Mapping a critical point on pion condensate line in isospin phase diagram from Lee-Yang zeros

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FONDAZIONE BRUNO KESSLER | ECT* EUROPEAN CENTRE FOR THEORETICAL STUDIES IN NUCLEAR PHYSICS AND RELATED AREAS

CRC-TR 211 Strong-interaction matter under extreme conditions

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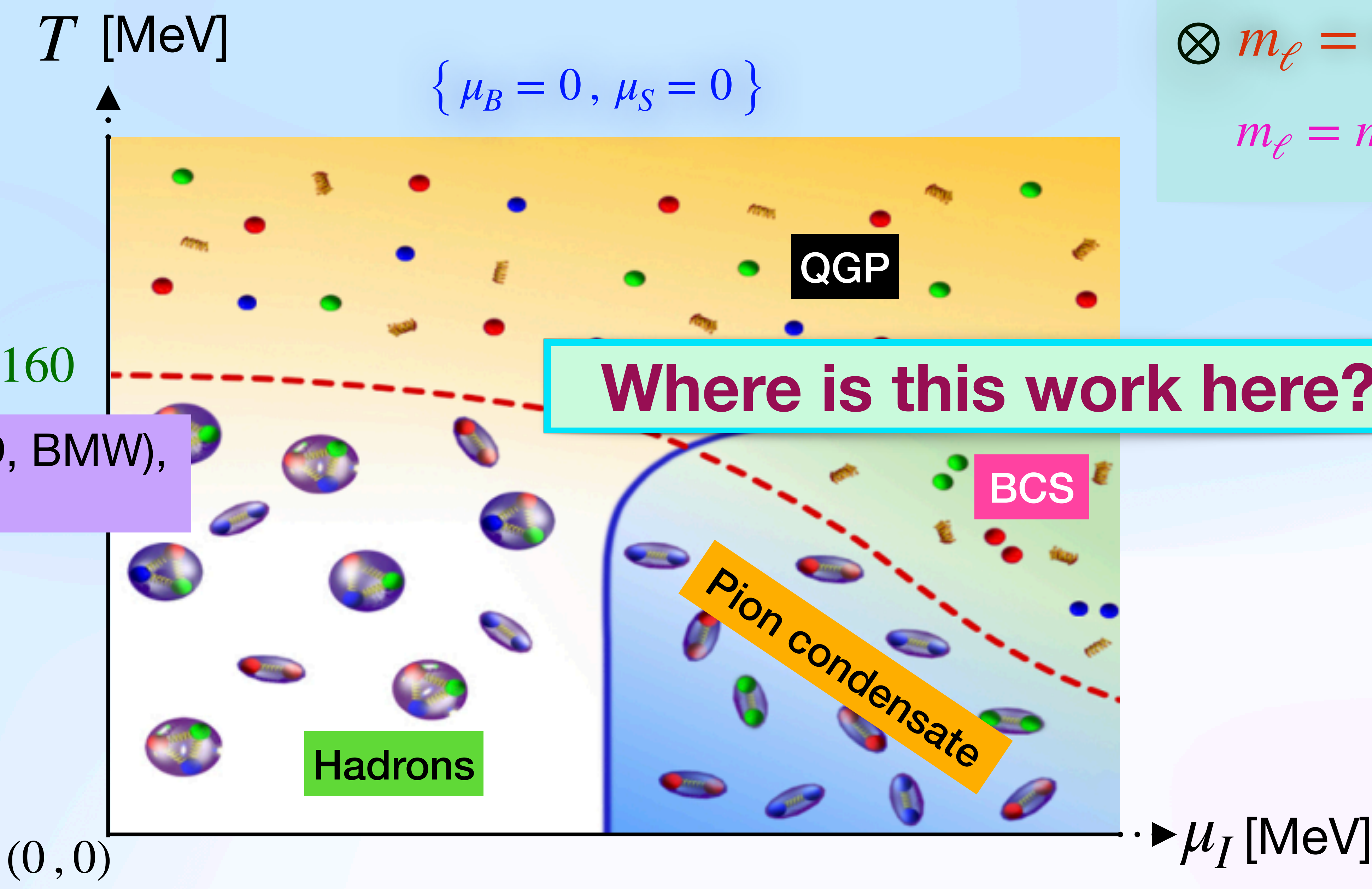
Plan of talk

- Motivation and domain of this work
- Core methodology and working lattices
- Zeroes, results and their stability
- Mapping a critical point on the critical line
- Radius of convergence and subsequent observations
- Overlap problem and its severity
- Conclusion and Outlook

Motivation

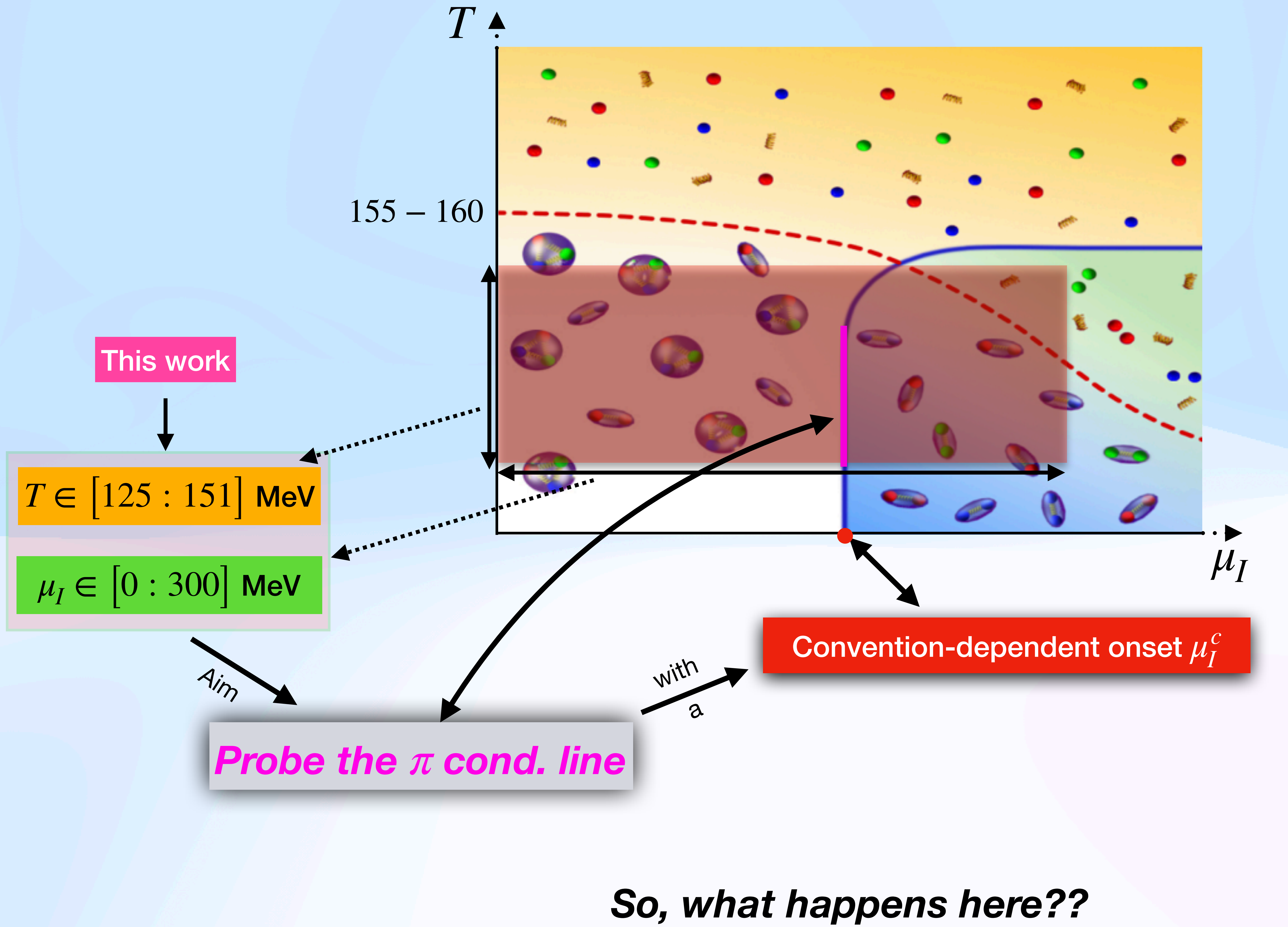


Lattice (HotQCD, BMW), FRG results

$\otimes\ m_\ell = m_s \,/\, 27$

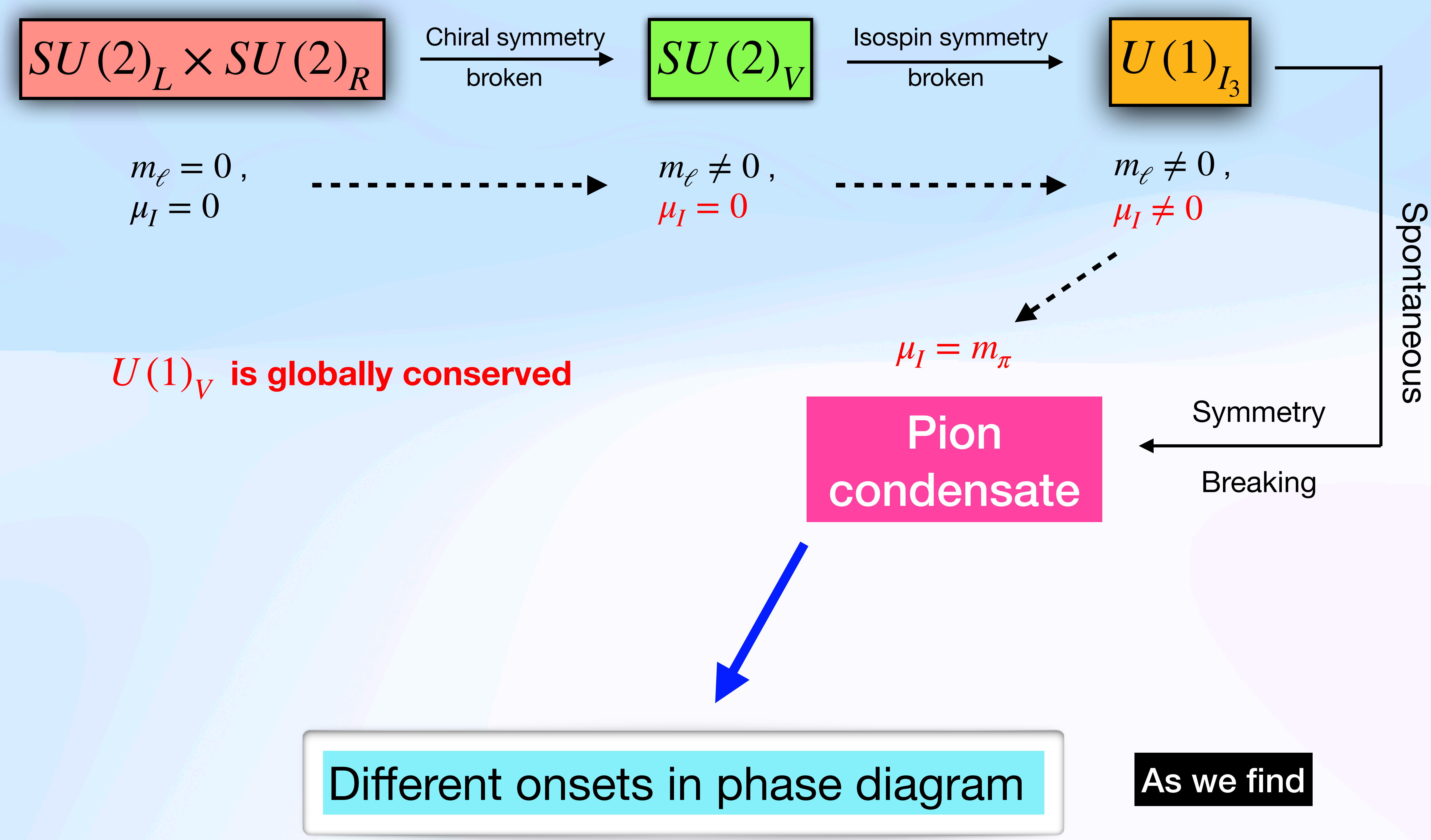
$m_\ell = m_u = m_d$

Brandt et. al. : PRD 97 (2018) 5, 054514, arXiv : 1712.08190 [hep-lat]



T
155 – 160
μ_I
This work
$T \in [125 : 151]$ MeV
$\mu_I \in [0 : 300]$ MeV
Aim
Probe the π cond. line
with a
Convention-dependent onset μ_I^c
So, what happens here??

The Theory : Symmetry



The prevailing notions so far

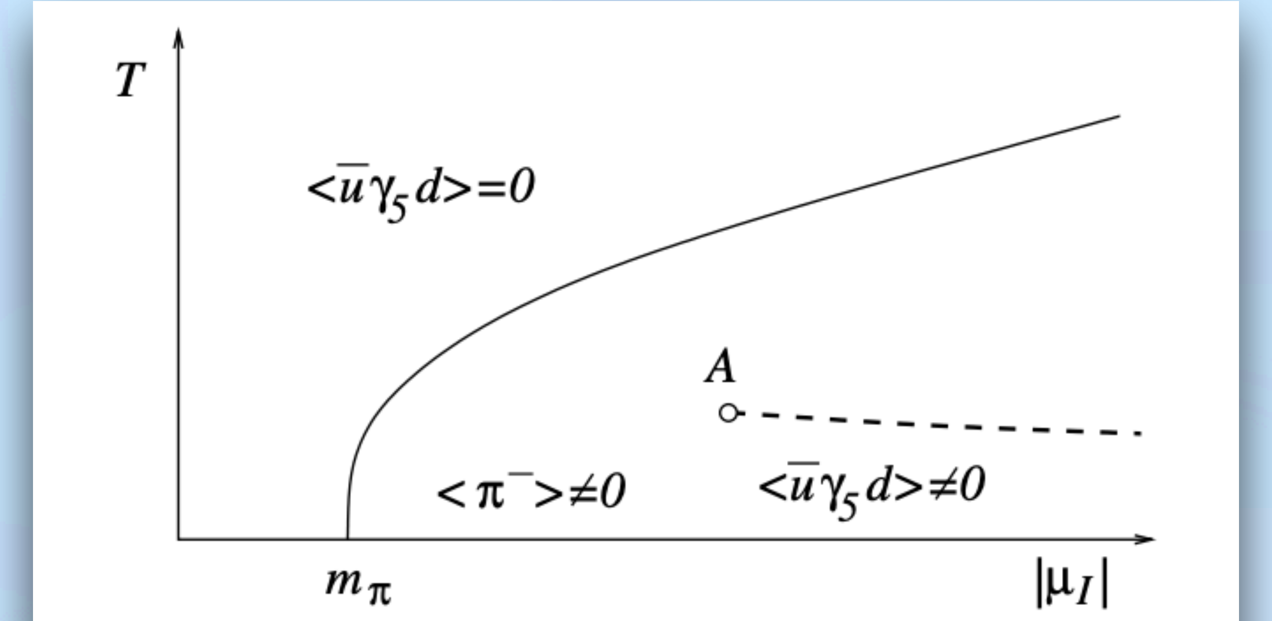
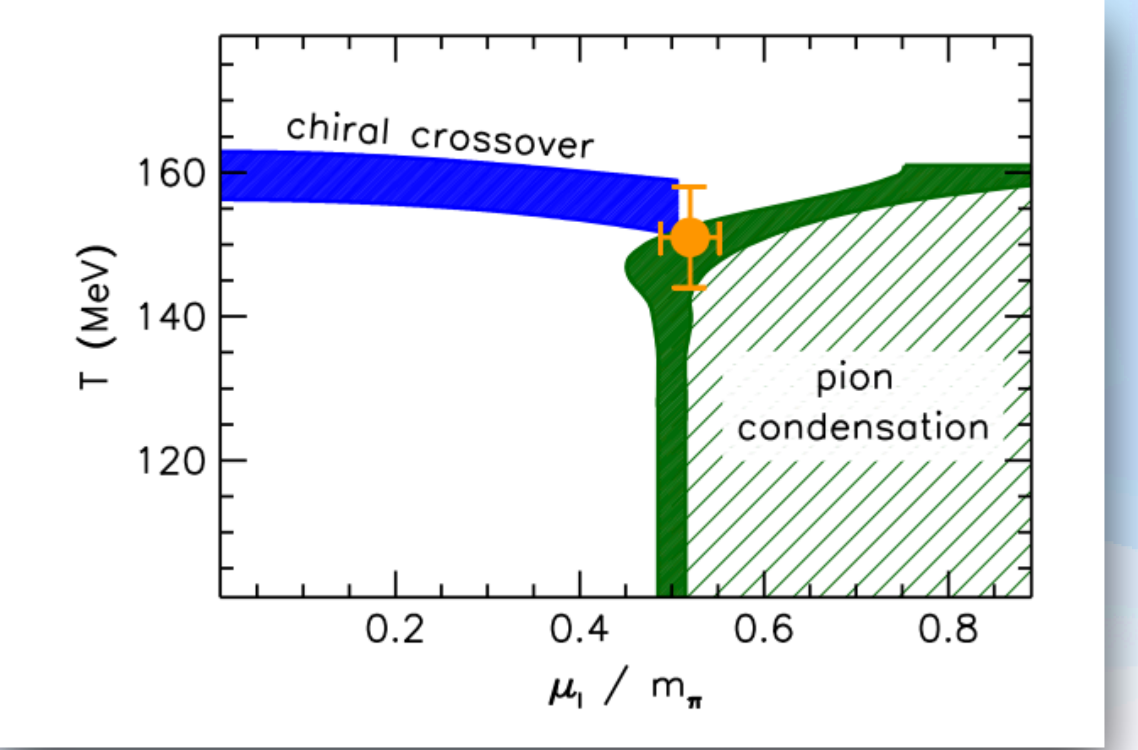


FIG. 1. Phase diagram of QCD at finite isospin density.



Son, Stephanov
PRL

Brandt et. al.
PRD

$$\mu_I = \mu_u - \mu_d$$

$\Downarrow$

$$\mu_I^c = m_\pi$$

Adopted here in this work

$$\mu_I = \frac{\mu_u - \mu_d}{2}$$

$\Downarrow$

$$\mu_I^c = \frac{m_\pi}{2}$$

So, what we do here??

On Lattice (Lattice QCD)

In complex isospin densities

Determine the **Lee-Yang zeros** of the **QCD partition function**

No assumption of the nature of phase transition

Closest zero and

Radius of convergence (RoC**)**

as $f\left(\mu_I, T\right)$

Regulator-free

Can RoC be indicator of the condensate critical line ???

See **manifestations of** this **pion condensate onset**

Implications of this **radius of convergence**

WORKING FORMULATION??

Working formulation

To evaluate the zeros, we use the familiar Newton-Raphson formulation :

$$\mu_{n+1} = \mu_n - \frac{\mathscr{Z}(\mu_n)}{\mathscr{Z}'(\mu_n)} , \quad \mu \equiv \mu_I$$

- $\mathscr{Z}'(\mu_n) = \left[\partial \mathscr{Z} / \partial \mu\right]_{\mu=\mu_n}$ (Isospin partition function $\longrightarrow \mathscr{Z}$)
- $\mu_n \, (n \geq 0 \in Z^+) \longrightarrow$ estimate obtained on n^{th} iteration . What iteration???
- Start from an initial guess μ_0 and continue iteratively , unless $|\, \mu_{n+1} - \mu_n \,| \leq \epsilon$

Tolerance

In this work, we choose $\epsilon = 0.002$

So, how is $\mathscr{Z}(\mu_I)$ defined here???

$$\mathcal{Z}\left(\mu_I, V, T\right) = \int DU e^{-S_g[U]} \left[\det M\left(\mu_I, V, T\right) \right]$$

$$= \left\langle \exp\left[\sum_{n=1}^{N} \frac{\mu_I^n}{n!} D_n^{(u)} \right] \right\rangle$$

Unbiased exponential resummation

SM, Hegde ; **PRD 108 (2023) 3, 034502**, arXiv : **2302.06460 [hep-lat]**

$\langle O \rangle$ calculated on gauge ensemble generated at $\mu_I = 0$

$$D_n = \frac{\partial^n}{\partial \mu_I^n} \ln \left[\det M \right] \Big|_{\mu_I = 0}$$

, and $D_n^{(u)}$ contains **unbiased corrections**

These corrections are important to reproduce the exact Taylor coefficients order-by-order. (Upto fourth order here)

SM, Hegde, Schmidt ; **PRD 106 (2022), 3, 034504**, arXiv : **2205.08517 [hep-lat]**

And now the working lattice …

Lattice setup

- (2+1)-flavor Highly Improved Staggered Quark (HISQ) action
- Working lattices $\longrightarrow$ $32^3 \times 8$ lattices (higher volumes in future)
- Total of $20K$ **configurations** $\left(N_{conf} = 20K \right)$
- Working temperatures $\longrightarrow$ $125 \leq T \leq 171$ MeV
- Physical light and strange quark masses, for each temperature $\left(m_\ell = m_s / 27\right)$

How do we implement the simulations???

Implementation

- Choose the initial complex μ_0 from the complex set

$$S = \{\mu_0 : 0 \leq \mathrm{Re}\left(\mu_0\right) \leq 2 \ , \ 0 \leq \mathrm{Im}\left(\mu_0\right) \leq 2\}$$

in steps of 0.1 on each direction , along $\mathrm{Re}\left(\mu_0\right)$ & $\mathrm{Im}\left(\mu_0\right)$

- Choose the tolerance value $\epsilon = 0.002$

- Choose the upper bound of the number of iterations

$$N_{max} = 10^8$$

with $N_{conf} = 20K, N_B = 50$ (bootstrap samples)

And :

WORKFLOW

Start with a μ_0 value $\in S$

$\mu_{NR}^{(1)}$ $\mu_{NR}^{(2)}$ .... $\mu_{NR}^{(49)}$ $\mu_{NR}^{(50)}$

$\mu_{NR}^{(b)} \longrightarrow$ estimate of the Newton-Raphson root in b^{th} bootstrap sample

Final estimates :

$$\mu_{NR}\left(\mu_0\right) = \frac{1}{50}\sum_{b=1}^{50}\mu_{NR}^{(b)}$$

← Mean

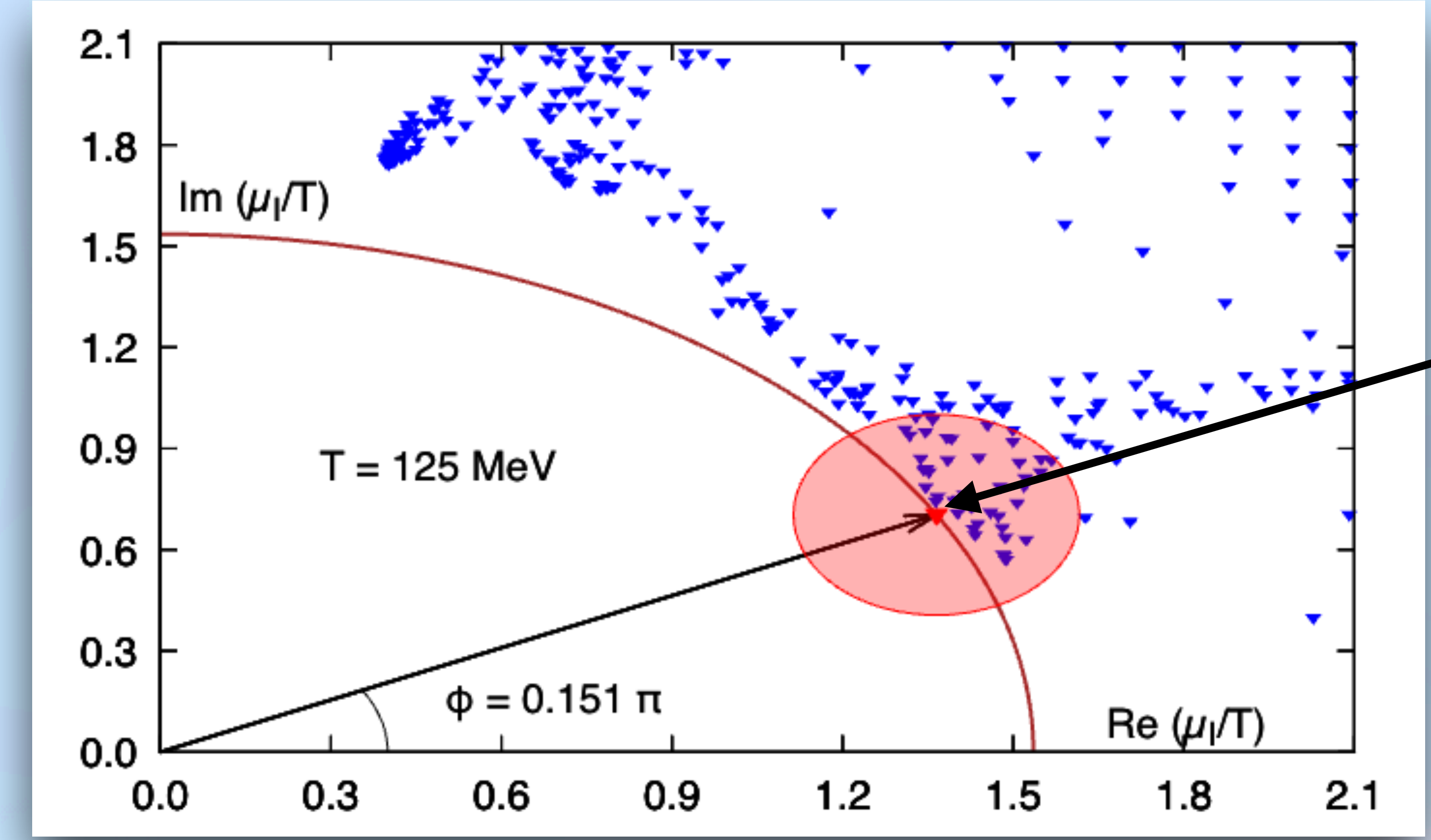
$$\sigma_{NR}^2\left(\mu_0\right) = \frac{1}{50}\sum_{b=1}^{50}\left\{\mu_{NR}^{(b)} - \mu_{NR}\left(\mu_0\right)\right\}^2$$

Variance

Follow this for all the other μ_0

RESULTS ???

Results



Complex μ_I plane

Nearest Lee-Yang zero μ_I^0

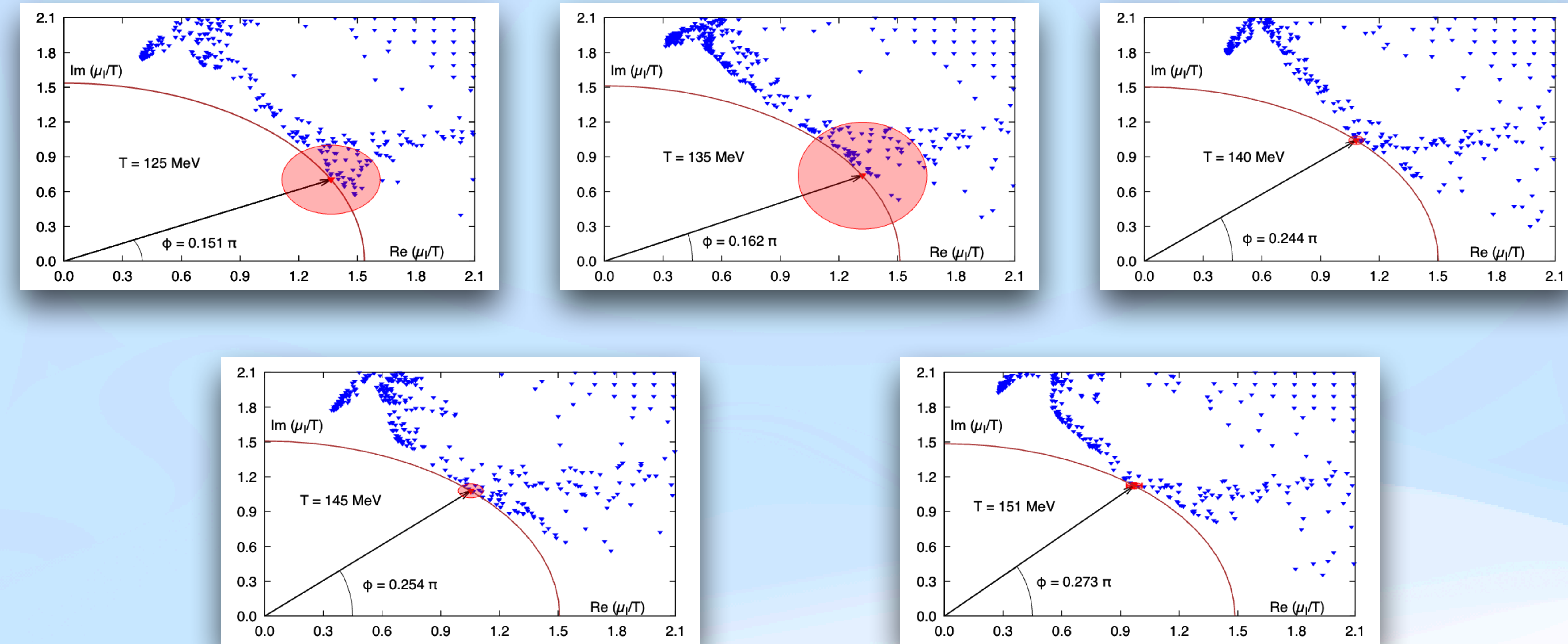
↓

Closest to point of expansion → origin $(0\,,0)$ here

- Individual error bars on roots are **only** shown here for μ_I^0.
- Elliptical representation of errors : major and minor axes.
- This line shows **one quarter** of **the circle of convergence**.
- μ_I^0 makes an **angle** ϕ (radian units) **with the real μ_I axis.**

$$\phi = \tan^{-1}\left[\frac{\mu_{I,i}^T}{\mu_{I,r}^T}\right]$$

Other $T's$??

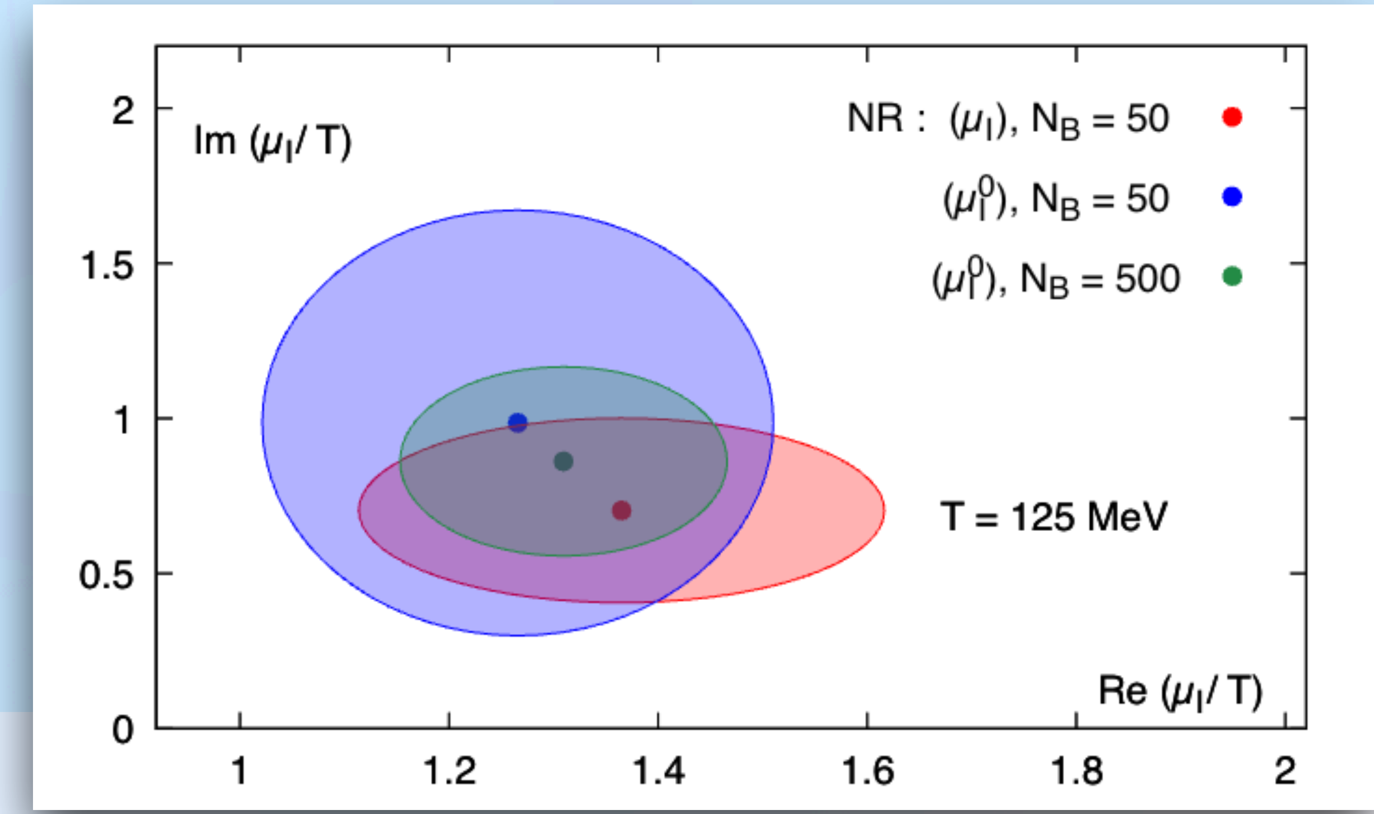


- μ_I^0 **approaching real axis** with reducing T (reducing angle ϕ)
- Expecting to find **real** $\mu_I^0 \Rightarrow$ **genuine critical point** at **lower** T

How stable (reliable) are these results ??

However

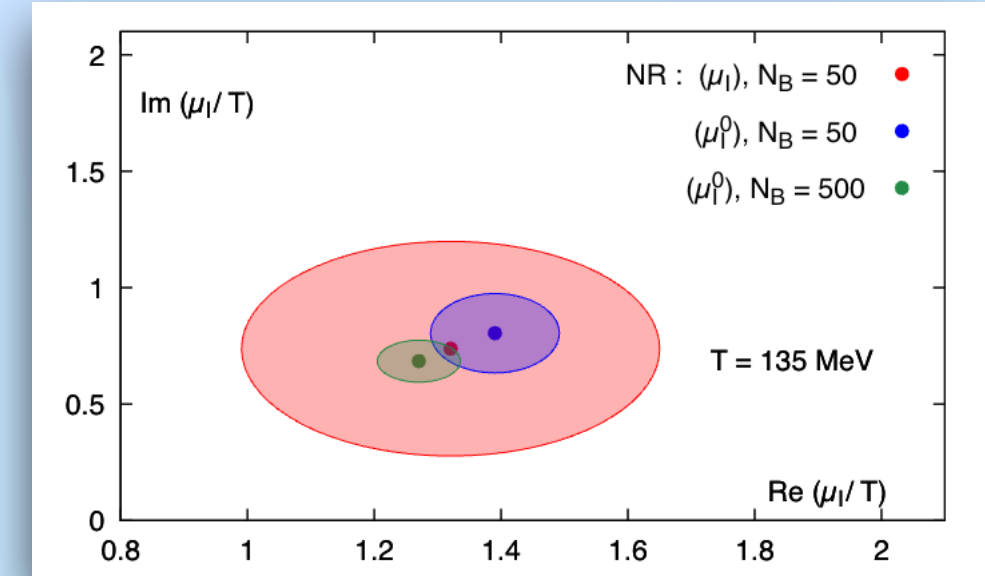
Stability of results

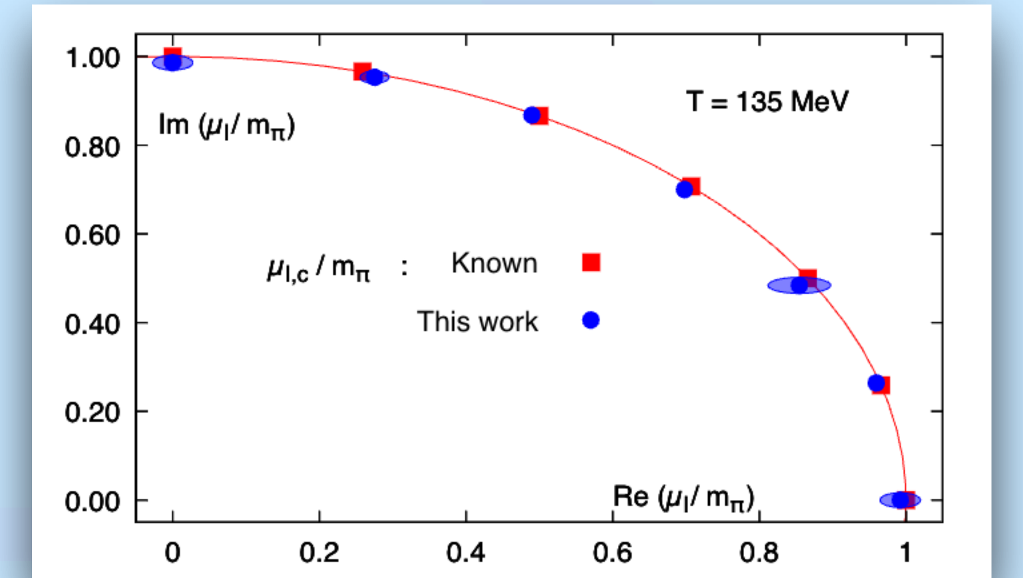


New estimates $\longleftarrow$ using μ_I^0 **as initial guesses**

- **Good agreement (overlap)** with the **old** $\Longleftrightarrow$ **new estimates** of the roots / zeros
- The present estimates of zeros **are reliable** !!!

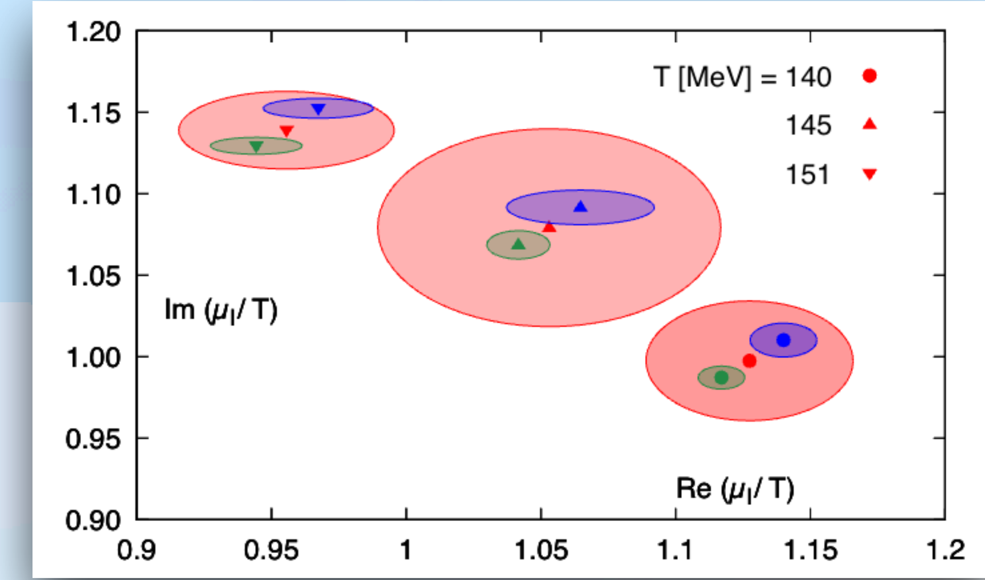
What about other T ??





All possible critical points **for**
$T = 135$ **MeV as** initial guesses

$$\mu_0 = (m_\pi \cos\phi\,, m_\pi \sin\phi) \;\;,\;\; \phi = \left\{ \frac{n\pi}{12} \;\;,\;\; 1 \le n \le 6 \right\}$$

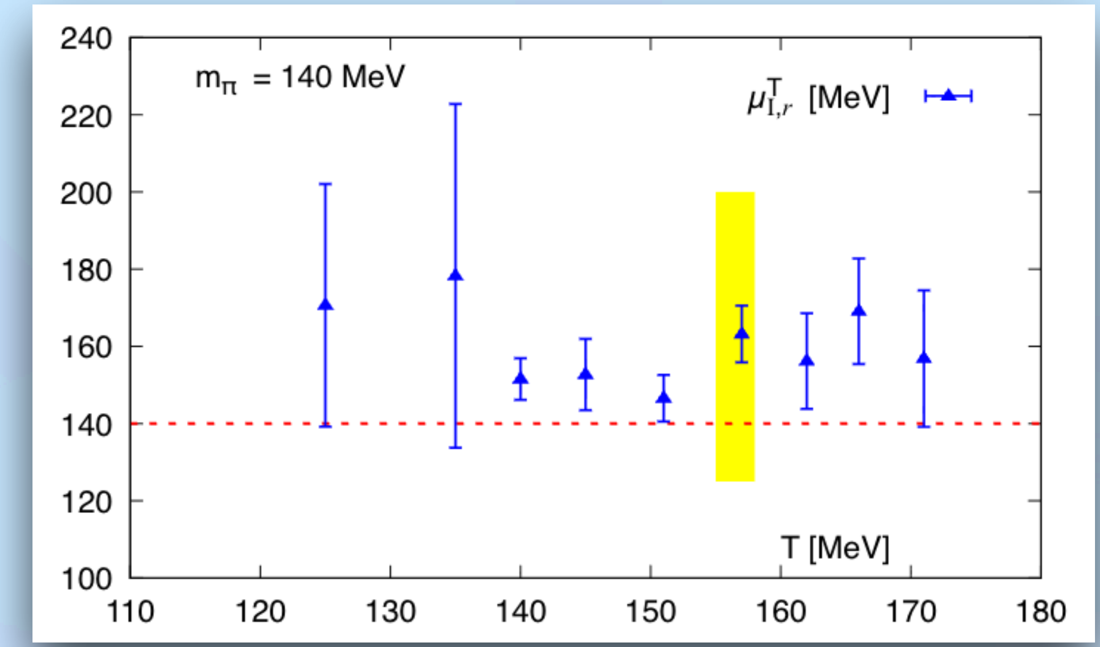


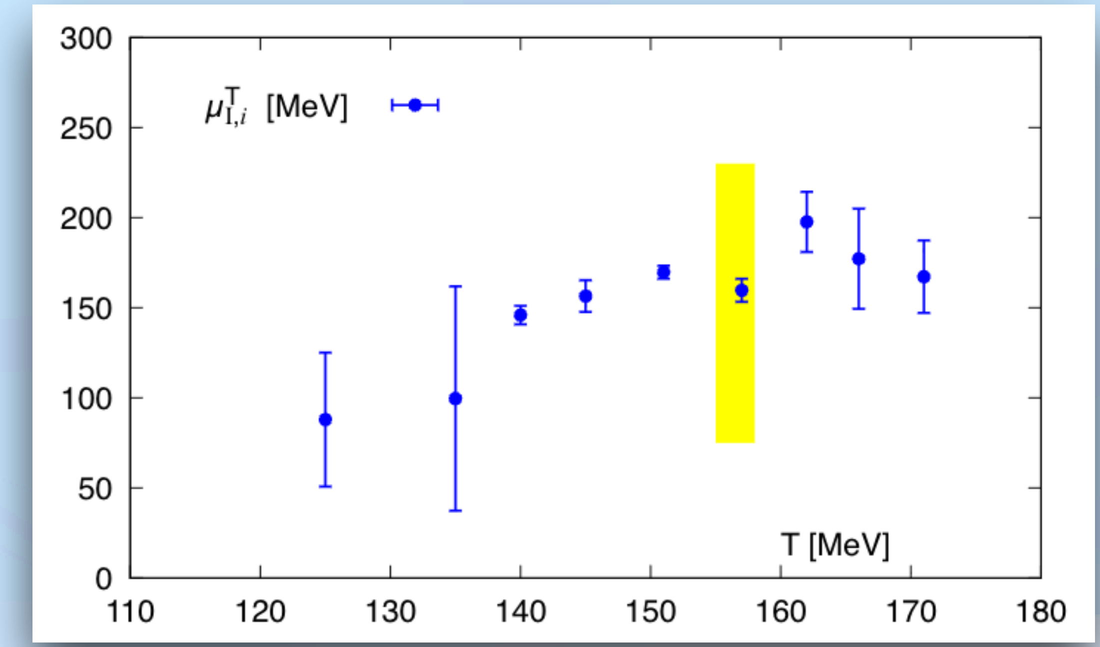
Commendable agreement upto $\underline{T = 151}$ MeV

Reliable therefore, at least for this work

$\therefore$ ***Let' s analyse these 0' s***

Real and imaginary parts of μ_I^0





- $\mu^T_{I,r} \sim$ Re $\left(\mu_I^0\right)$ close to m_π for **lower values of T**

- **Monotonically reducing $\mu^T_{I,i}$** $\sim$ Im $\left(\mu_I^0\right)$ with **decreasing T**

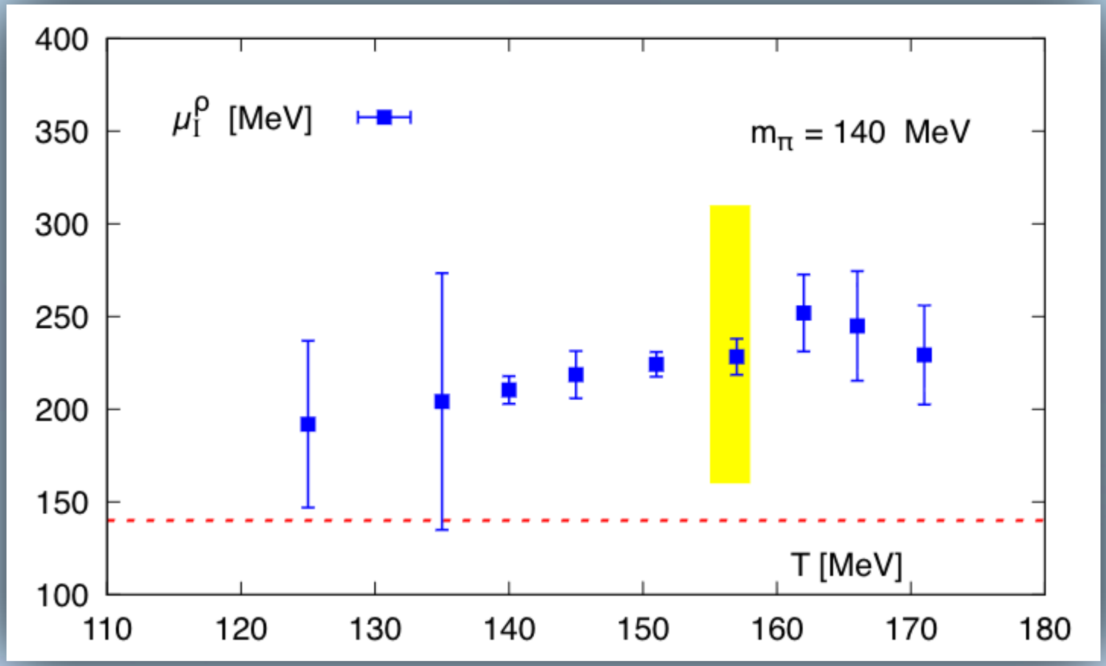
Also, **Increasing errors** for $\mu^T_{I,r}$ and $\mu^T_{I,i}$ for $T < 140$ MeV

↓

signs of possible **"vicinity of critical point(s)"**

What about the coveted RoC ???

Radius of convergence (RoC) :



(Note, $|\mu_I^0| = \mu_I^\rho / T$)

$$\mu_I^\rho = \sqrt{\left(\mu_{I,r}^T\right)^2 + \left(\mu_{I,i}^T\right)^2}$$

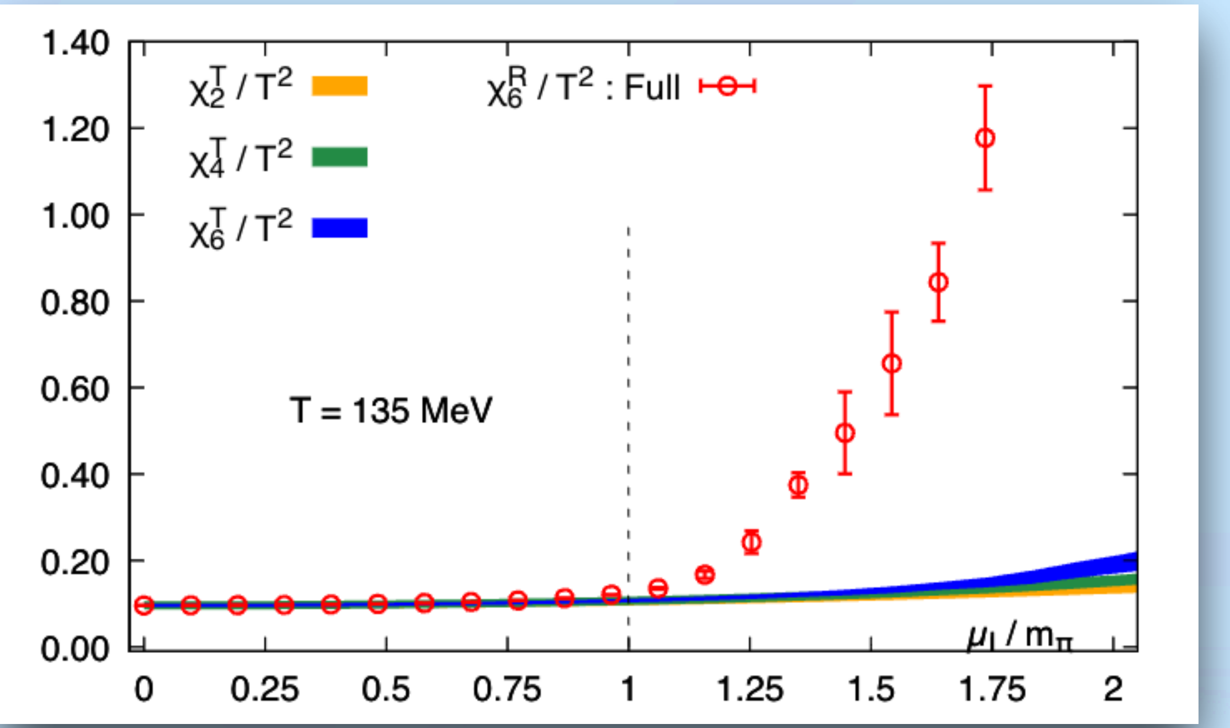
Dimensional (in MeV units)

- **Monotonic reduction** : μ_I^ρ decrease with T for $125 \leq T \leq 151$ MeV

Qualitatively consistent with χPT predictions

- $\mu_I^\rho \sim m_\pi$, within errors for $T = 135$ MeV

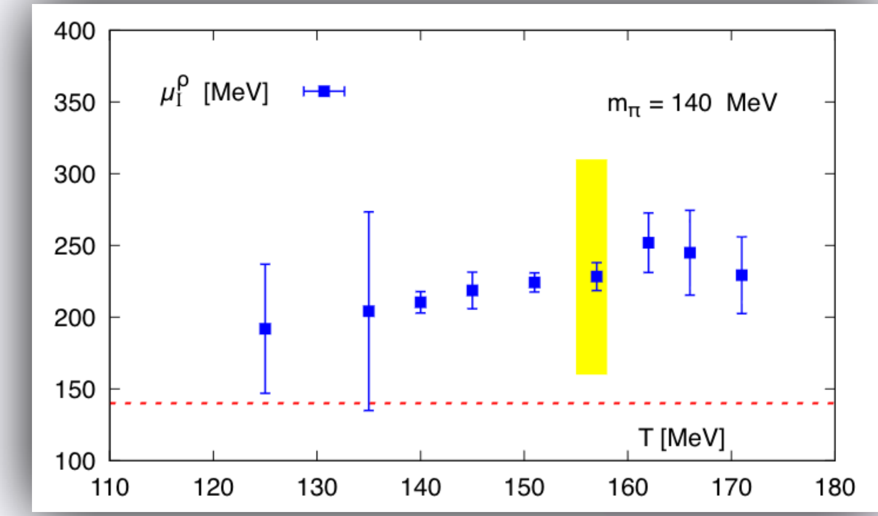
Can it indicate possible critical point / line???



- Resumed susceptibility χ_6^R **deviates sharply** from Taylor counterpart results for $\mu_I > m_\pi$ (**good agreement** for $\mu_I < m_\pi$)

similar to

Borsanyi et. al. : **PRD 109, 054509 (2024)**, arXiv : **2308.06105 [hep-lat]**

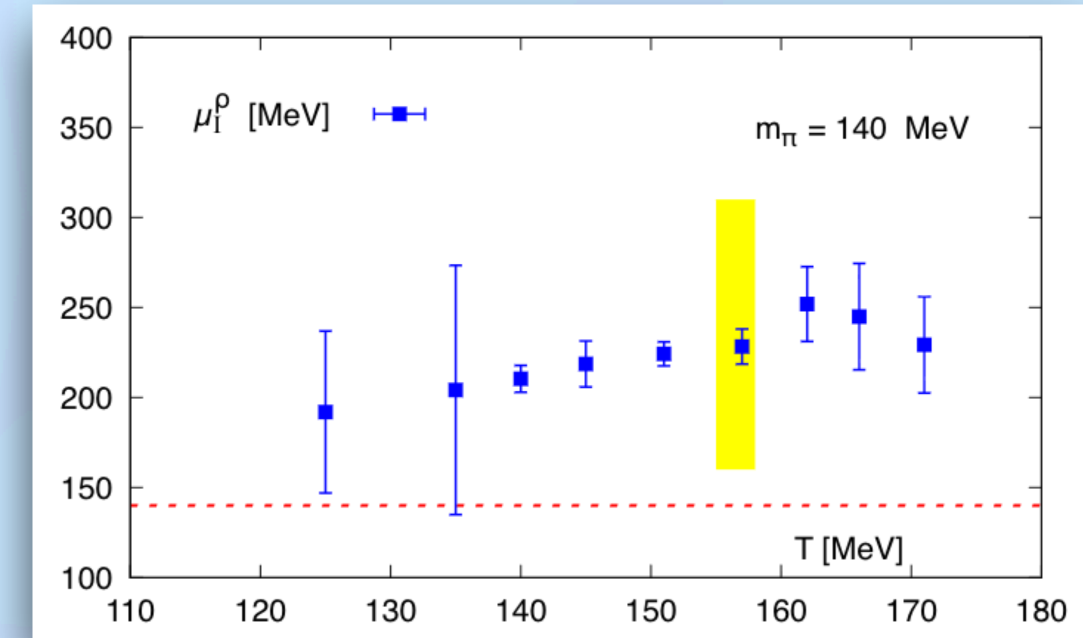


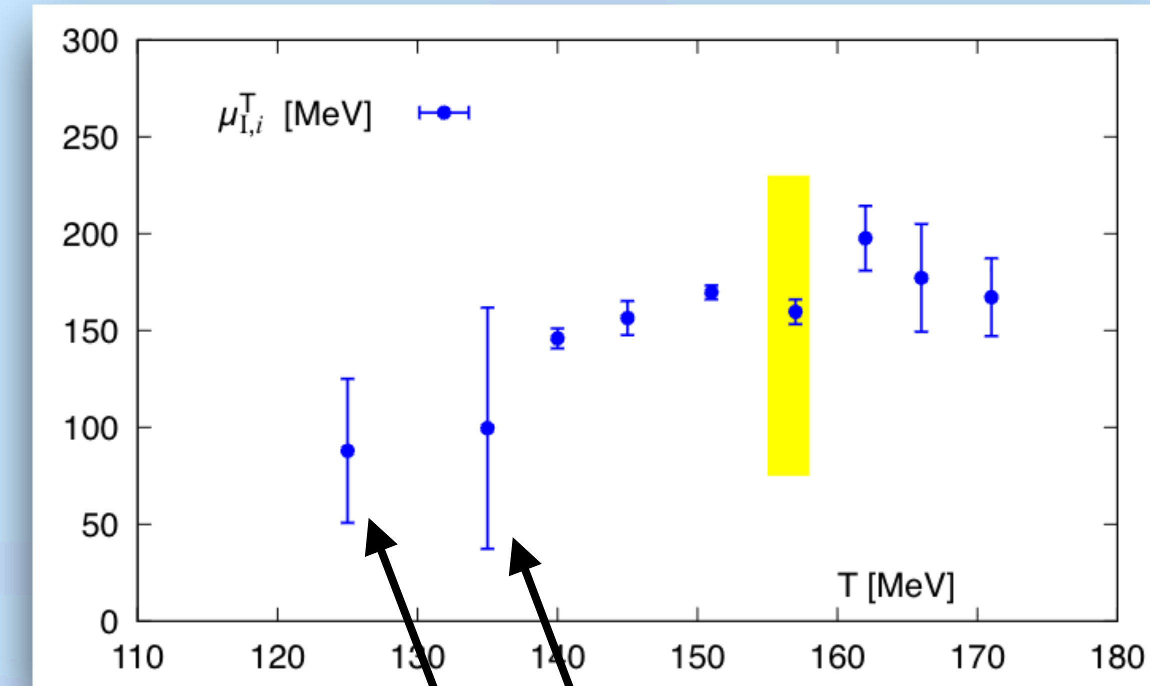
Manifesting **divergence**

Possible phase transition signatures

RoC : $\mu_I^\rho \longrightarrow$ **good indicator** of π-boundary , at least in low T

However





$\mu_{I,i}^T \neq 0$ for $T \leq 135$ MeV

Non-zero **imaginary** part

$\left(\mu_I^c, T_I^c\right) = \left(m_\pi, 135\right)$ is

NOT

A CRITICAL POINT

So, analyse lower T...

But

No further data for $T < 125$ MeV available on these lattices

↓

Extrapolate $\mu_{I,i}^{T}$ and μ_{I}^{ρ} in T, to lower T

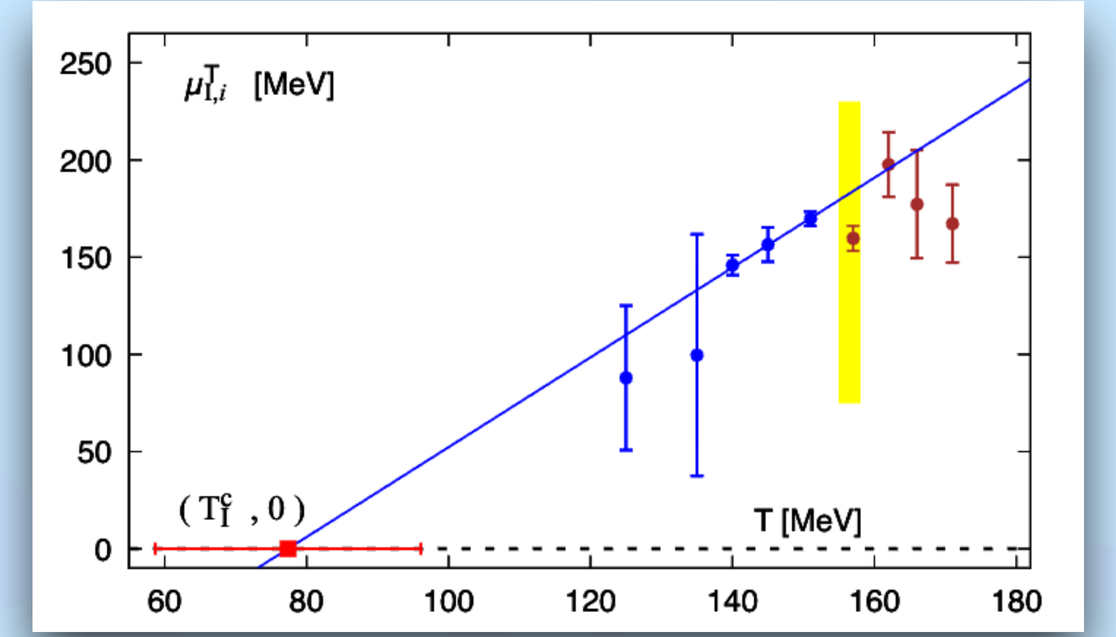
↓

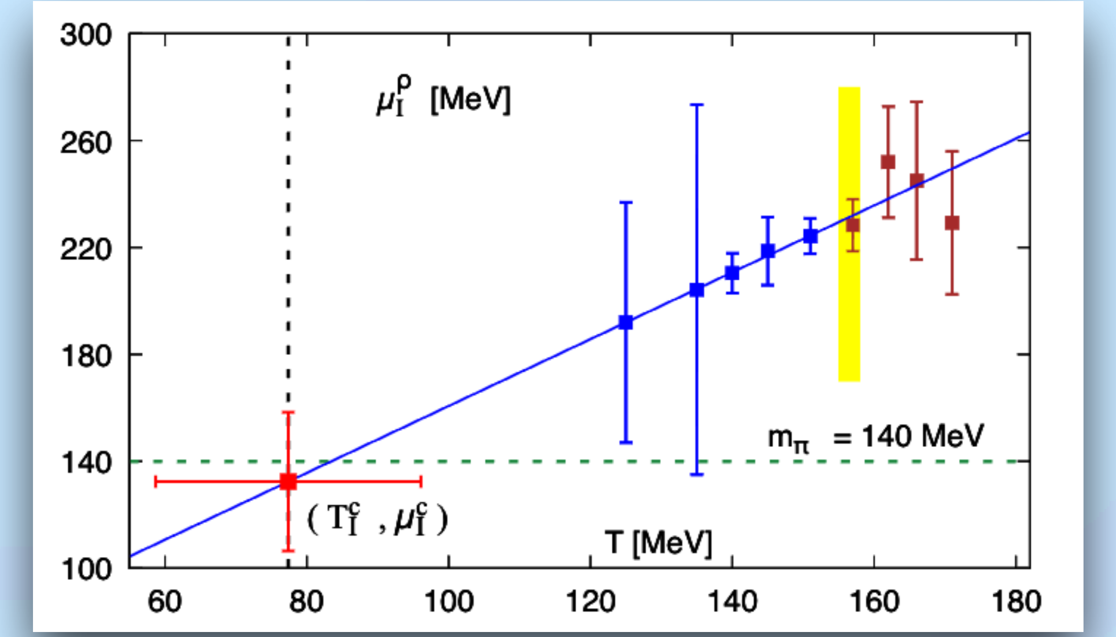
1. Find : $T = T_I^c \;\ni\; \mu_{I,i}^{T} = 0$, and hence, $\mu_I^0\left(T_I^c\right) \in \mathbb{R}$

2. Find : μ_I^{ρ} at $T = T_I^c$: $\Rightarrow \mu_I^c$ $\{$ since, at $T = T_I^c$, $\mu_I^{\rho} = \mathrm{Re}\left(\mu_I^0\right)$ $\}$

↓

Thus, obtain an estimate of a critical point $\left(\mu_I^c\,,T_I^c\right)$





Linear extrapolation of $\mu^T_{I,i}$ and μ^ρ_I + only in $T \in [125 : 151]$ MeV

$$\left(\mu^c_I\,,\,T^c_I\right) := \left(132.40 \pm 25.91\ ,\ 77.38 \pm 18.71\right) \text{ MeV}$$

This lies on the present state-of-the-art pion condensate line in the QCD isospin phase diagram. $\Longleftrightarrow$ Nomenclature with $\mu^c_I = m_\pi$ **(One of the achievements).**

within error bars

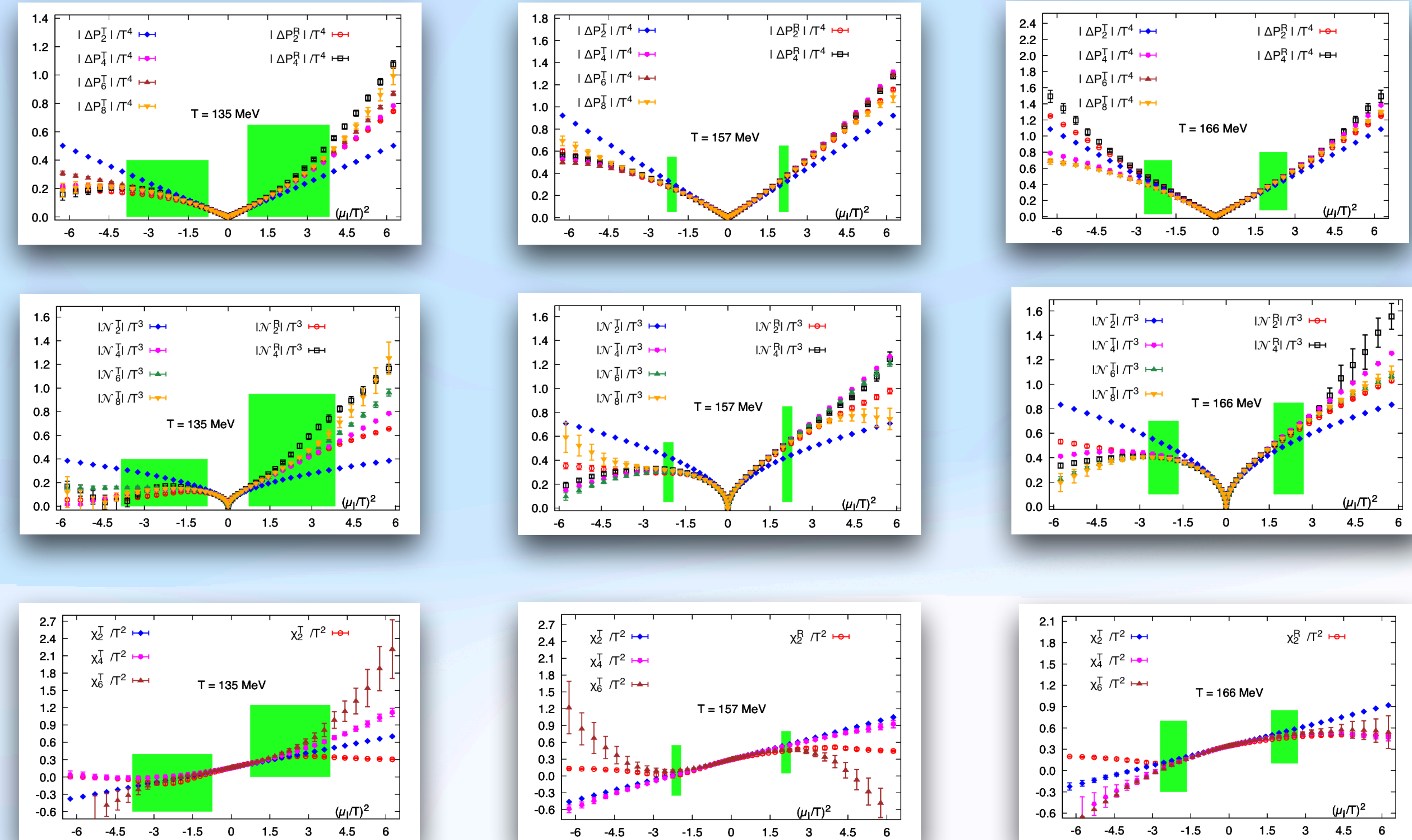
Son, Stephanov : *PRL*

Andersen et. al. (2024) : *PRD*

Andersen et. al. (2025) : *PRD*

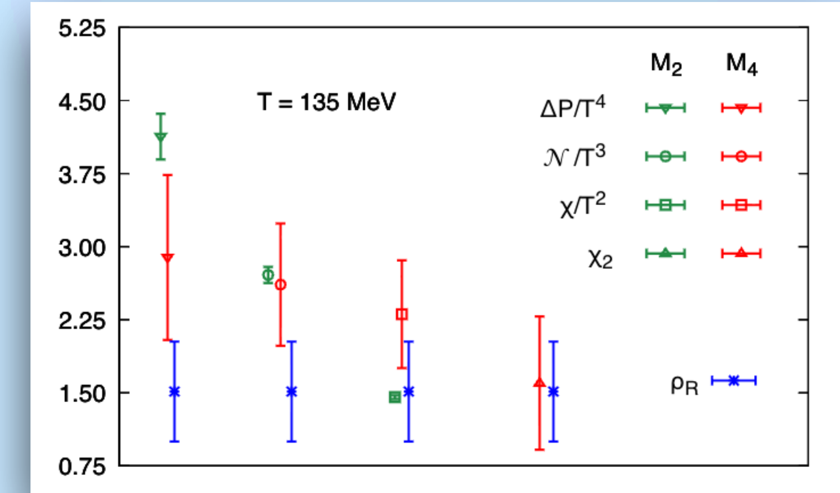
$\mu_I = \mu_u - \mu_d$

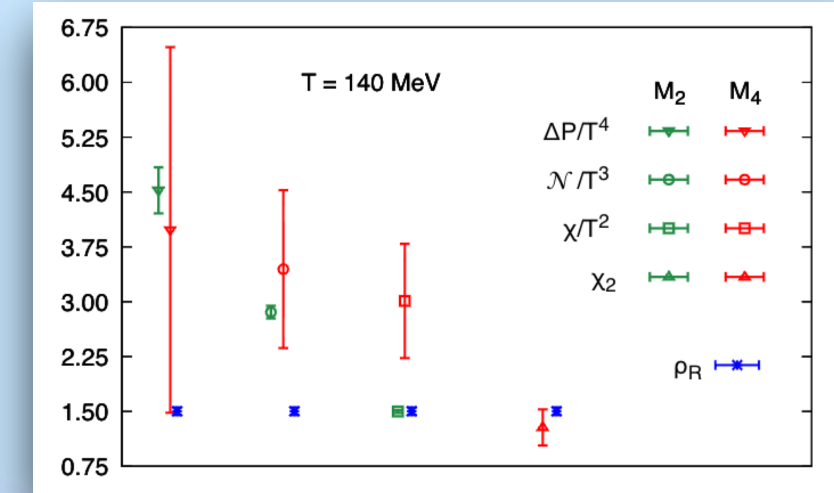
Some observations related to this RoC ...

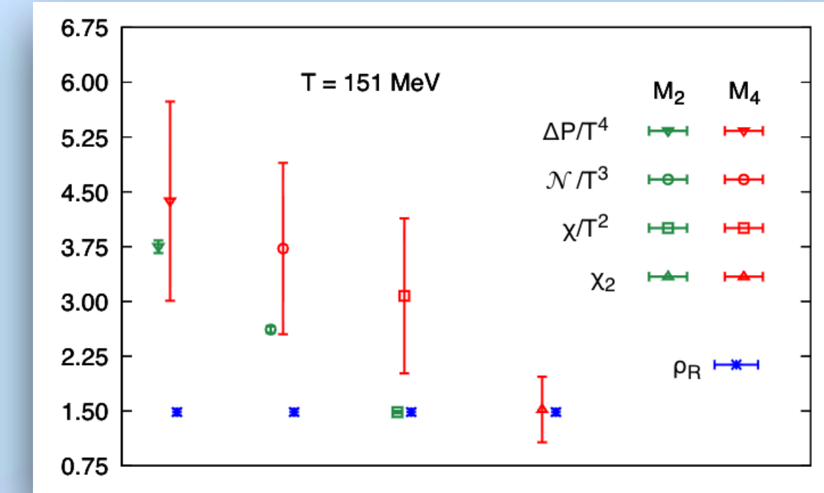


- Order-by-order deviations **beyond** μ_I^ρ → for **different** T; for **both Re** (μ_I) **and Im** (μ_I)

- Qualitatively, deviations → $\chi > \mathcal{N} > \Delta P$







$$\rho_R = \left|\mu_I^0\right|$$

$$M_n = \left| \frac{c_{2n+2}\, c_{2n-2} - c_{2n}^2}{c_{2n+4}\, c_{2n} - c_{2n+2}^2} \right|^{1/2}$$

$$\frac{\Delta P}{T^4} = \sum_{n=1}^{N} c_n \left(\frac{\mu_I}{T}\right)^n$$

More order-by-order stable → Mercer-Roberts estimates M_n

- M_n estimates **approach** $\rho_R = \mu_I^\rho / T$

For all the three T

For higher order μ_I derivatives of ΔP

Overlap problem

Quantified by **Kurtosis κ**

$$\kappa = M_4 / \sigma^4$$

$$M_4 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^4$$

Here, $N = N_{conf}$

$$\sigma = \left(\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2\right)^{1/2}$$

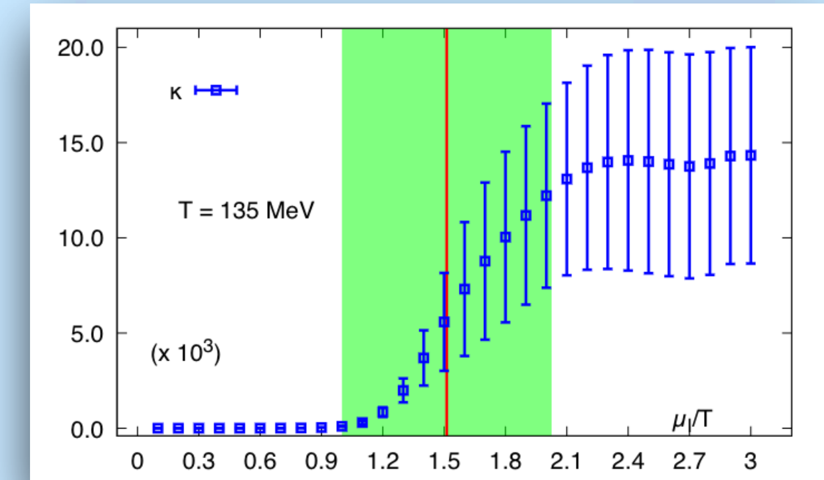
$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

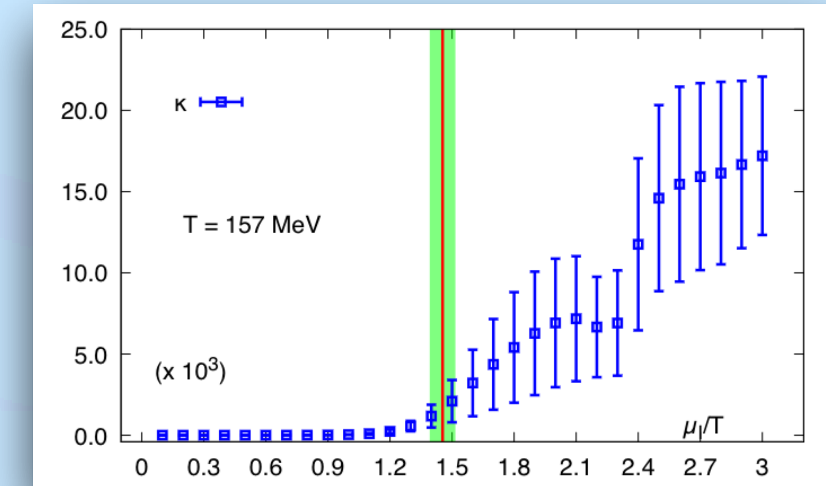
Results???

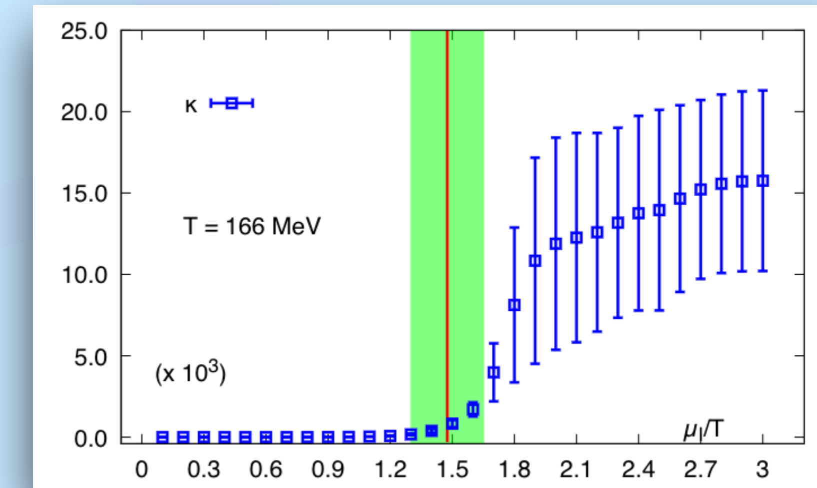
describing distribution D

$$D = \left\{ R_i(\mu_I) : i = 1,2,\cdots,N_{conf} \right\}, \quad R(\mu_I) = \det M(\mu_I) / \det M(0)$$

Overlap problem







- **Controllable until** Radius of convergence (**drastic after that**) → for all the three T .
- Indicates the **efficacy of $\mu_I = 0$ extrapolations** to **determine finite μ_I observables** .
- This truly **breaks down** and is **unreliable** **beyond** the Radius of convergence μ_I^ρ

Summary

- Present estimates of Lee-Yang zeros $\rightarrow$ map to a critical point $\Rightarrow$ exists on the **present state-of-the-art π-cond. line** (**within error bars**).

- **Early indications of a phase transition** here, from monotonic reduction of Im $\left(\mu_I^0\right)$ with reducing T, while Re $\left(\mu_I^0\right)$ approaching m_π .

- The root estimate results so far, show stability and well-defined convergence.

- **Radius of convergence** (RoC) is a **good indicator** of **this π-condensate line**, with **divergences** manifesting **beyond this RoC,** for different observables.

- Good agreement : **RoC** $\Longleftrightarrow$ **MR** estimates for higher order observables.

- **Overlap problem** becoming noticeably **severe** **beyond this RoC**.

With lot of future things to do…

Future works and Outlook

- Simulations at **higher lattice volumes** (**larger N_σ for same N_τ**)

↓ **Finite-V analysis**

if these zeros truly **become real** → confirm a **phase transition** **,** and

NOT a crossover

- Also, how do the Im $\left(\mu_0\right)$ behave or, **scale** with volumes ?

↙ ↘

$1/V$ scaling — **First order**

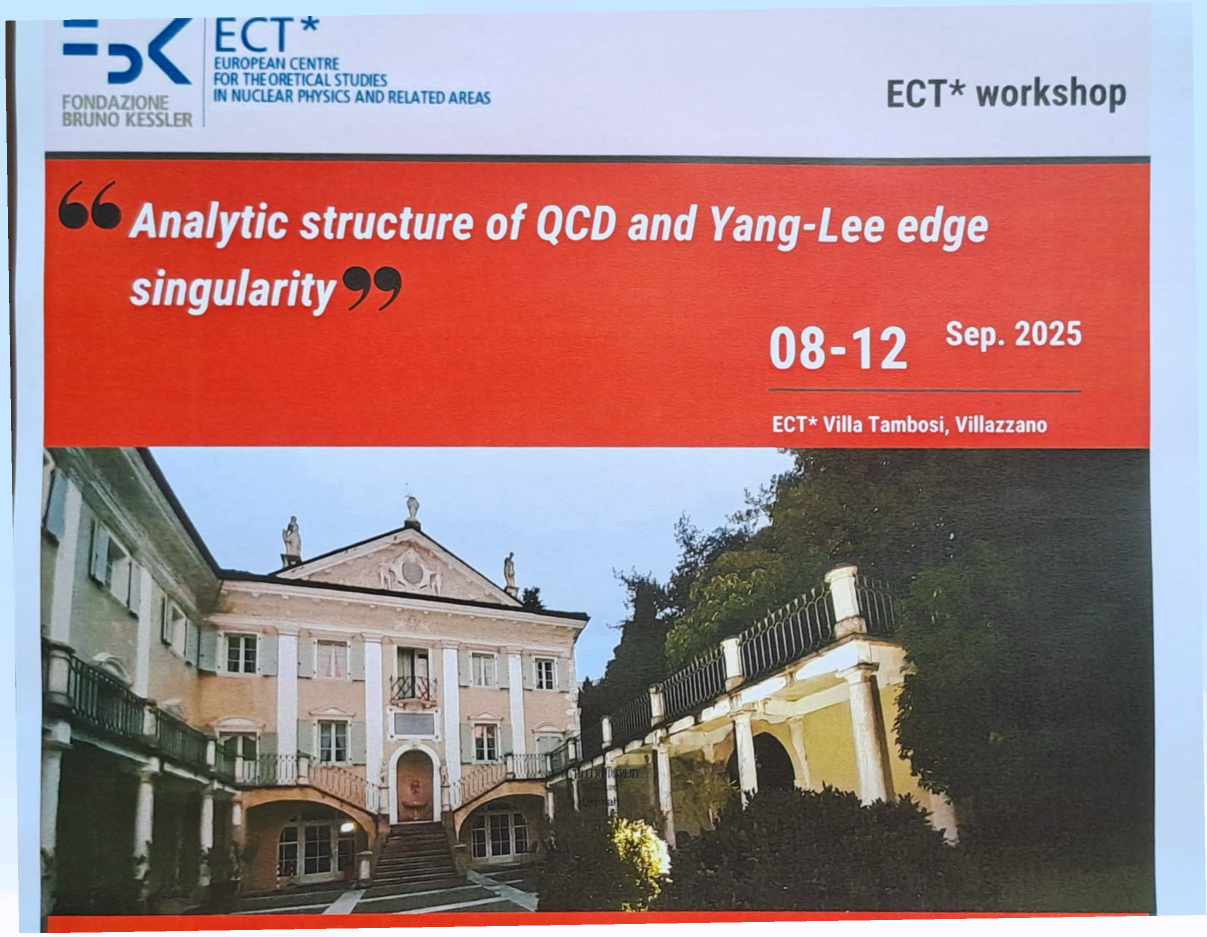
or

Critical-exponent dependent scaling — **Second order**

→
- Estimate of T_I^c in thermodynamic limit

→
- Universality class determination, therefore

From a LY zero perspective



THANK YOU FOR YOUR ATTENTION

Backup slides

The nomenclature of isospin

$$B = \frac{1}{3}\left(N_u + N_d + N_s\right)\ ,\ \ S = -N_s\,,\ \ I = \frac{1}{2}\left(N_u - N_d\right)$$

$$\mu_B = \frac{3}{2}(\mu_u + \mu_d) \qquad\qquad \mu_I = \left(\mu_u - \mu_d\right)$$

$B_u = B_d = B_s = 1/3$ (Baryons are $3-$quark systems)

$|I_p| = |I_n| = 1/2$, since $2I + 1 = 2$ (proton and neutron)

Thus, $I_u = -I_d$ and $|I_u| = |I_d| = 1/2$