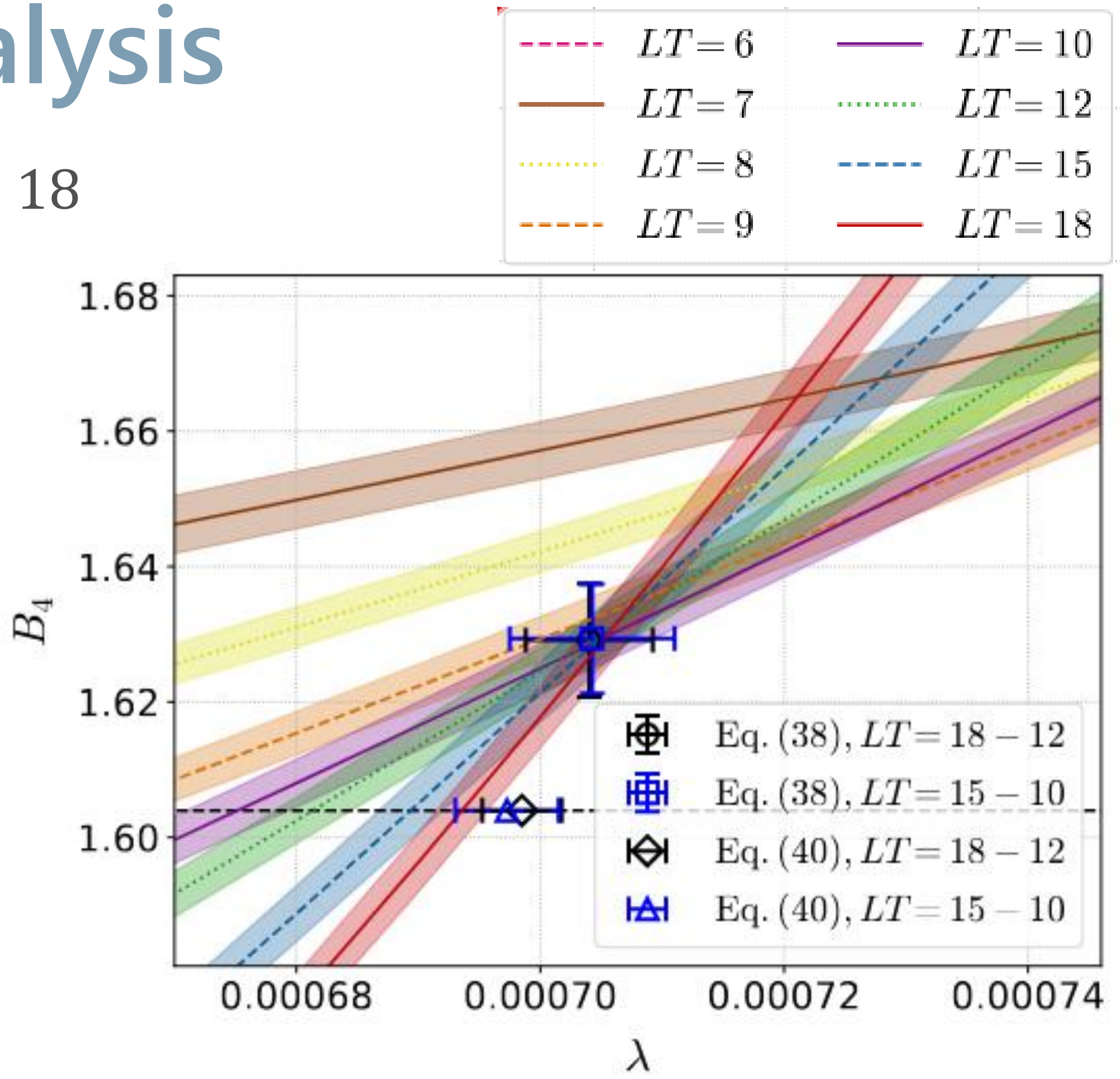


Ashikawa+ ('24)

Binder Cumulant Analysis

Numerical results at $N_t = 6$, $LT = 6 - 18$

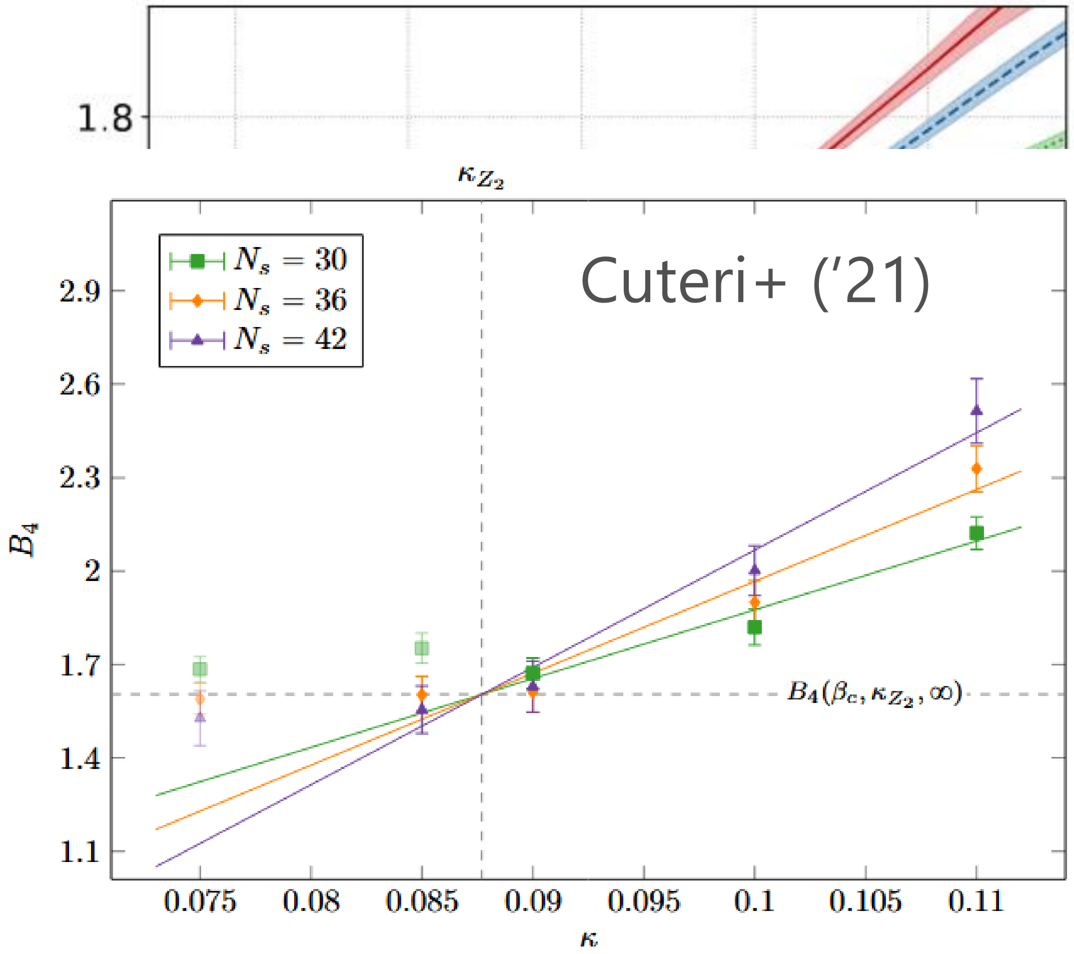




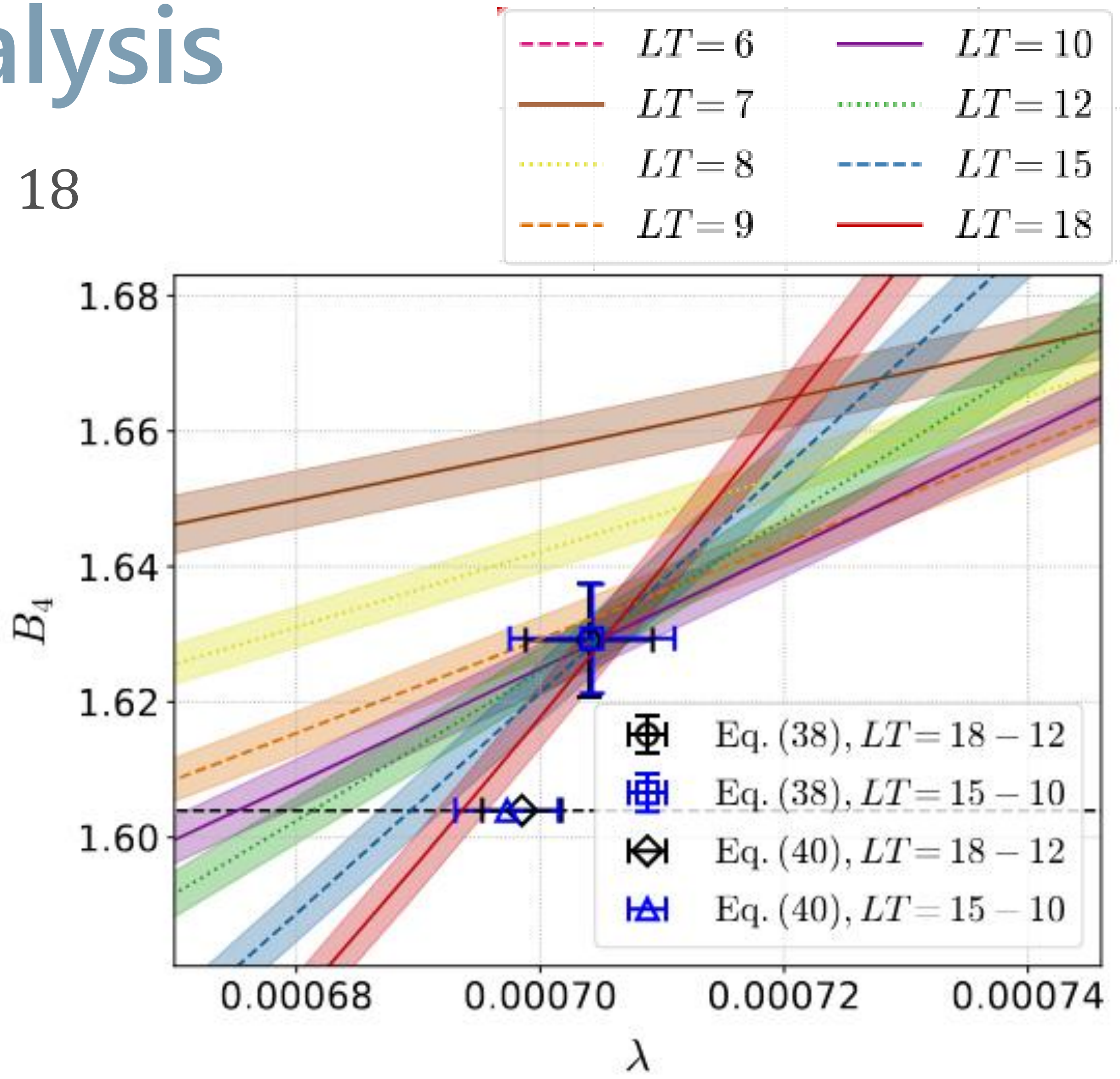
$$\lambda = 2^{N_t+2} N_c N_f \kappa^{N_t}$$

Binder Cumulant Analysis

Numerical results at $N_t = 6$, $LT = 6 - 18$



Ashikawa+ ('24)



$$\lambda = 2^{N_t+2} N_c N_f \kappa^{N_t}$$

LYZ Ratio Method for HQ-CP

Zeros on the complex-λ plane

Partition-function ratio by reweighting

$$\frac{Z(\lambda)}{Z(\mathrm{Re}\lambda)} = \frac{\langle e^{-S(\lambda)+S(\lambda_0)}\rangle_0}{\langle e^{-S(\mathrm{Re}\lambda)+S(\lambda_0)}\rangle_0}$$

LYZ Ratio Method for HQ-CP

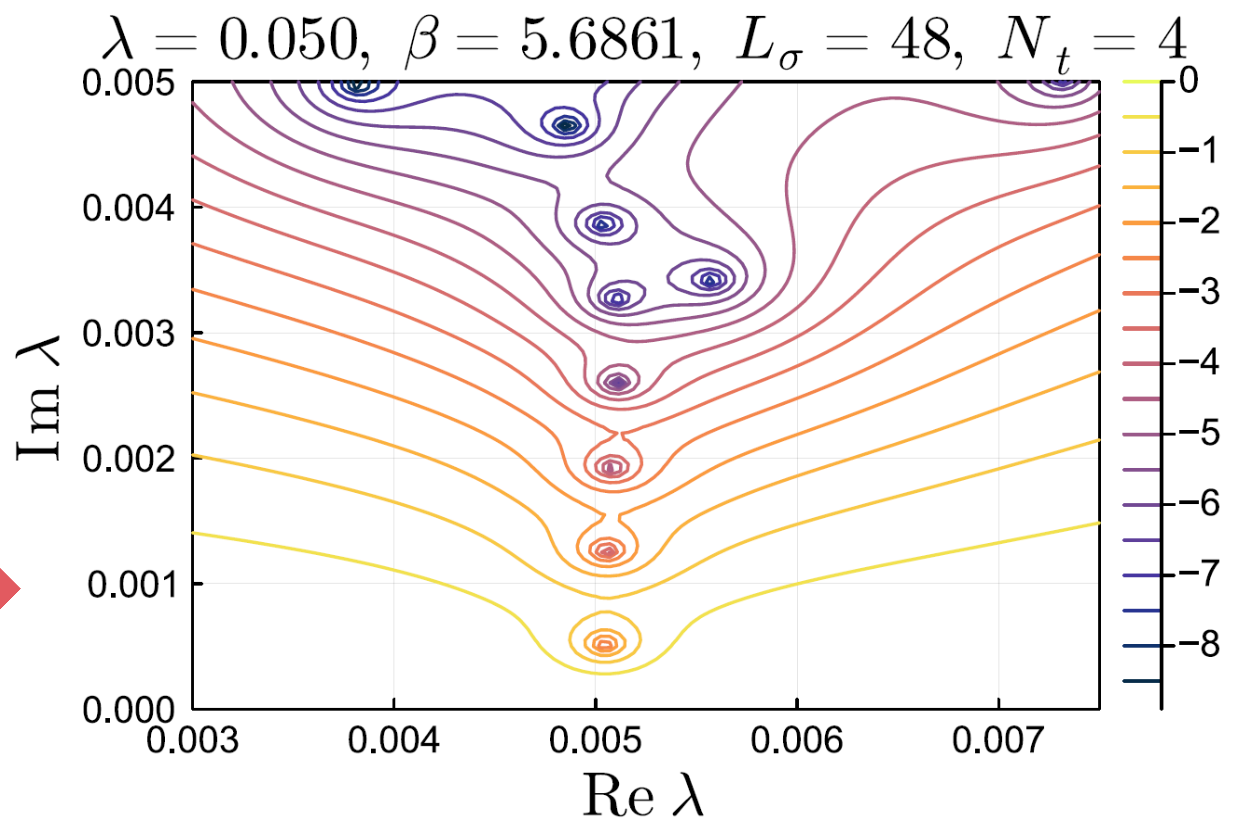
Zeros on the complex-λ plane

Partition-function ratio by reweighting

$$\frac{Z(\lambda)}{Z(\mathrm{Re}\lambda)} = \frac{\langle e^{-S(\lambda)+S(\lambda_0)}\rangle_0}{\langle e^{-S(\mathrm{Re}\lambda)+S(\lambda_0)}\rangle_0}$$

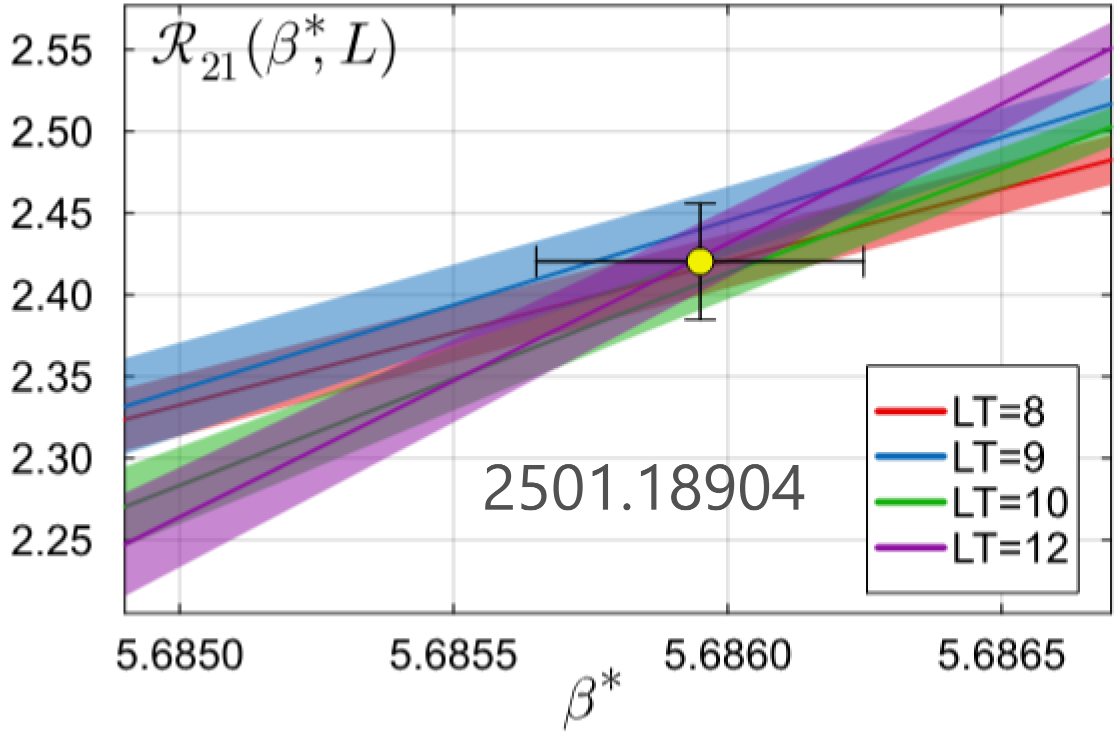
LYZ are nicely obtained up to the 4th one.

Contour plot of $|Z(\lambda)/Z(\mathrm{Re}\lambda)|$

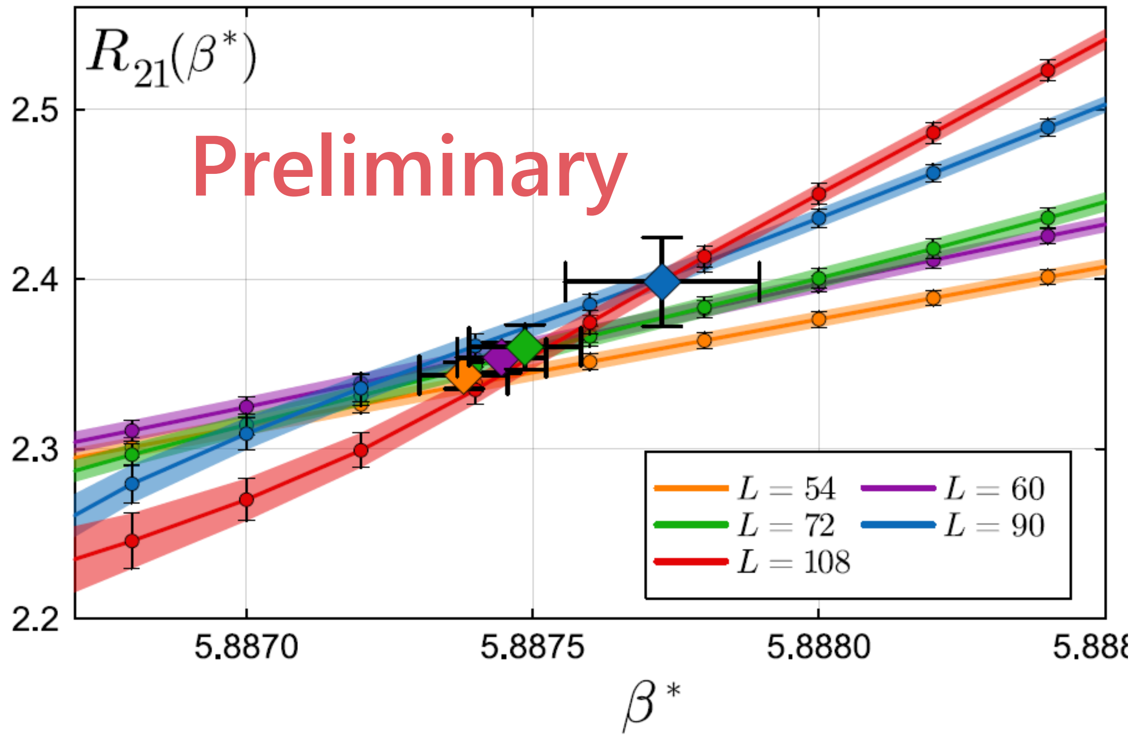


LYZ Ratio

$N_t = 4$

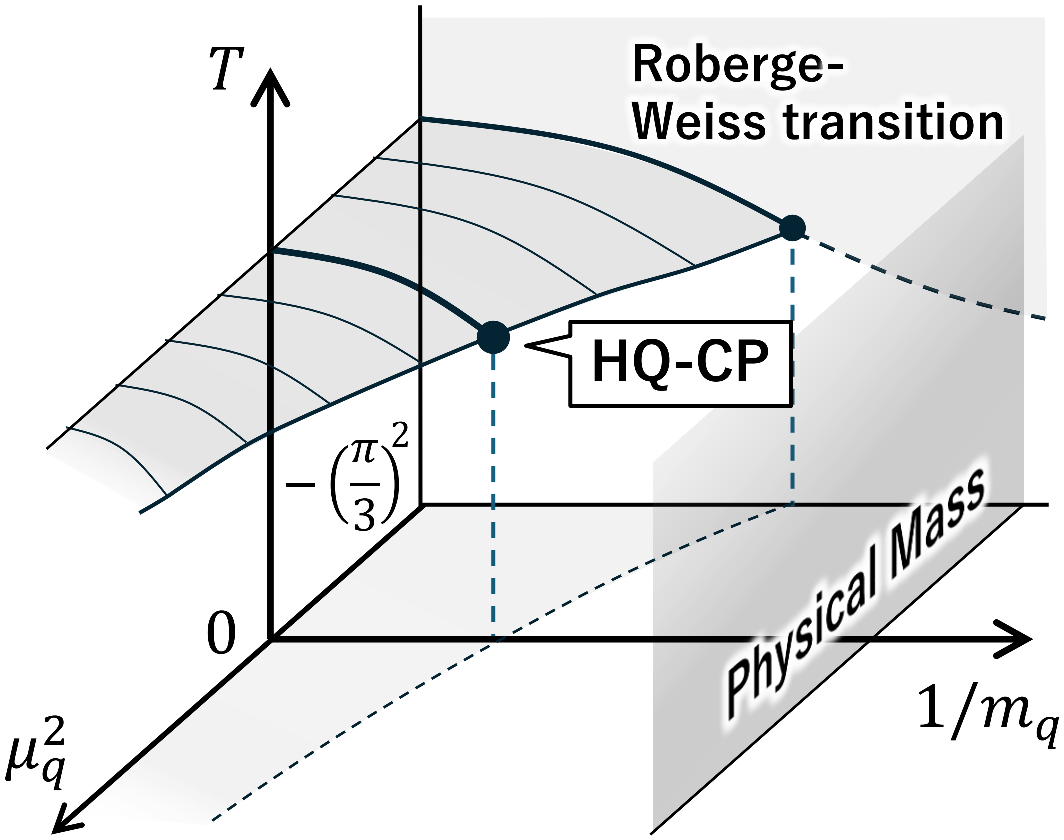


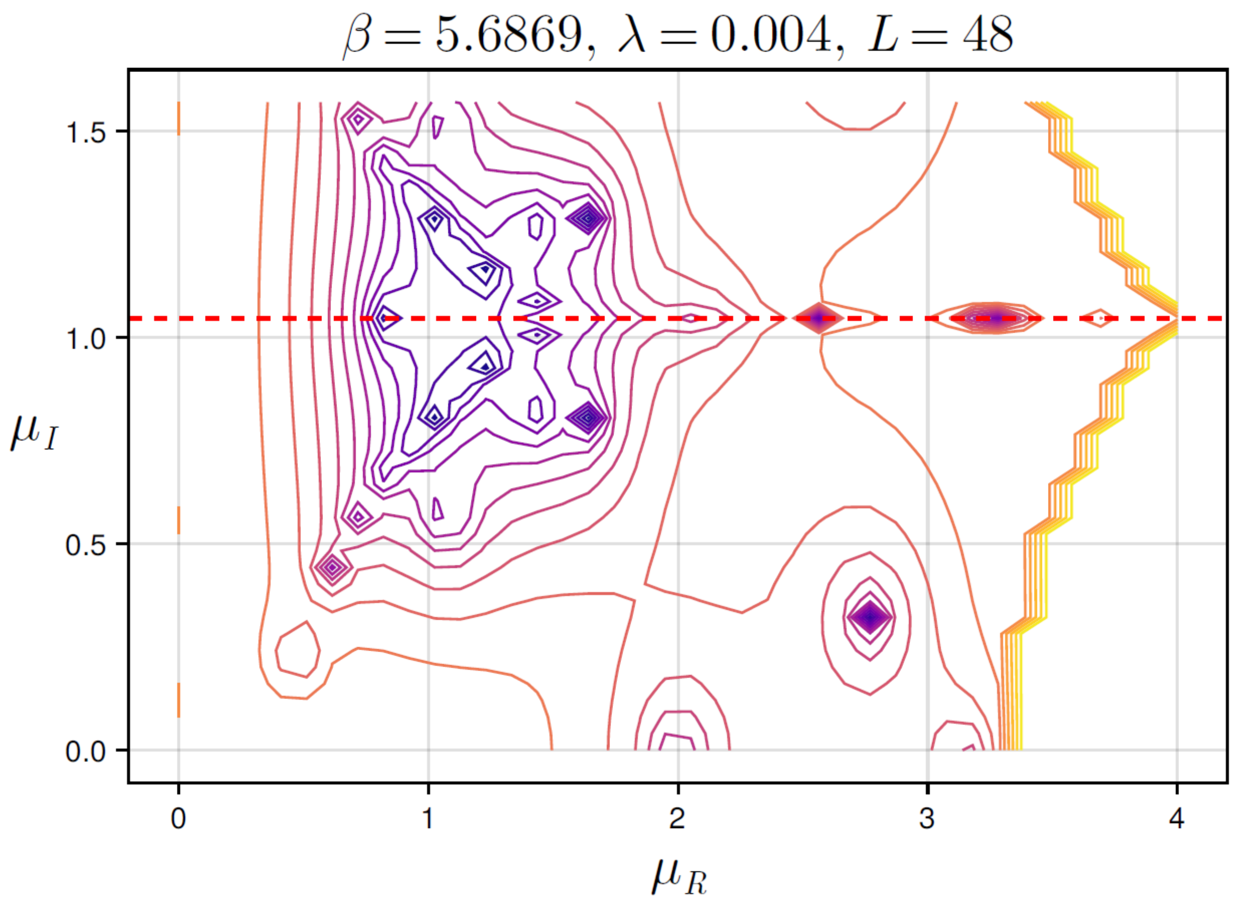
$N_t = 6$



Comparable statistics with the Binder-cumulant method.

LYZ on Complex-μ Plane





Summary

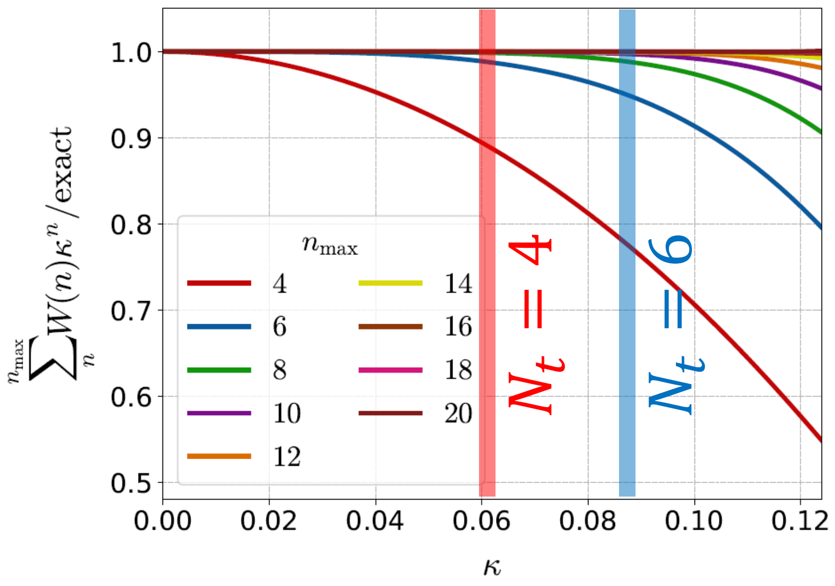
- LYZ-ratio method is a powerful method to locate a CP in general systems.
  - It has several advantages compared to Binder-cumulant method: suppression of scaling violation, non-linearity, etc.
- The LYZs in the heavy-quark region of QCD are successfully found using the numerical simulations based on the hopping-parameter expansion.
  - The LYZR method works well!
- Outlook: nonzero μ_q, RW transition, etc.

Convergence of HPE

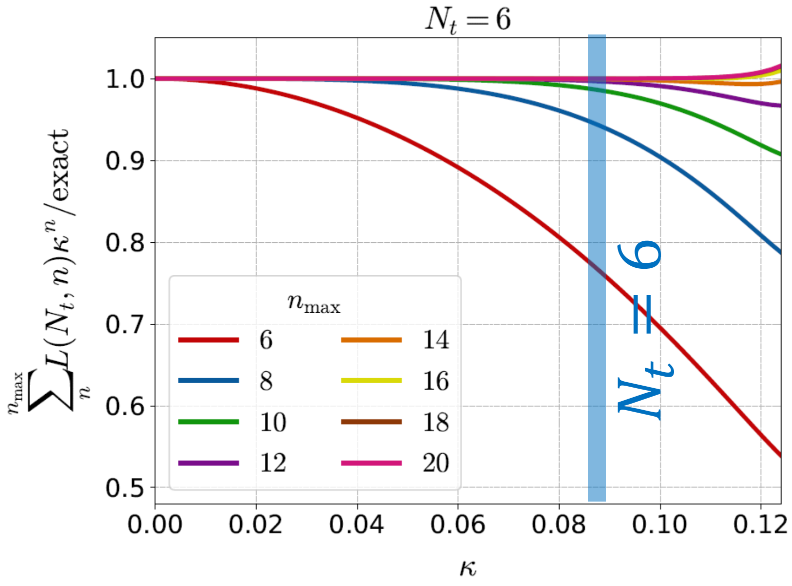
Wakabayashi+ ('22)

- HPE of free lattice field (U=1)

Wilson-loop-type



Polyakov-loop-type



$N_t = 4$ $\kappa_c = 0.0602(4)$ Kiyohara+,'21

$N_t = 6$ $\kappa_c = 0.0877(9)$ Cuteri+, '21

NNLO and higher

Wakabayashi+ ('22)