Mapping a critical point on pion condensate line in isospin phase diagram from Lee-Yang zeros

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FONDAZIONE BRUNO KESSLER ECT* EUROPEAN CENTRE FOR THEORETICAL STUDIES IN NUCLEAR PHYSICS AND RELATED AREAS

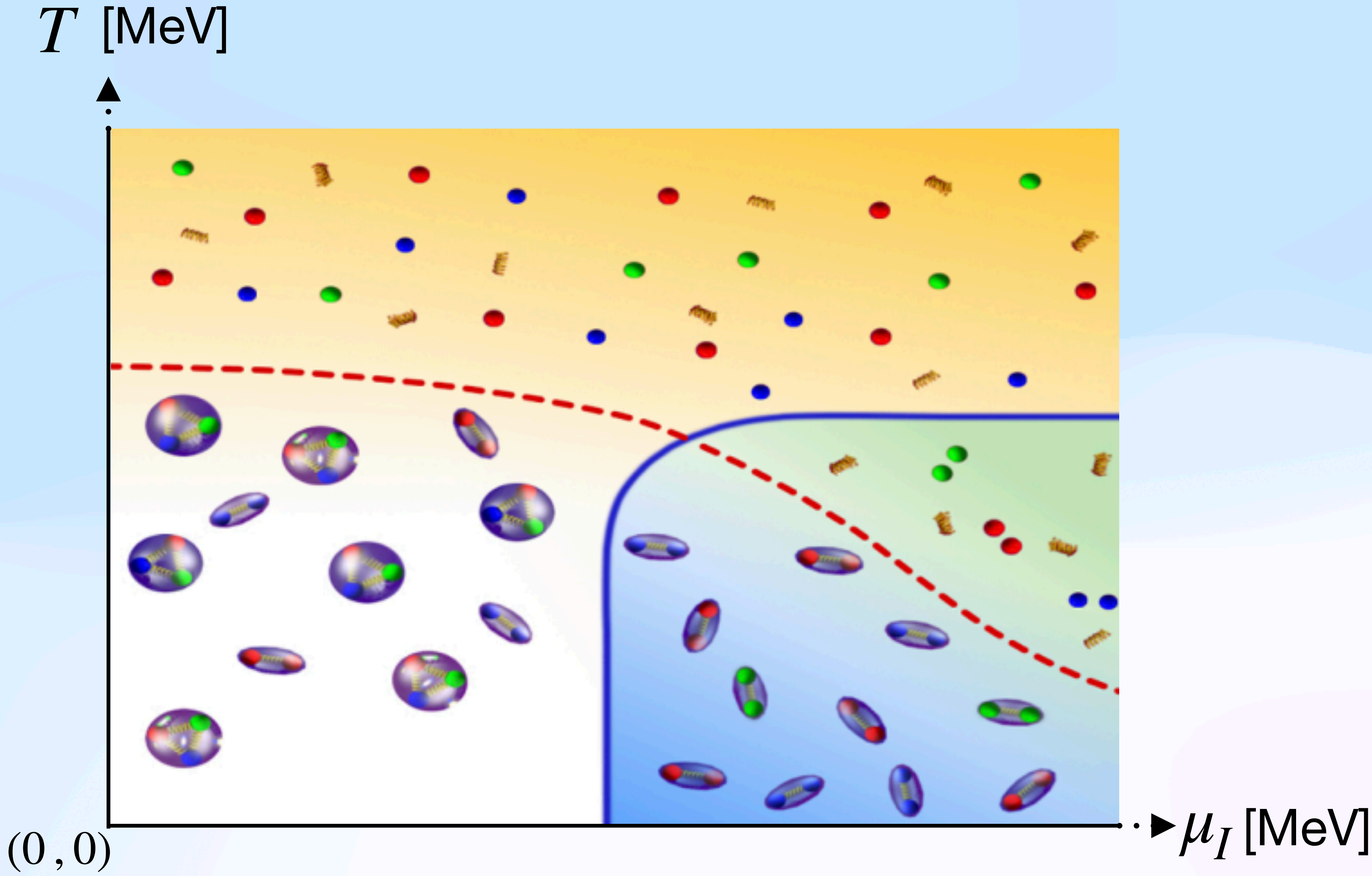
CRC-TR 211 Strong-interaction matter under extreme conditions

UNIVERSITÄT BIELEFELD Faculty of Physics

Plan of talk

- Motivation and domain of this work
- Core methodology and working lattices
- Zeroes, results and their stability
- Mapping a critical point on the critical line
- Radius of convergence and subsequent observations
- Overlap problem and its severity
- Conclusion and Outlook

Motivation



Brandt et. al. : **PRD 97 (2018) 5, 054514**, arXiv : **1712.08190 [hep-lat]**

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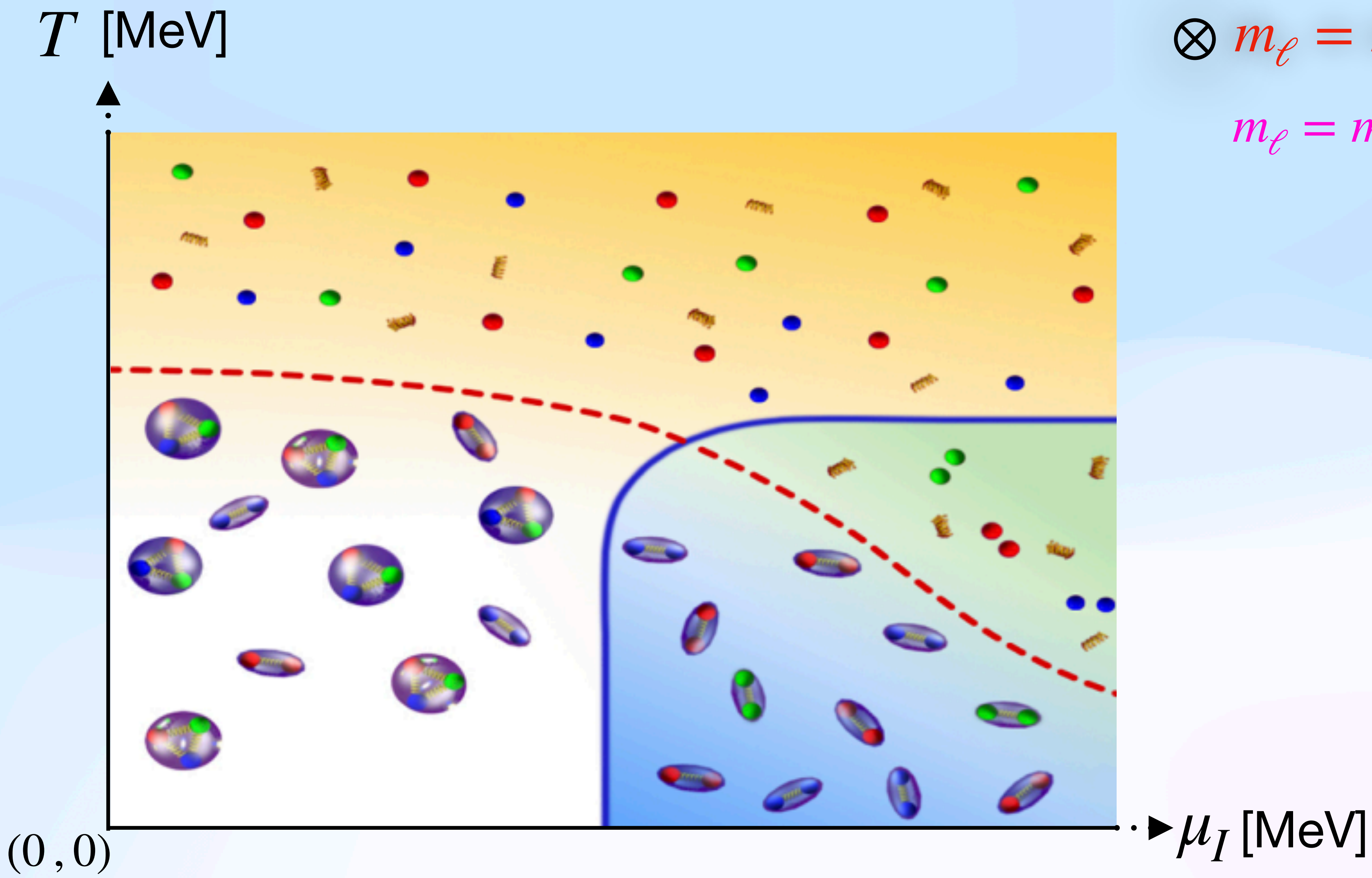

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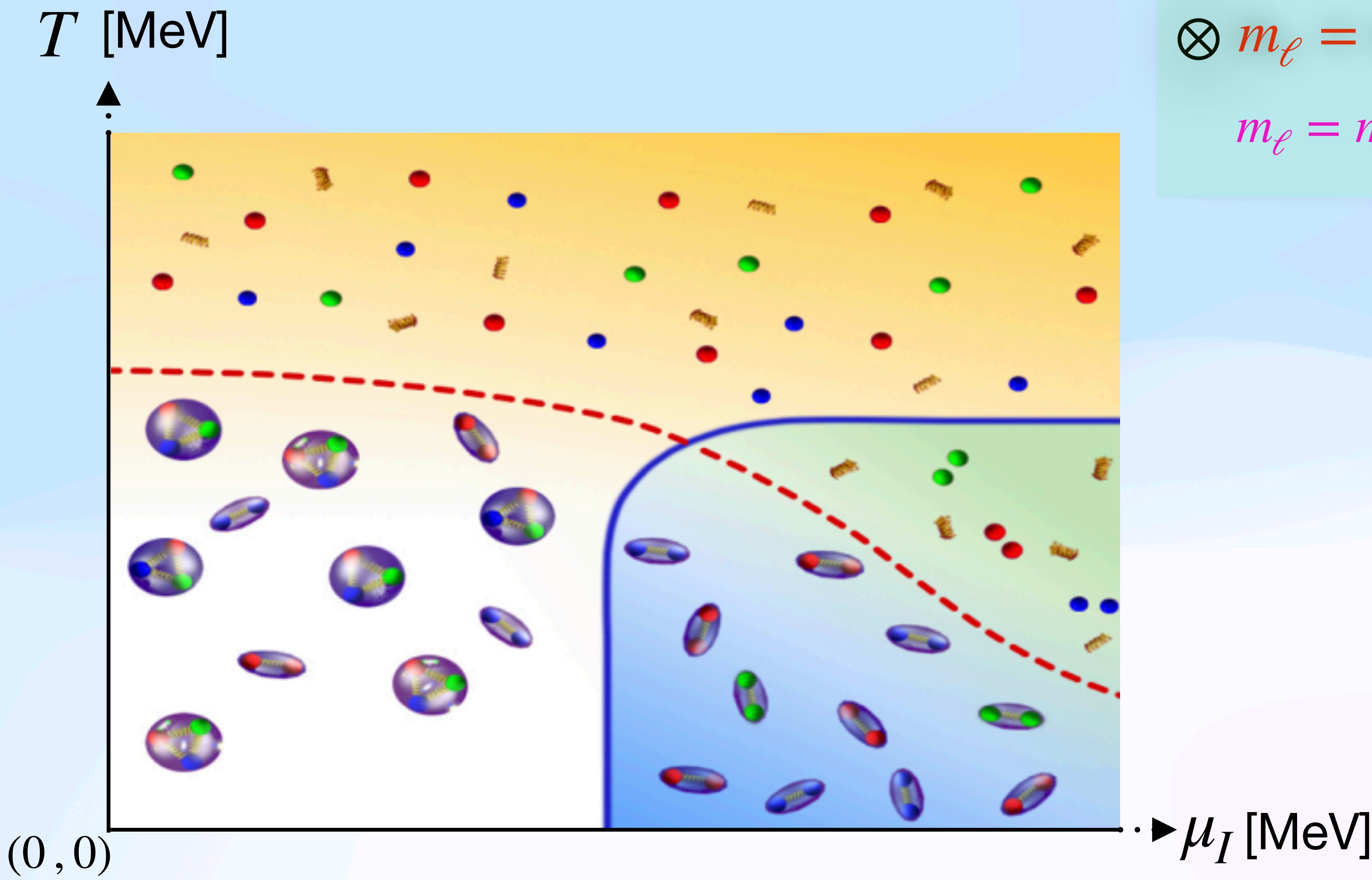

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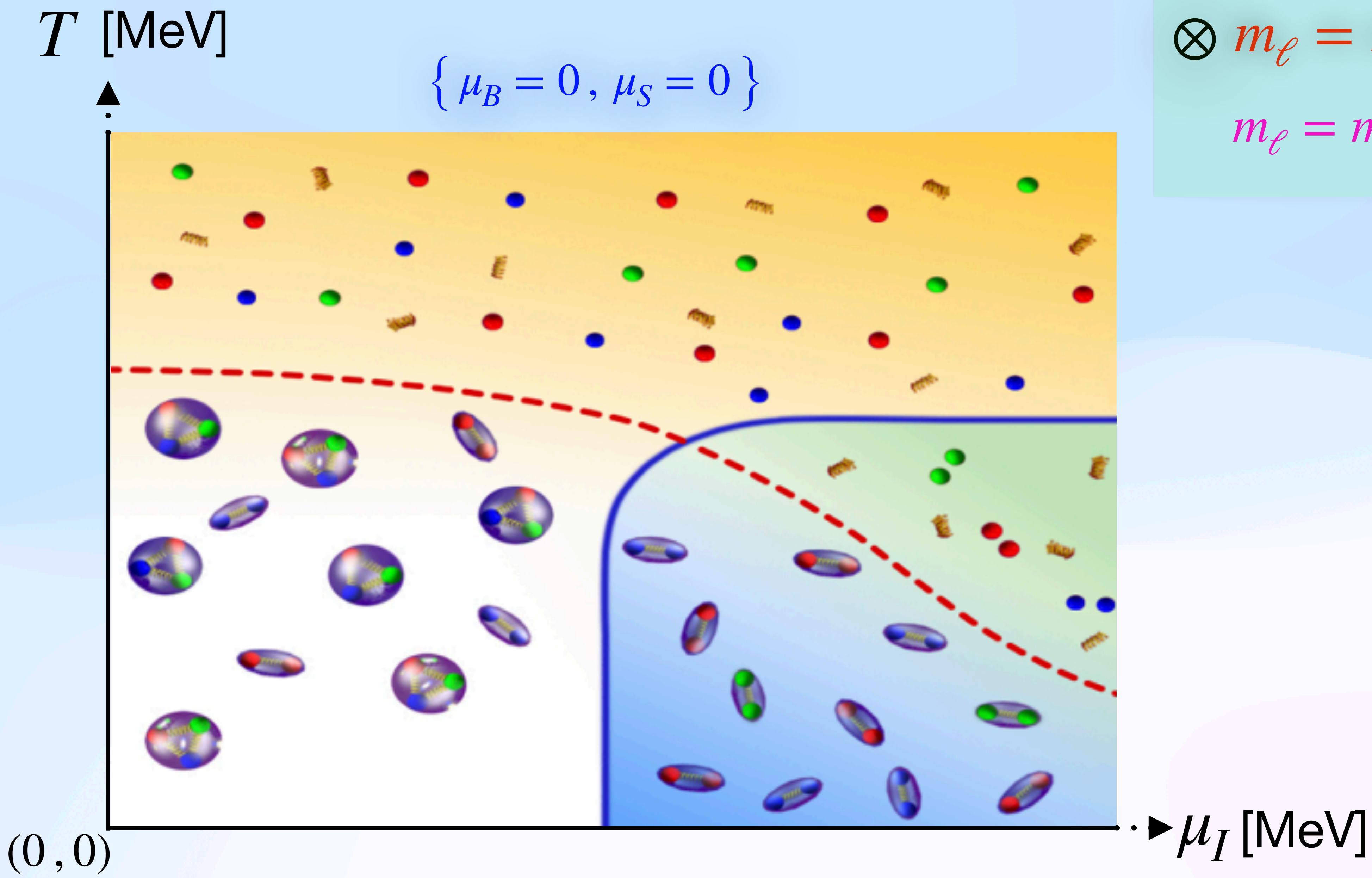

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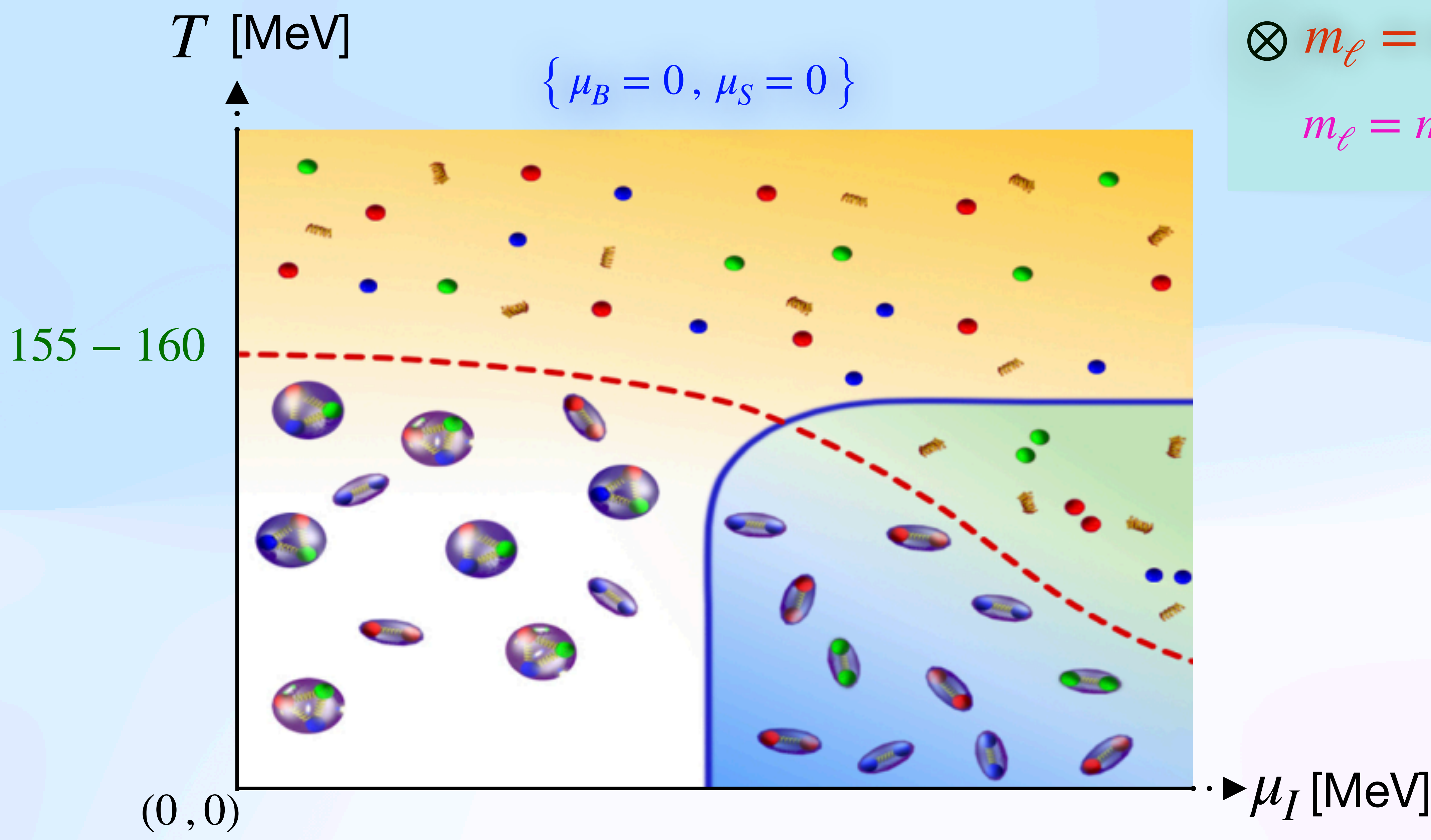

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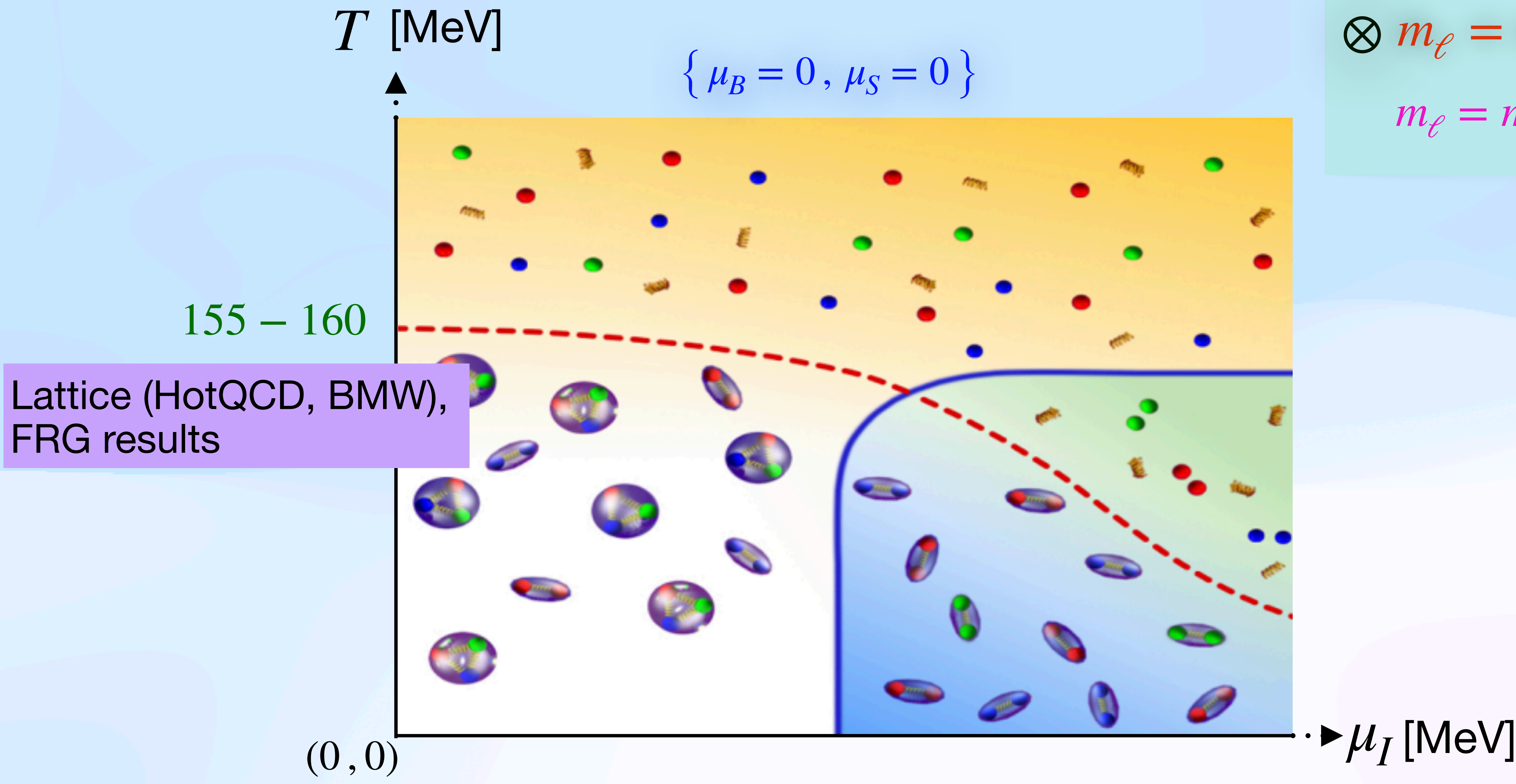

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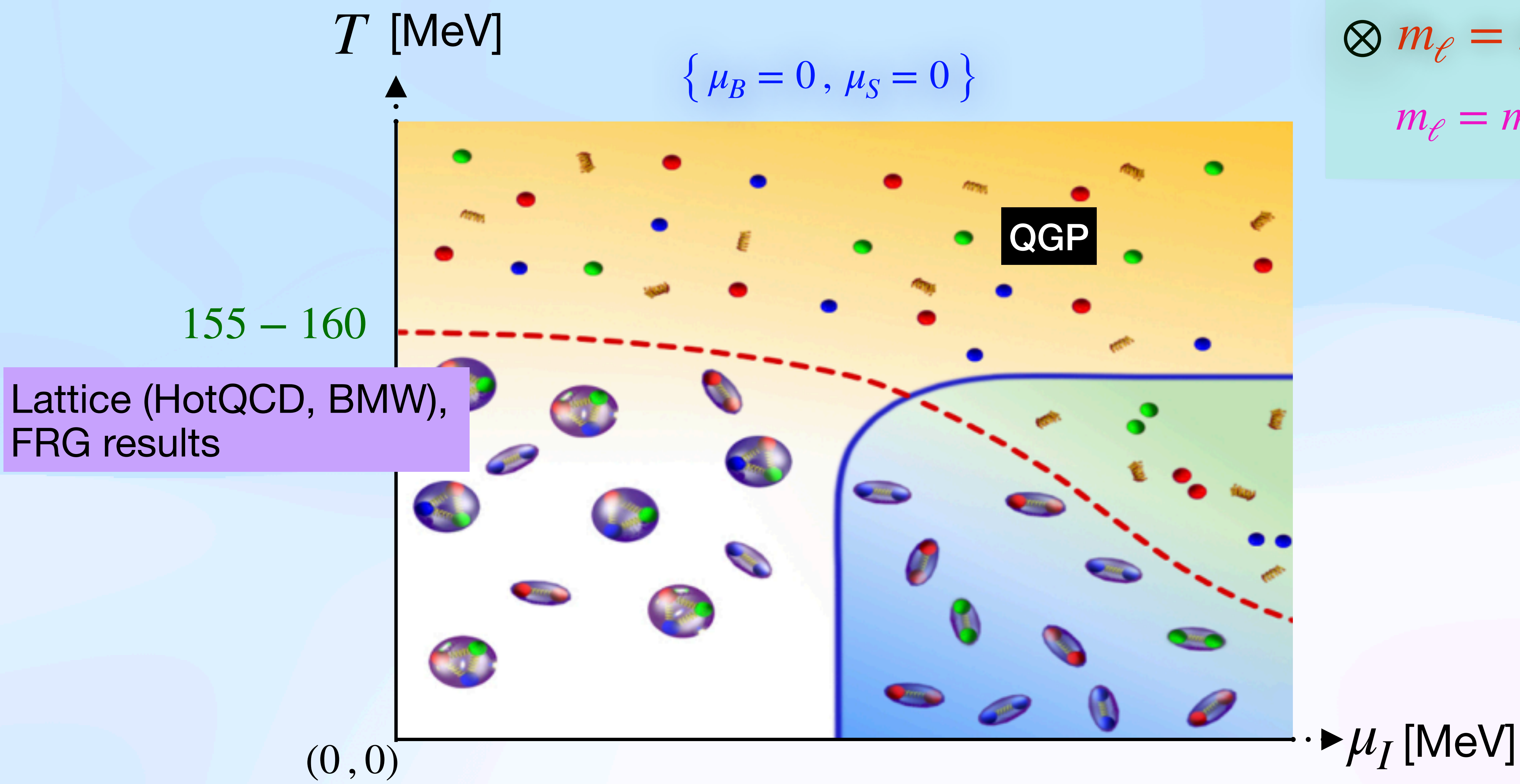

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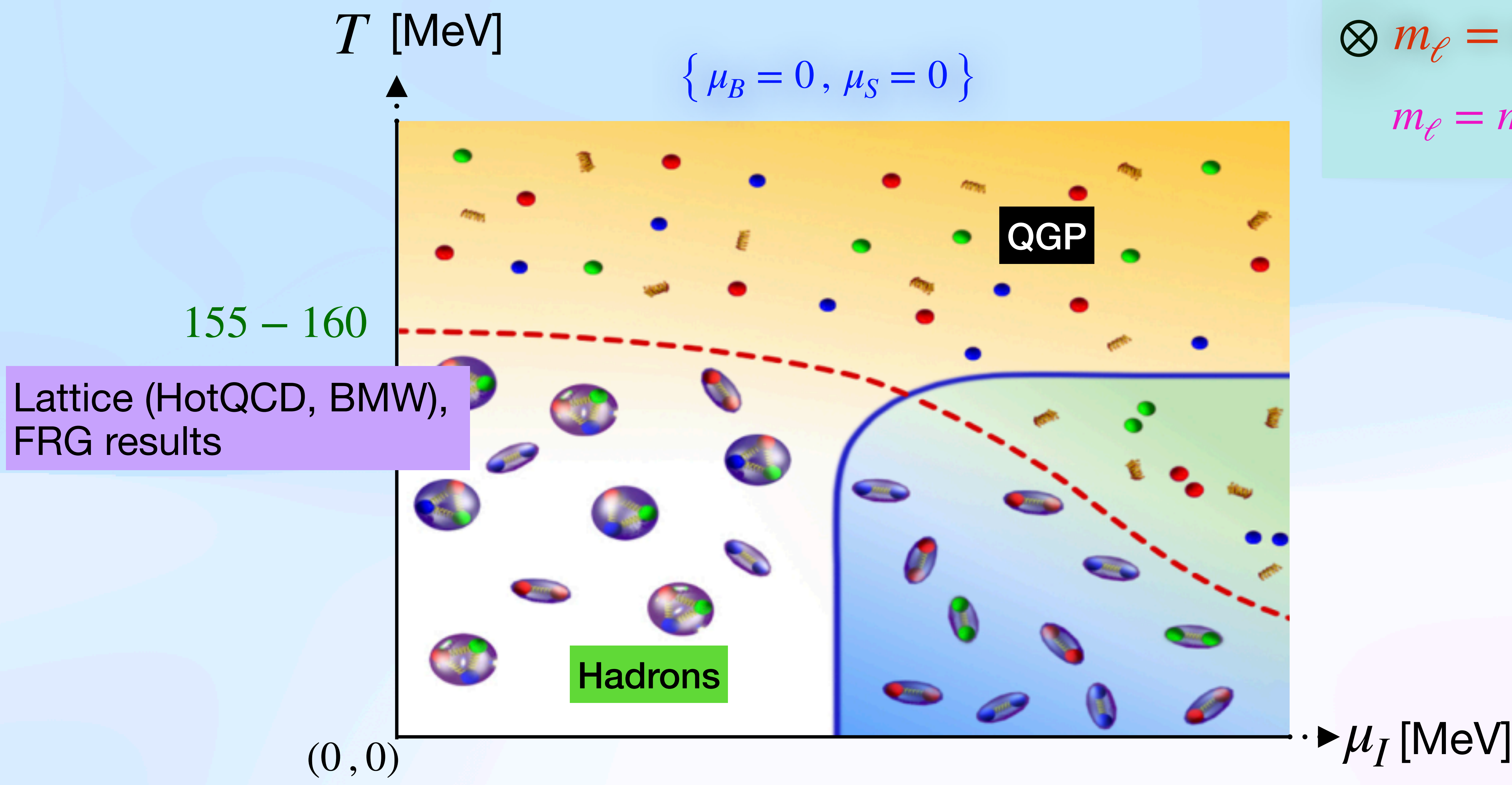

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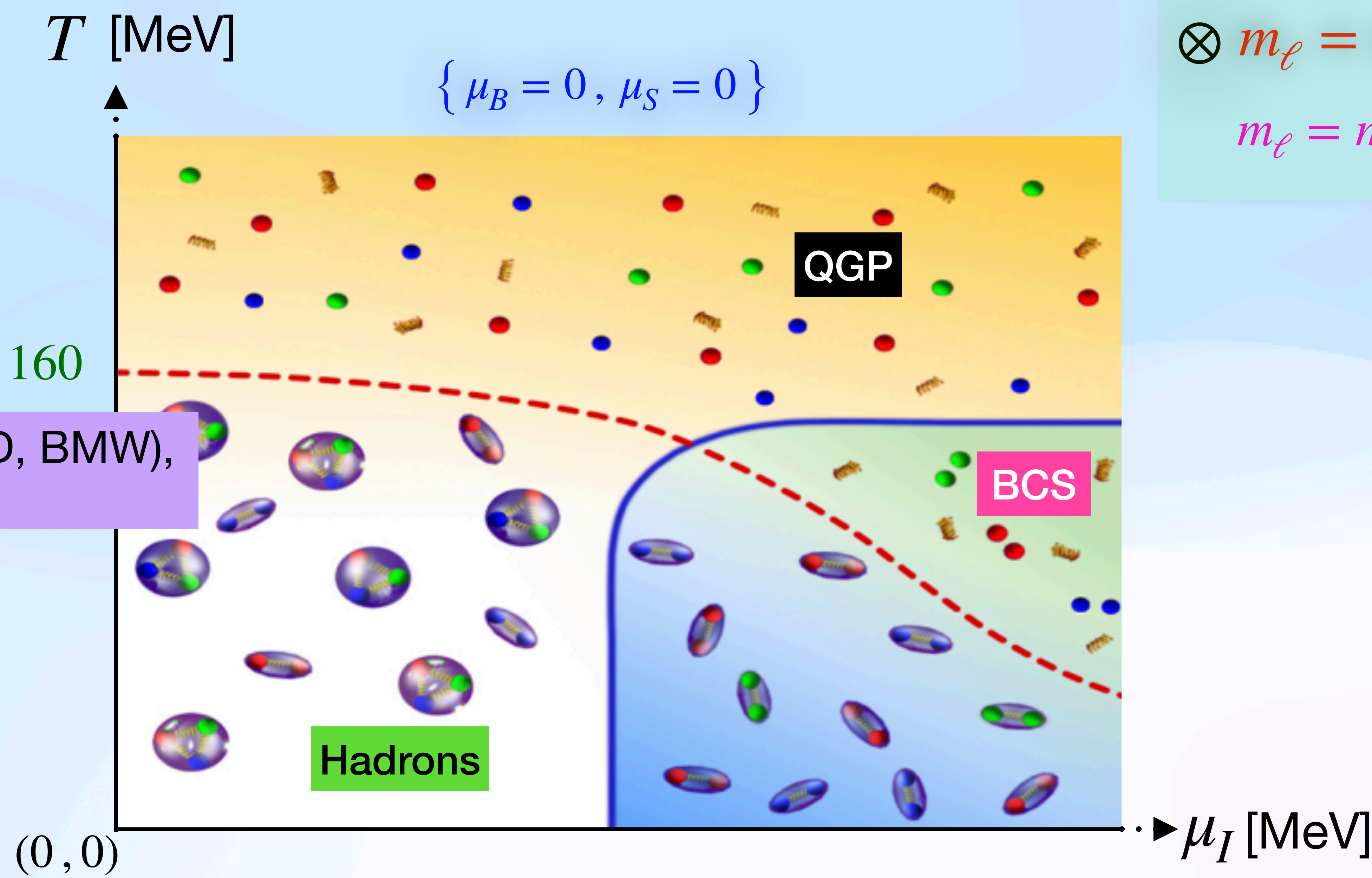

Lattice (HotQCD, BMW), FRG results

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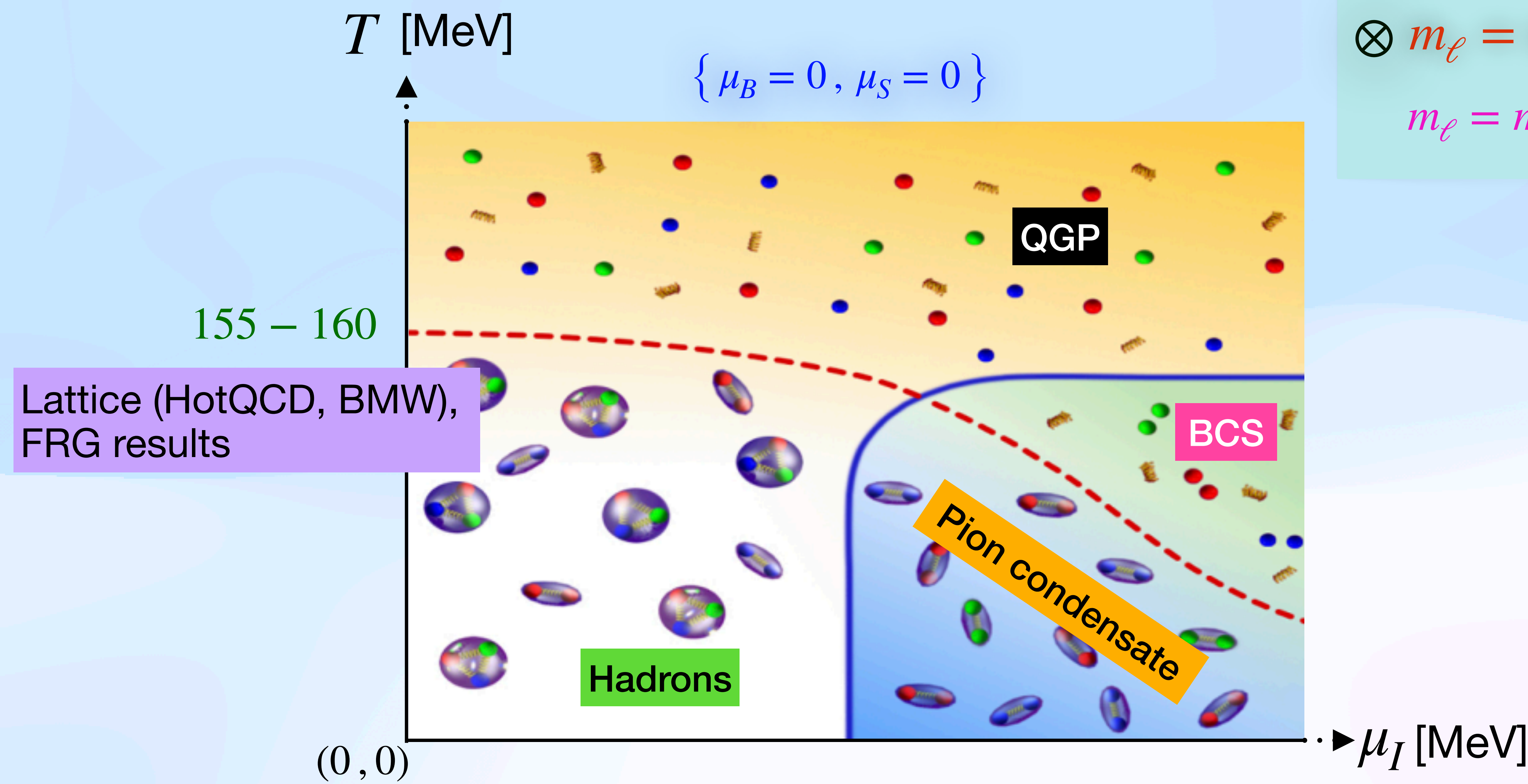

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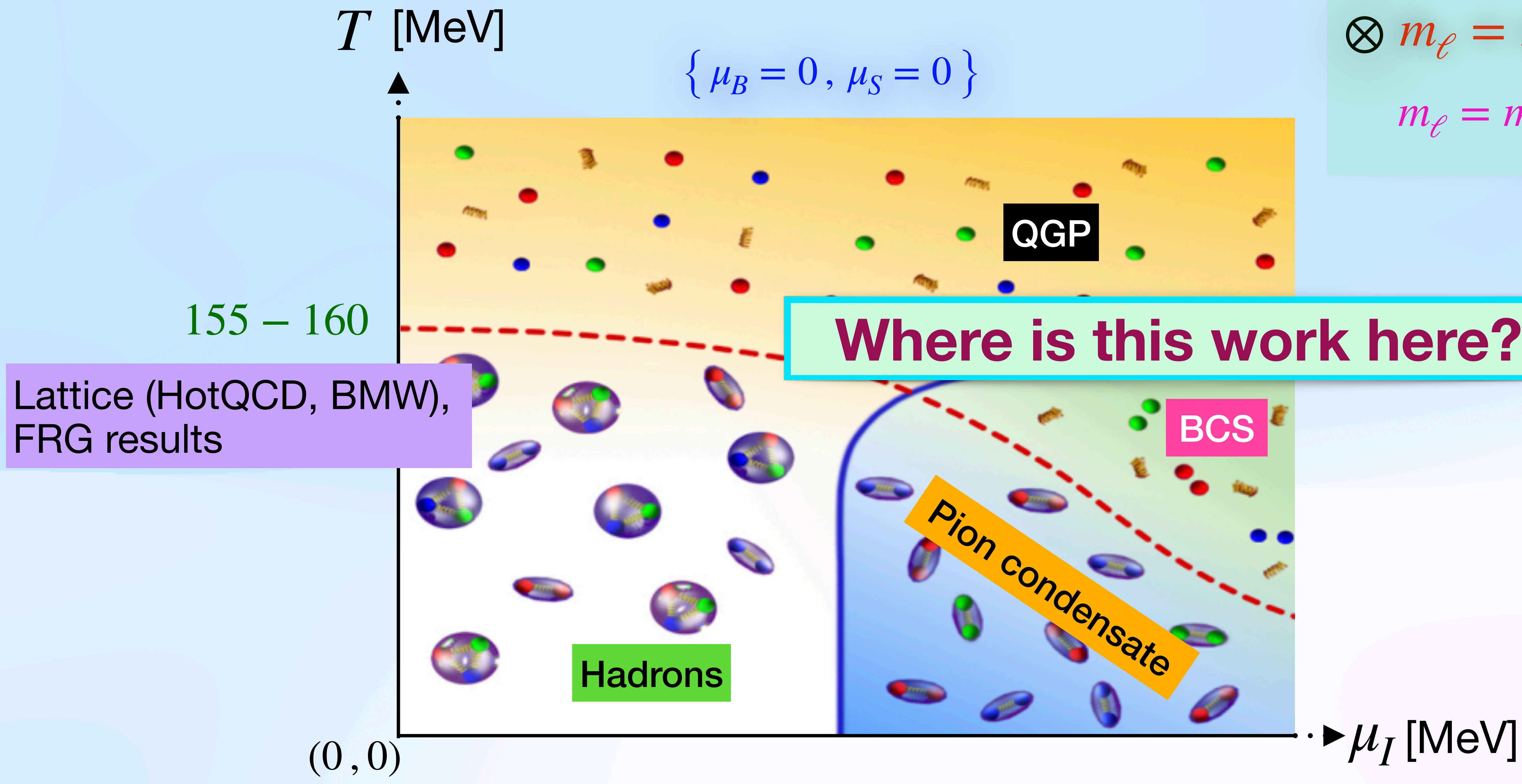

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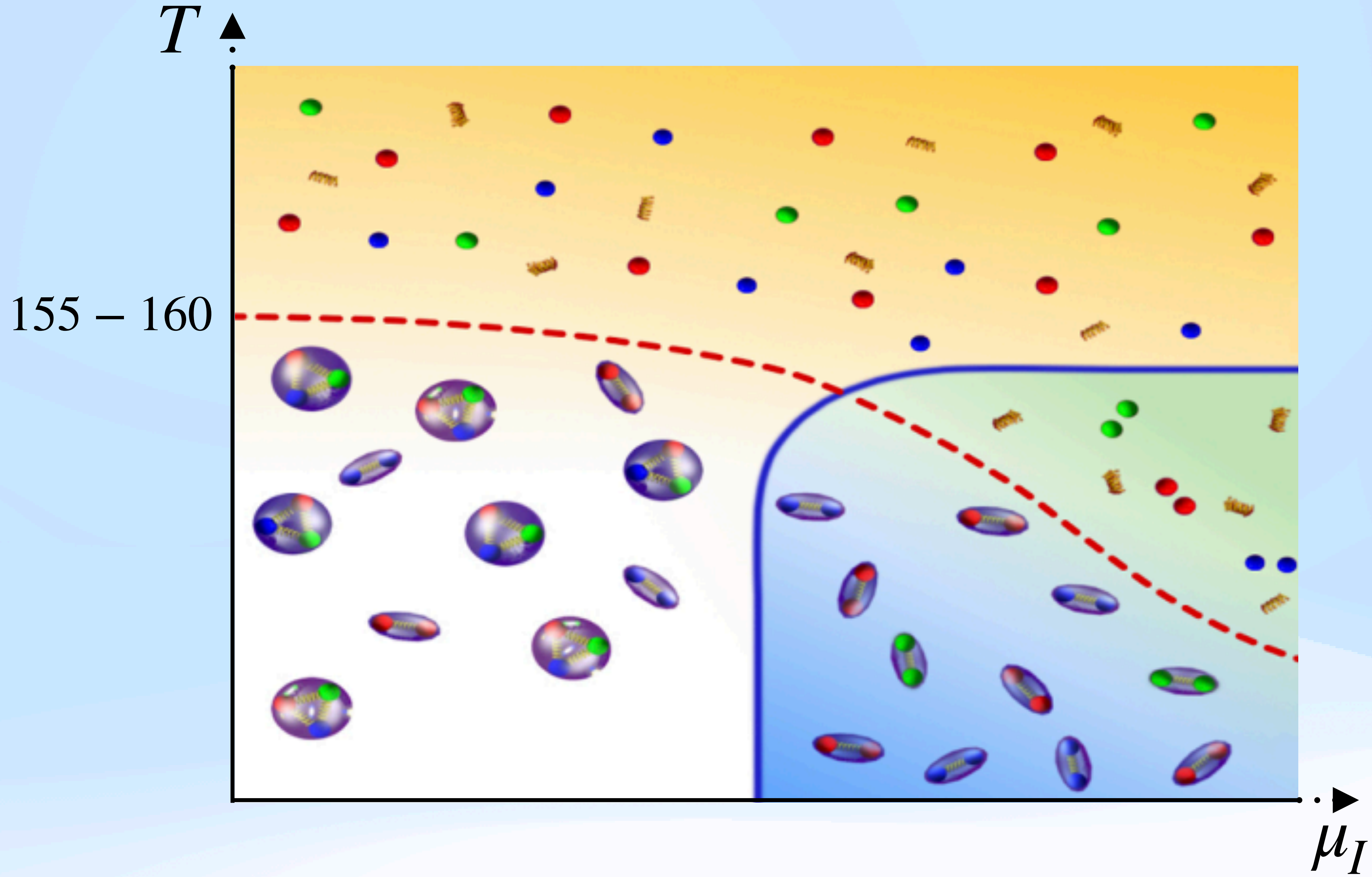

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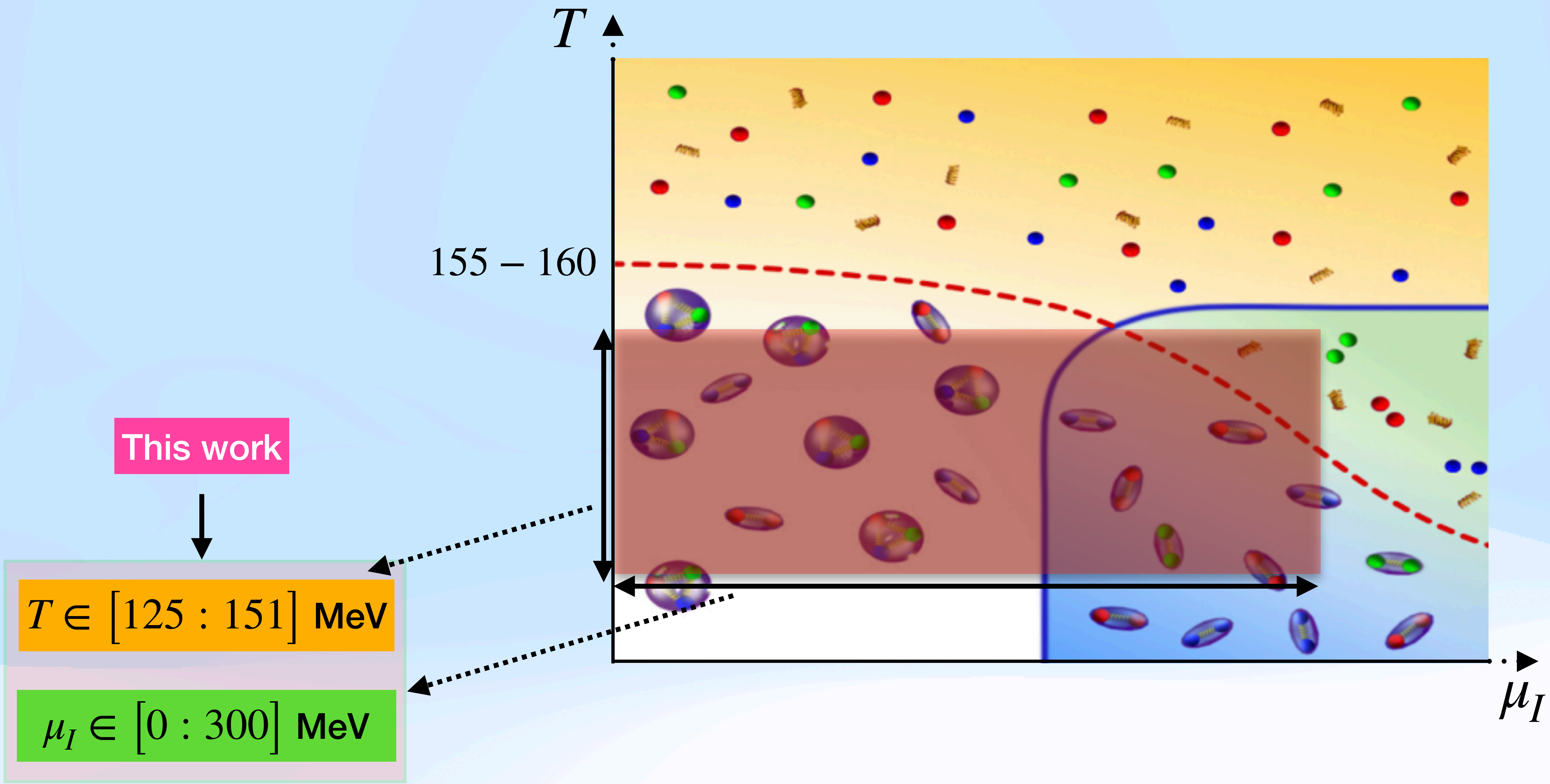
155 – 160
μ_I



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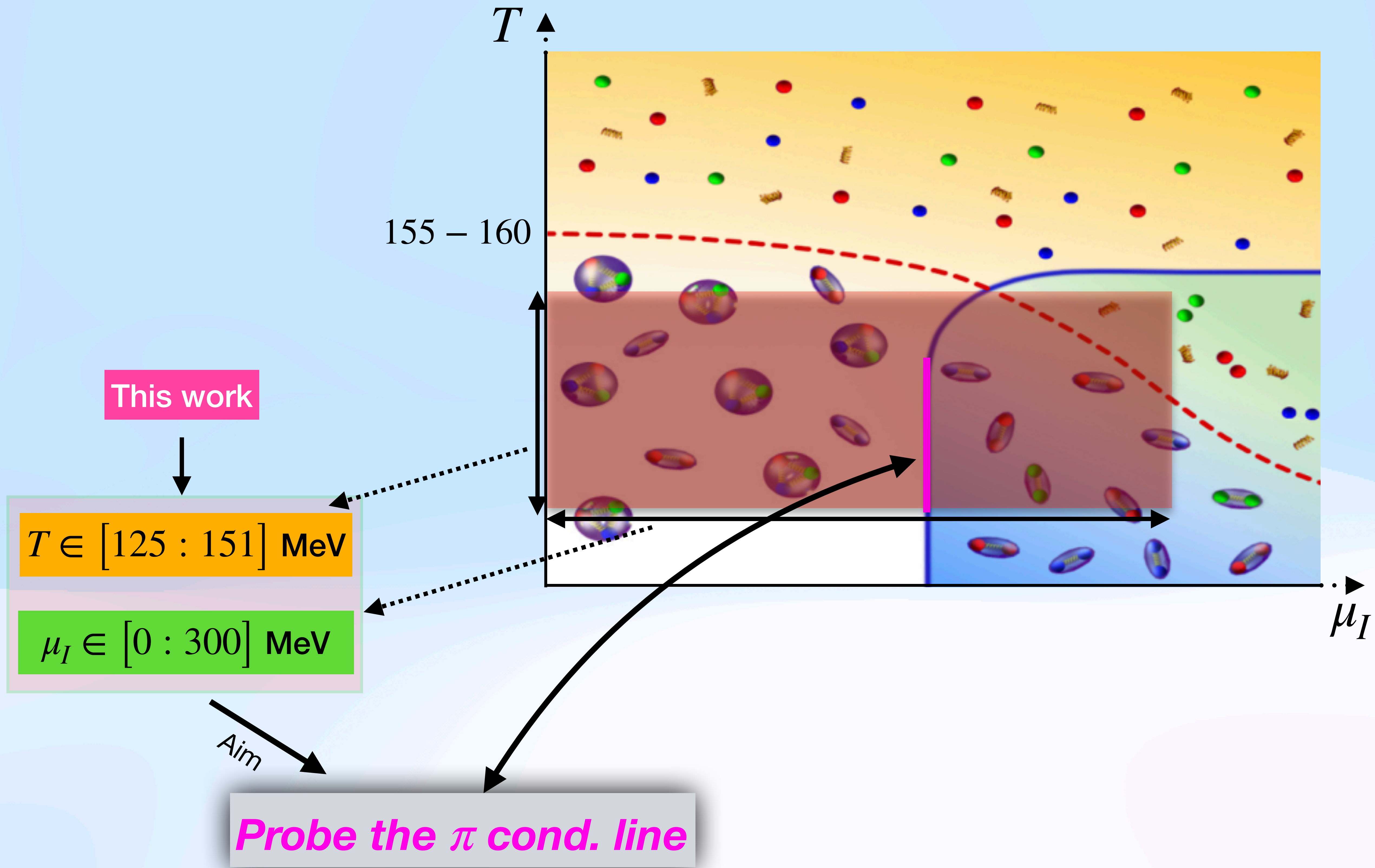

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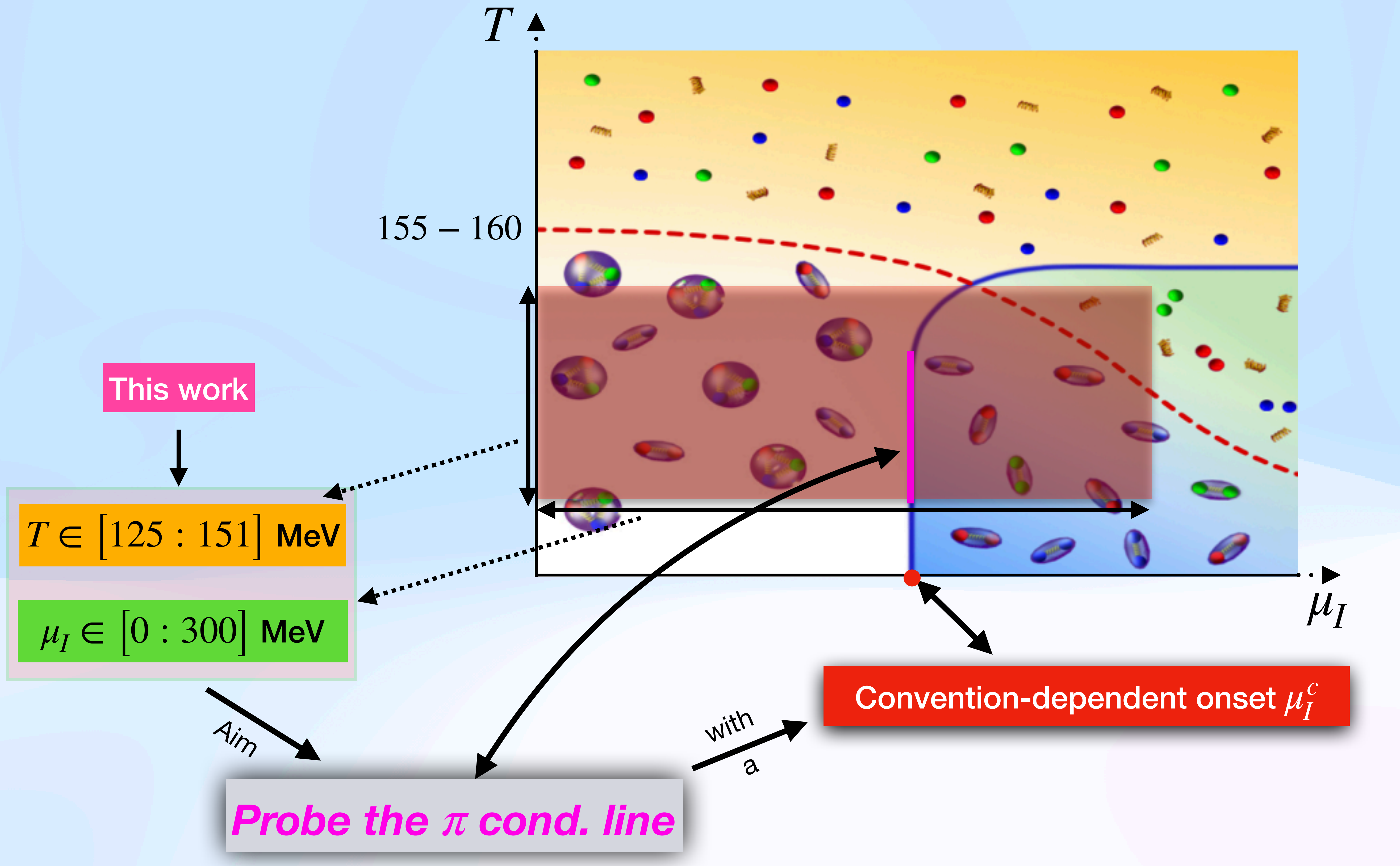
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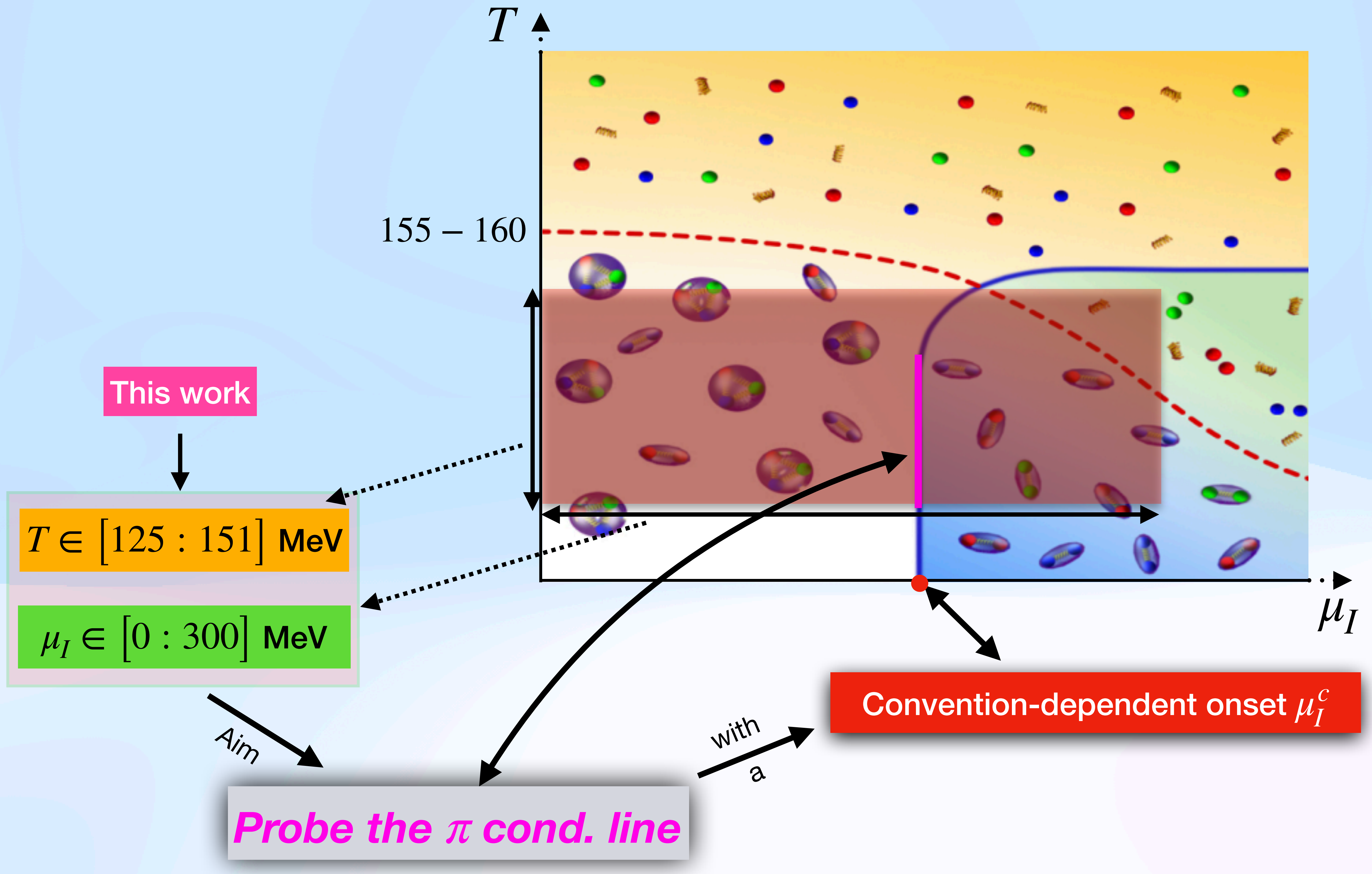

So, what happens here??

The Theory : Symmetry

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$SU(2)_L \times SU(2)_R$ → (Chiral symmetry broken) → $SU(2)_V$

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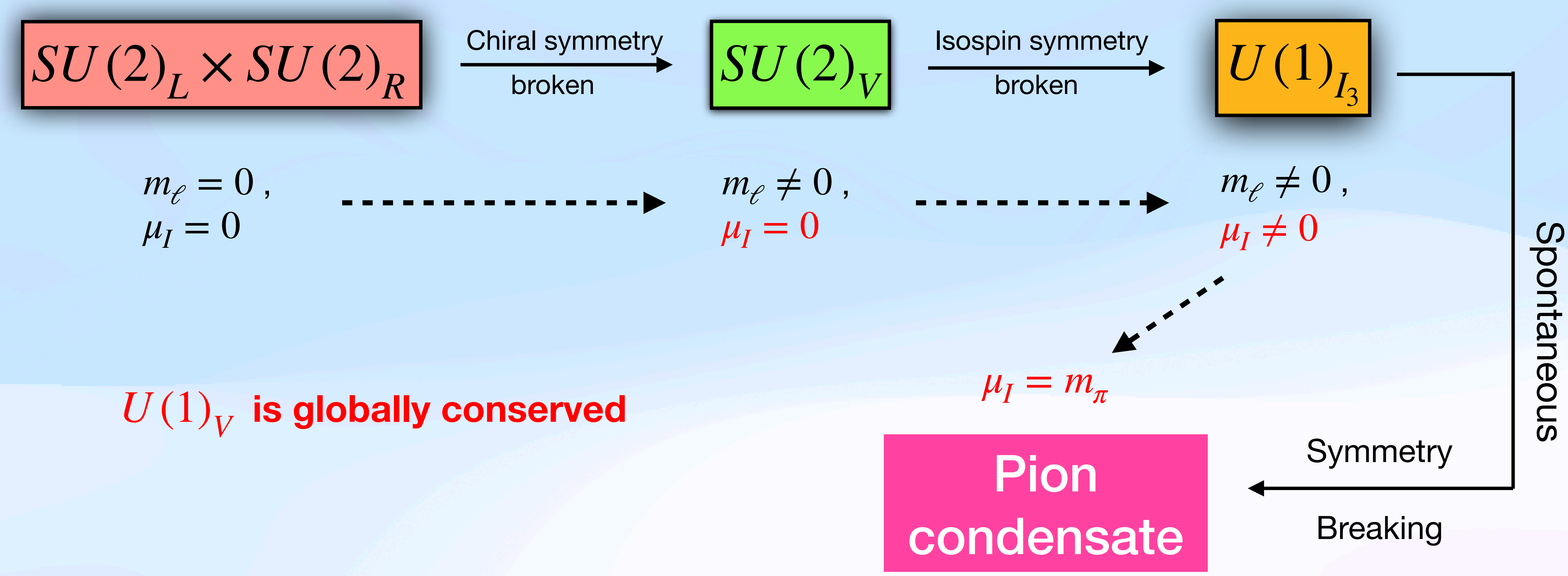

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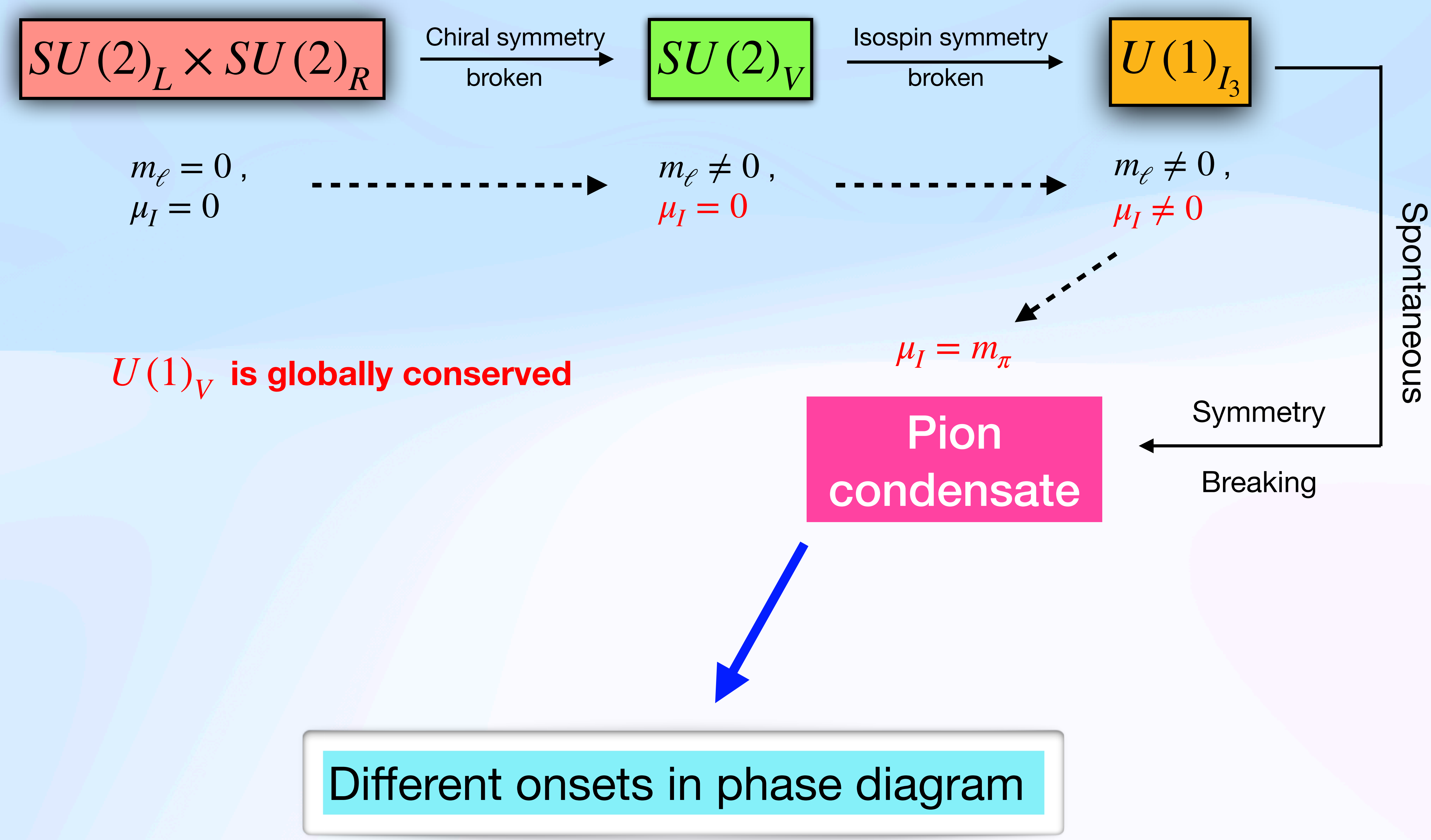

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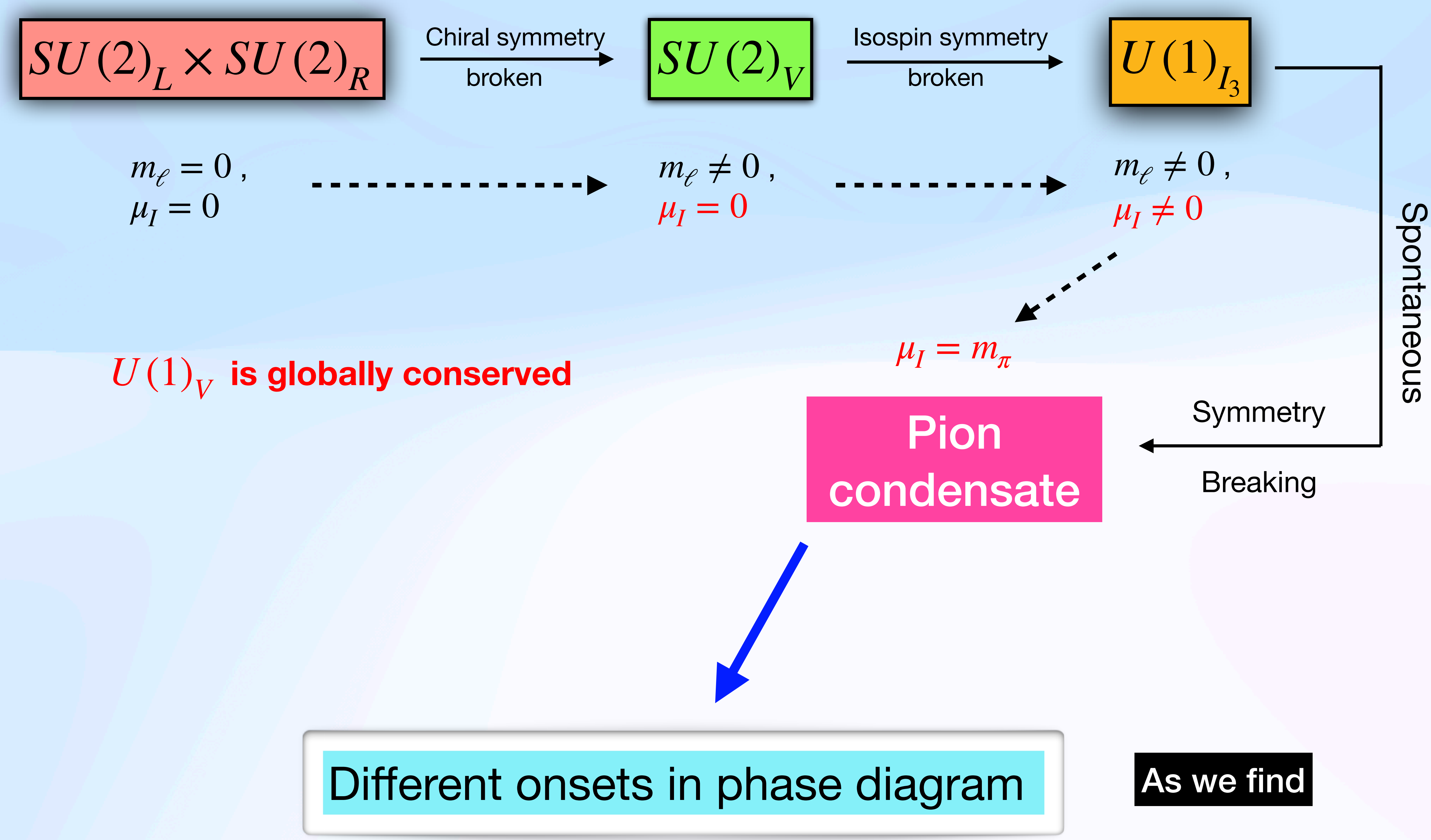

The prevailing notions so far

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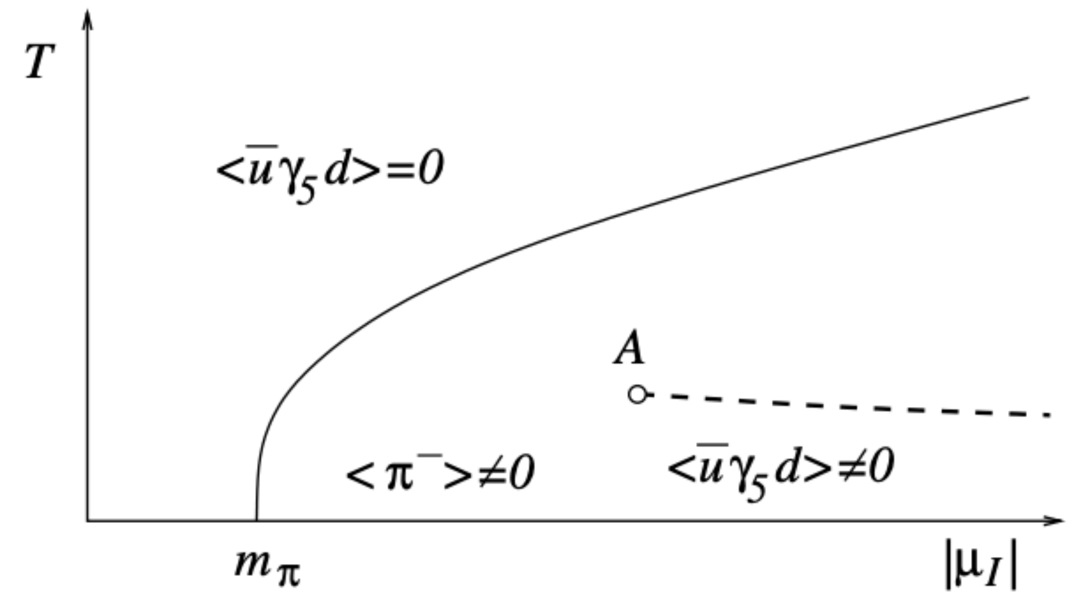

FIG. 1. Phase diagram of QCD at finite isospin density.

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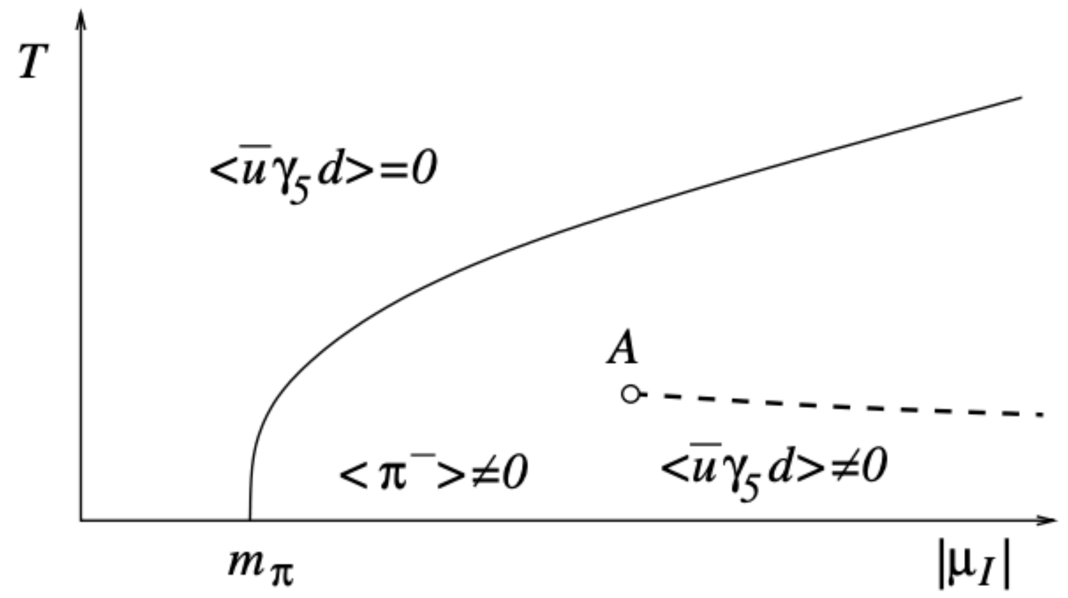

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Son, Stephanov
PRL

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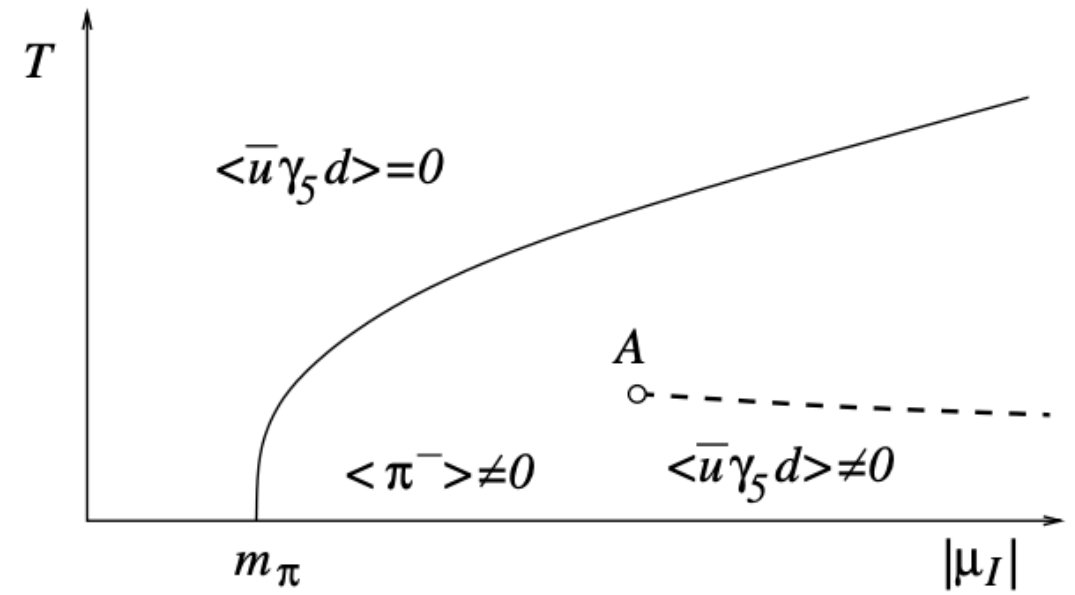

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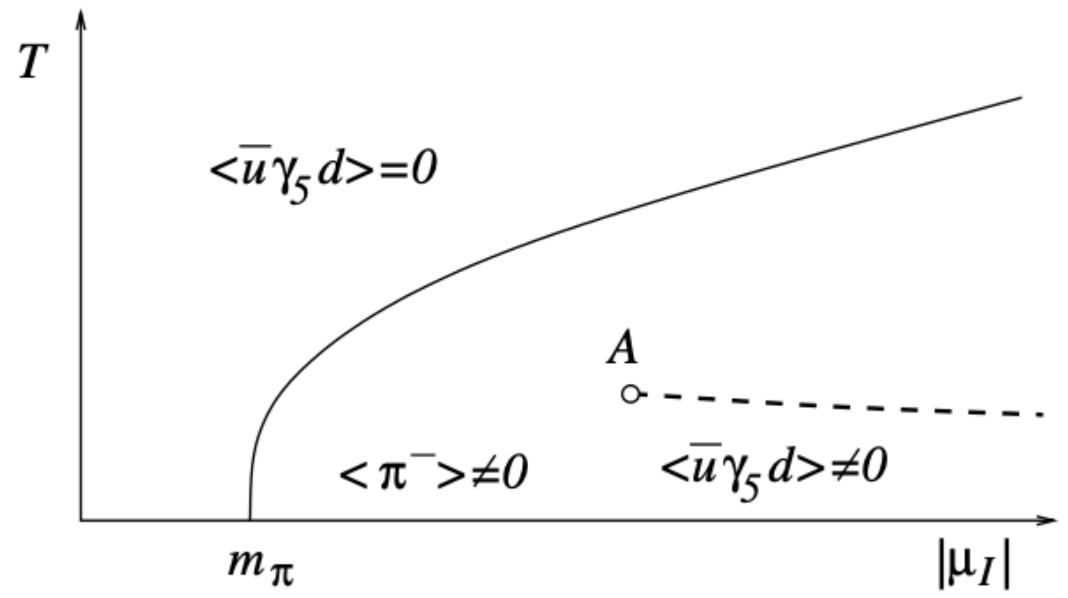

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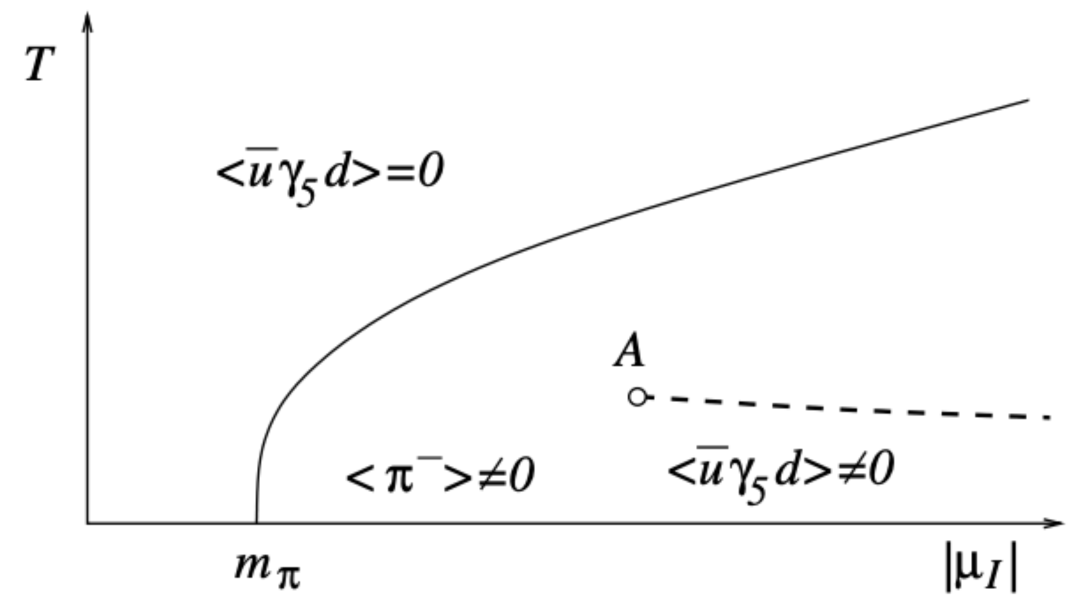

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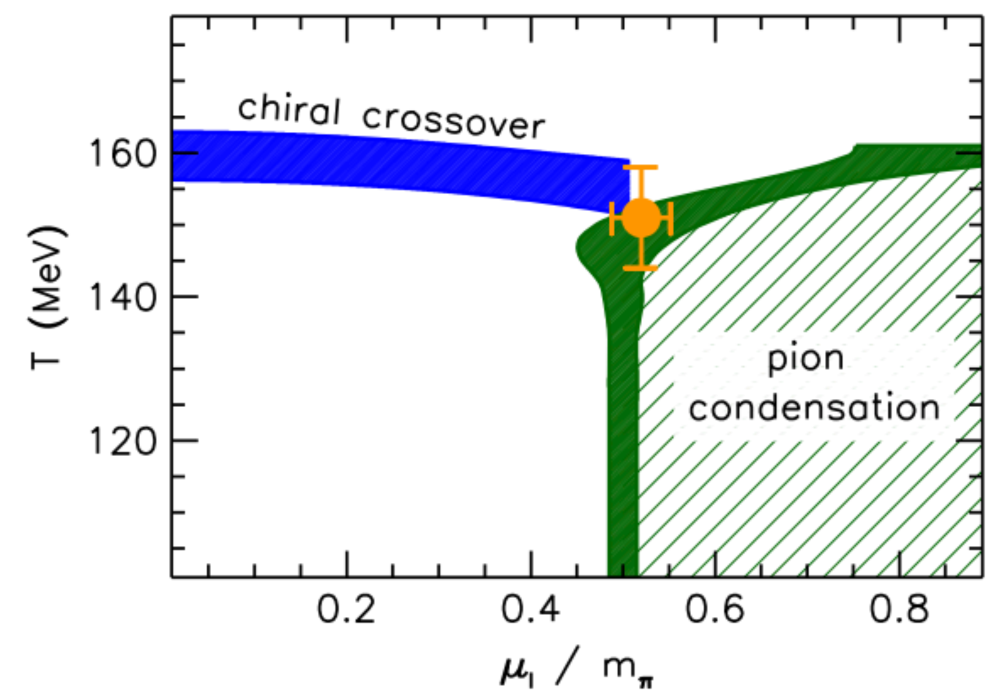

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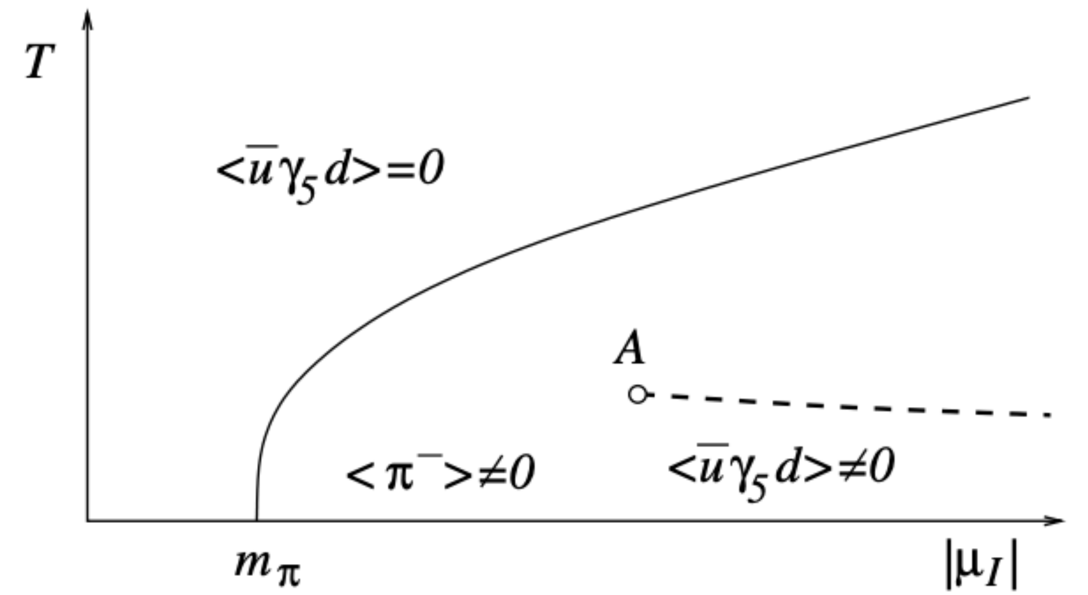

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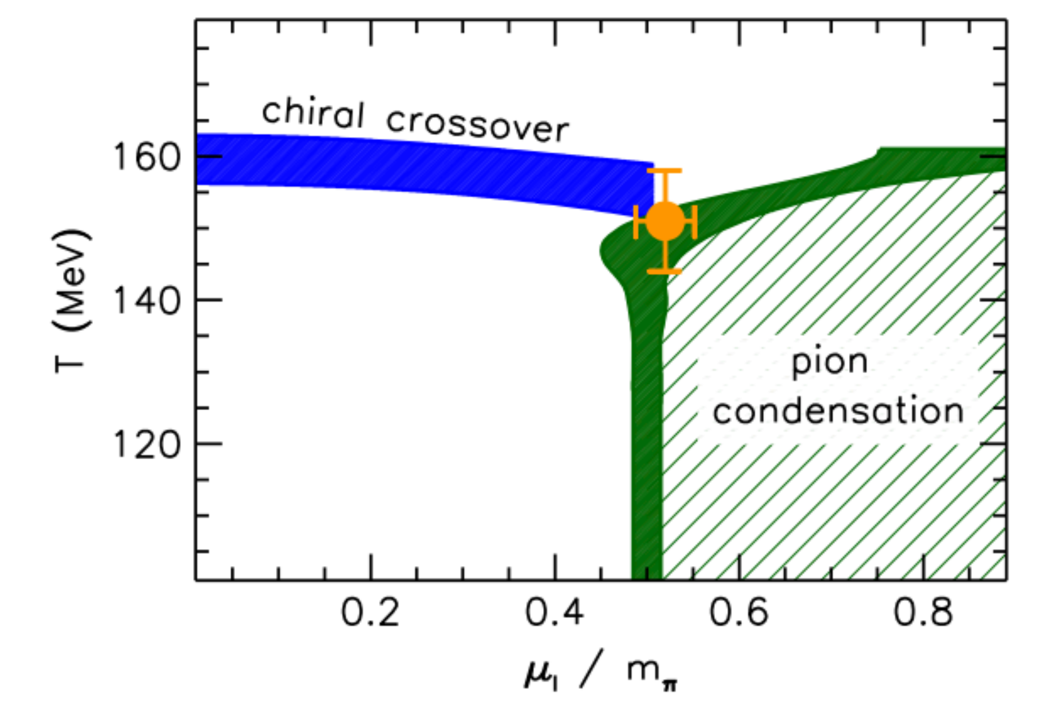

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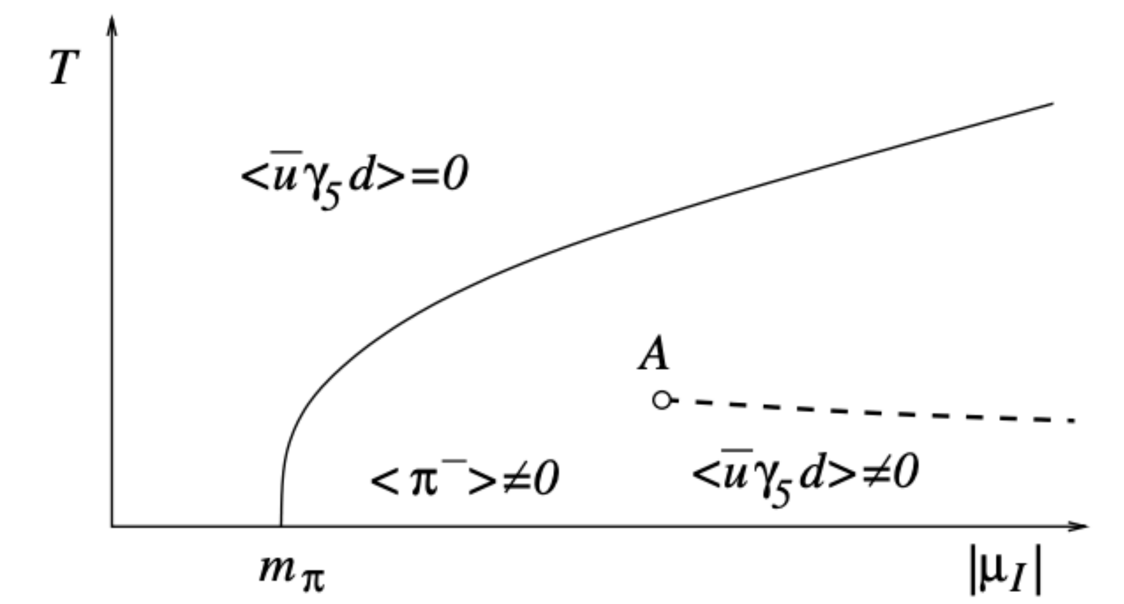

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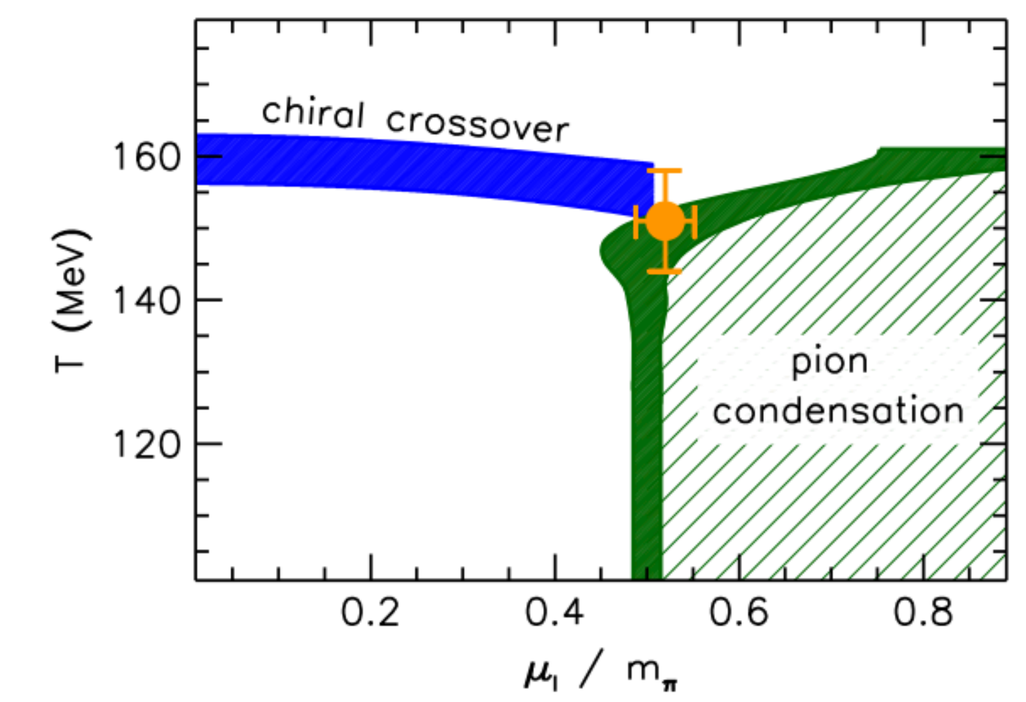

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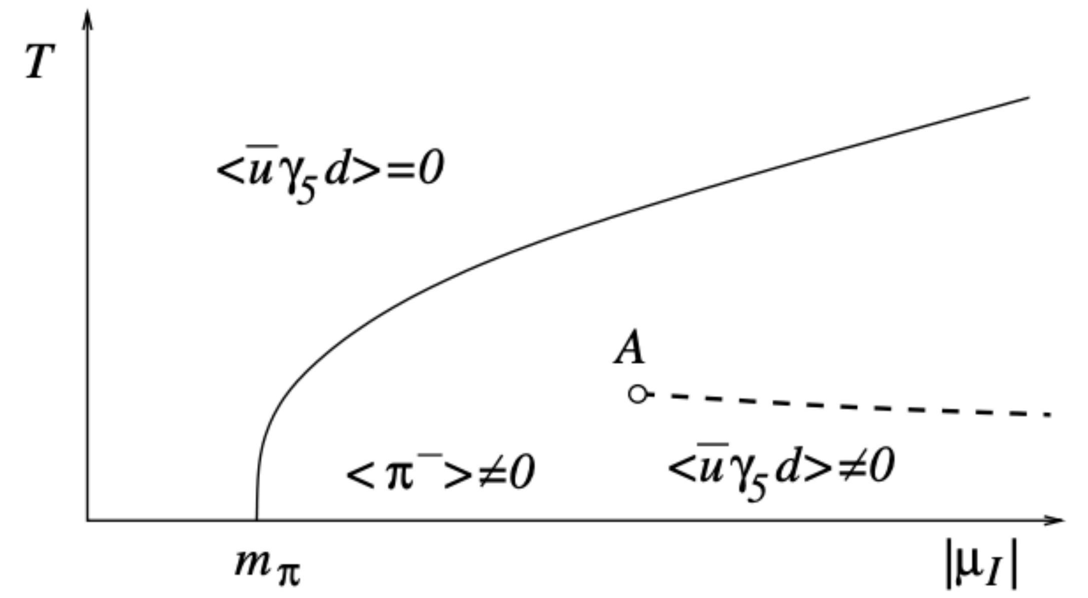

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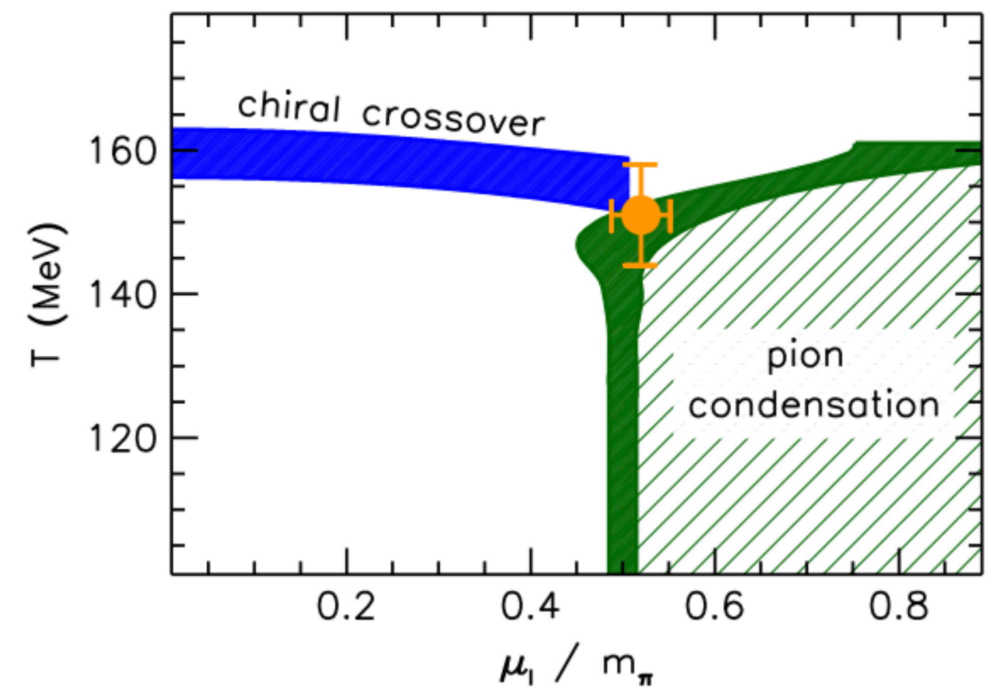

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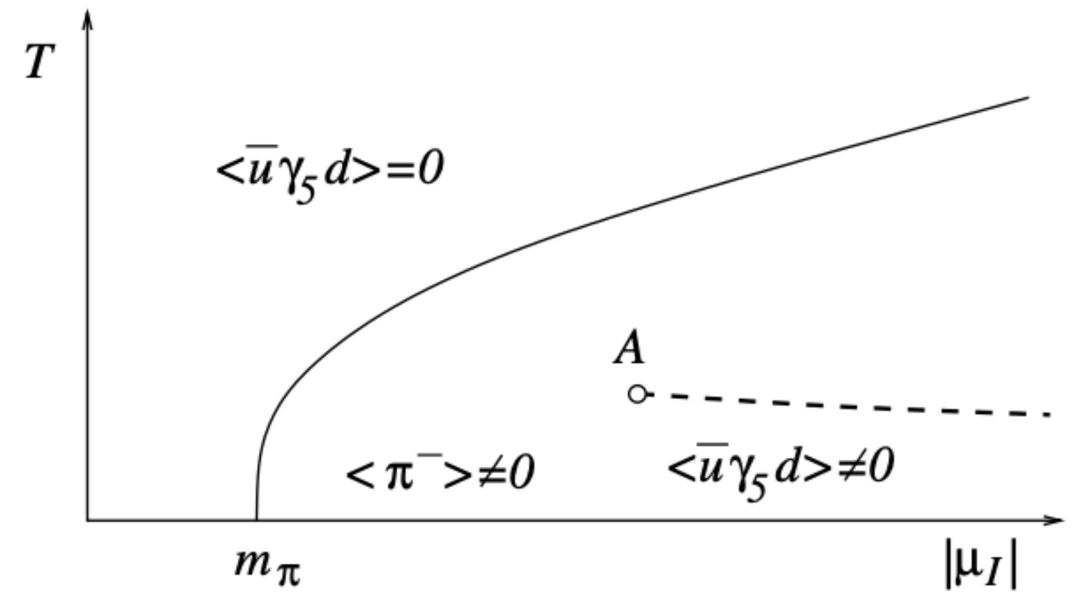

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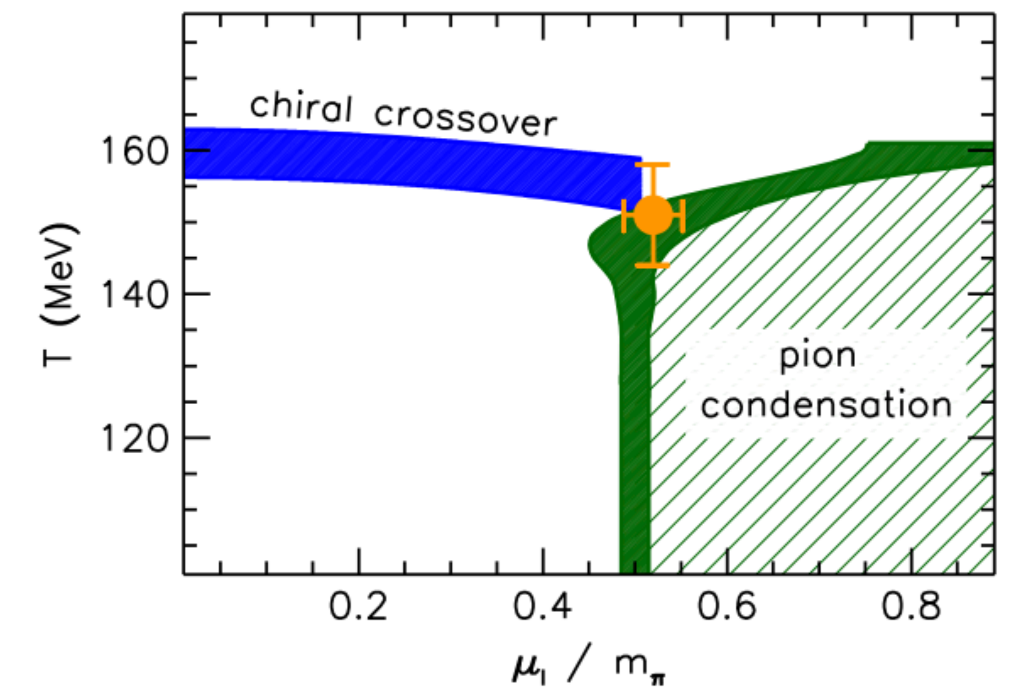

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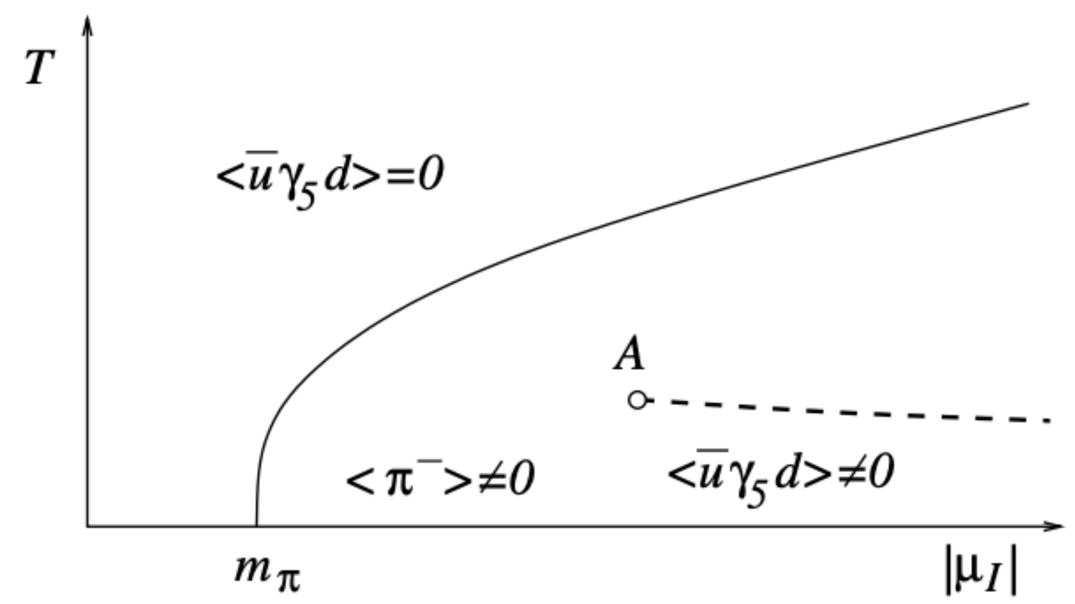

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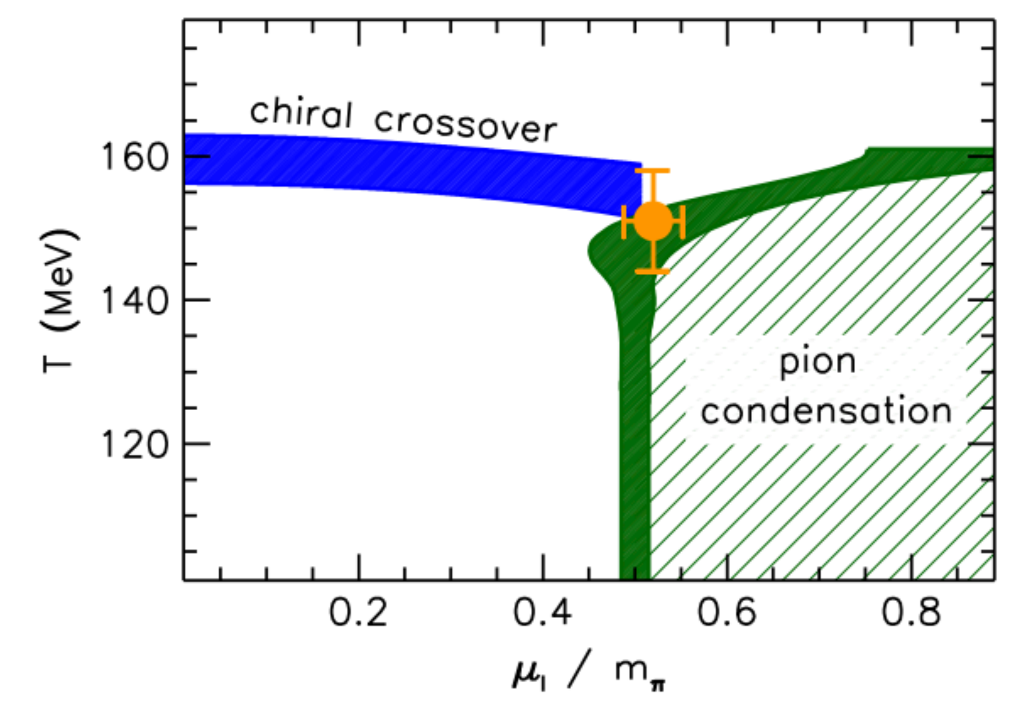

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So, what we do here??

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In complex isospin densities

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Closest zero and

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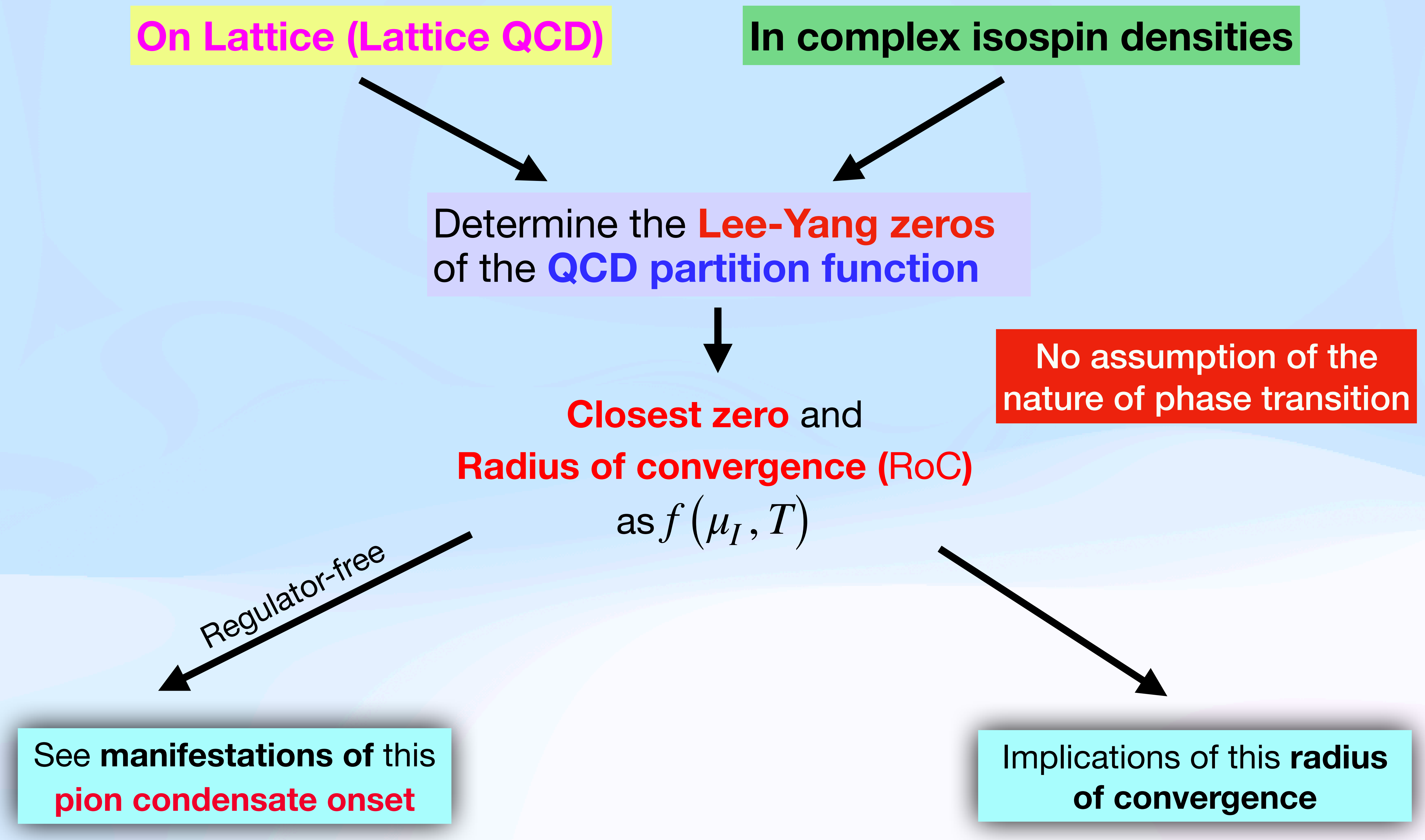
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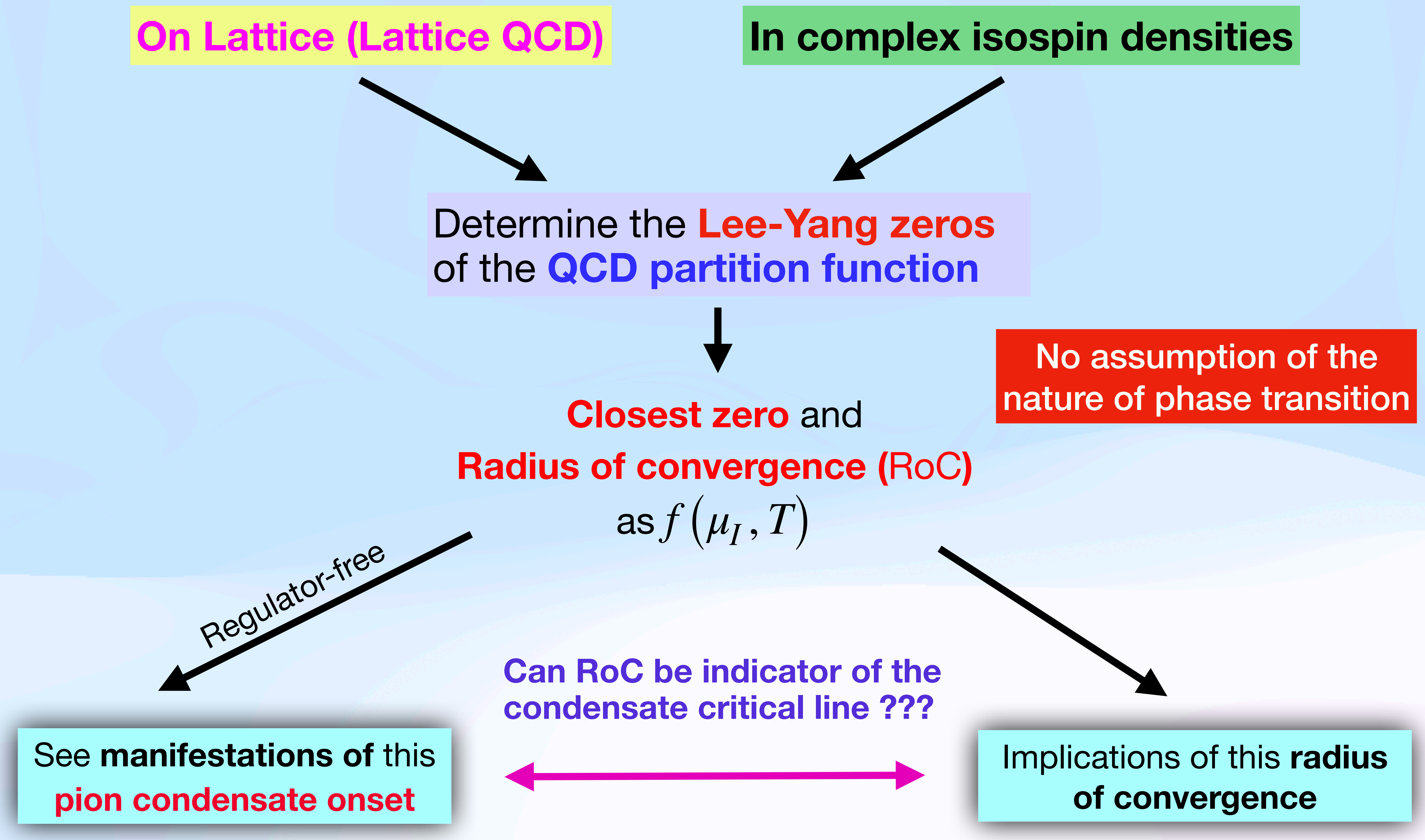
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WORKING FORMULATION??

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So, how is $\mathscr{Z}(\mu_I)$ defined here???

$$\mathcal{Z}\left(\mu_I, V, T\right) = \int DU e^{-S_g[U]} \left[\det M\left(\mu_I, V, T\right)\right]$$

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Unbiased exponential resummation

SM, Hegde ; **PRD 108 (2023) 3, 034502**, arXiv : **2302.06460 [hep-lat]**

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These corrections are important to reproduce the exact Taylor coefficients order-by-order. (Upto fourth order here)

SM, Hegde, Schmidt ; **PRD 106 (2022), 3, 034504**, arXiv : **2205.08517 [hep-lat]**

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And now the working lattice …

Lattice setup

- (2+1)-flavor Highly Improved Staggered Quark (HISQ) action
- Working lattices $\longrightarrow$ $32^3 \times 8$ lattices (higher volumes in future)
- Total of $20K$ **configurations** $\left(N_{conf} = 20K \right)$
- Working temperatures $\longrightarrow$ $125 \leq T \leq 171$ MeV
- Physical light and strange quark masses, for each temperature $\left(m_\ell = m_s/27\right)$

How do we implement the simulations???

Implementation

Implementation

- Choose the initial complex μ_0 from the complex set

Implementation

- Choose the initial complex μ_0 from the complex set

$$S = \{\mu_0 : 0 \leq \text{Re}\left(\mu_0\right) \leq 2 \ , \ 0 \leq \text{Im}\left(\mu_0\right) \leq 2\}$$

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in steps of 0.1 on each direction , along $\text{Re}\left(\mu_0\right)$ & $\text{Im}\left(\mu_0\right)$

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with $N_{conf} = 20K,\, N_B = 50$ (bootstrap samples)

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And :

WORKFLOW

WORKFLOW

Start with a μ_0 value $\in S$

WORKFLOW

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$\mu_{NR}^{(1)}$

WORKFLOW

Start with a μ_0 value $\in S$

$\mu_{NR}^{(1)}$

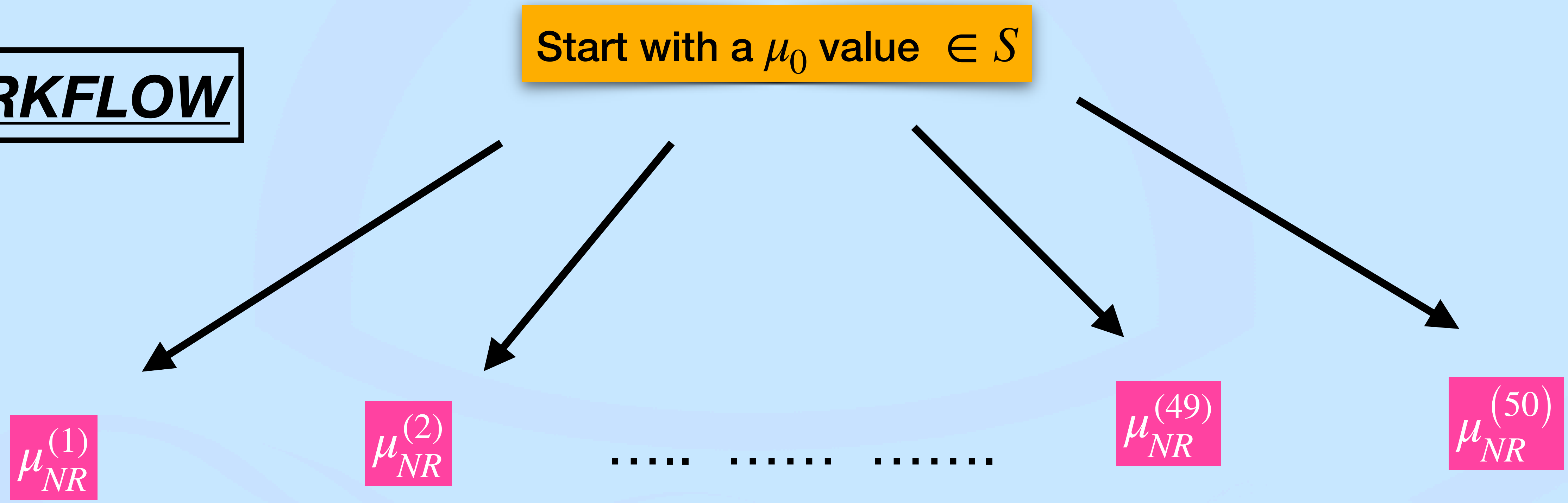
$\mu_{NR}^{(2)}$

WORKFLOW

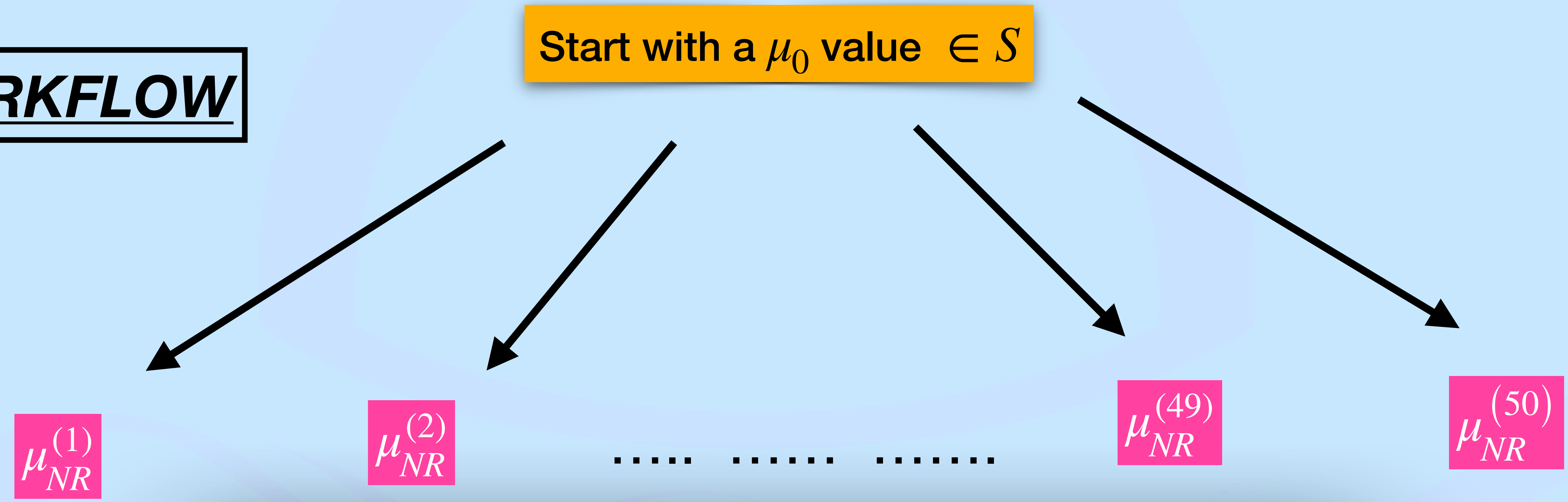
Start with a μ_0 value $\in S$

$\mu_{NR}^{(1)}$ $\mu_{NR}^{(2)}$ ….. …… …….

WORKFLOW

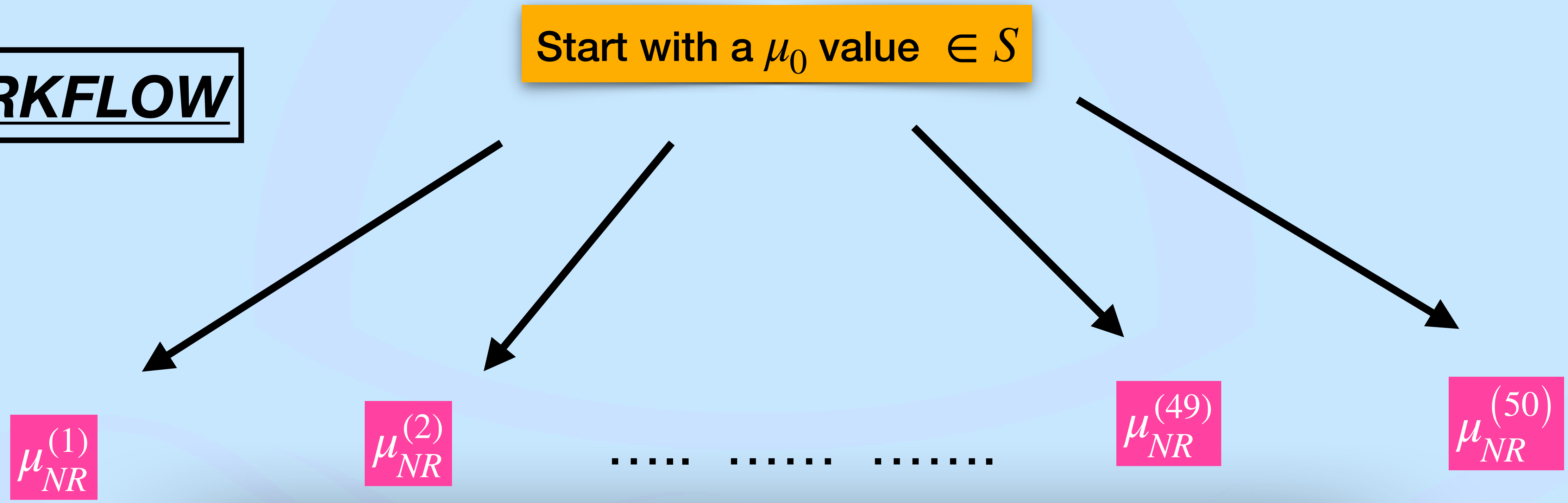


WORKFLOW



$\mu_{NR}^{(b)} \longrightarrow$ estimate of the Newton-Raphson root in b^{th} bootstrap sample

WORKFLOW



$\mu_{NR}^{(b)} \longrightarrow$ estimate of the Newton-Raphson root in b^{th} bootstrap sample

Final estimates :

WORKFLOW

Start with a μ_0 value $\in S$

$\mu_{NR}^{(1)}$ $\mu_{NR}^{(2)}$ ….. …… ……. $\mu_{NR}^{(49)}$ $\mu_{NR}^{(50)}$

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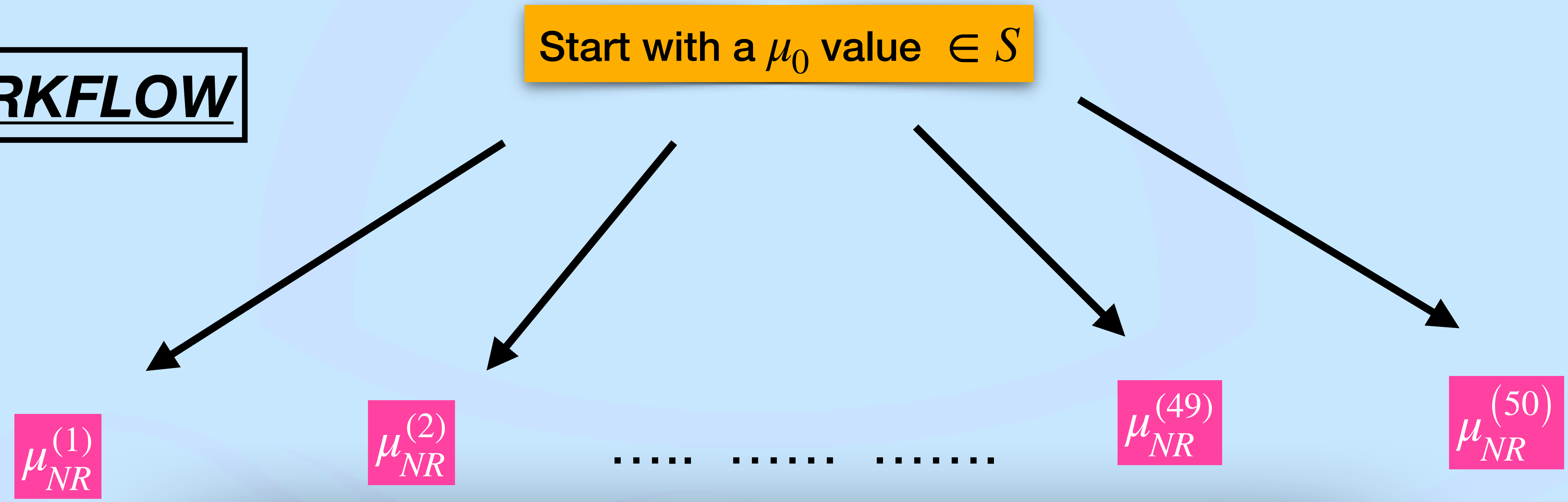
Final estimates :

$$\mu_{NR}\left(\mu_0\right) = \frac{1}{50}\sum_{b=1}^{50}\mu_{NR}^{(b)}$$

WORKFLOW



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$$\sigma_{NR}^2\left(\mu_0\right) = \frac{1}{50}\sum_{b=1}^{50}\left\{\mu_{NR}^{(b)} - \mu_{NR}\left(\mu_0\right)\right\}^2$$

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Variance

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Variance

Follow this for all the other μ_0

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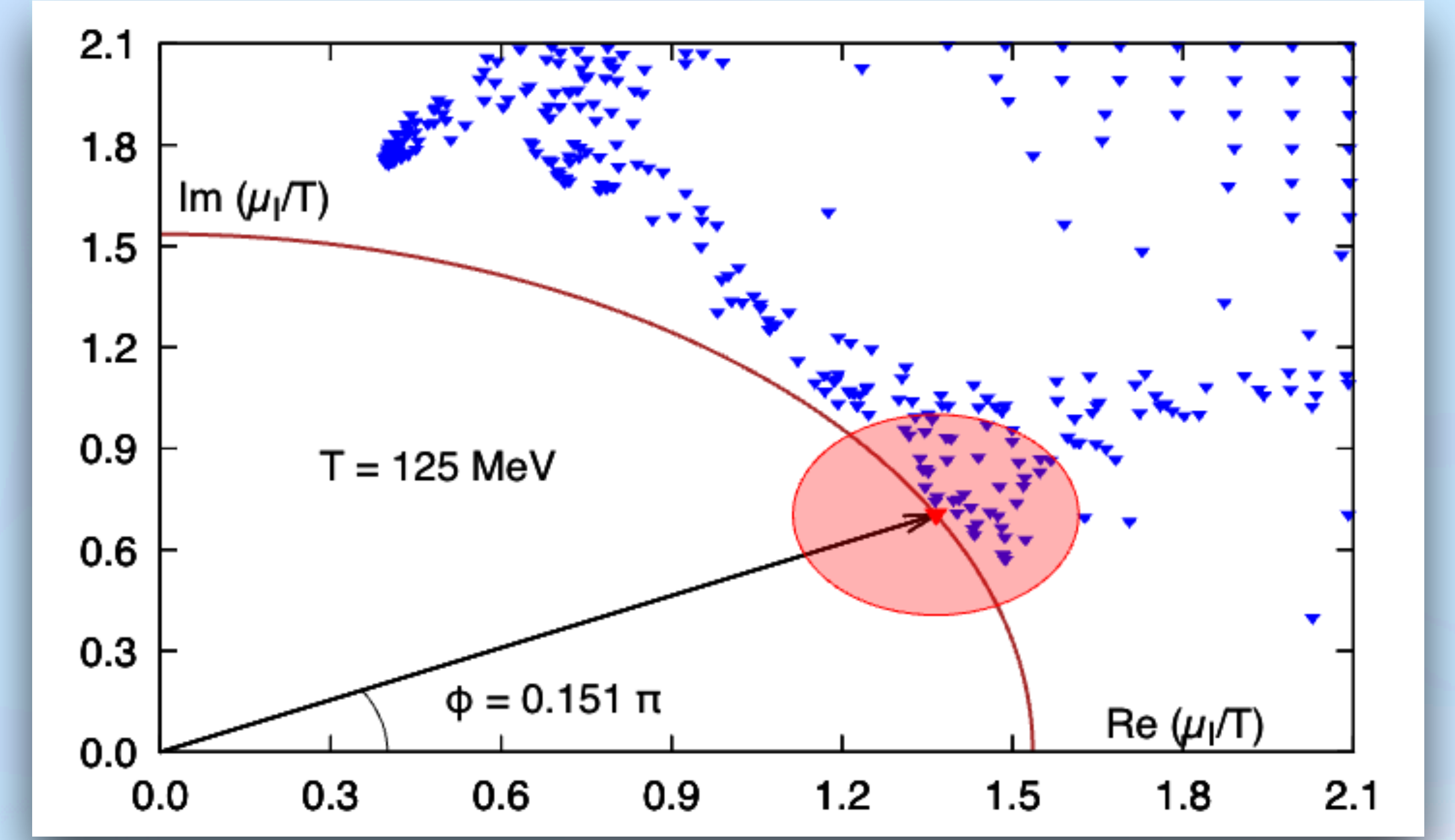
Follow this for all the other μ_0

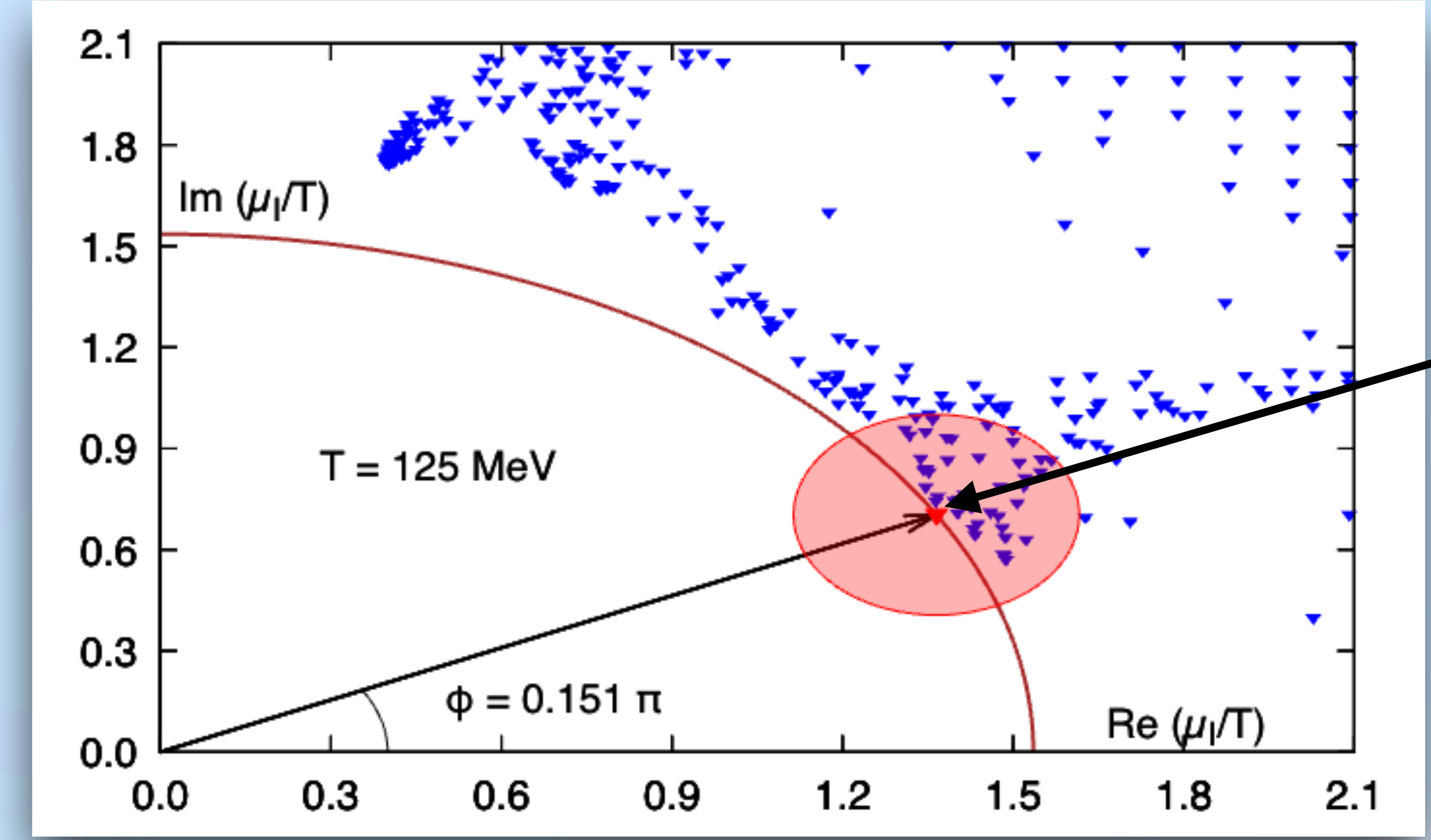
RESULTS ???

Results



Complex μ_I plane

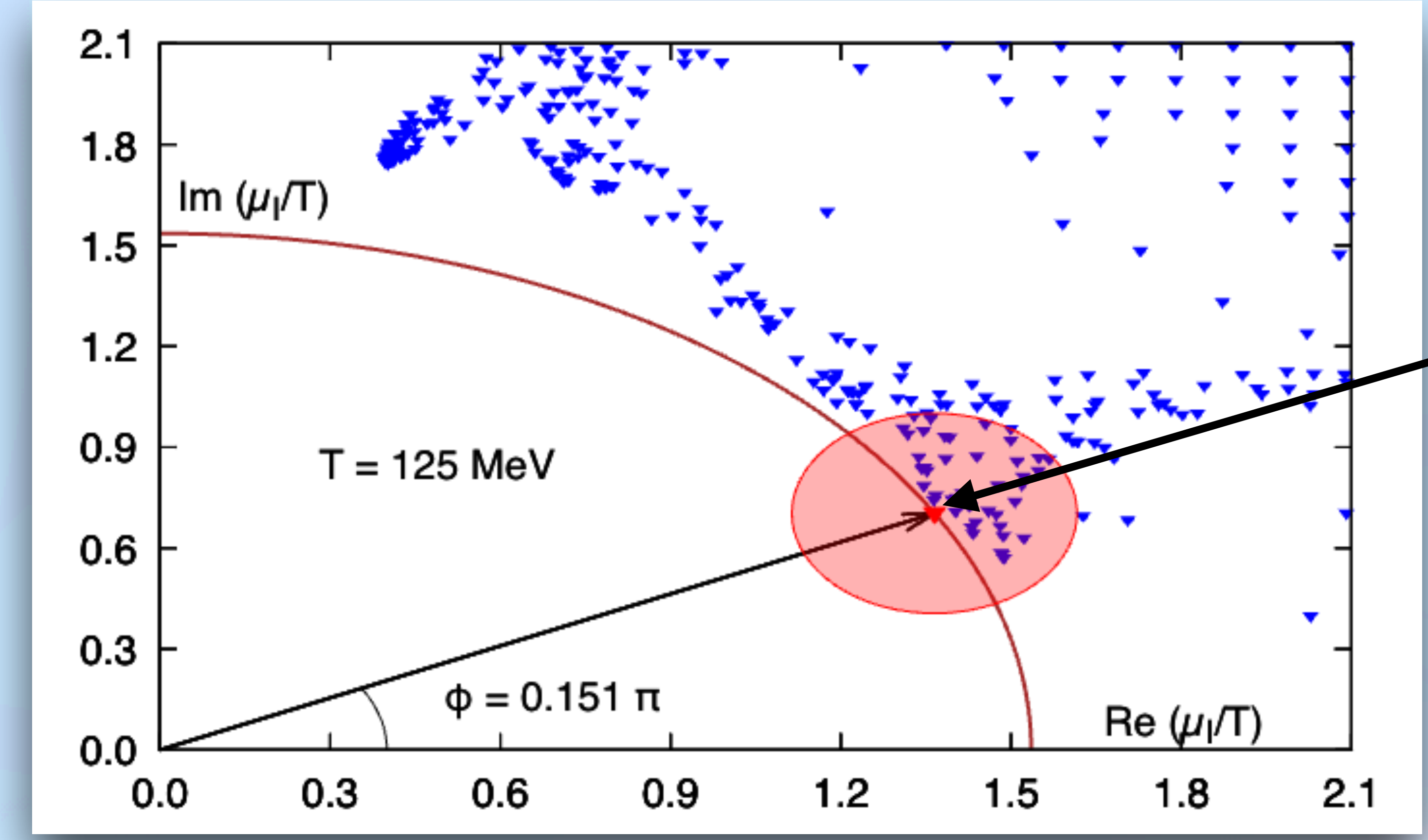
Results



Complex μ_I plane

Nearest Lee-Yang zero μ_I^0

Results



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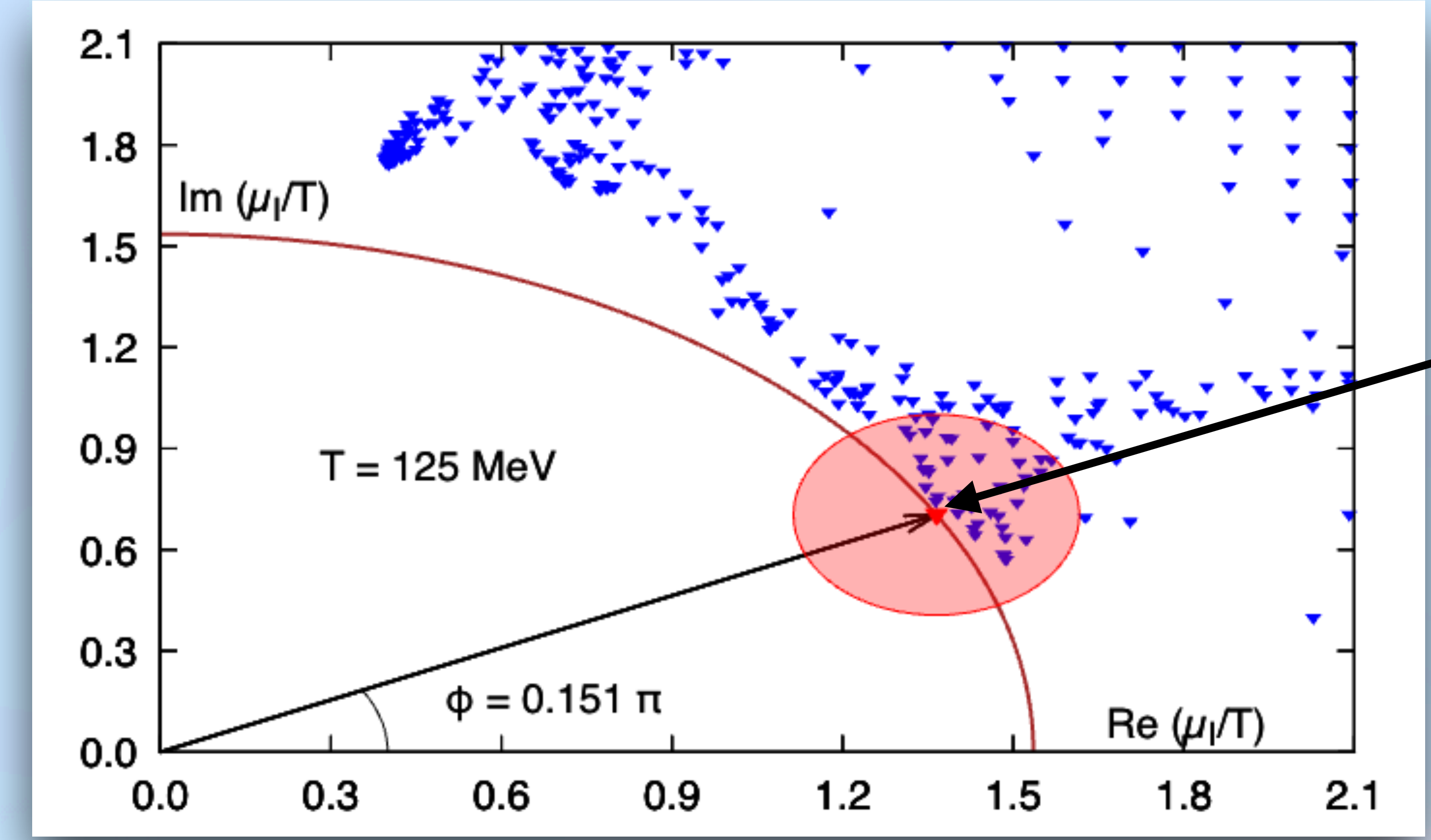

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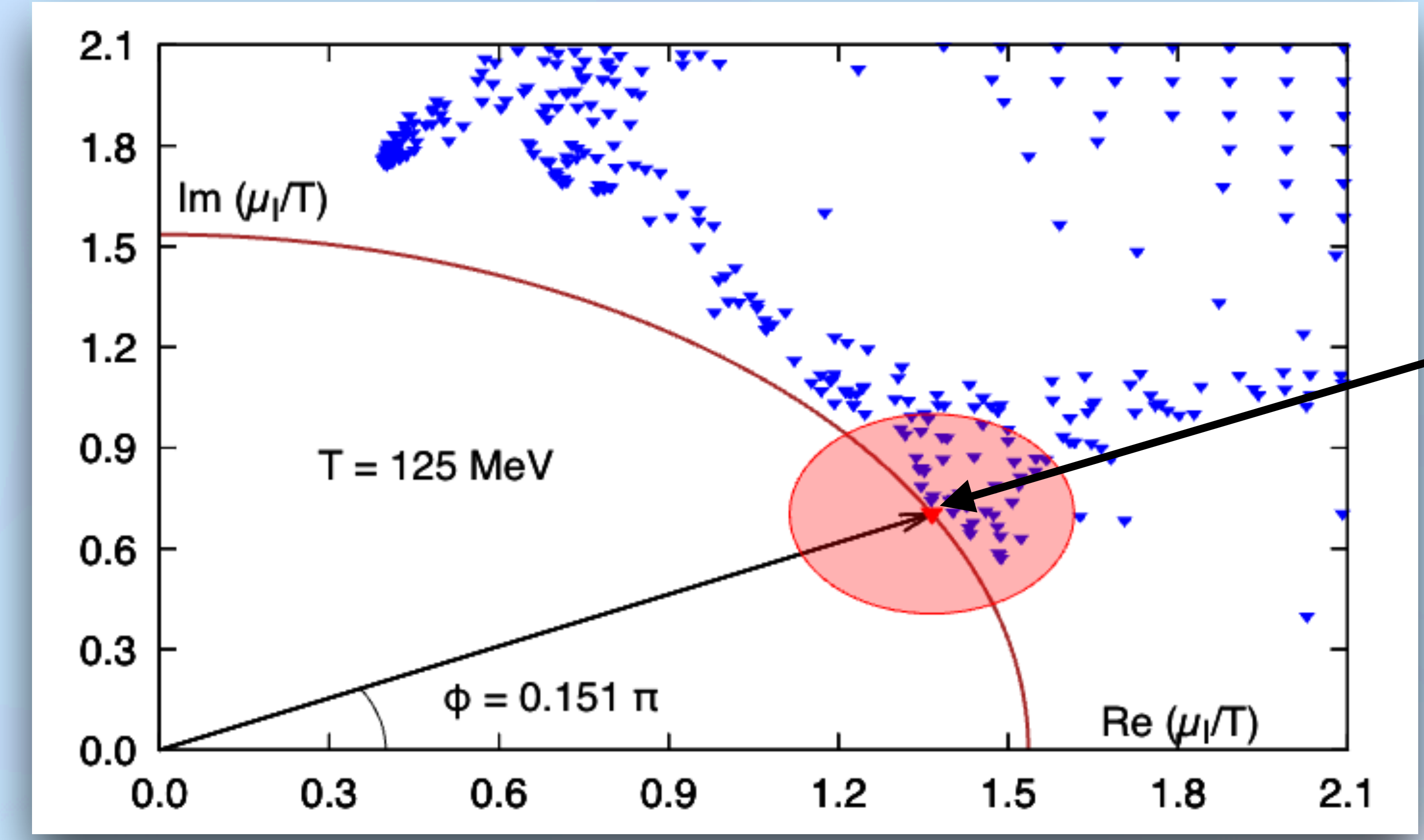
↓

Closest to point of expansion

→ origin $(0\,,0)$ here

Results



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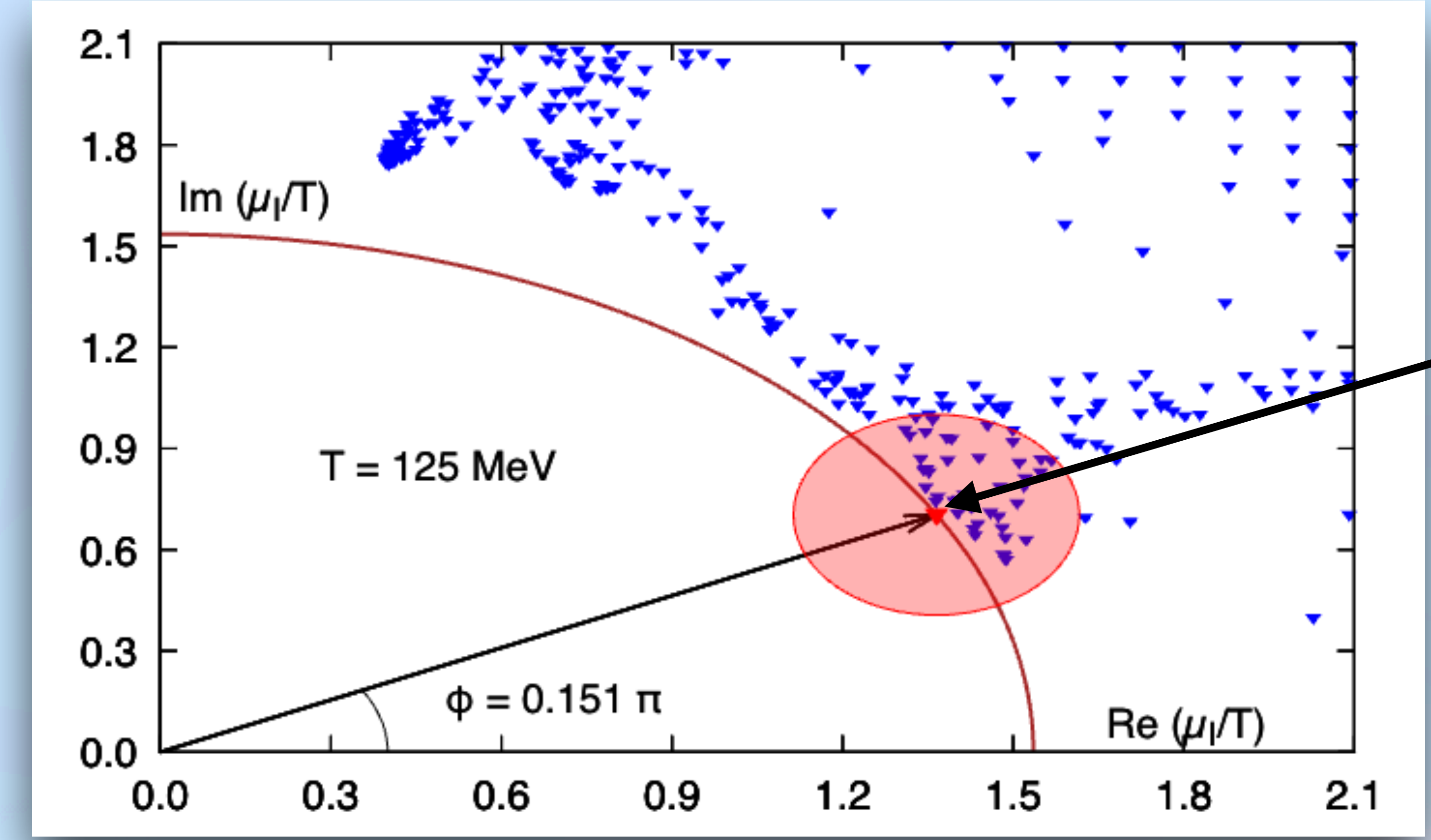
Nearest Lee-Yang zero μ_I^0

↓

Closest to point of expansion
→ origin $(0\,,0)$ here

- Individual error bars on roots are **only** shown here for μ_I^0.
- Elliptical representation of errors : major and minor axes.
- This line shows **one quarter** of **the circle of convergence**.
- μ_I^0 makes an **angle** ϕ (radian units) **with the real** μ_I **axis.**

Results



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$$\phi = \tan^{-1}\left[\frac{\mu_{I,i}^T}{\mu_{I,r}^T}\right]$$

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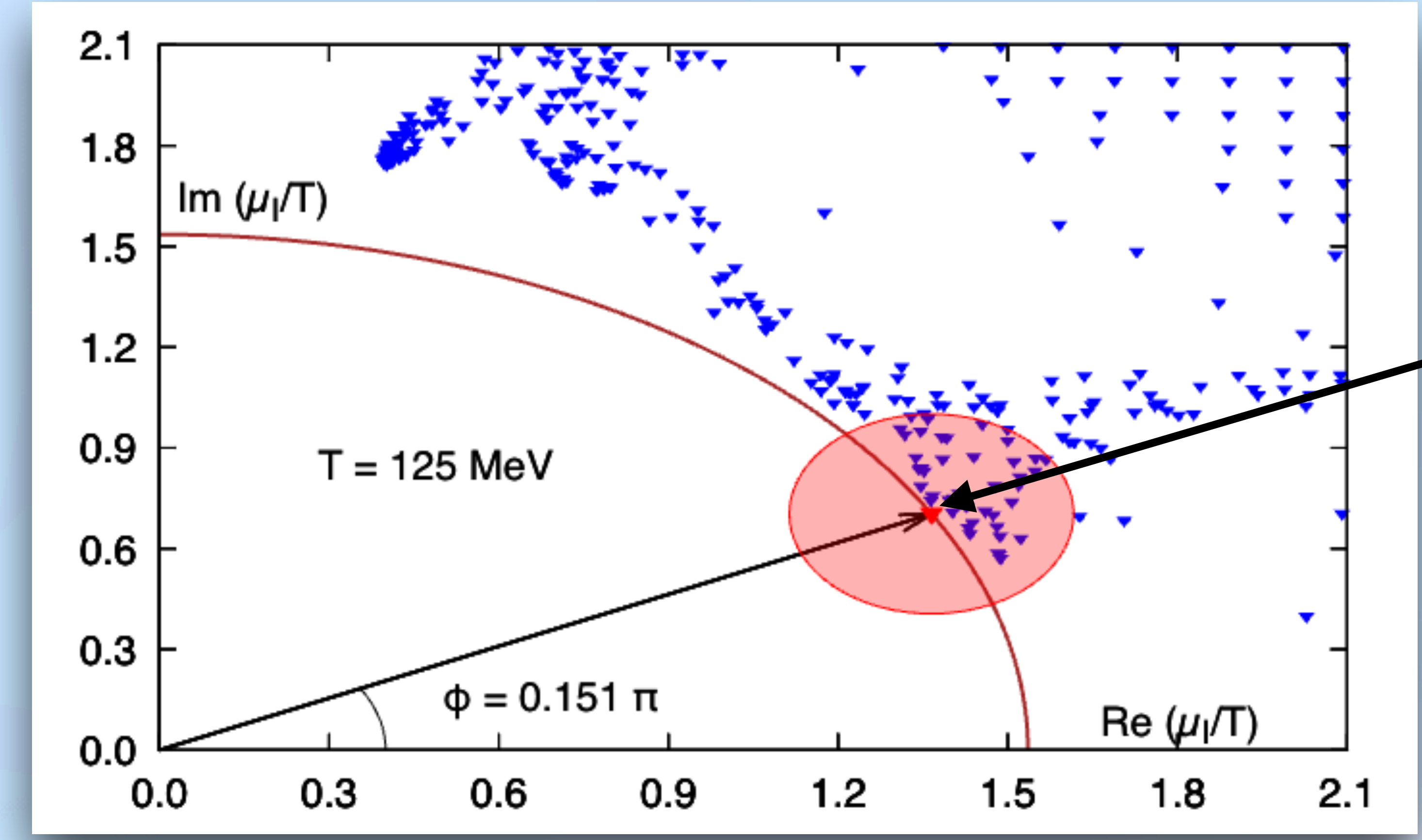

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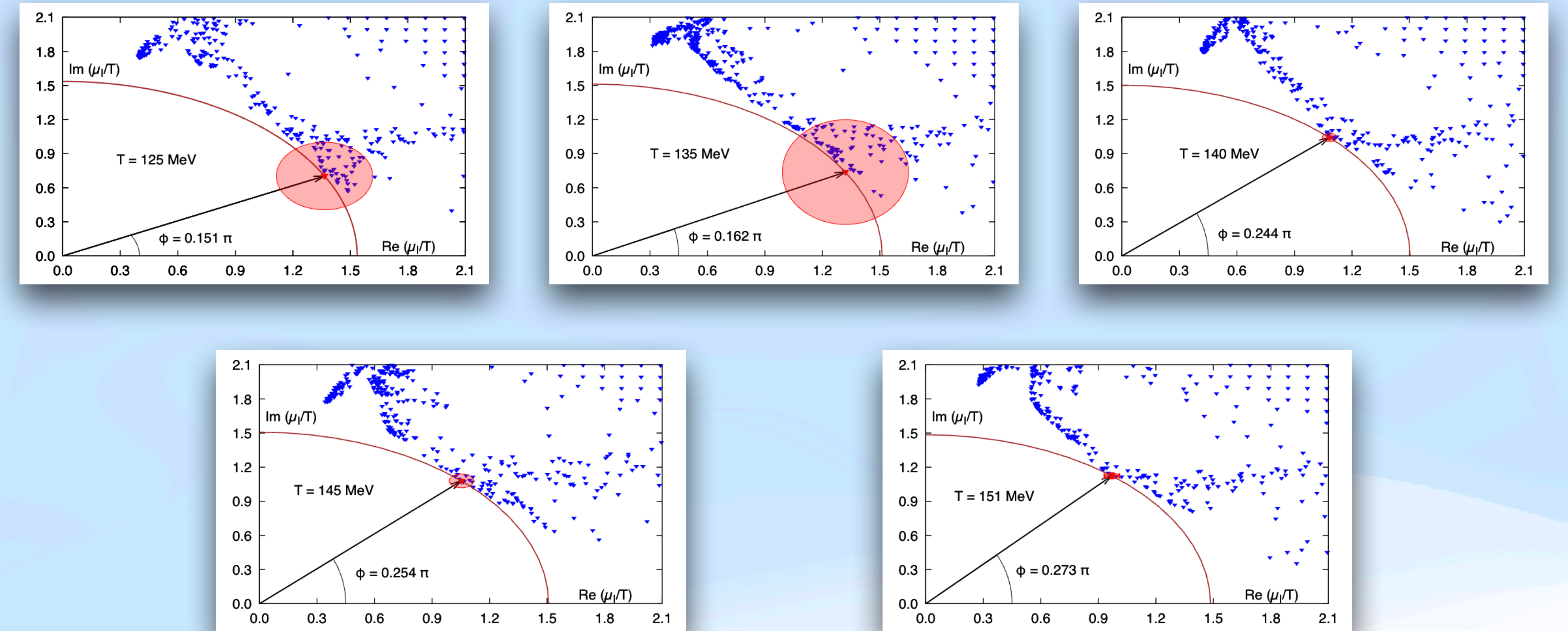
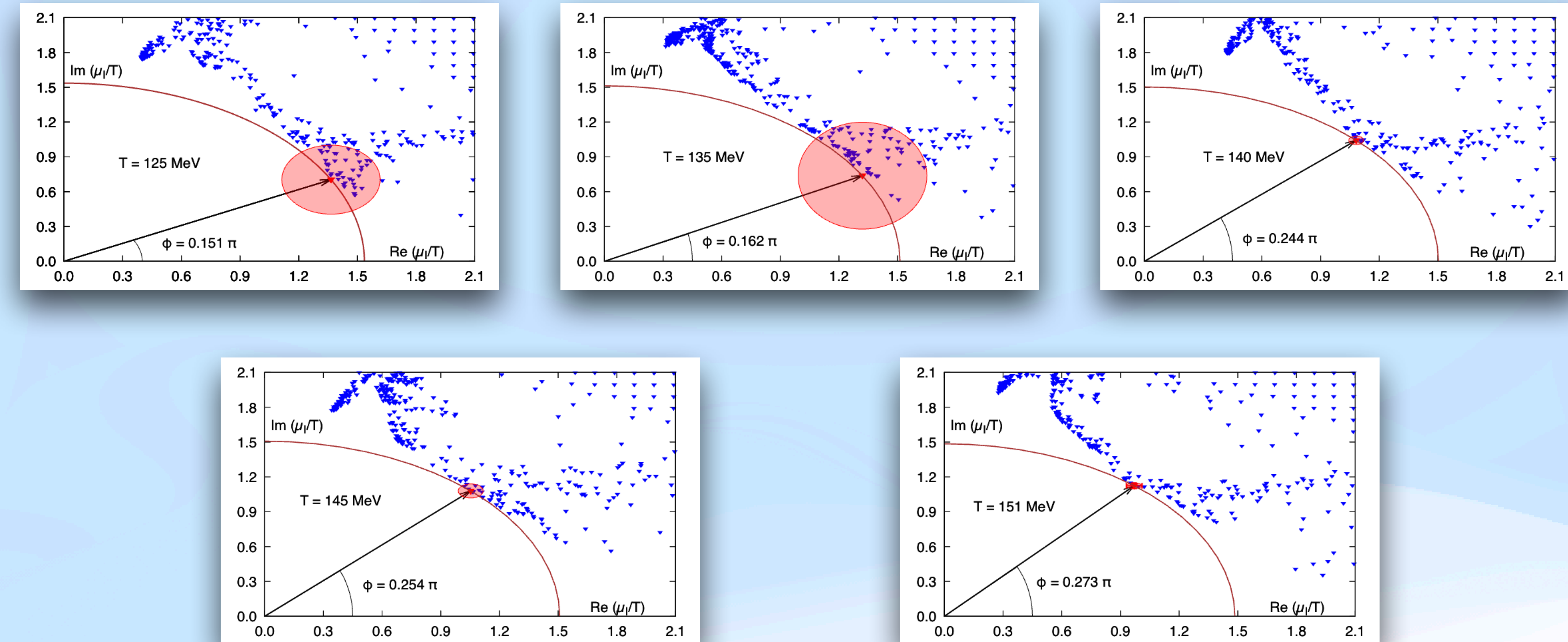
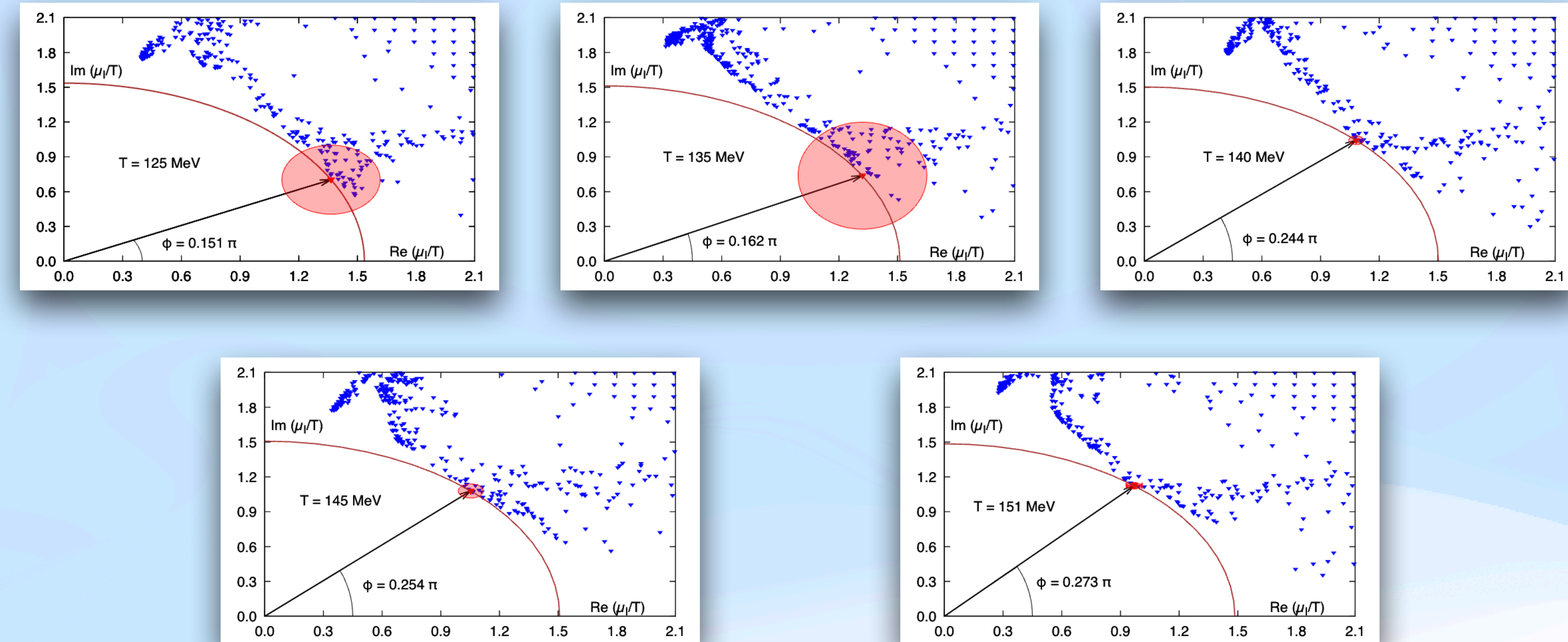
Other $T's\,??$



T = 125 MeV
ϕ = 0.151 π
T = 135 MeV
ϕ = 0.162 π
T = 140 MeV
ϕ = 0.244 π
T = 145 MeV
ϕ = 0.254 π
T = 151 MeV
ϕ = 0.273 π
Im (μ_I/T)
Re (μ_I/T)



- μ_I^0 **<u>approaching real axis</u>** with <u>reducing</u> T (reducing angle ϕ)
- Expecting to find **real** $\mu_I^0 \Rightarrow$ **<u>genuine critical point</u>** at **lower** T



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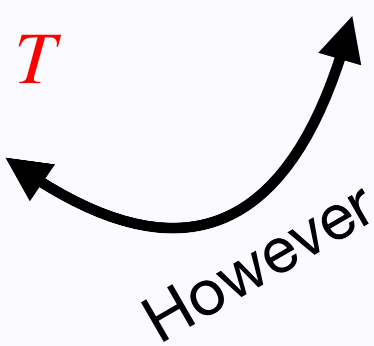

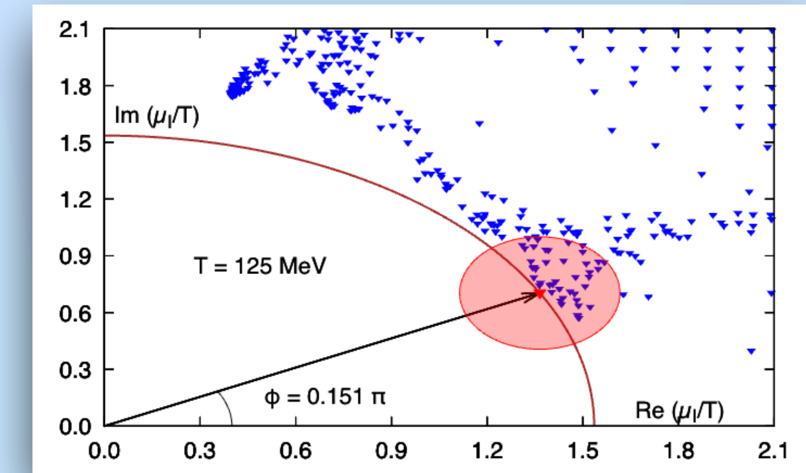

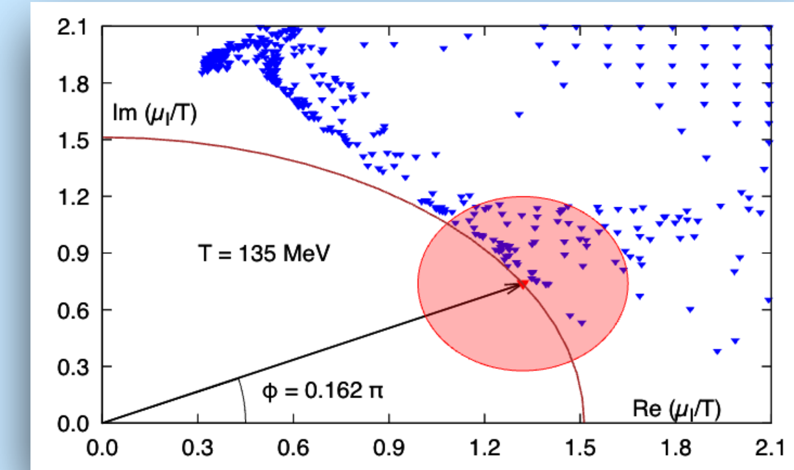

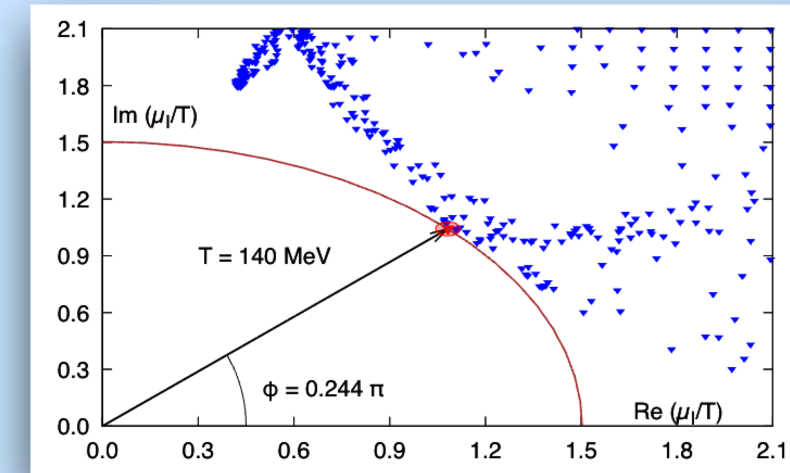

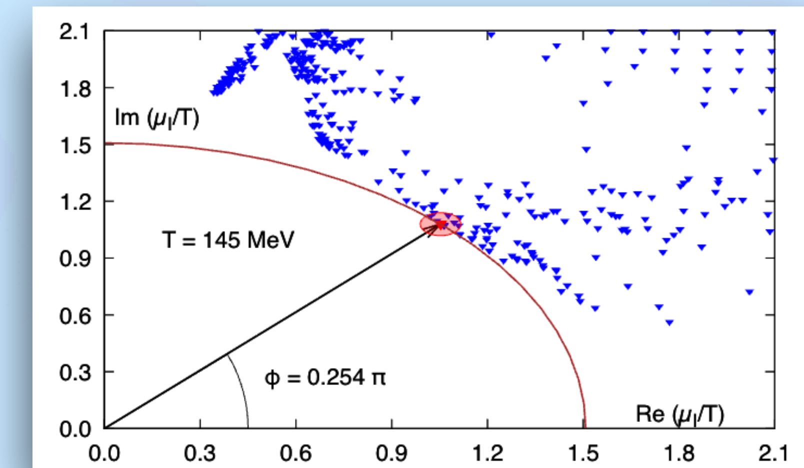

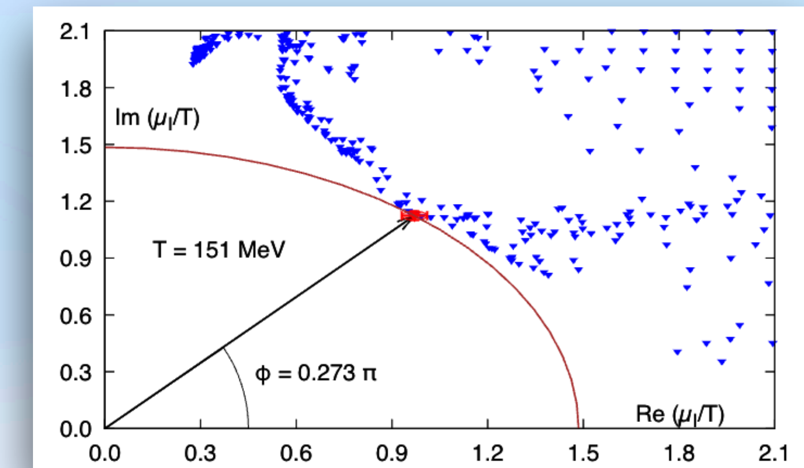

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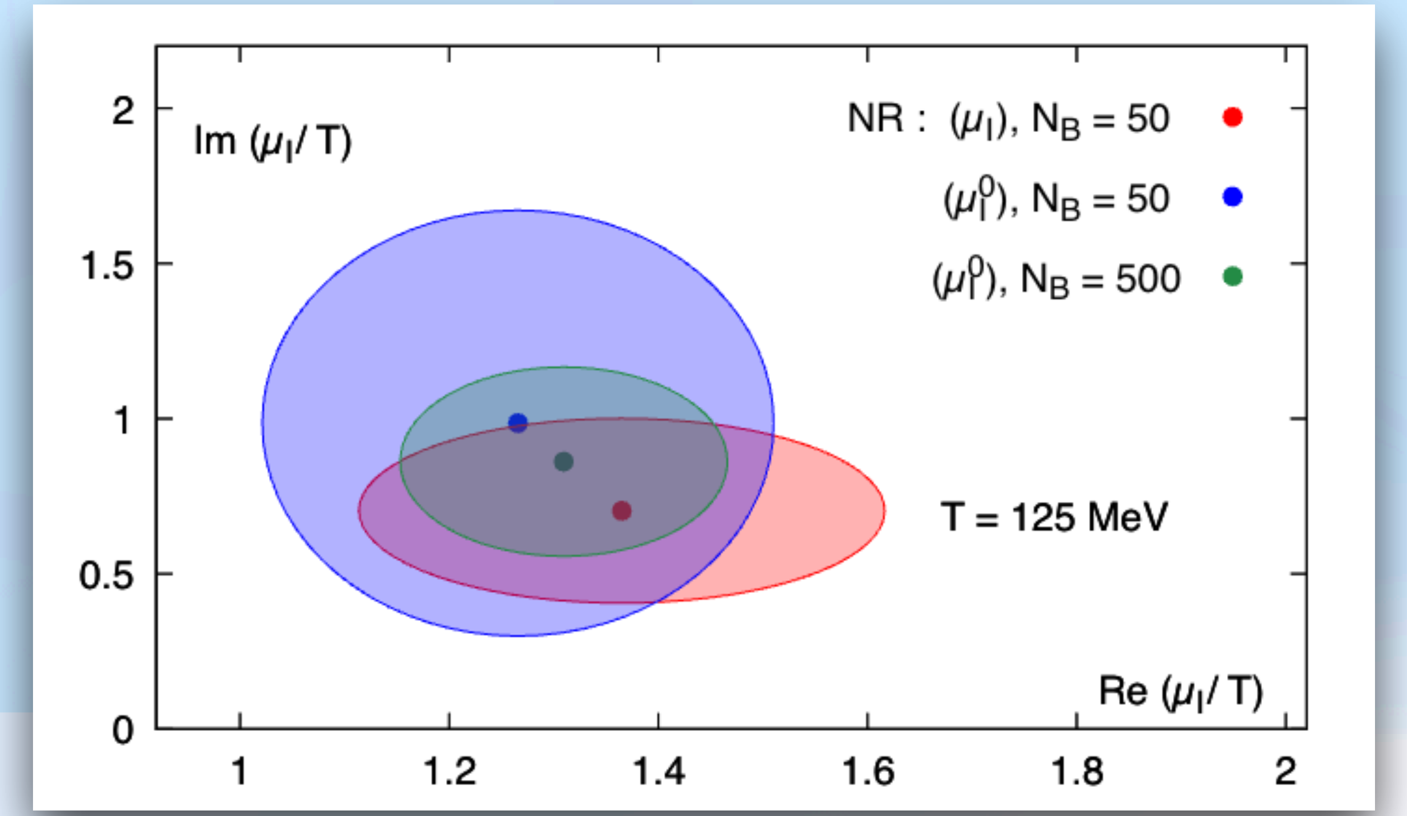
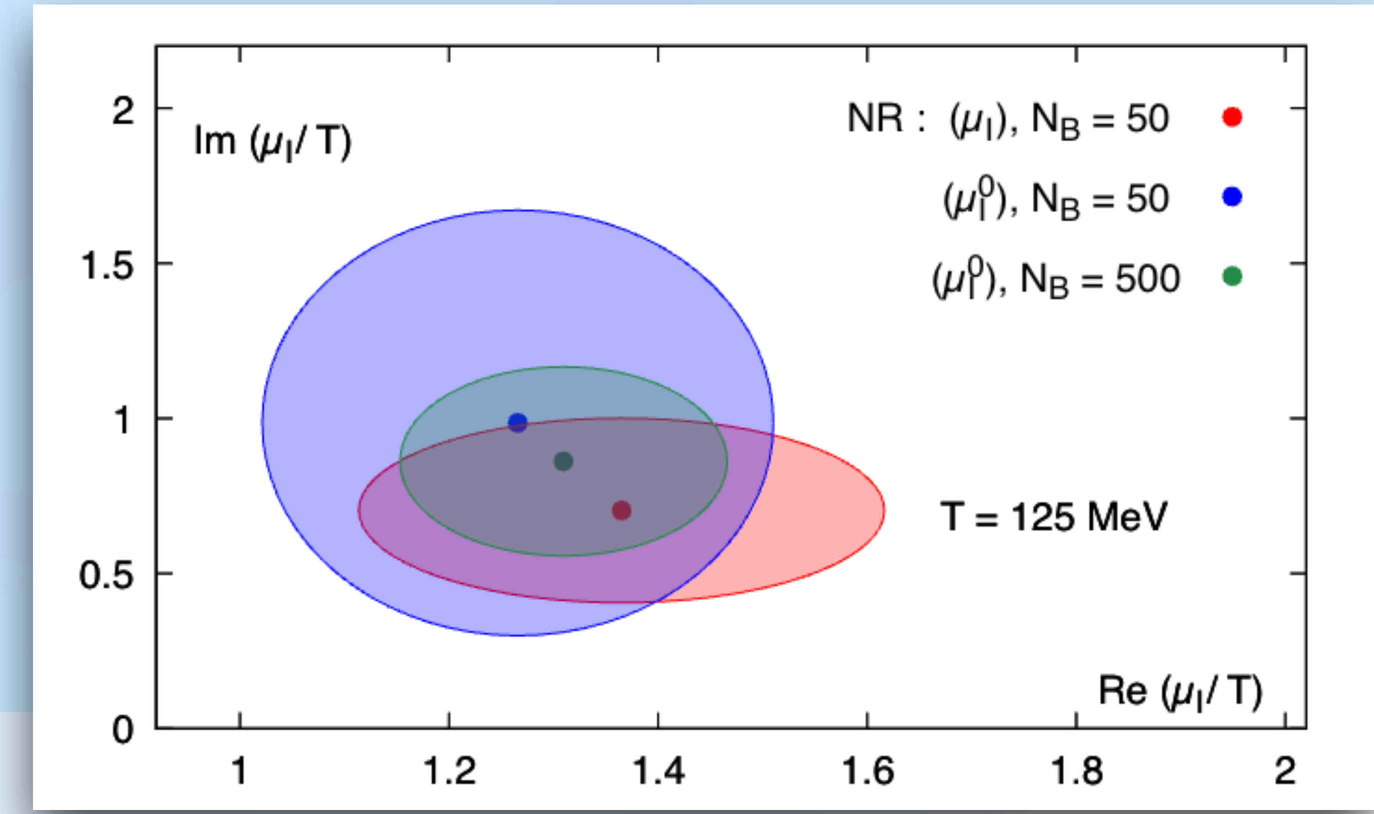
How stable (reliable) are these results ??

However

Stability of results

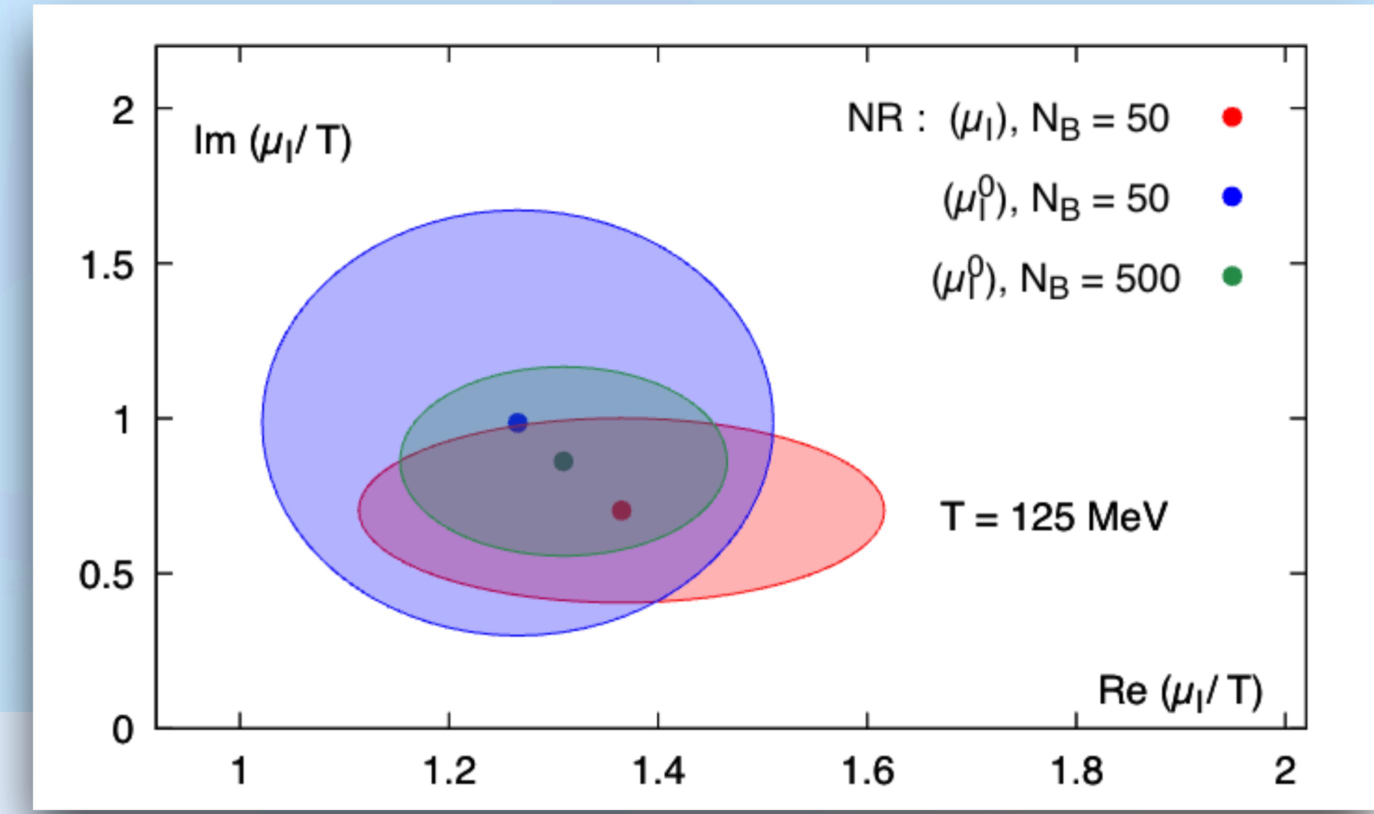


Stability of results



New estimates ⟵ using μ_I^0 **as initial guesses**

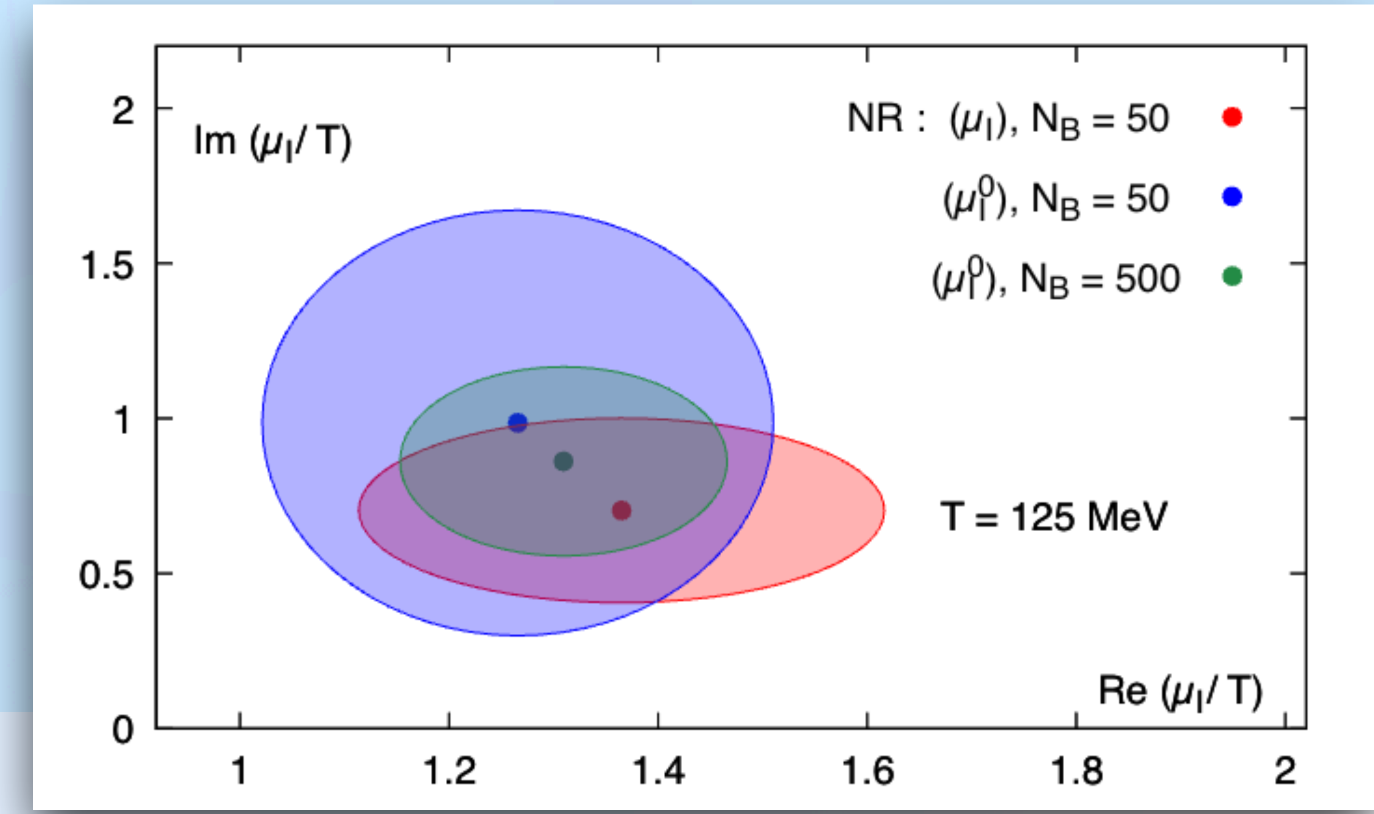
Stability of results



New estimates $\longleftarrow$ using μ_I^0 **as initial guesses**

- **Good agreement (overlap)** with the **old** $\Longleftrightarrow$ **new estimates** of the roots / zeros
- The present estimates of zeros **are reliable** !!!

Stability of results



New estimates $\longleftarrow$ using μ_I^0 **as initial guesses**

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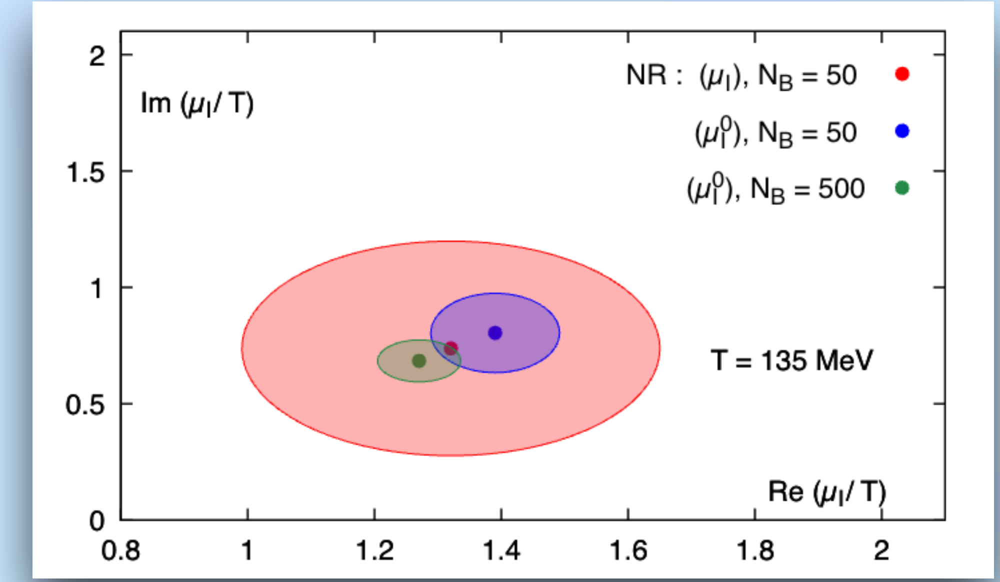
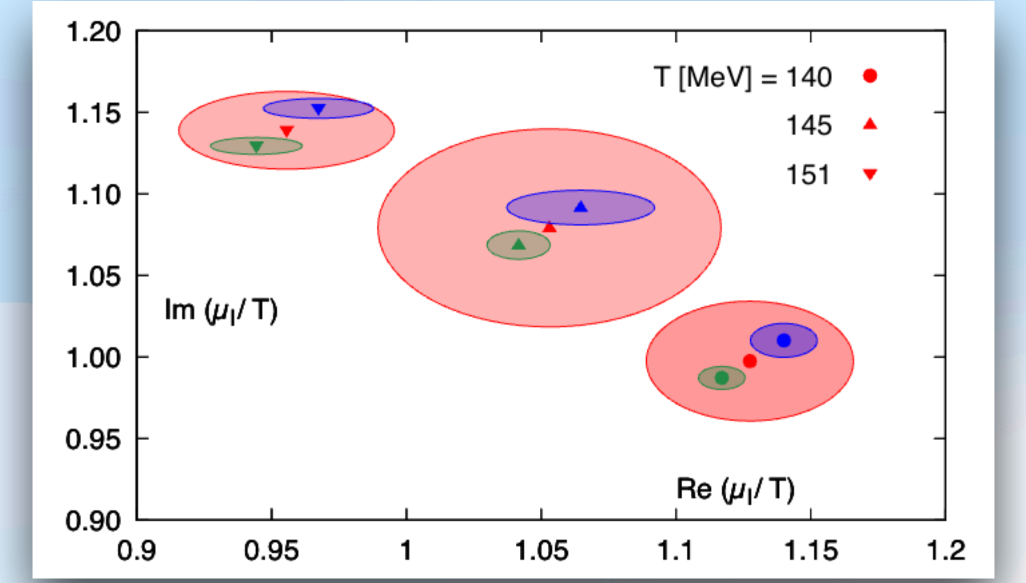
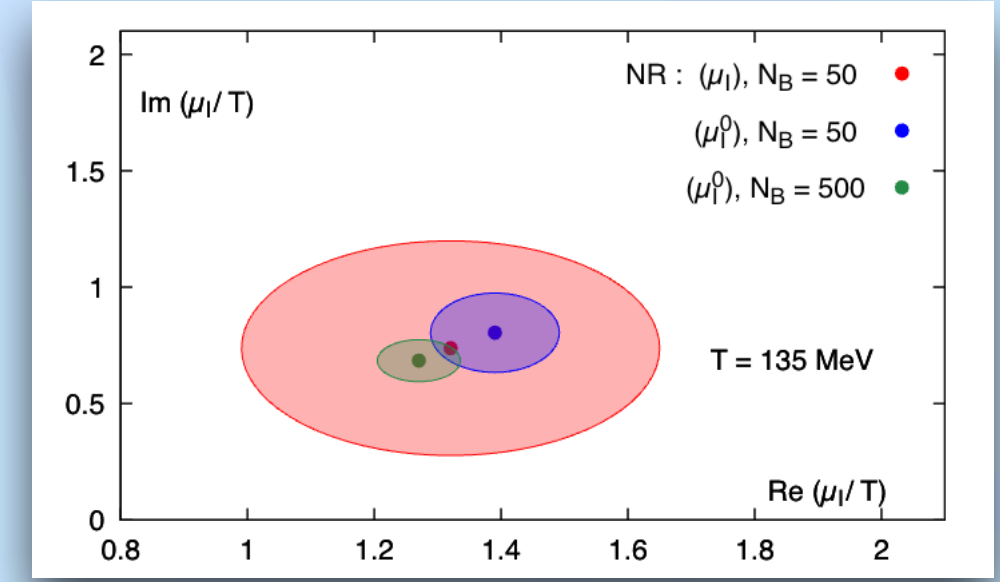
What about other T ??



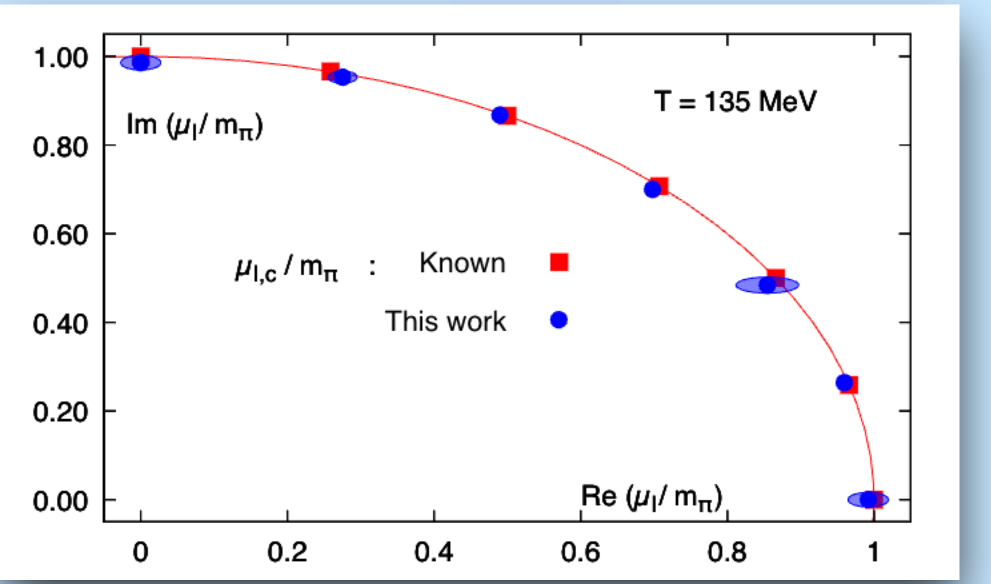
Im (μ_I/ T)
Re (μ_I/ T)
NR : (μ_I), N_B = 50
(μ_I^0), N_B = 50
(μ_I^0), N_B = 500
T = 135 MeV



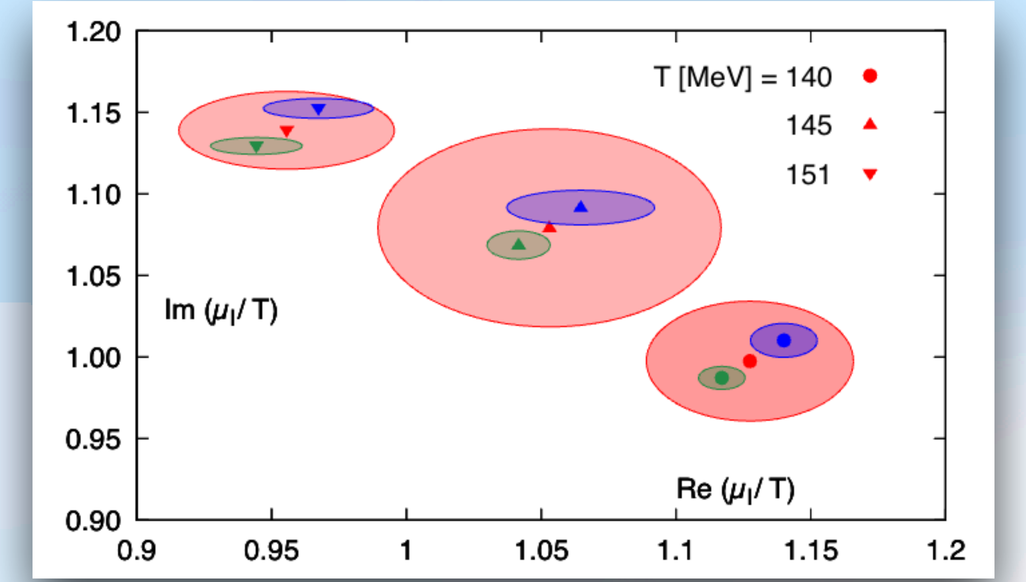
T [MeV] = 140
145
151
Im (μ_I/ T)
Re (μ_I/ T)



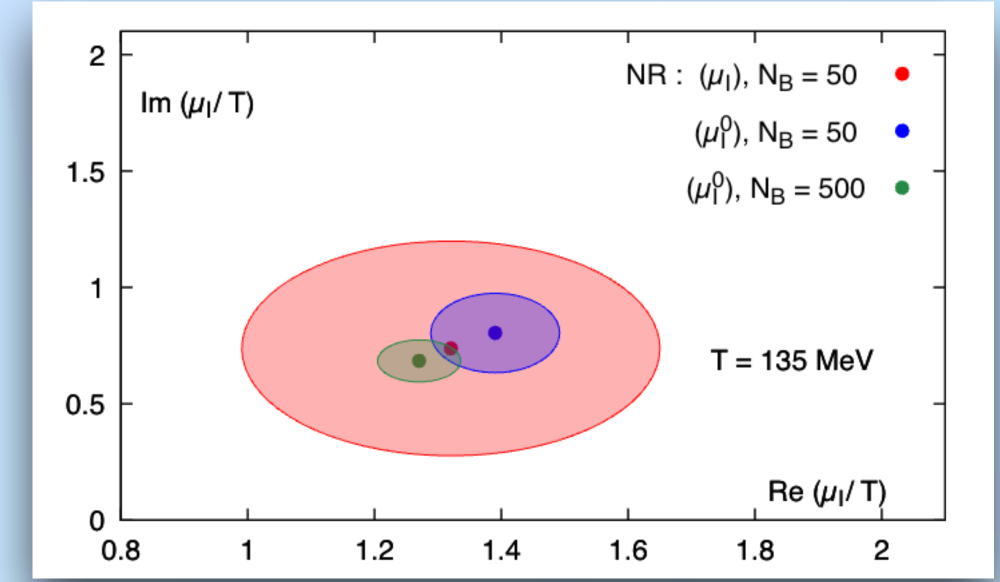
Im (μ_I/ T)
NR : $\langle\mu_I\rangle$, N_B = 50
$\langle\mu_I^0\rangle$, N_B = 50
$\langle\mu_I^0\rangle$, N_B = 500
T = 135 MeV
Re (μ_I/ T)

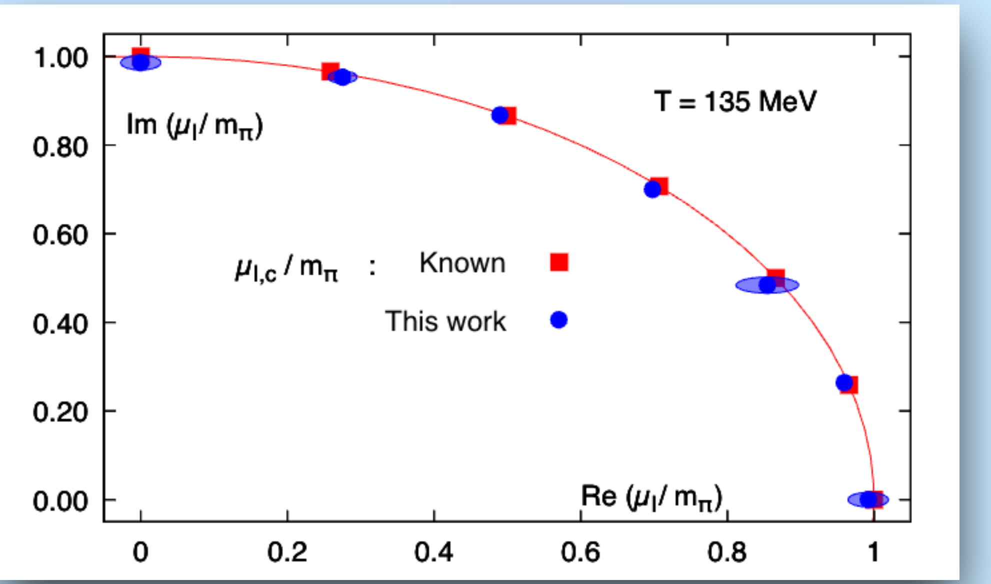


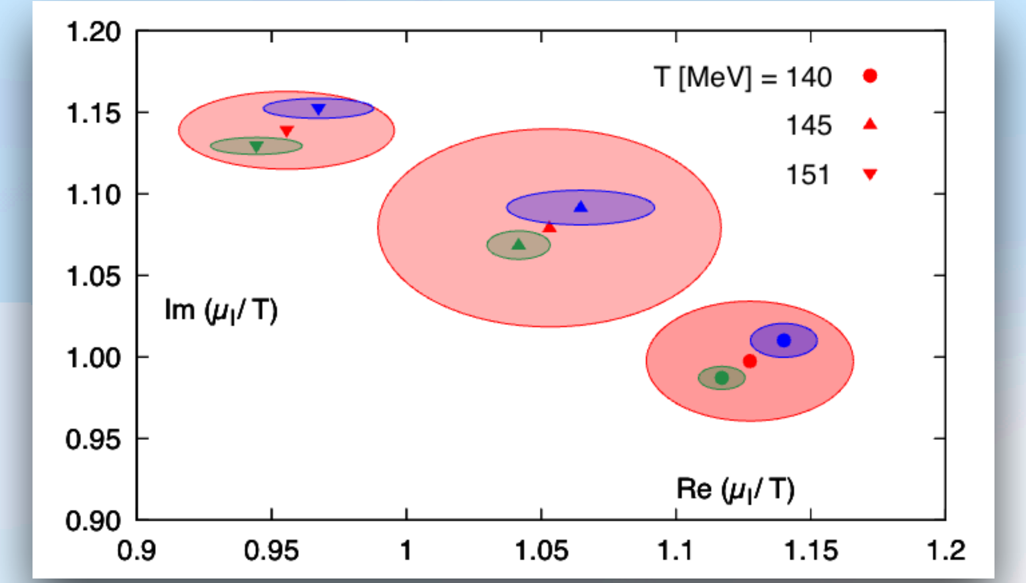
Im (μ_I/ m_π)
T = 135 MeV
$\mu_{I,c}$ / m_π : Known
This work
Re (μ_I/ m_π)



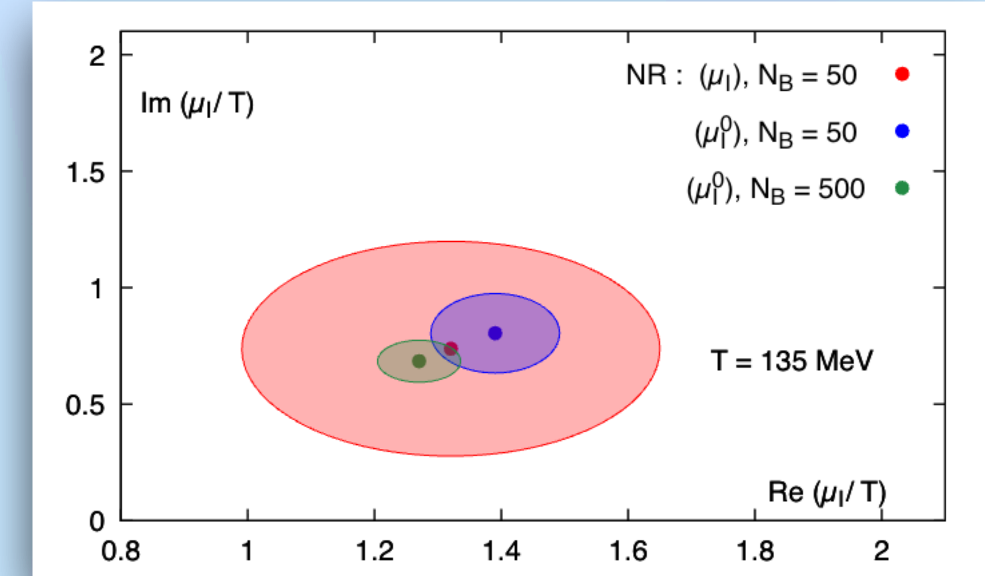
T [MeV] = 140
145
151
Im (μ_I/ T)
Re (μ_I/ T)

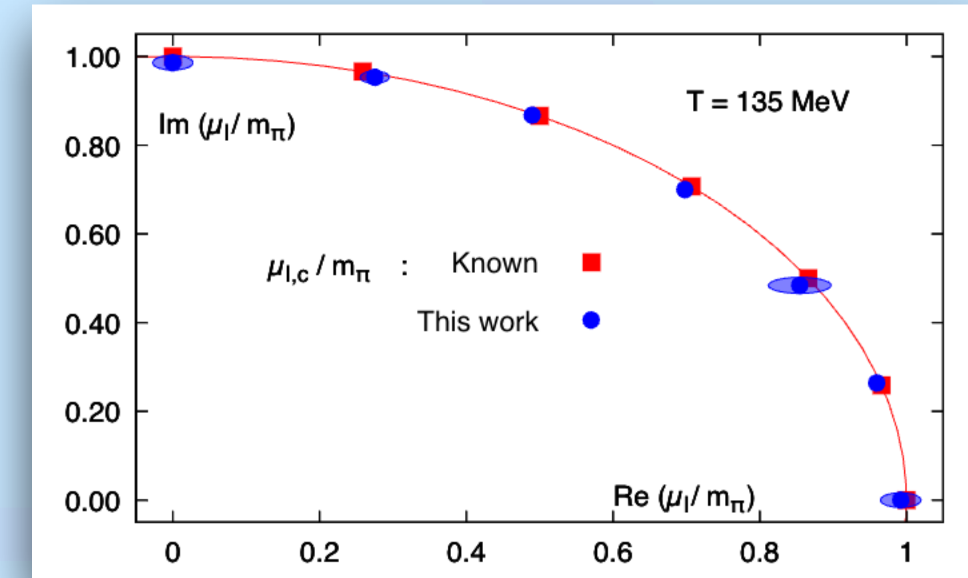


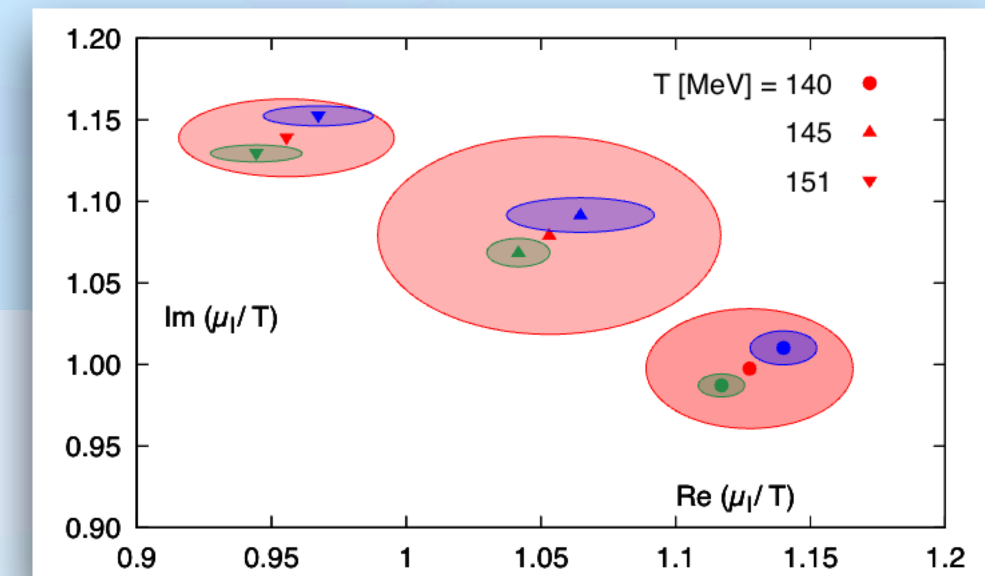




All possible critical points for $T = 135$ MeV as initial guesses

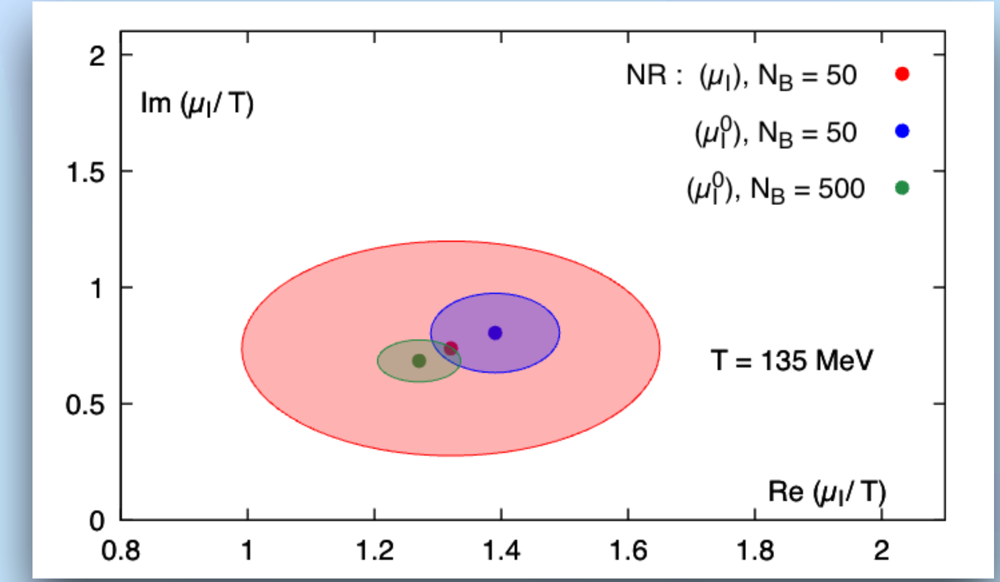


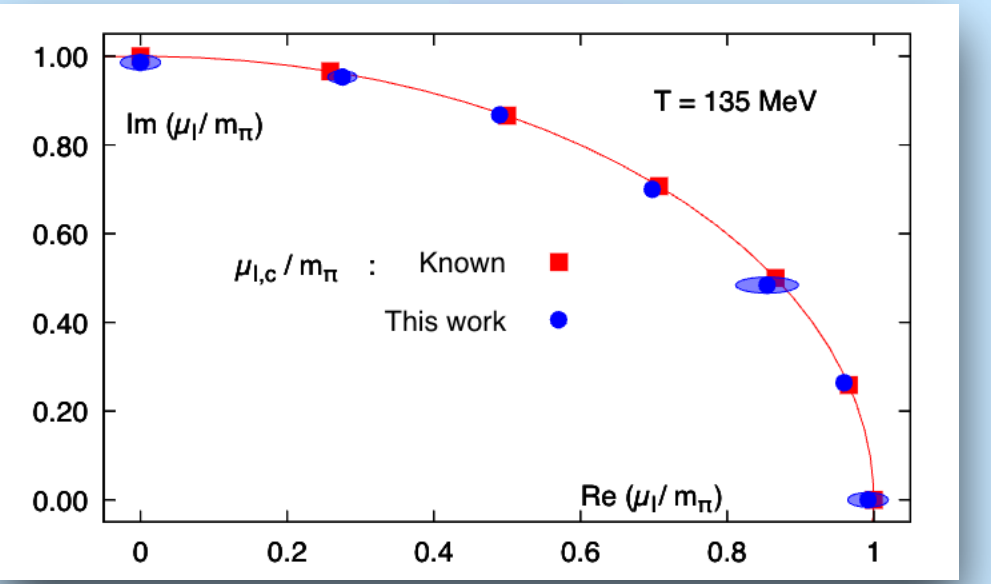


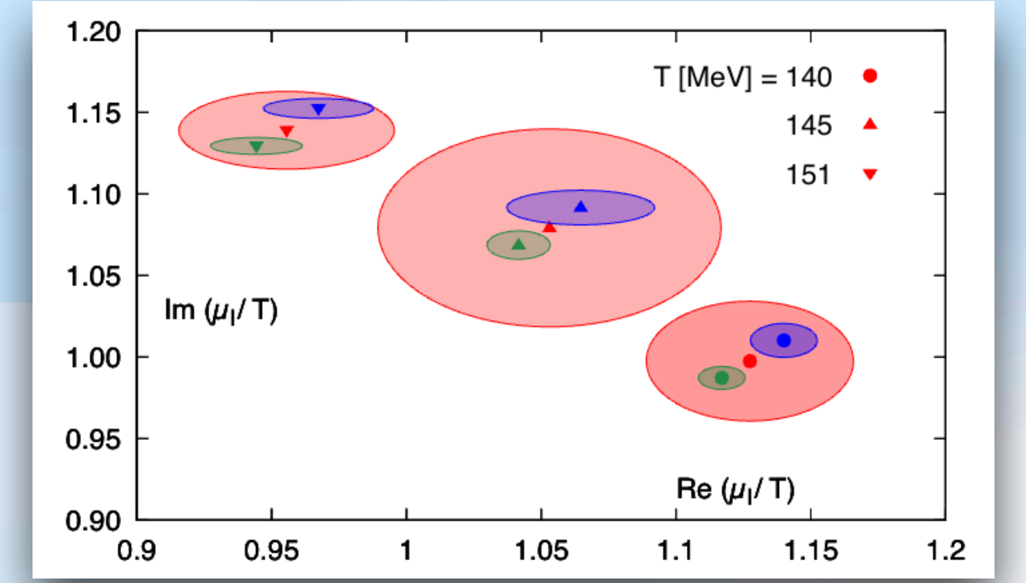


All possible critical points for $T = 135$ MeV as initial guesses

$$\mu_0 = (m_\pi \cos\phi\,, m_\pi \sin\phi)\ ,\quad \phi = \left\{ \frac{n\pi}{12}\ ,\quad 1 \leq n \leq 6 \right\}$$







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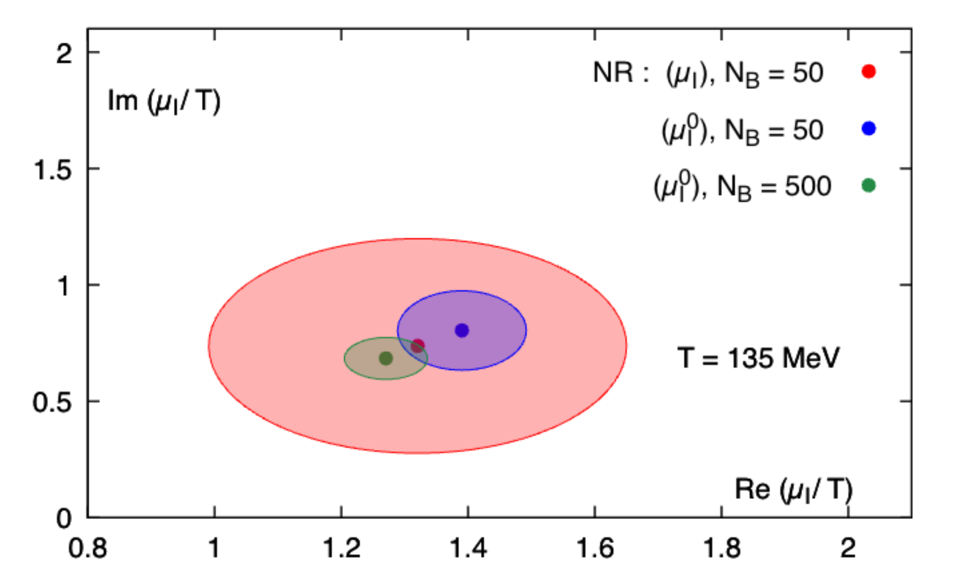

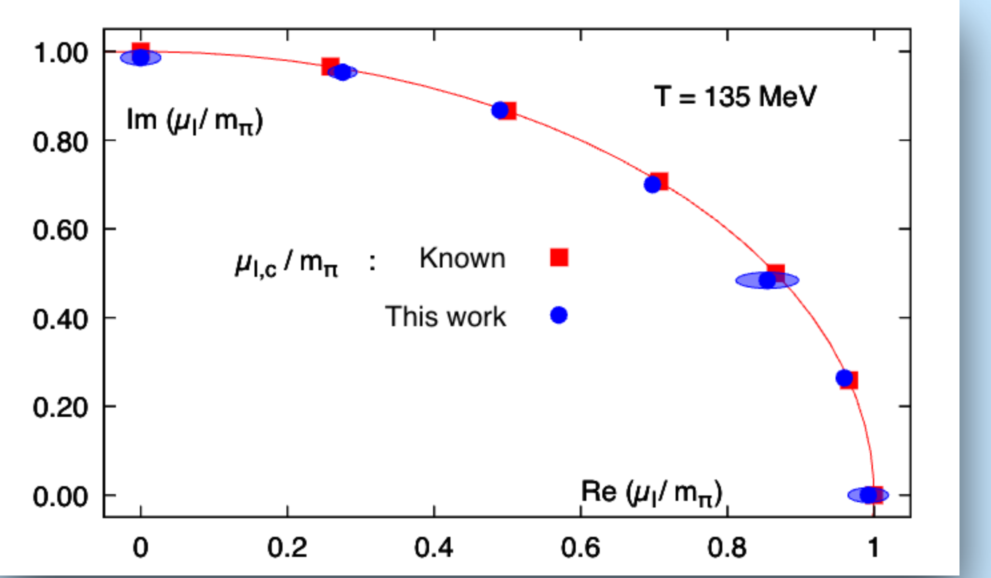

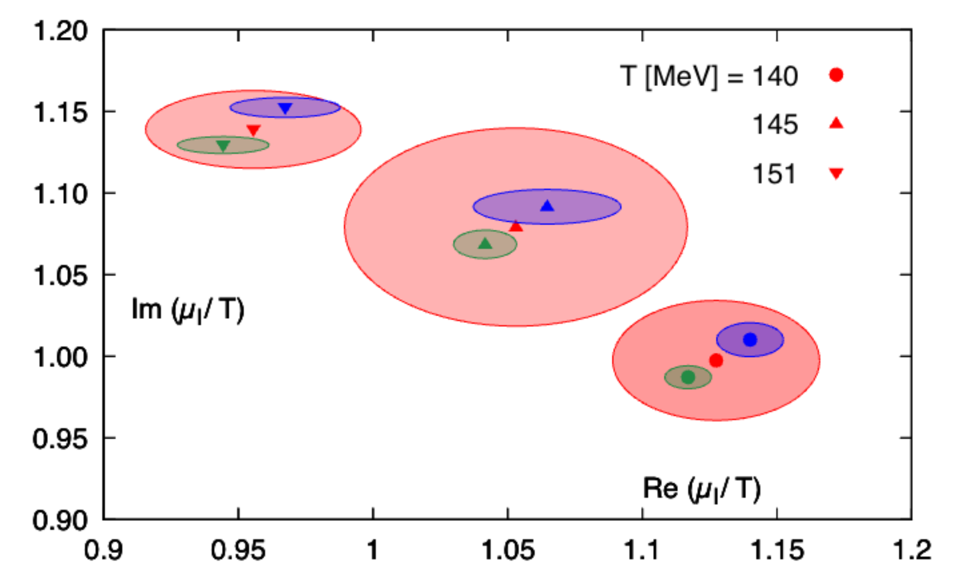

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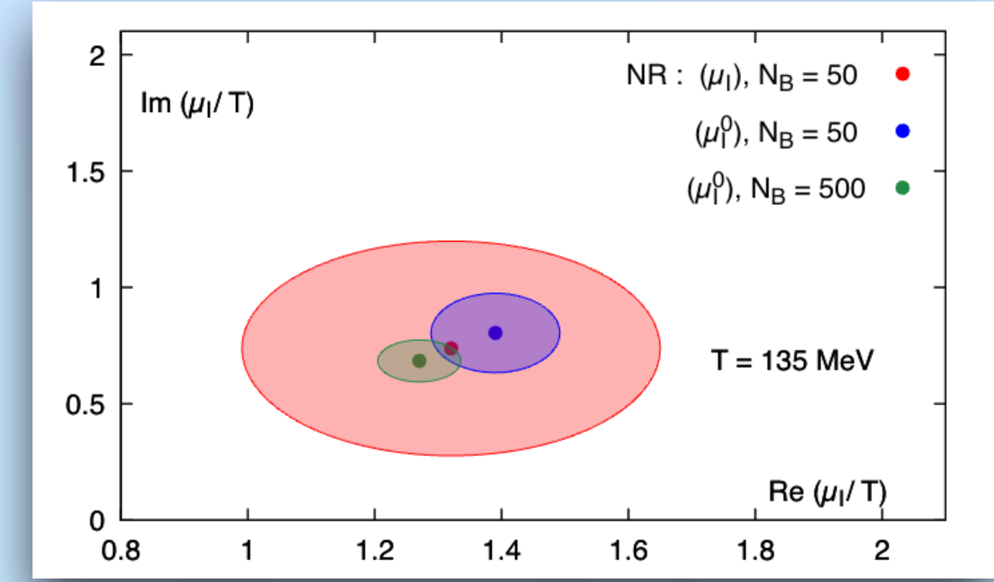
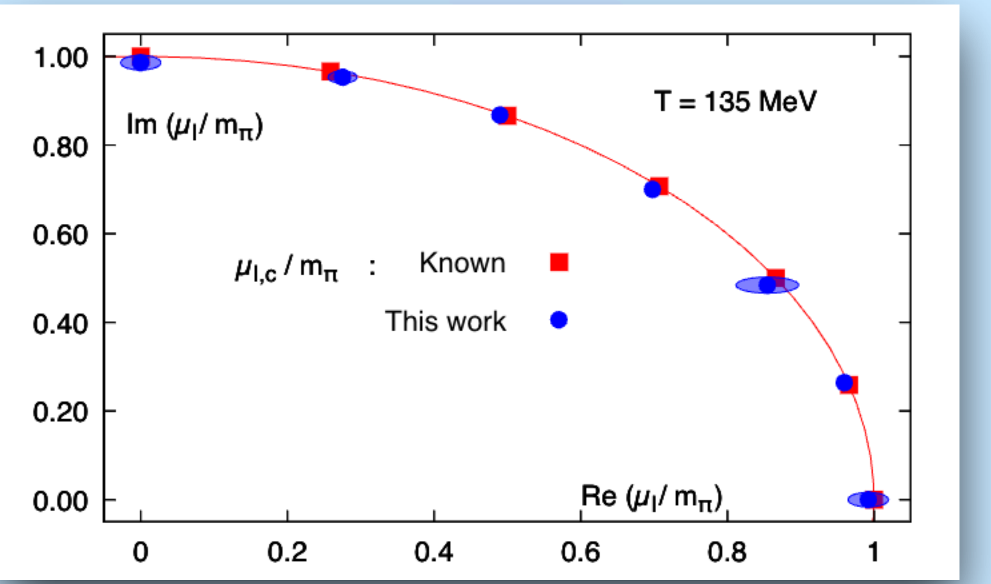
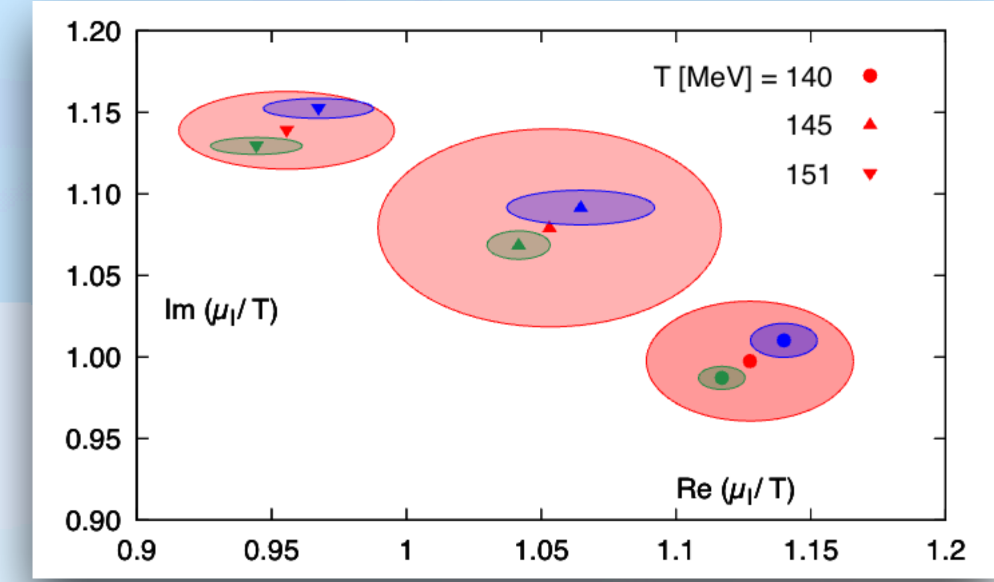
Commendable agreement upto $T = 151$ MeV







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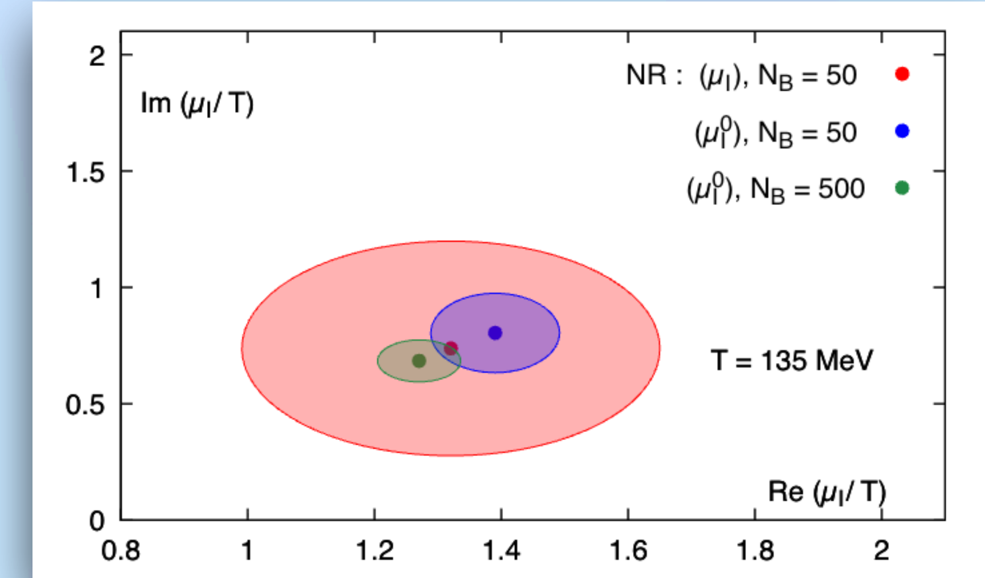
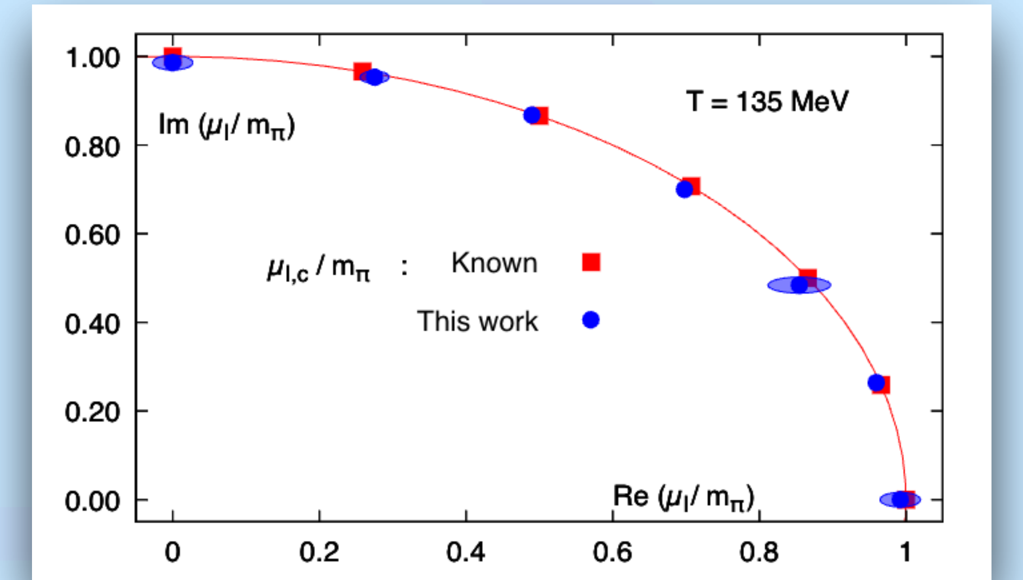
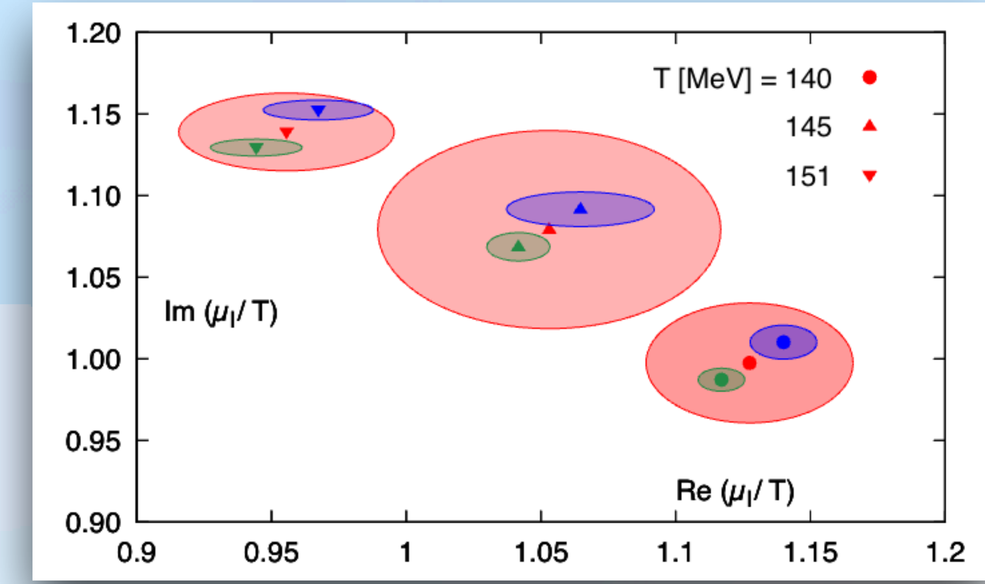
Reliable therefore, at least for this work







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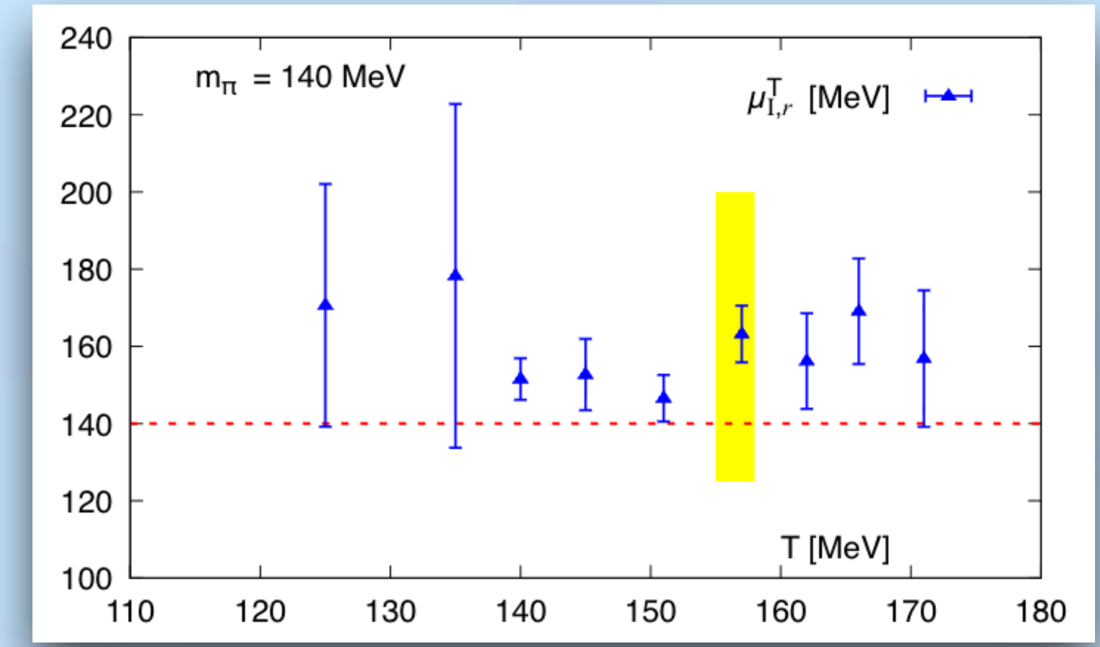
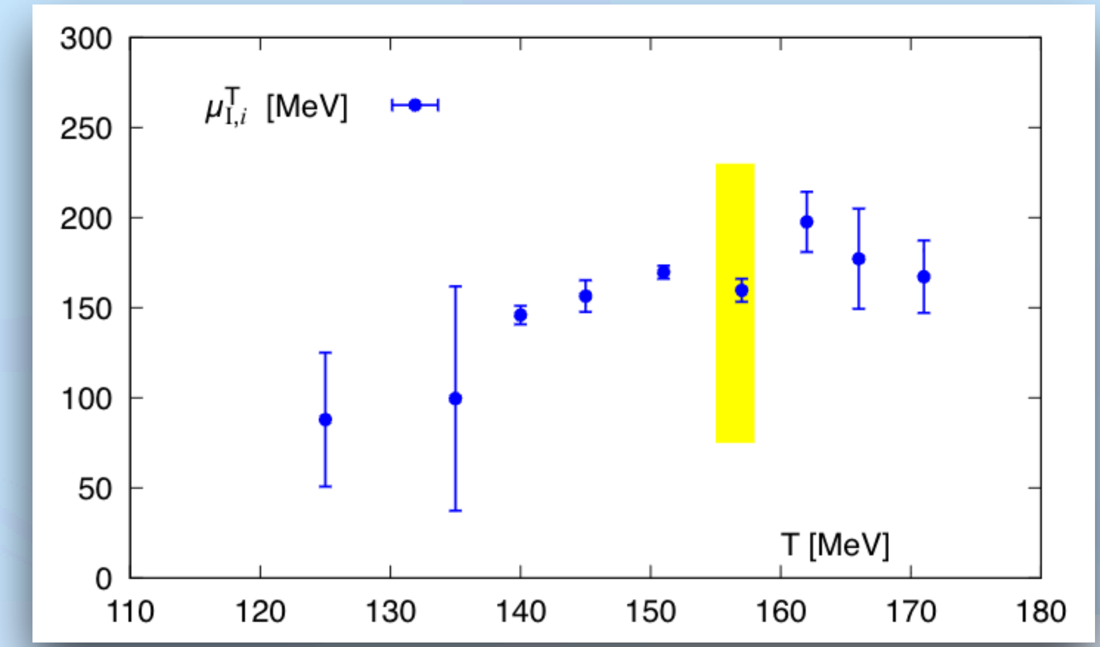
∴ ***Let' s analyse these 0' s***

Real and imaginary parts of μ_I^0





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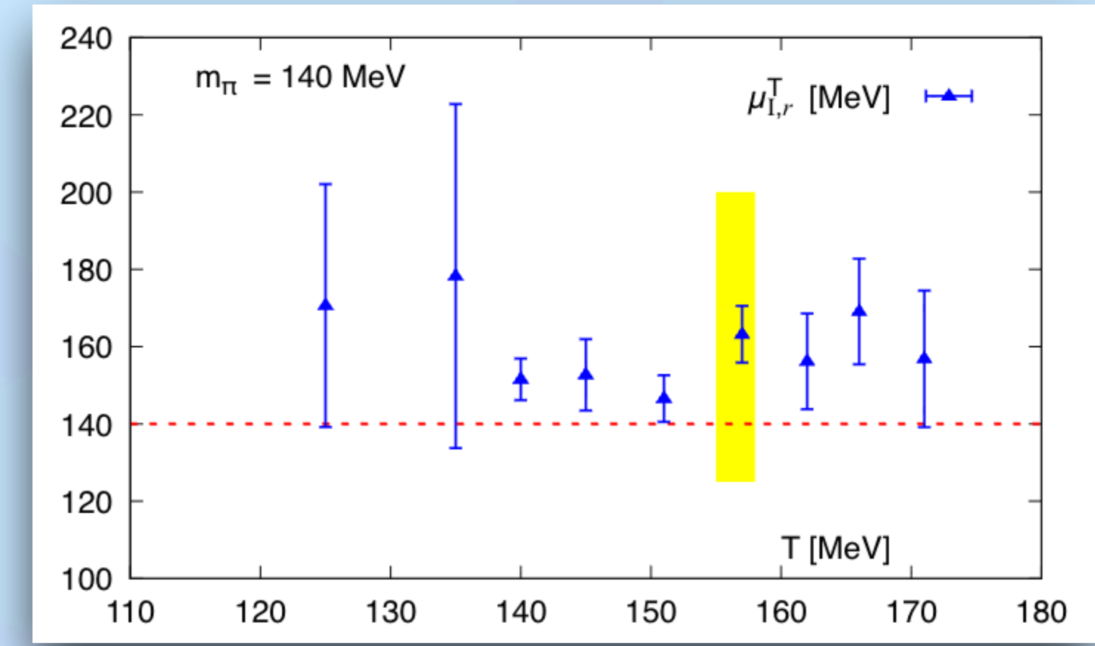
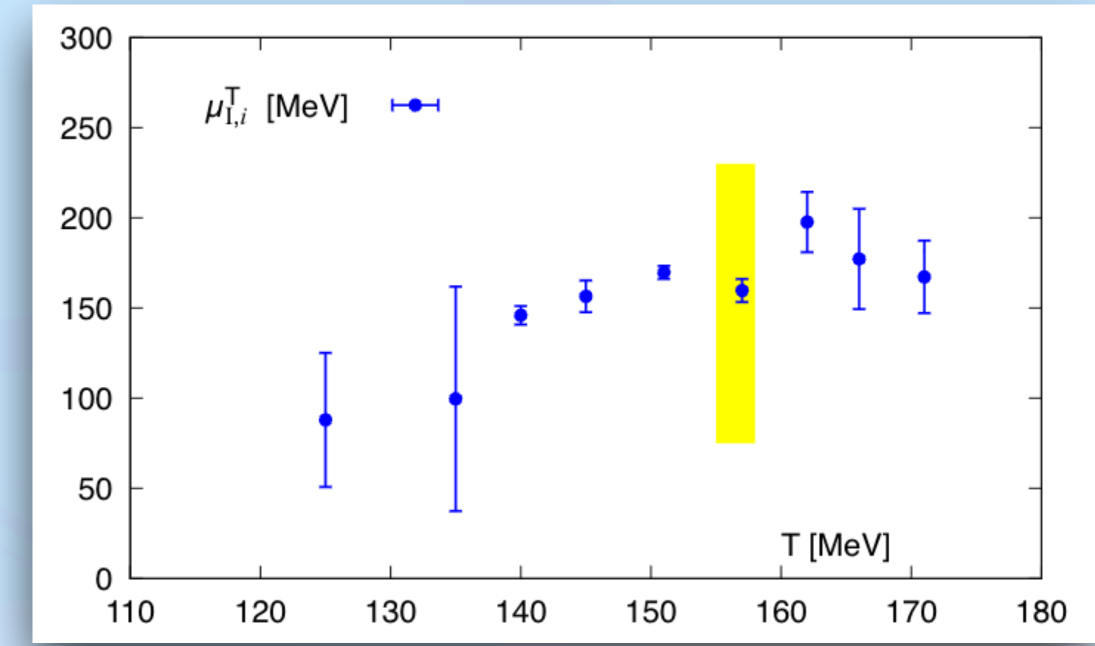
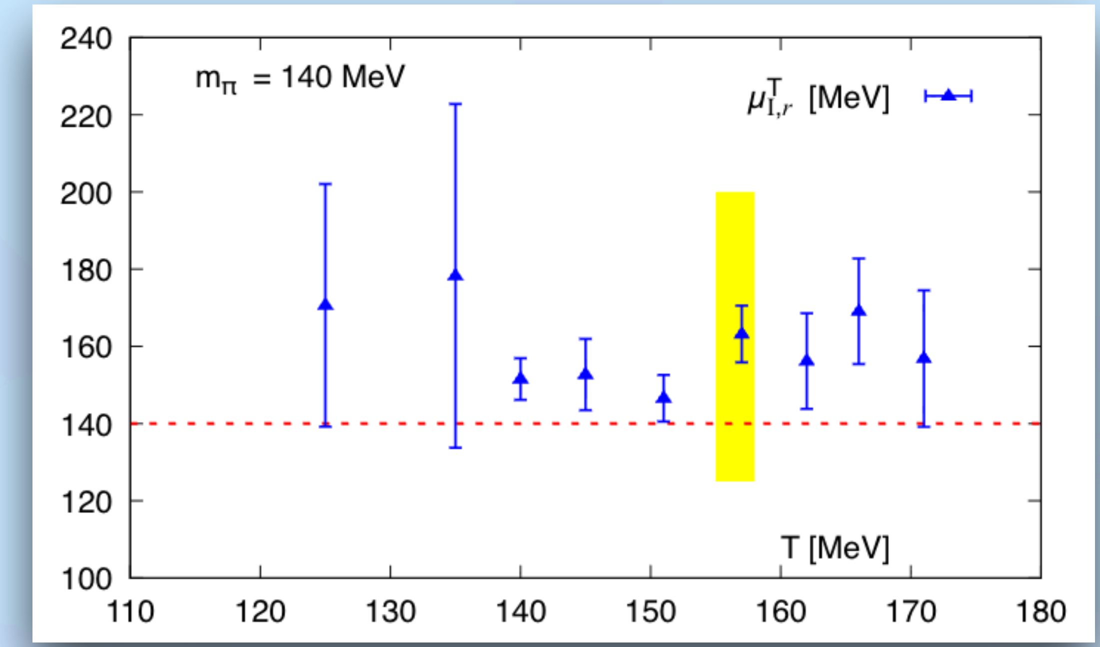
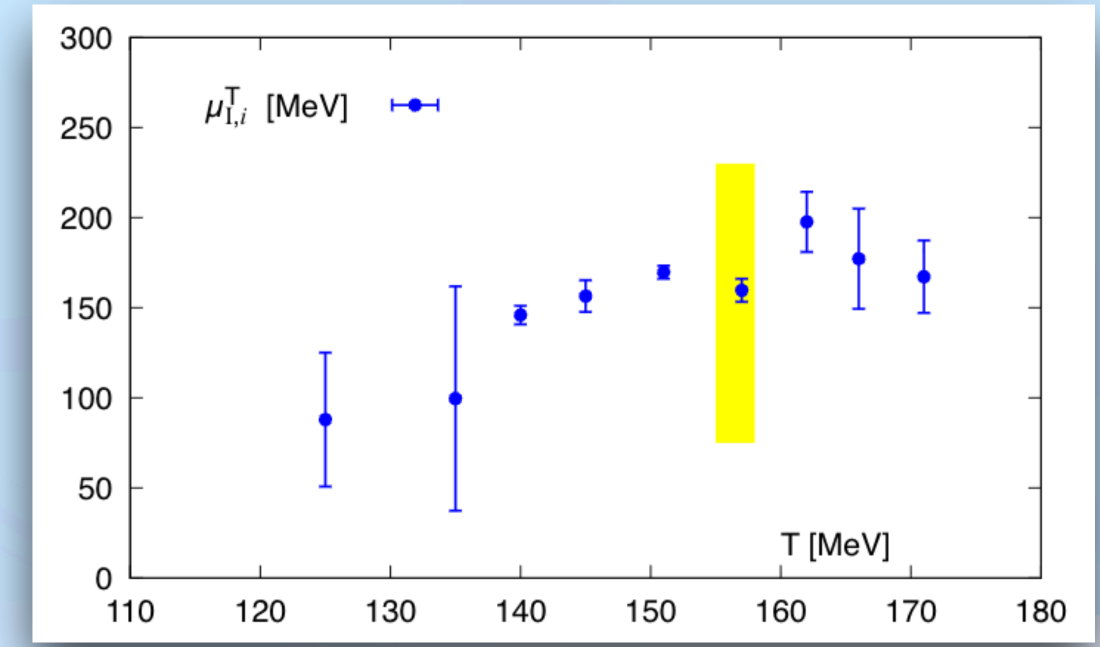




- $\mu^T_{I,r} \sim \text{Re}\left(\mu_I^0\right)$ close to m_π for **lower values of T**

Real and imaginary parts of μ_I^0





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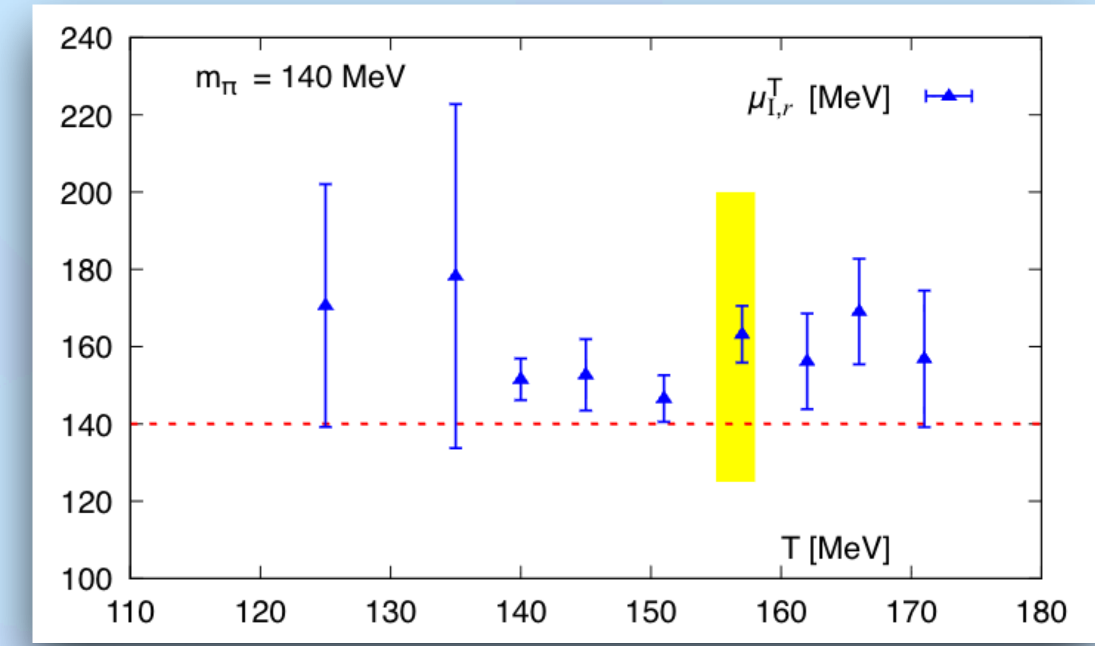
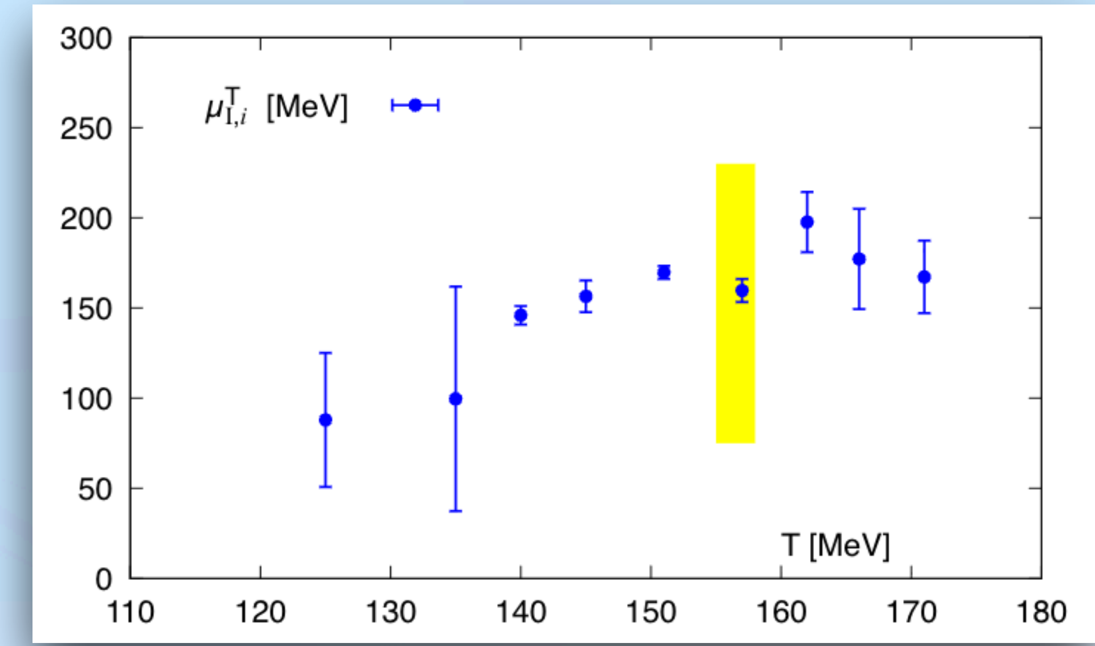
Real and imaginary parts of μ_I^0





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Also, **Increasing errors** for $\mu^T_{I,r}$ and $\mu^T_{I,i}$ for $T < 140$ MeV

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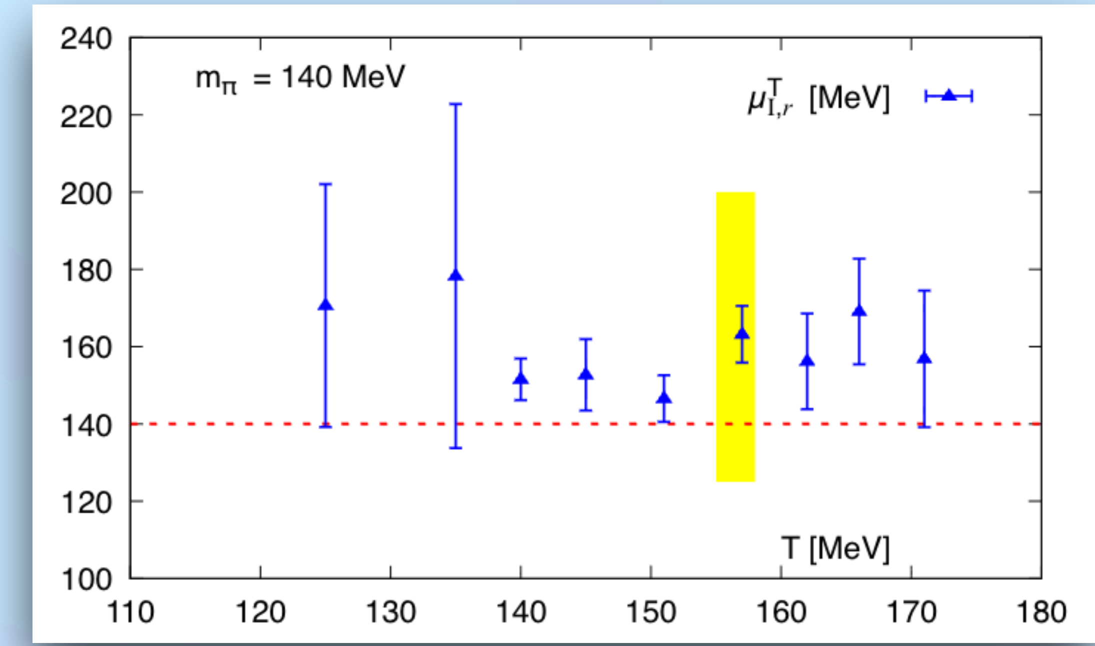

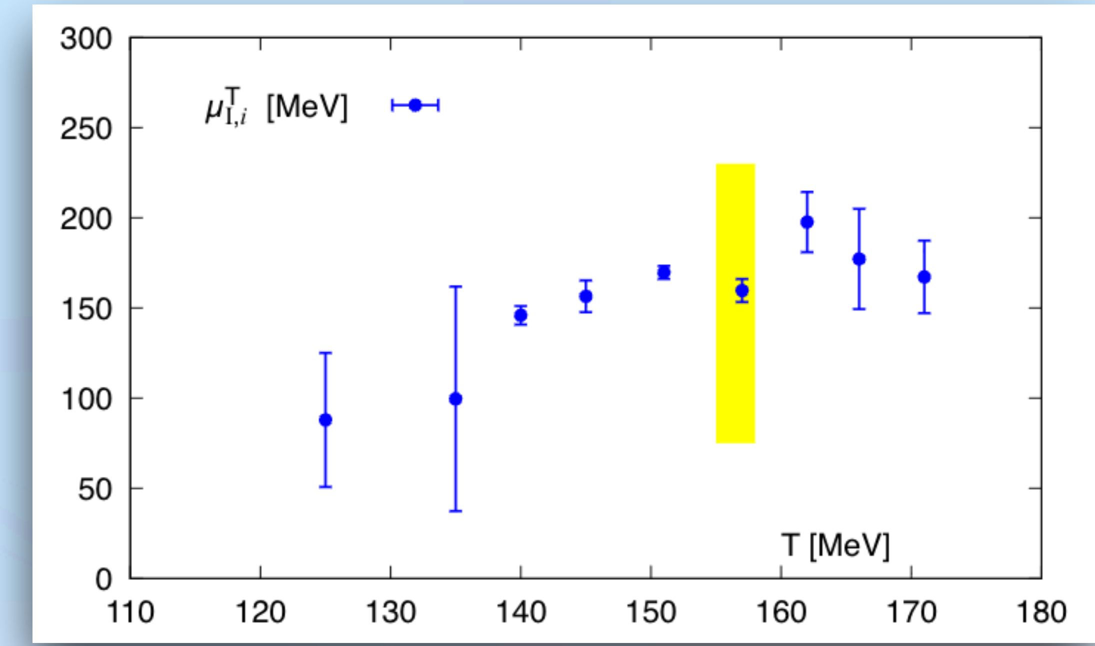

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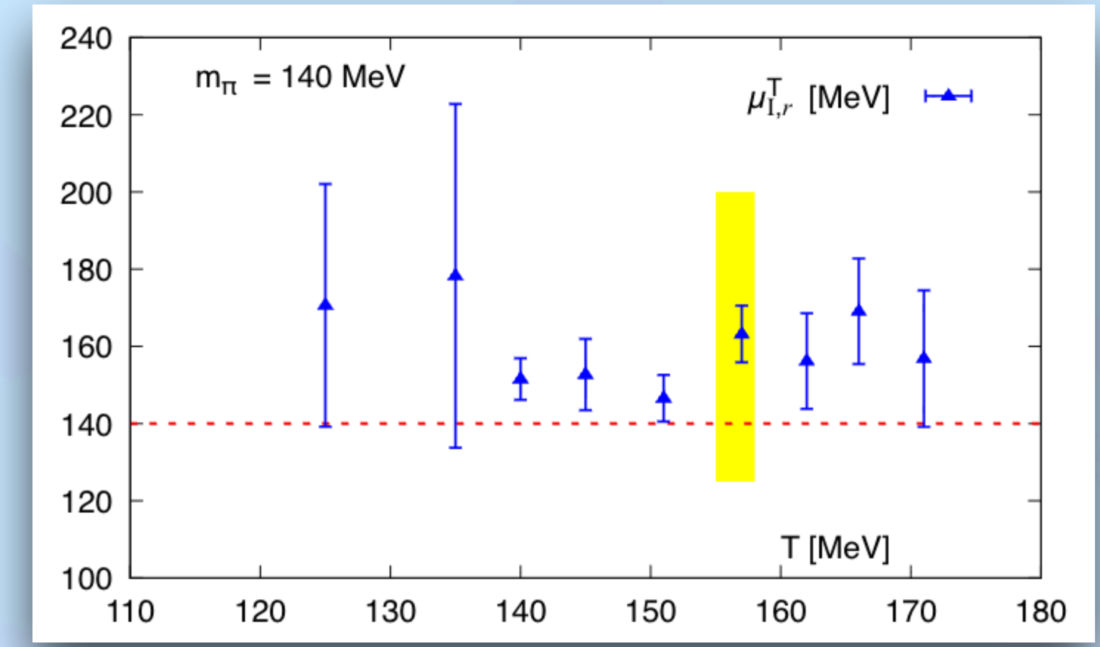

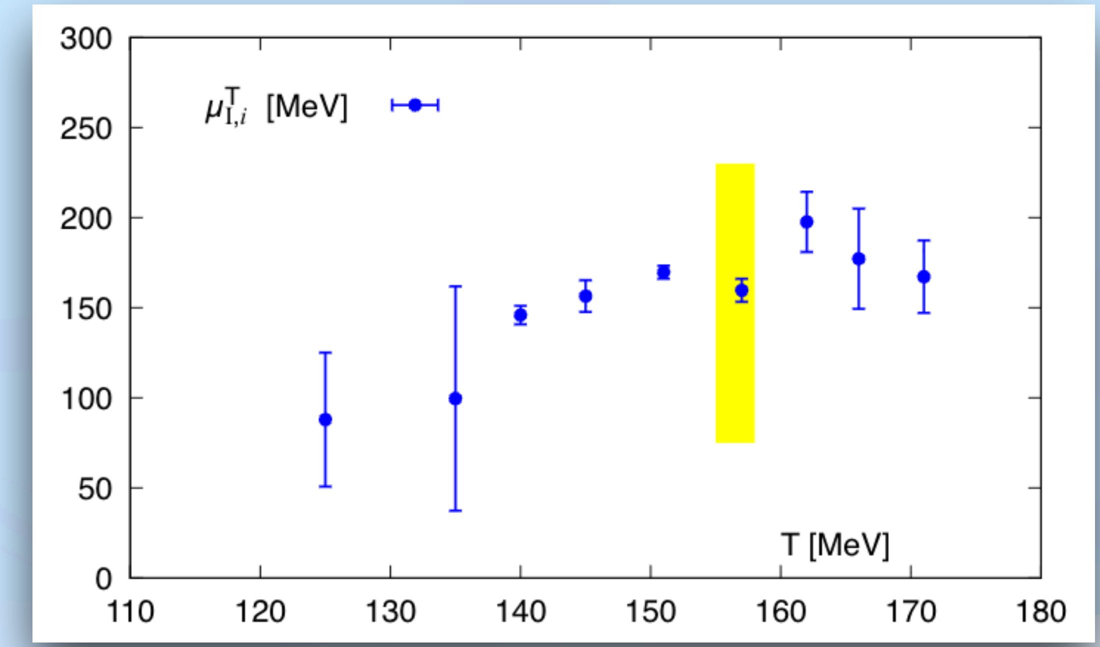

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↓

signs of possible **"vicinity of critical point(s)"**

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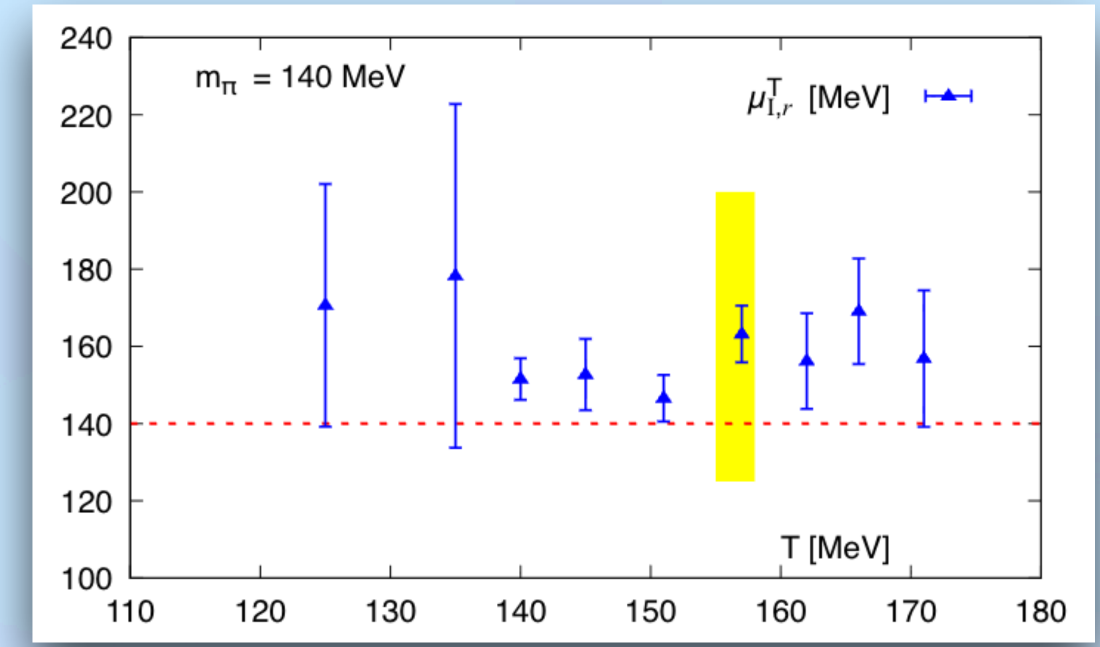

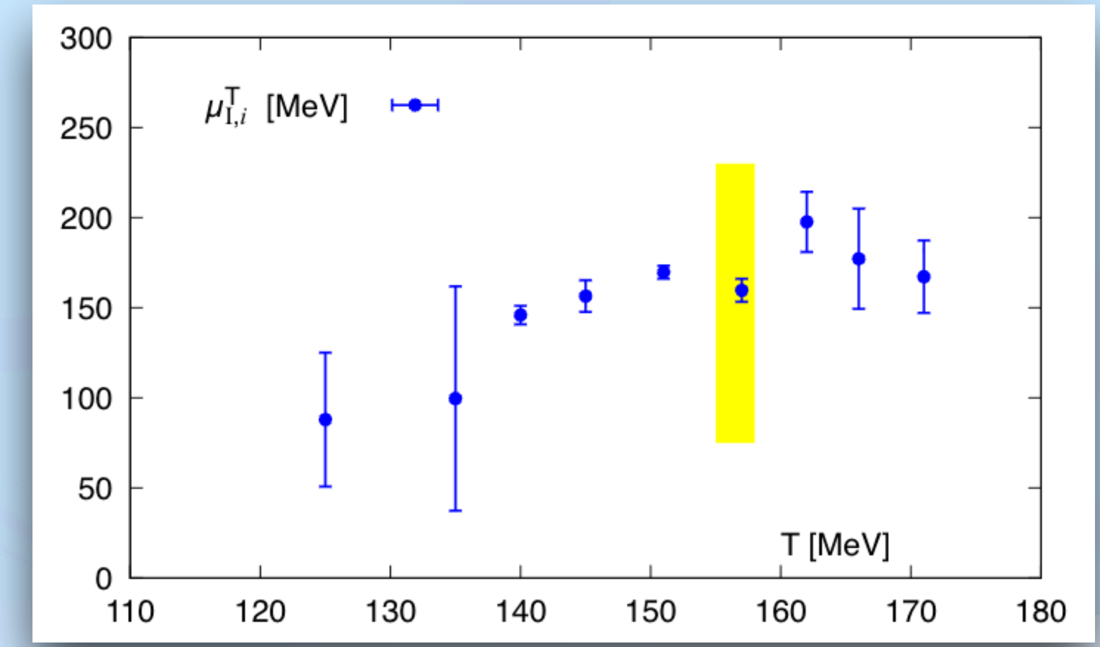

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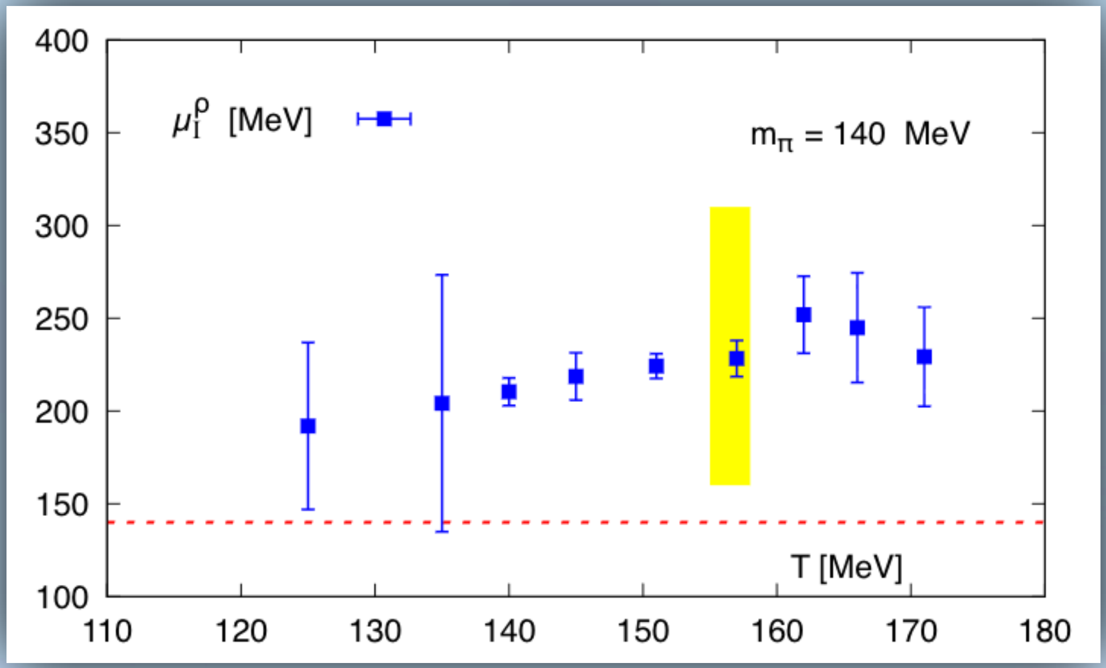
signs of possible **"vicinity of critical point(s)"**

What about the coveted RoC ???

Radius of convergence (RoC) :



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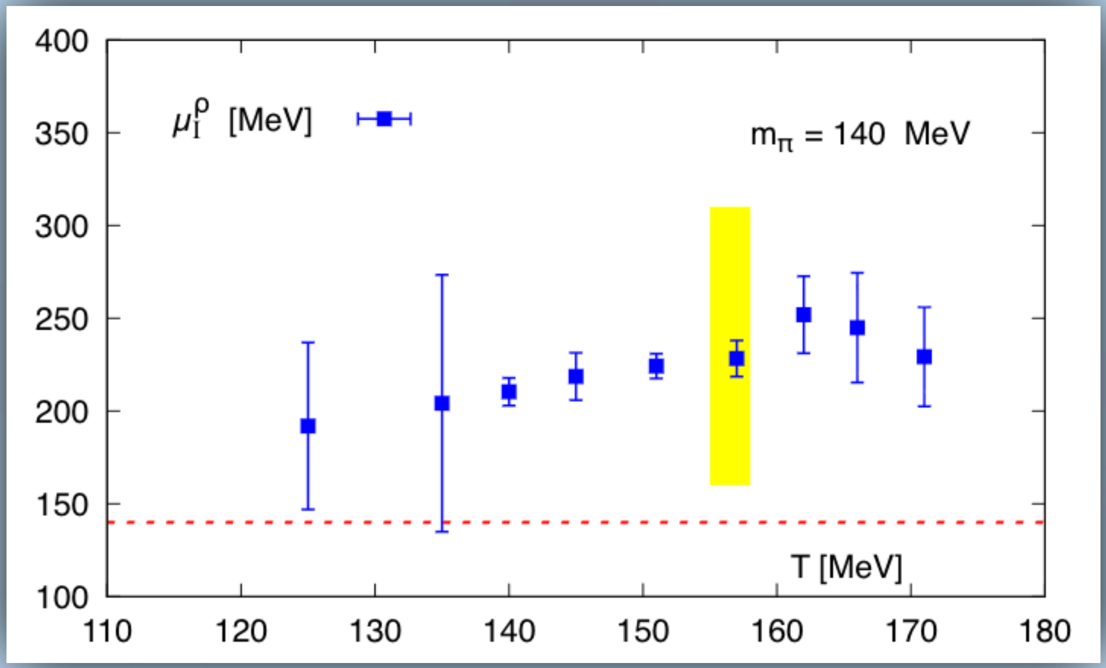

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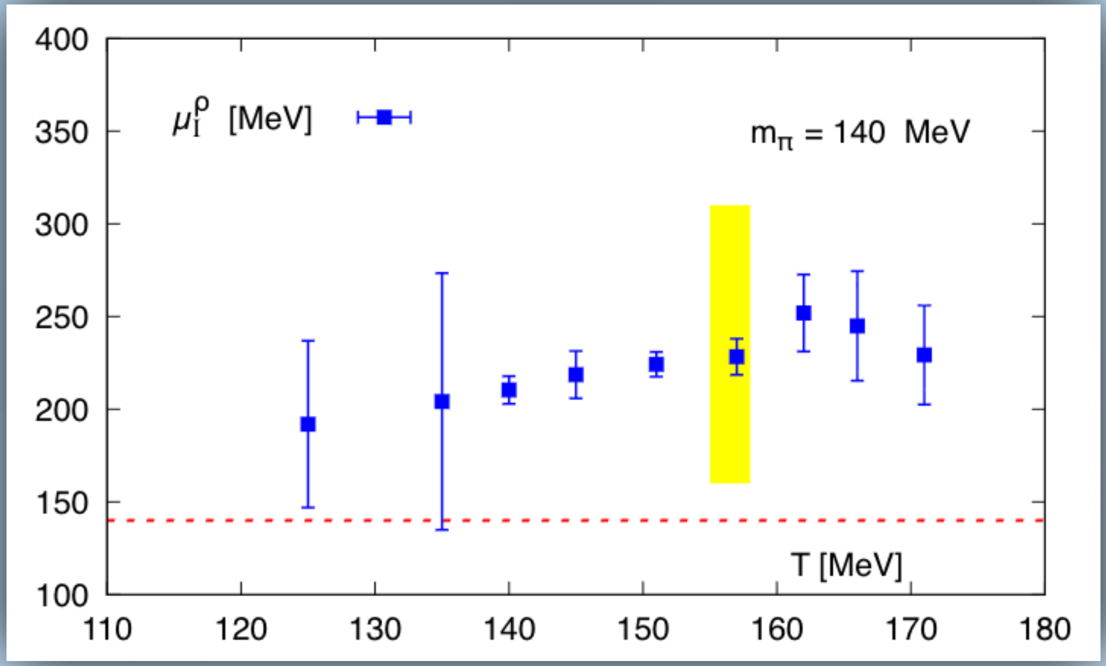

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Dimensional (in MeV units)

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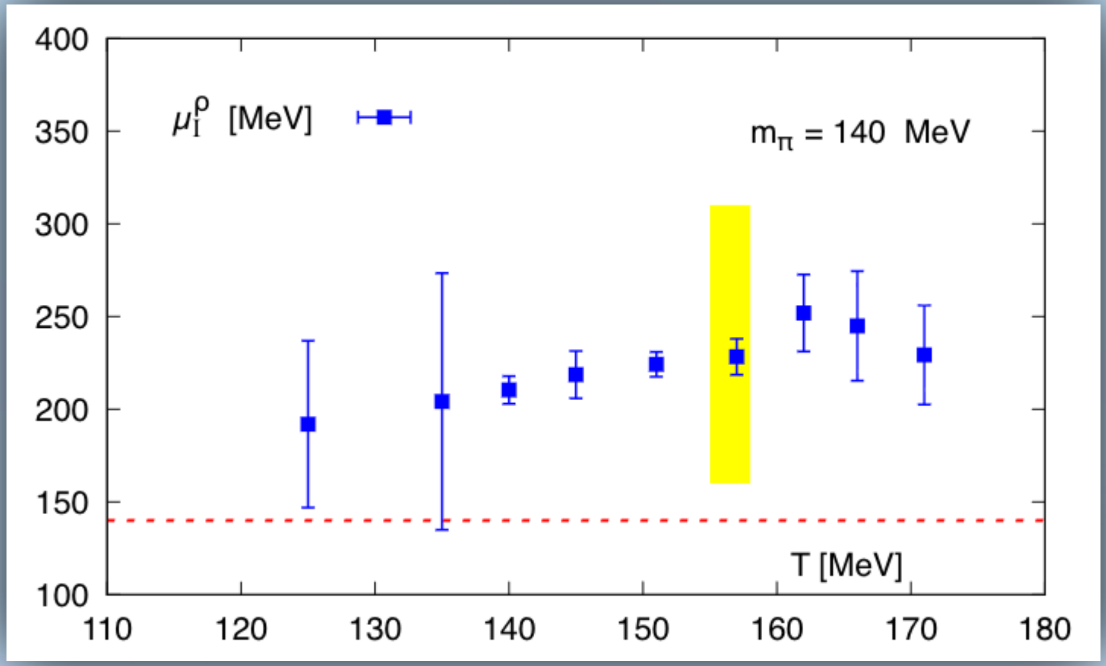

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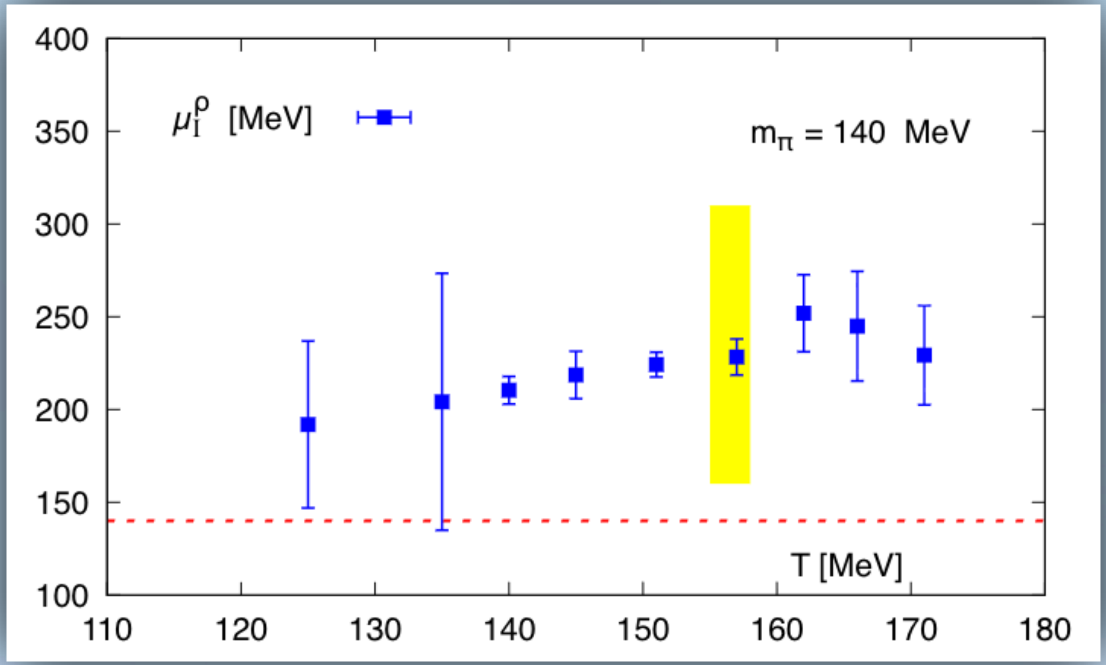

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Dimensional (in MeV units)

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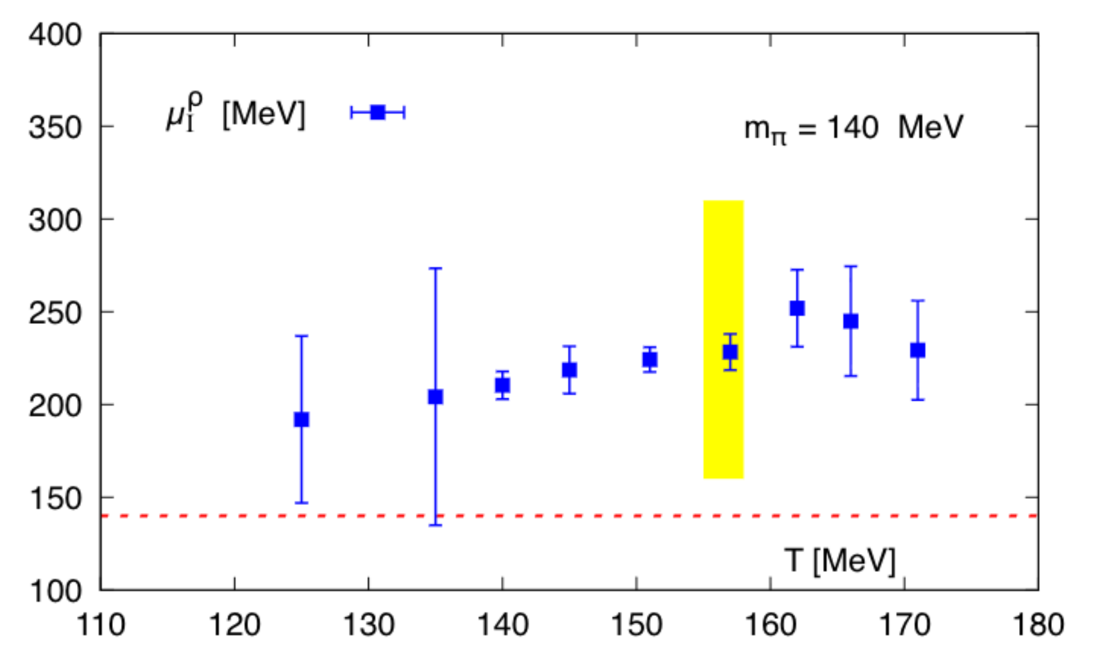

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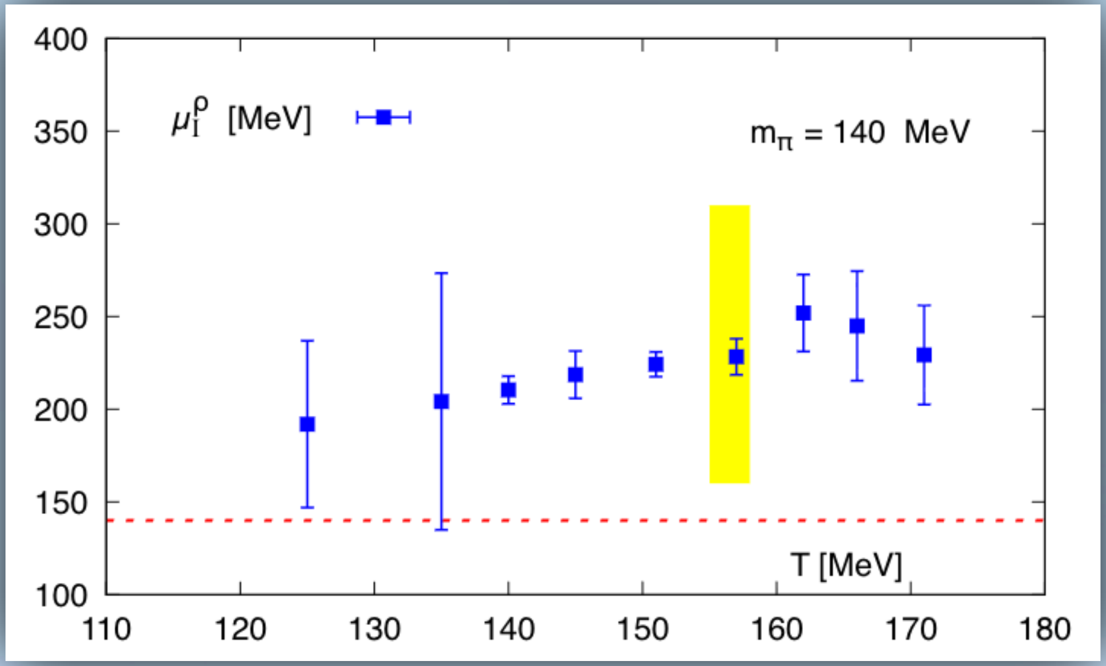

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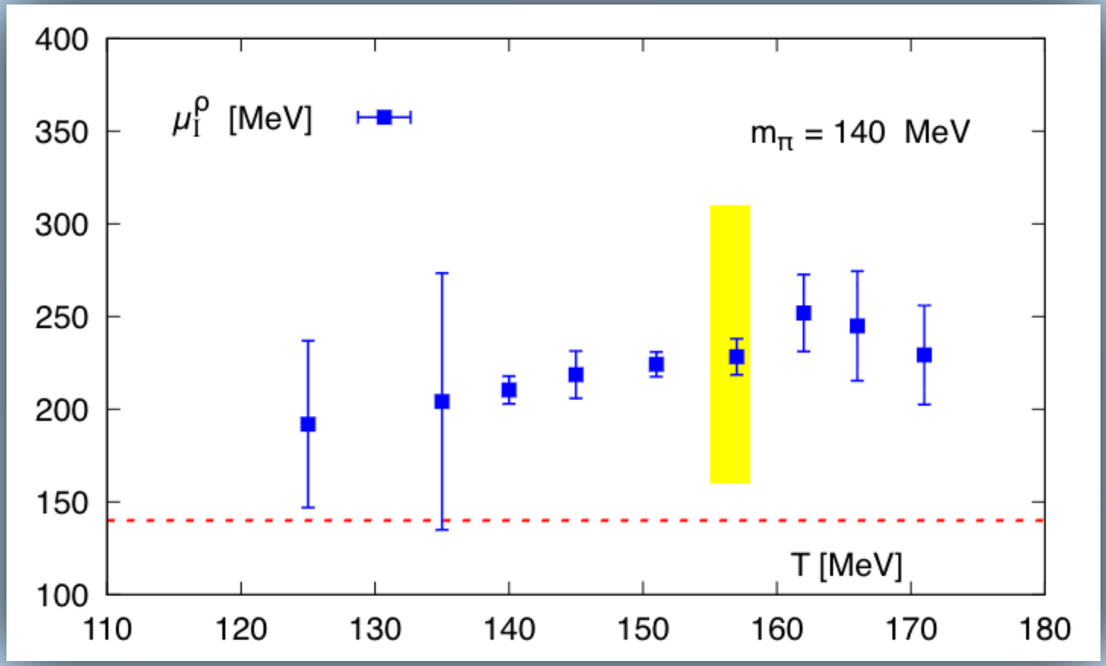

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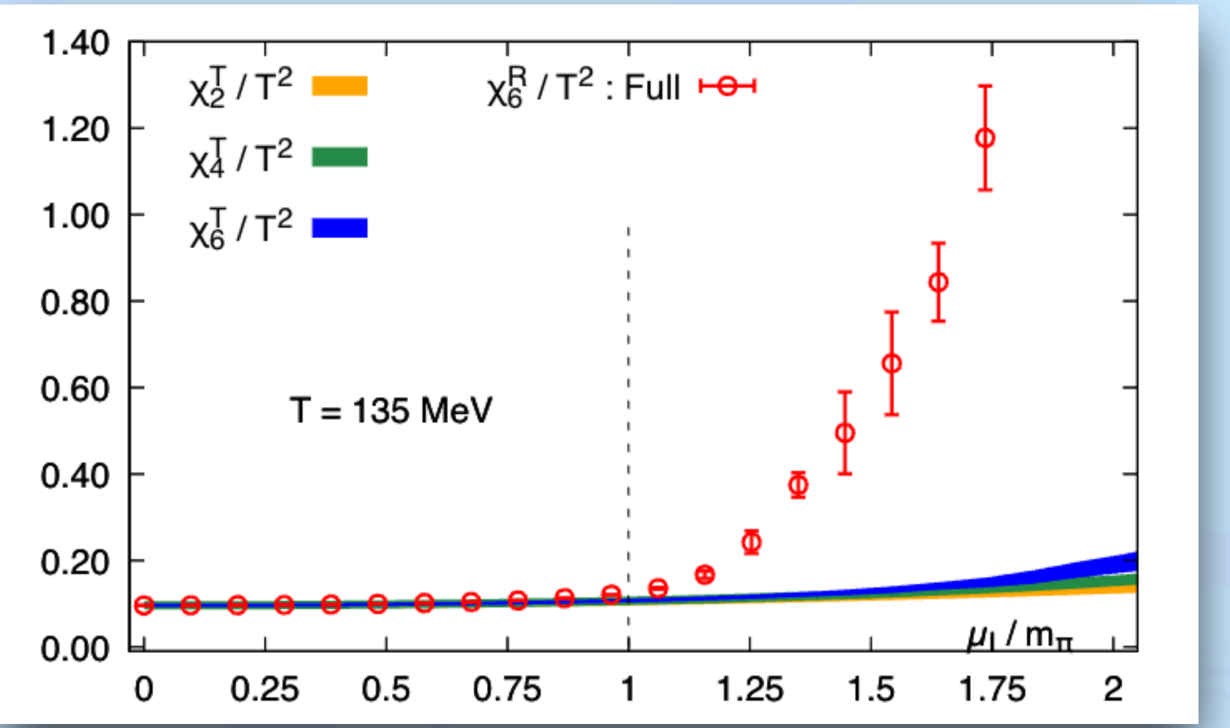
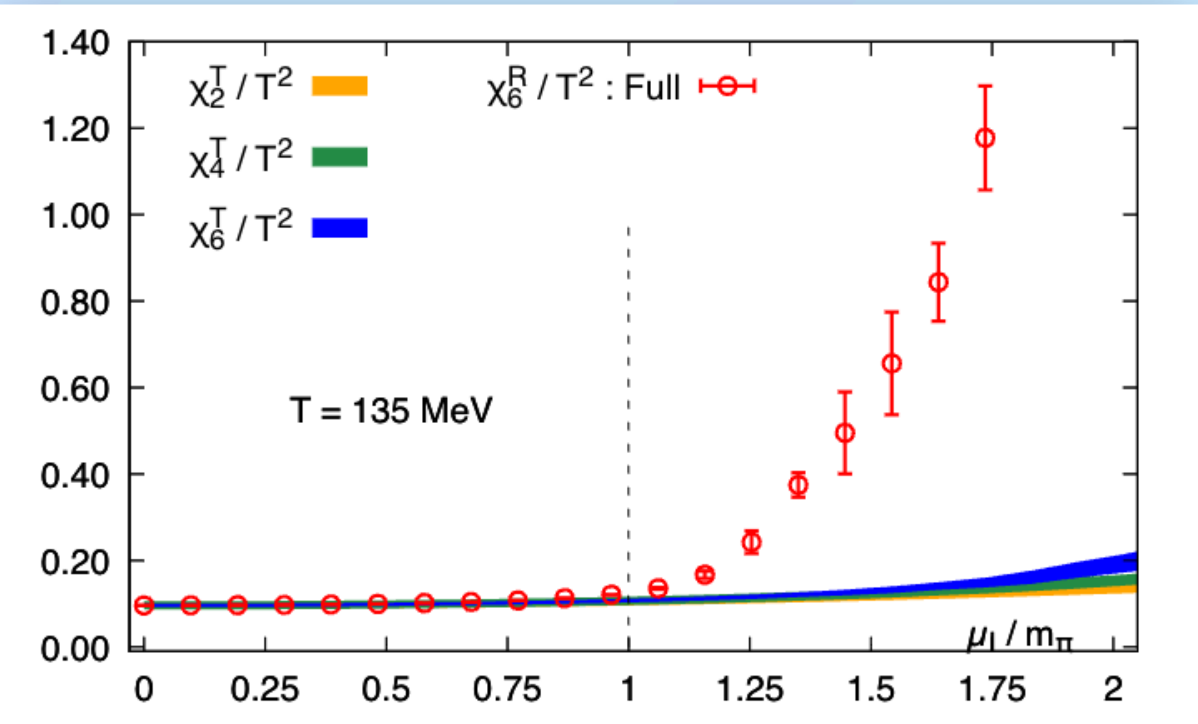
Can it indicate possible critical point / line???



1.40
1.20
1.00
0.80
0.60
0.40
0.20
0.00
0
0.25
0.5
0.75
1
1.25
1.5
1.75
2
χ_2^T / T^2
χ_4^T / T^2
χ_6^T / T^2
χ_6^R / T^2 : Full
T = 135 MeV
μ_I / m_π



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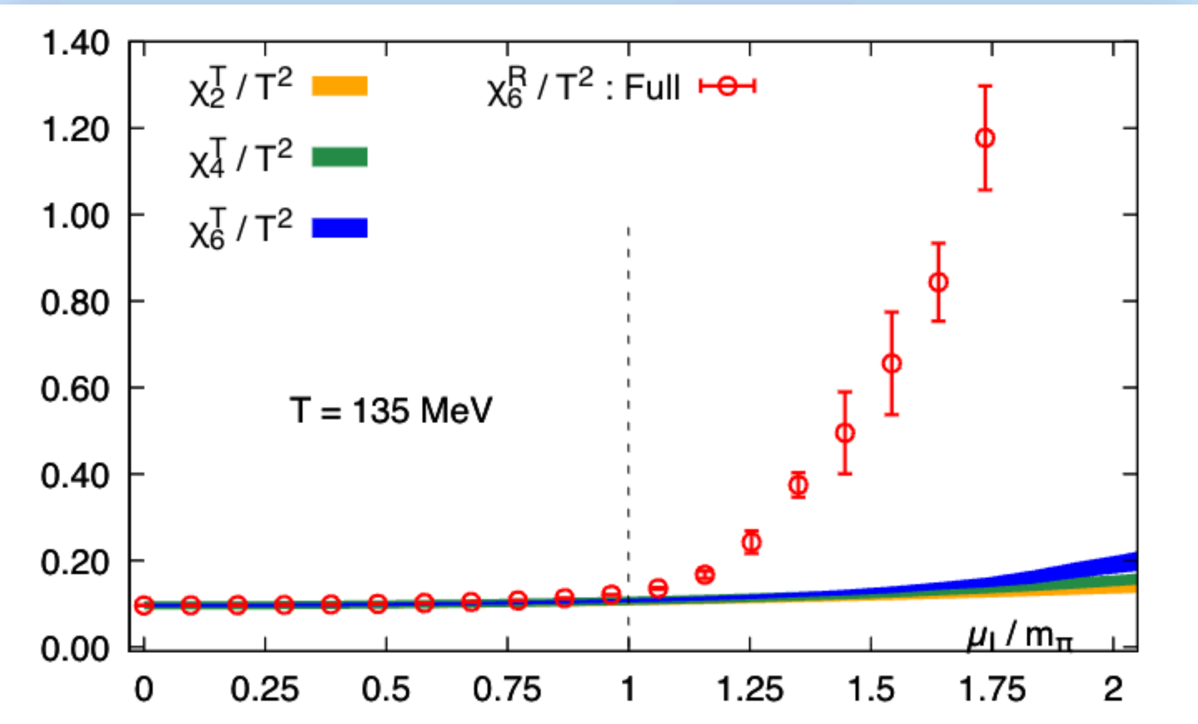

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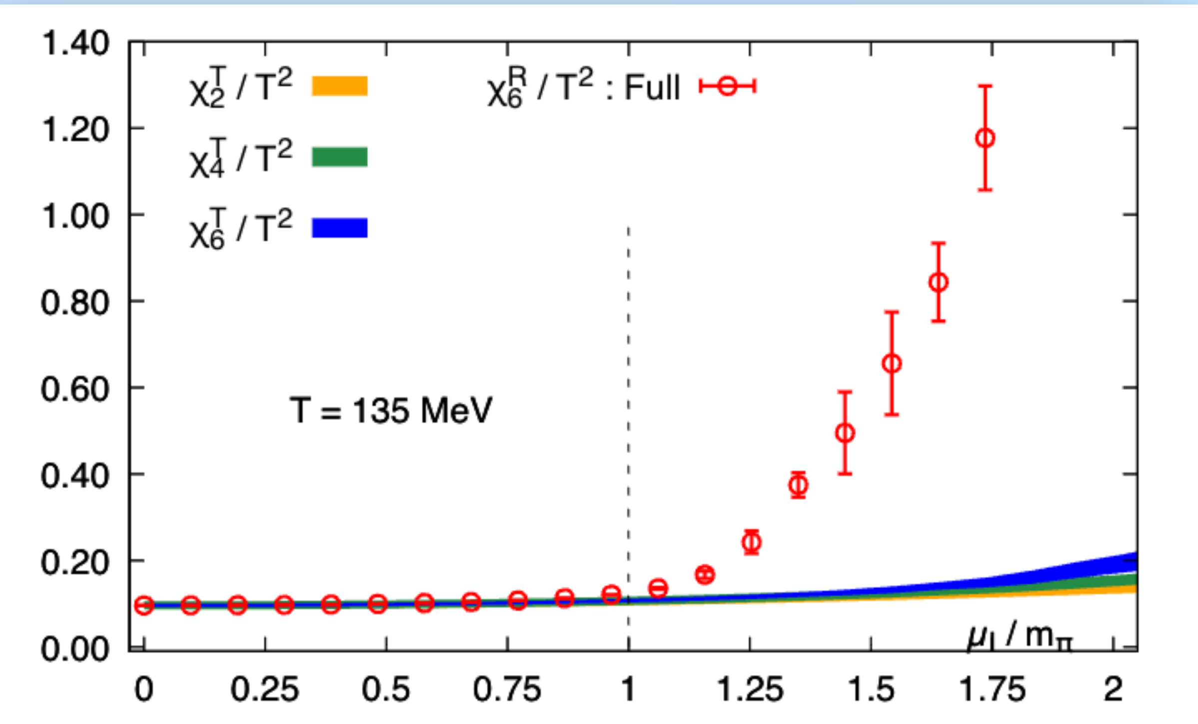

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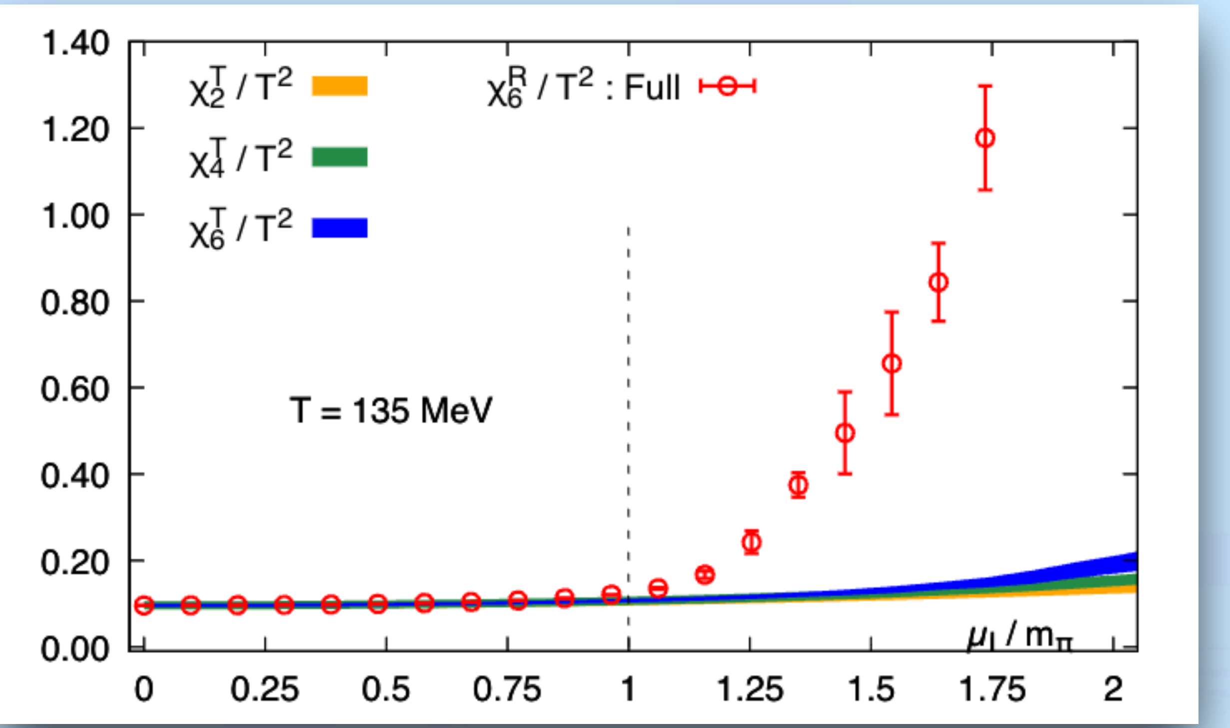

Manifesting **divergence**

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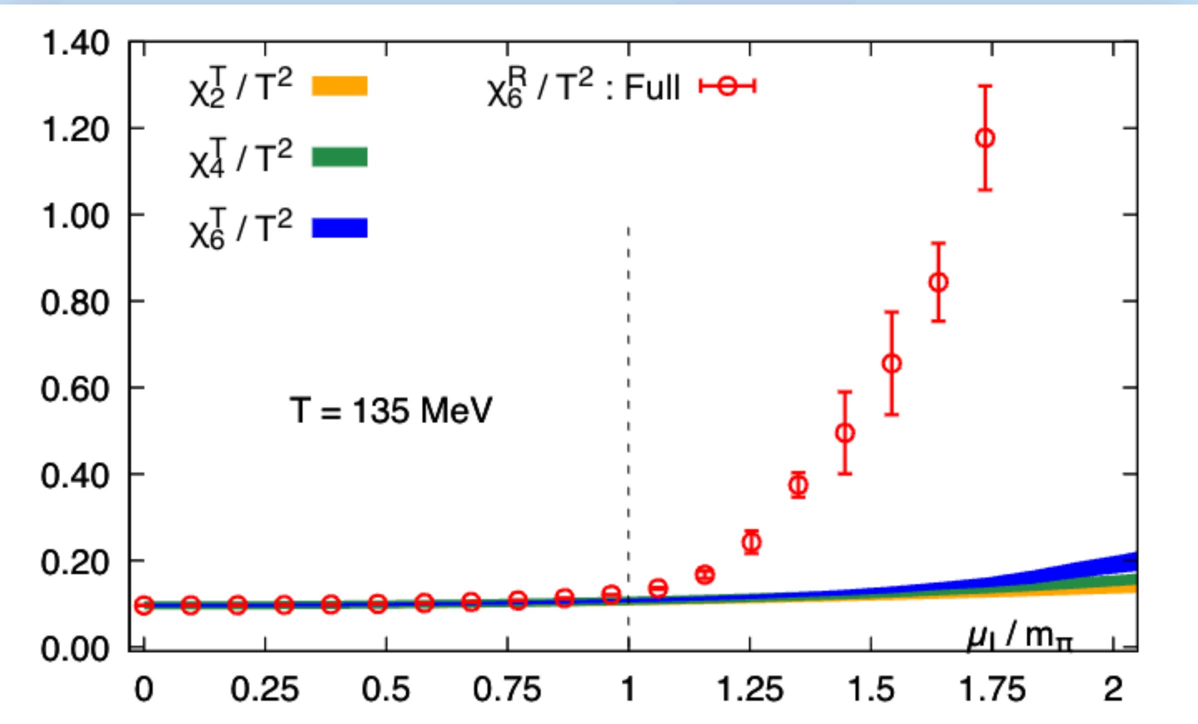

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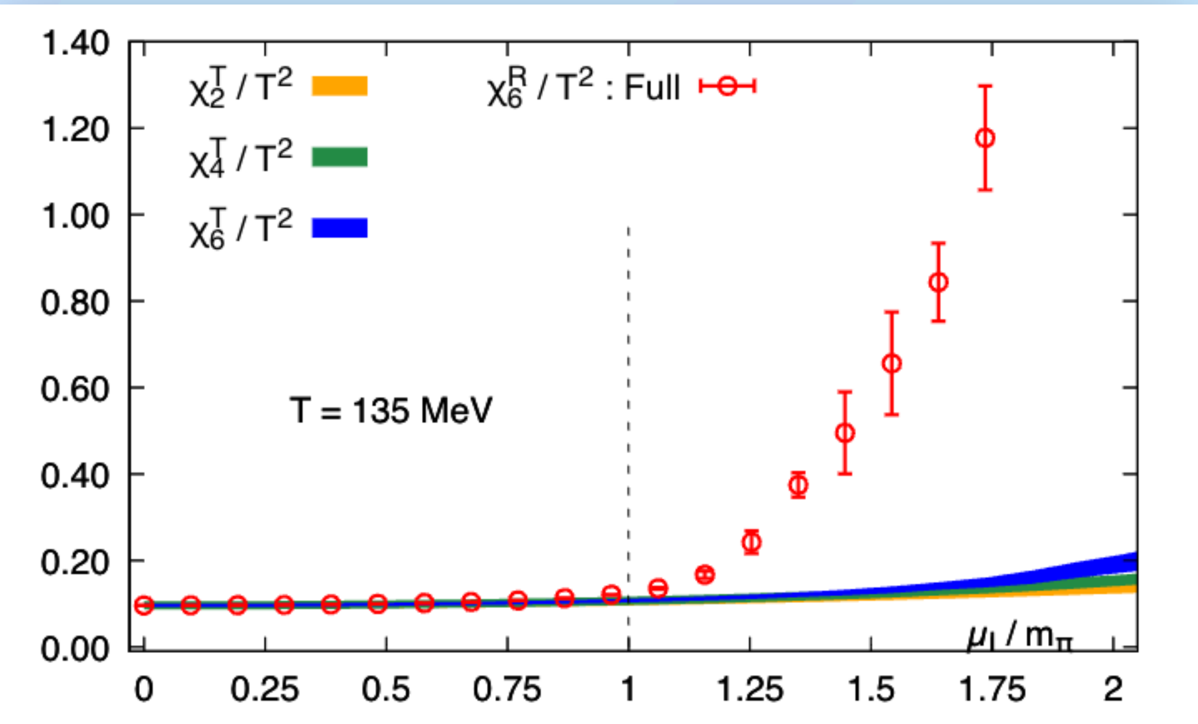
Manifesting **divergence**

Possible phase transition signatures



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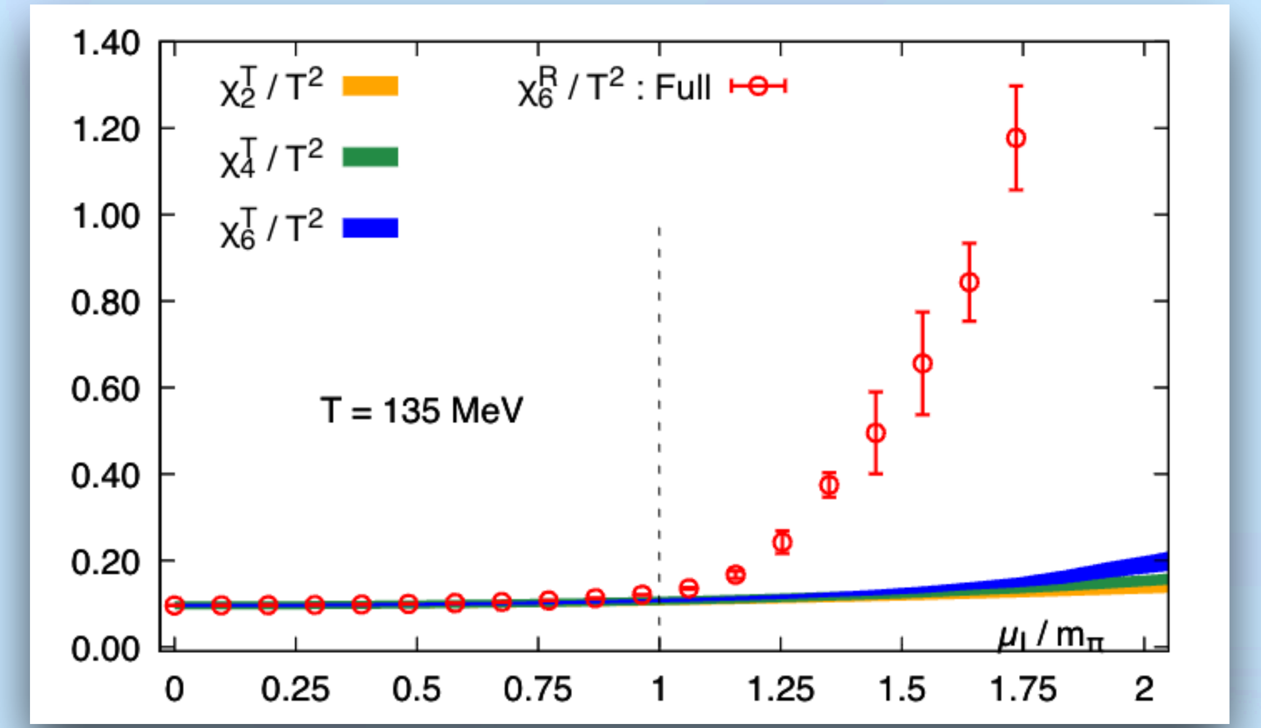

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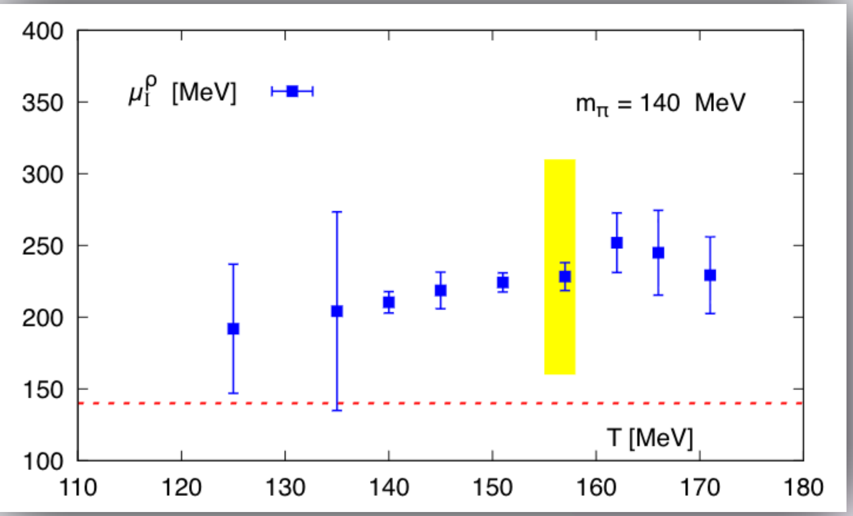

Manifesting **divergence**

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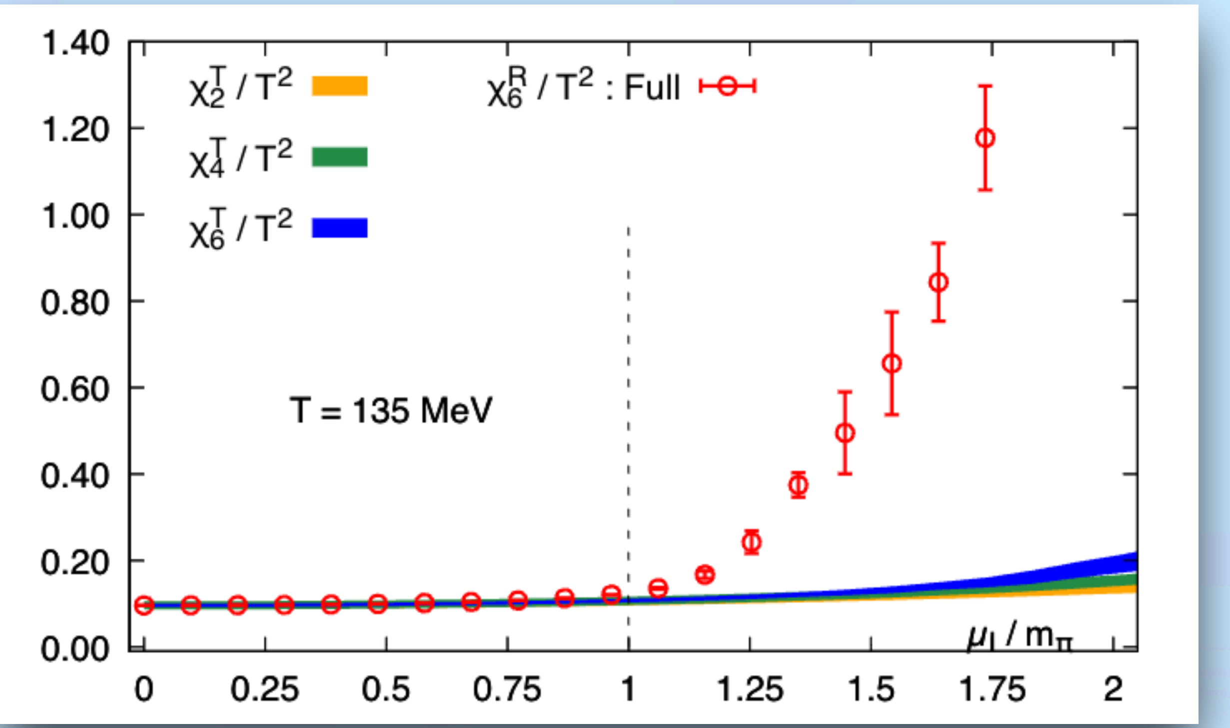
RoC : μ_I^ρ $\longrightarrow$ **good indicator** of π-boundary , at least in low T



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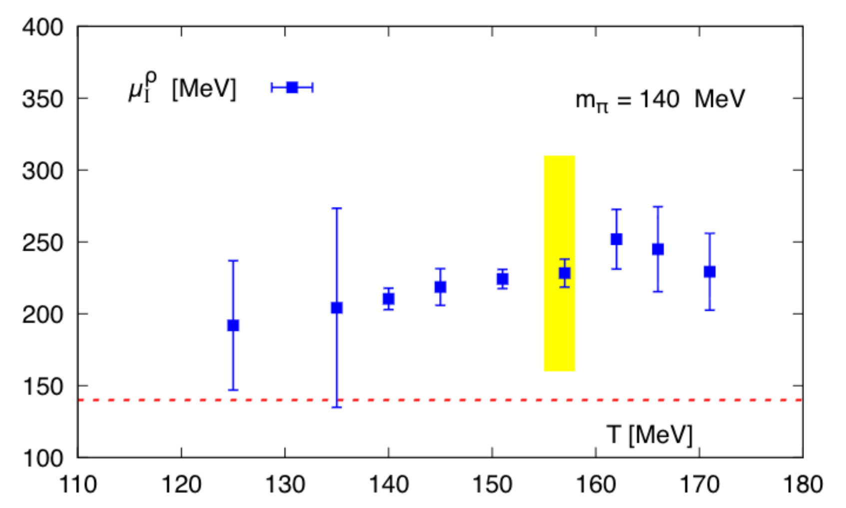

Manifesting **divergence**

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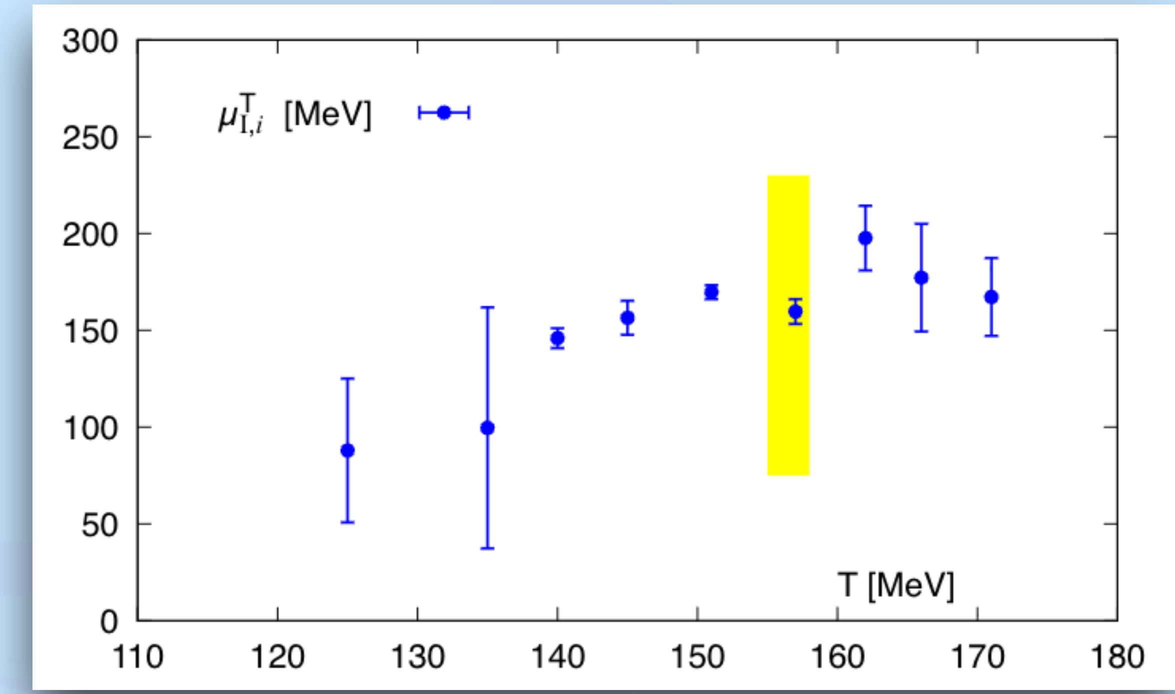
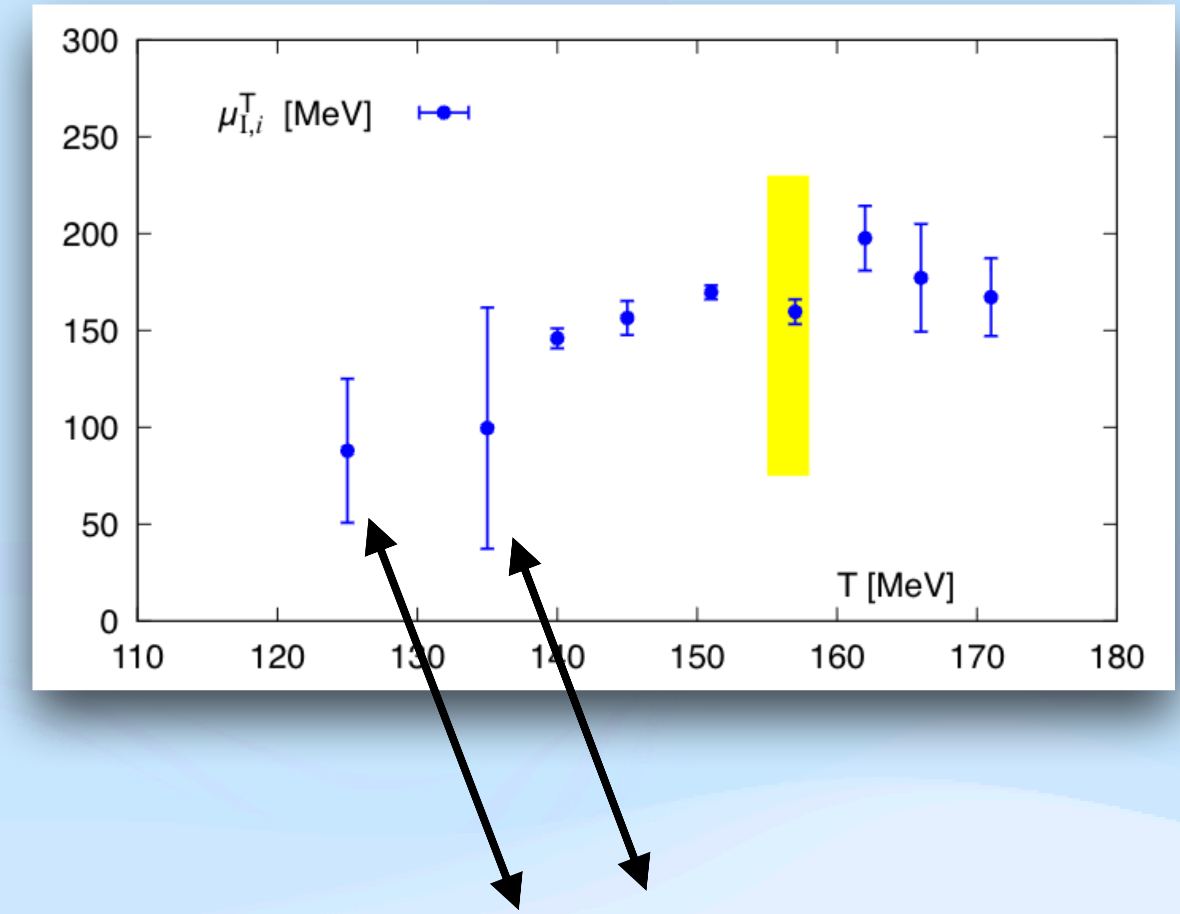
RoC : $\mu_I^\rho \longrightarrow$ **good indicator** of π-boundary , <u>at least in low T</u>

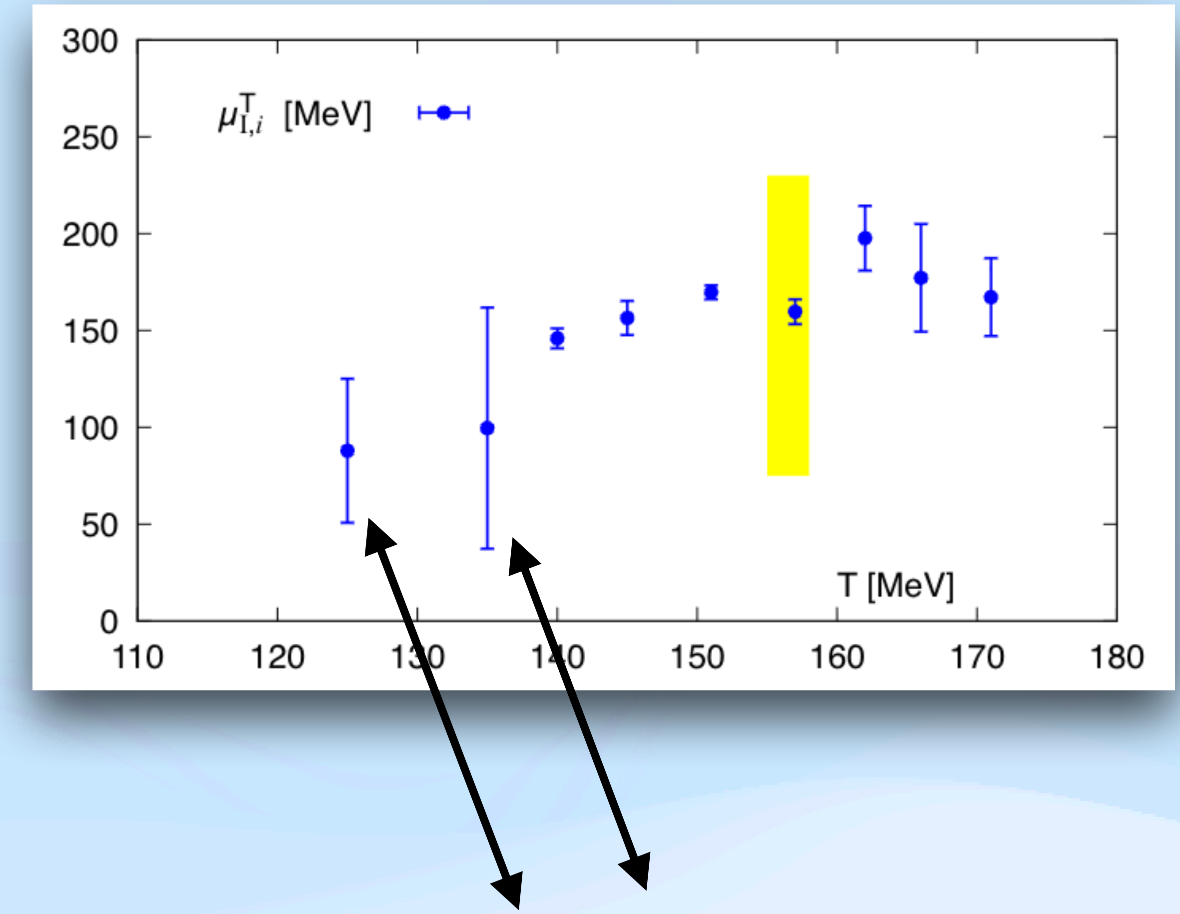
However



300
250
200
150
100
50
0
$\mu^{T}_{I,i}$ [MeV]
T [MeV]
110
120
130
140
150
160
170
180

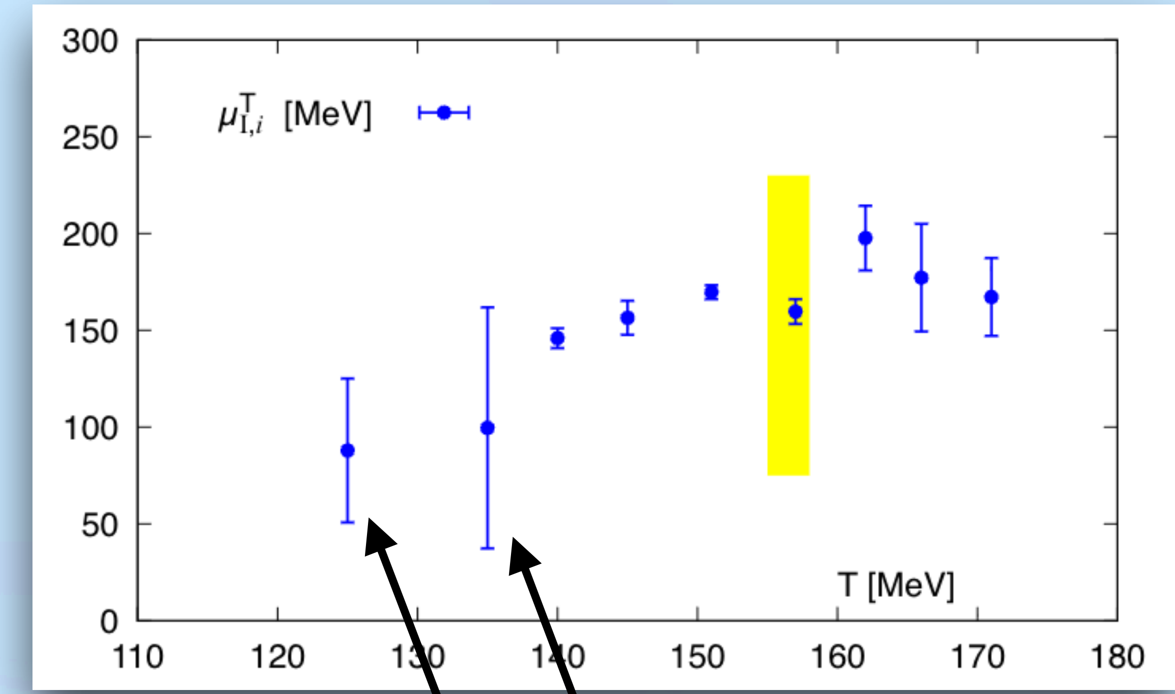


$\mu^{T}_{I,i}$ [MeV]
T [MeV]
300
250
200
150
100
50
0
110
120
130
140
150
160
170
180



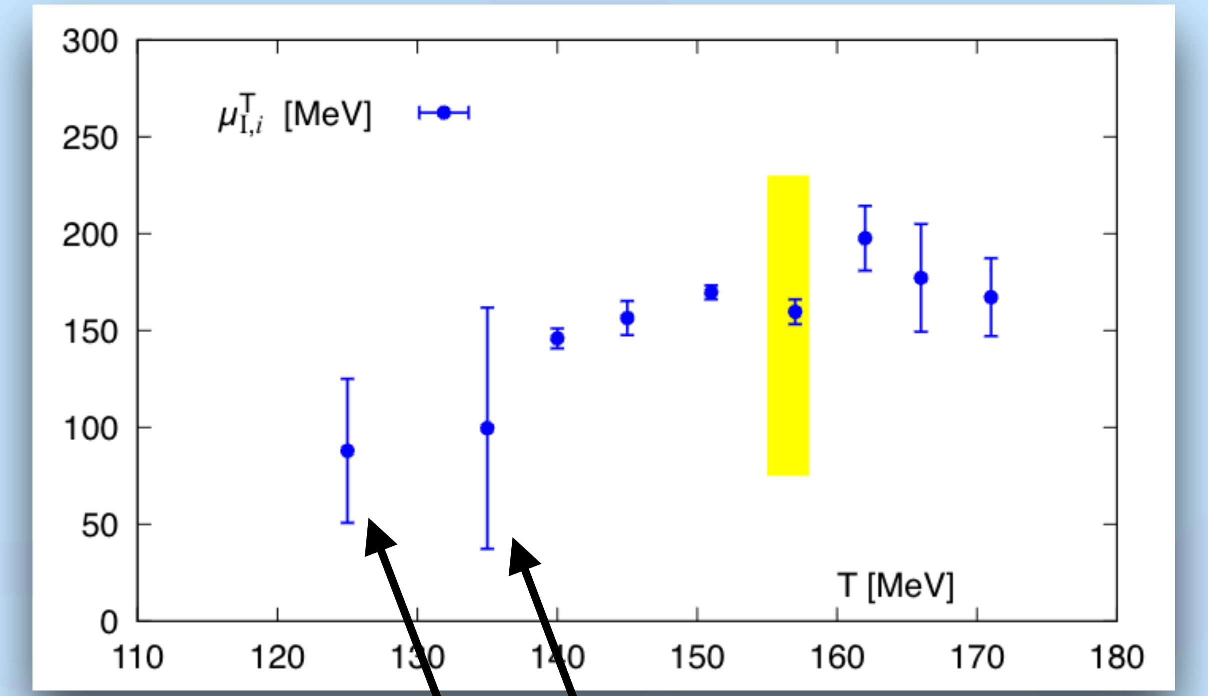
300
250
200
150
100
50
0
110
120
130
140
150
160
170
180
$\mu^{T}_{I,i}$ [MeV]
T [MeV]

$\mu^{T}_{I,i} \neq 0$



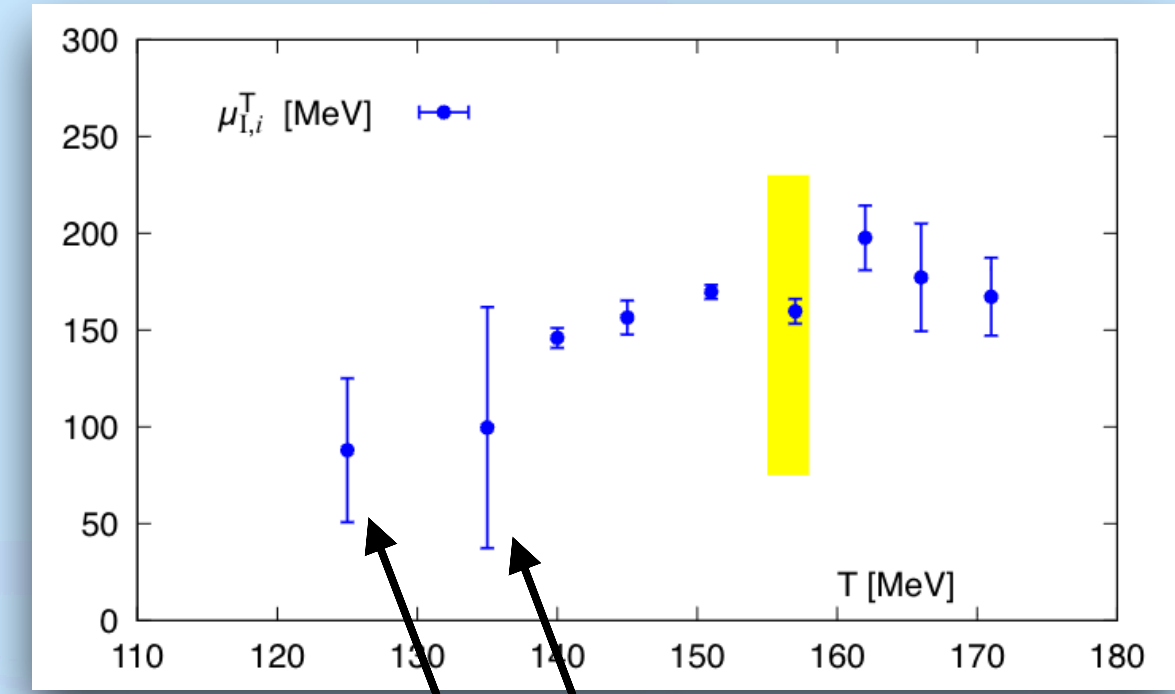
$\mu^{T}_{I,i} \neq 0$

for $T \leq 135$ MeV



$\mu_{I,i}^{T} \neq 0$ for $T \leq 135$ MeV

Non-zero **imaginary** part



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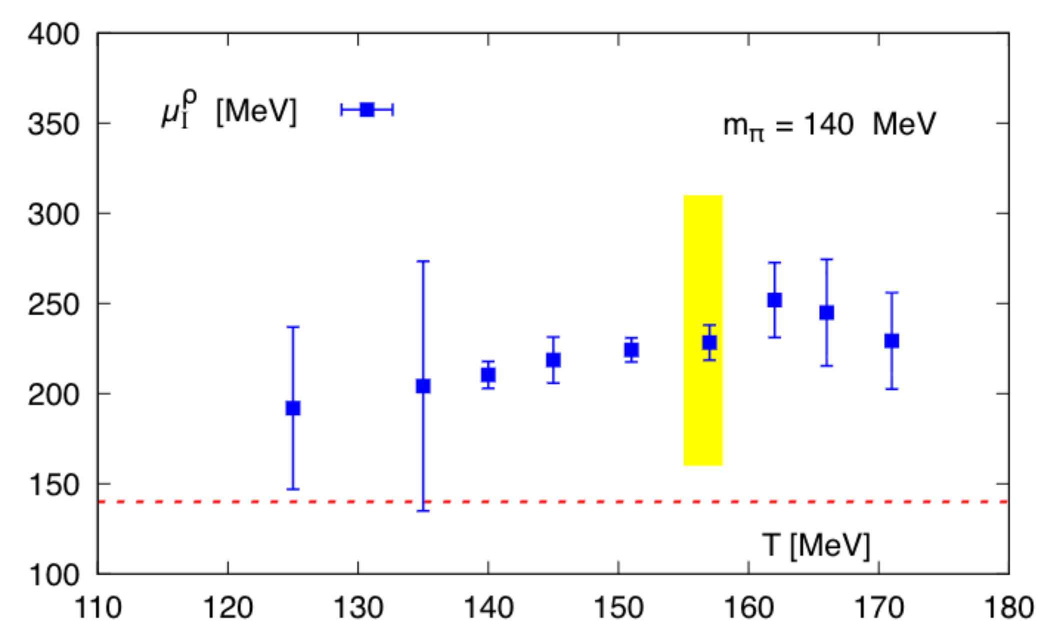
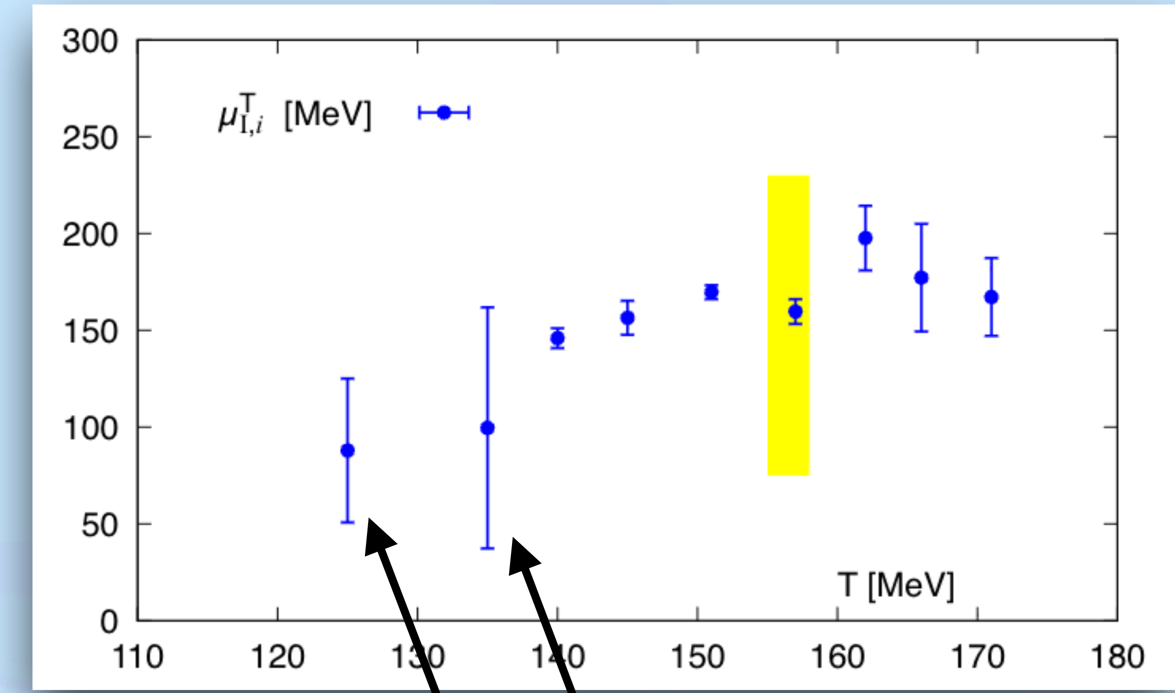
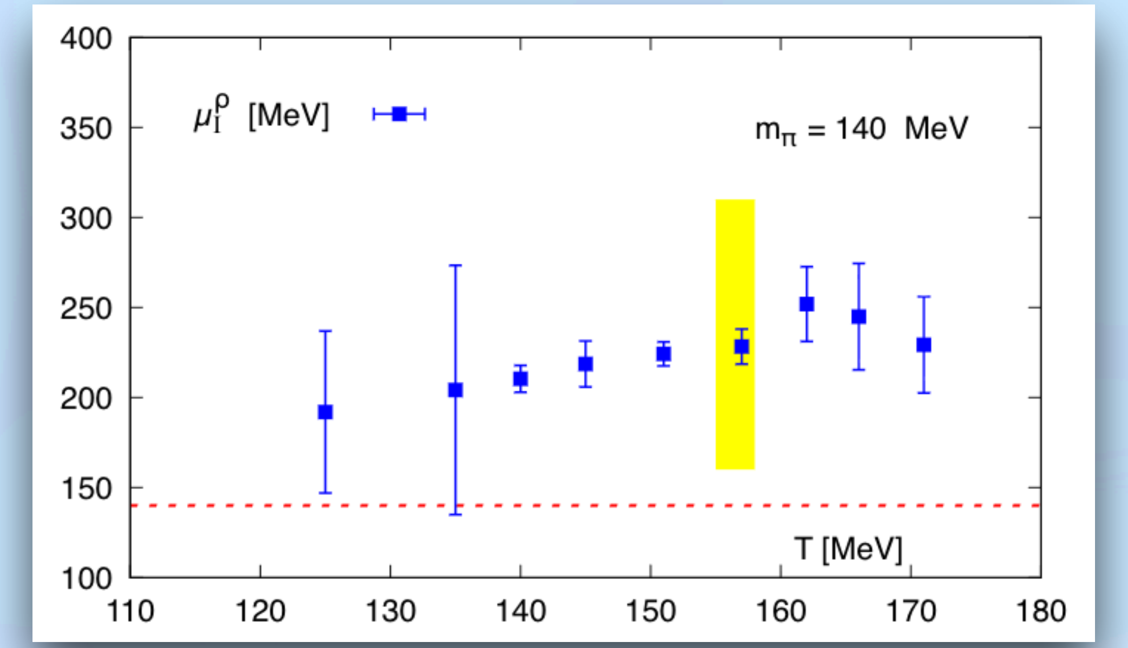


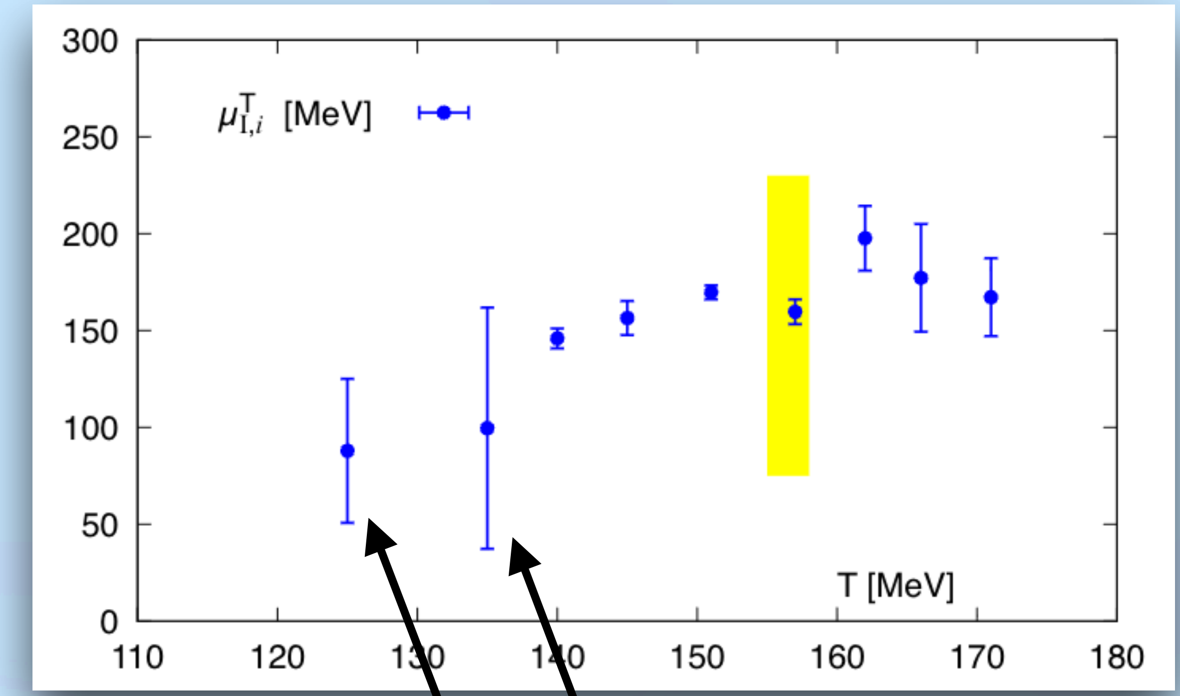


$\mu_{I,i}^T \neq 0$

for $T \leq 135$ MeV

Non-zero **imaginary** part





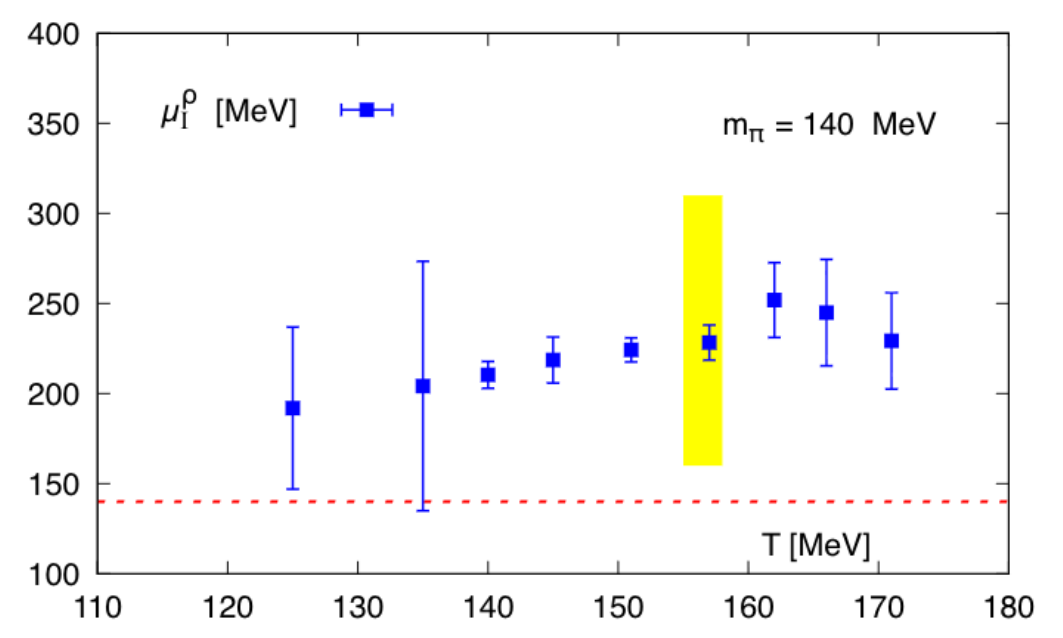
$\mu_{I,i}^T \neq 0$ for $T \leq 135$ MeV

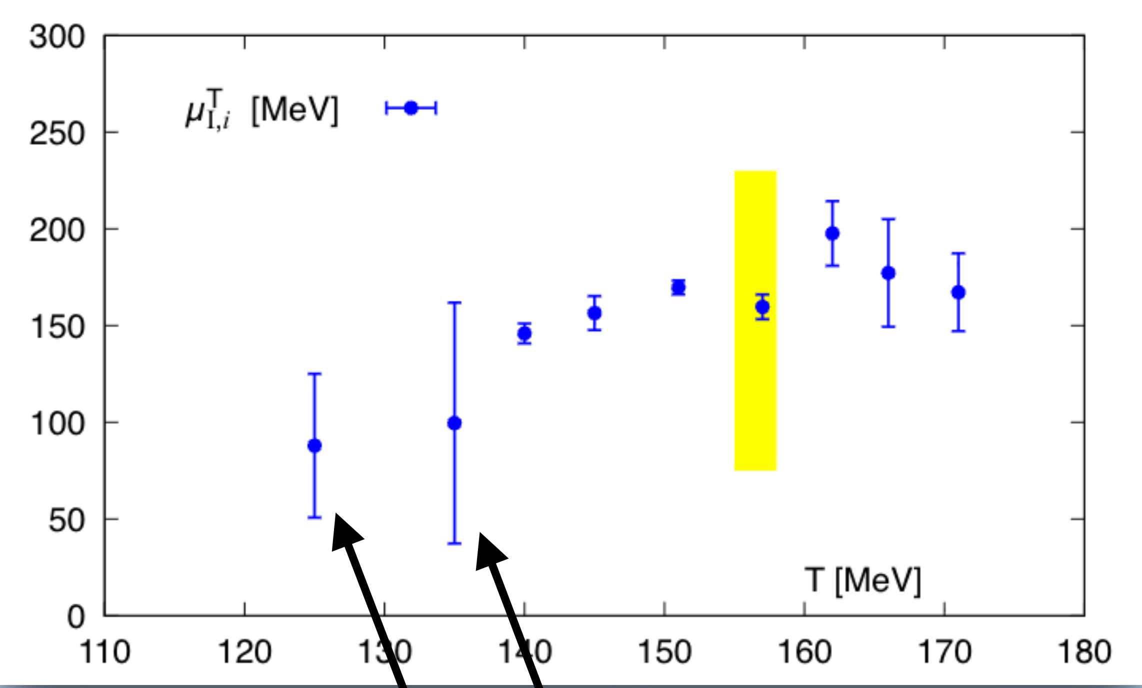
Non-zero **imaginary** part

$\left(\mu_I^c, T_I^c\right) = \left(m_\pi, 135\right)$ is

NOT

A CRITICAL POINT





$\mu^T_{I,i} \neq 0$ for $T \leq 135$ MeV

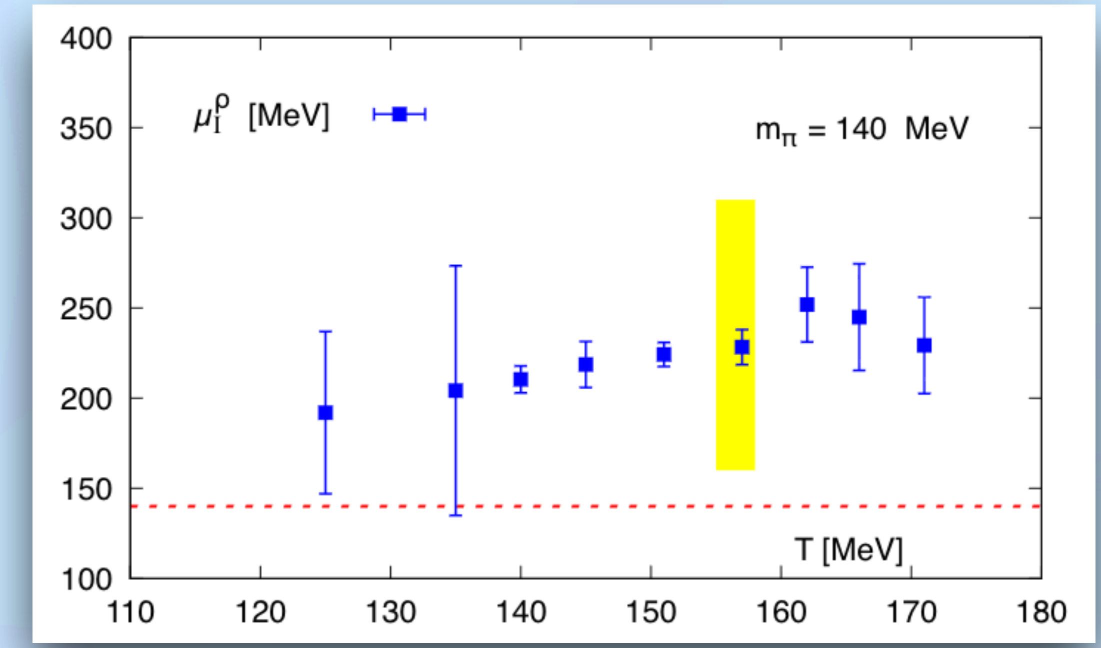
Non-zero **imaginary** part

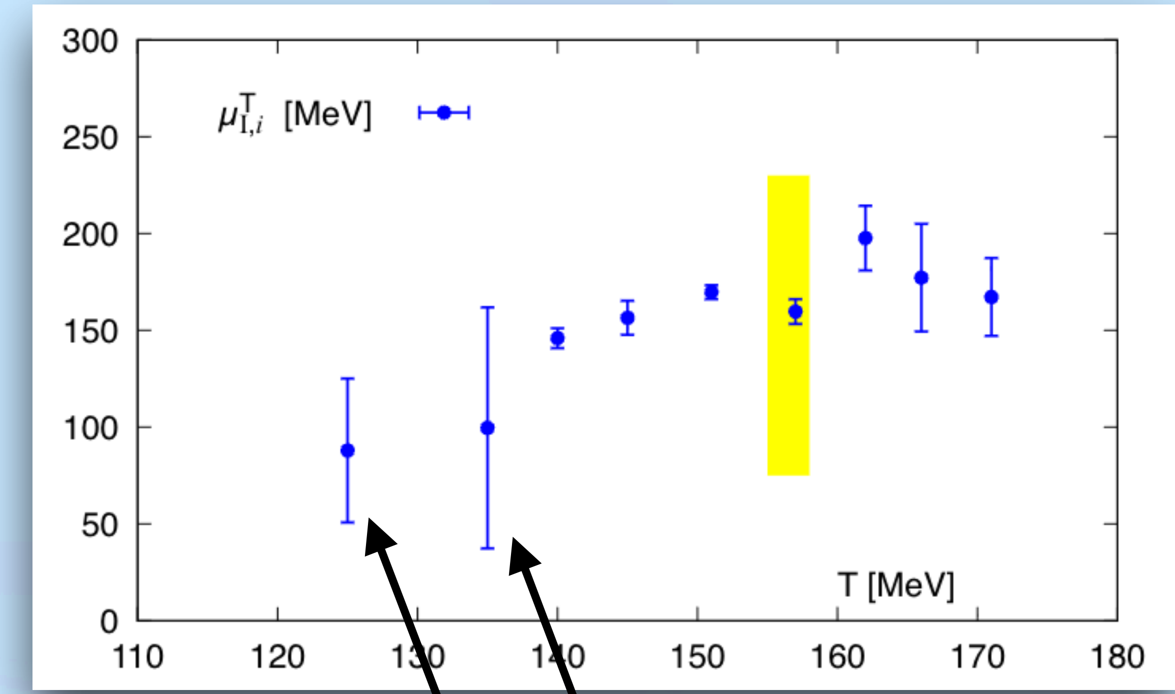
$(\mu_I^c, T_I^c) = (m_\pi, 135)$ is

NOT

A CRITICAL POINT

So, analyse lower T...





$\mu_{I,i}^T \neq 0$ for $T \lesssim 135$ MeV

Non-zero **imaginary** part

$(\mu_I^c, T_I^c) = (m_\pi, 135)$ is

NOT

A CRITICAL POINT

So, analyse lower T...

But

No further data for $T < 125$ MeV available on these lattices

No further data for $T < 125$ MeV available on these lattices

No further data for $T < 125$ MeV available on these lattices

↓

Extrapolate $\mu_{I,i}^{T}$ and μ_{I}^{ρ} in T, to lower T

No further data for $T < 125$ MeV available on these lattices

↓

Extrapolate $\mu^T_{I,i}$ and μ^ρ_I in T, to lower T

↓

No further data for $T < 125$ MeV available on these lattices

↓

Extrapolate $\mu^T_{I,i}$ and μ^ρ_I in T , to lower T

↓

1. Find : $T = T^c_I \ni \mu^T_{I,i} = 0$, and hence, $\mu^0_I \left(T^c_I\right) \in \mathbb{R}$

2. Find : μ^ρ_I at $T = T^c_I$: $\Rightarrow \mu^c_I$ $\{$ since, at $T = T^c_I$, $\mu^\rho_I = \mathrm{Re}\left(\mu^0_I\right)$ $\}$

No further data for $T < 125$ MeV available on these lattices

↓

Extrapolate $\mu^T_{I,i}$ and μ^ρ_I in T, to lower T

↓

1. Find : $T = T^c_I \ni \mu^T_{I,i} = 0$, and hence, $\mu^0_I \left(T^c_I\right) \in \mathbb{R}$

2. Find : μ^ρ_I at $T = T^c_I$: $\Rightarrow \mu^c_I$ $\left\{\text{since, at } T = T^c_I,\ \mu^\rho_I = \text{Re}\left(\mu^0_I\right)\right\}$

↓

No further data for $T < 125$ MeV available on these lattices

↓

Extrapolate $\mu_{I,i}^{T}$ and μ_{I}^{ρ} in T, to lower T

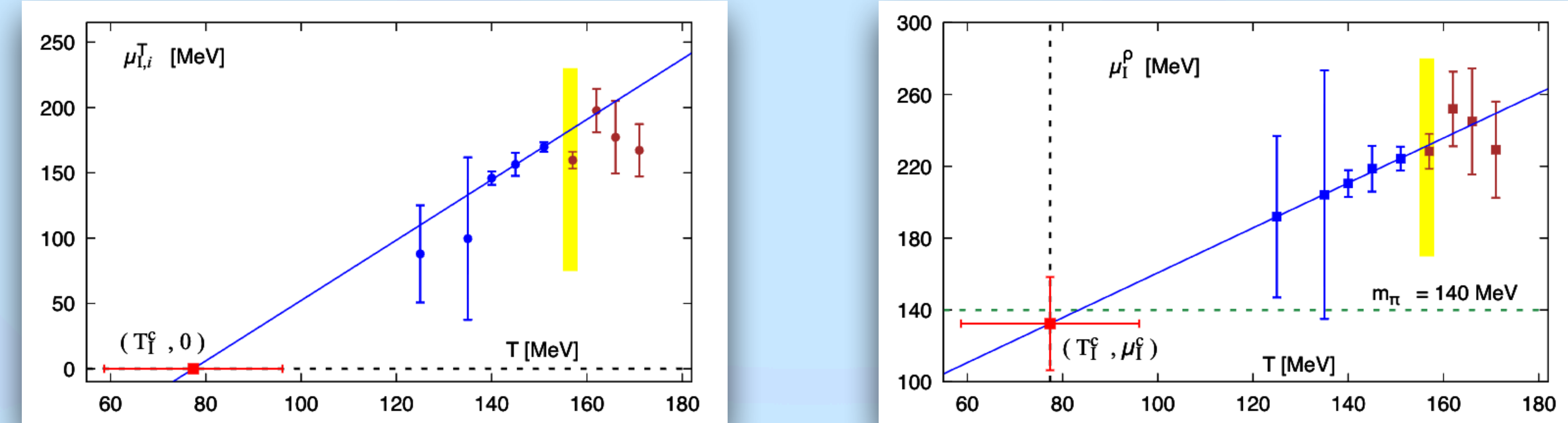
↓

1. Find : $T = T_I^c \ni \mu_{I,i}^{T} = 0$, and hence, $\mu_I^0\left(T_I^c\right) \in \mathbb{R}$

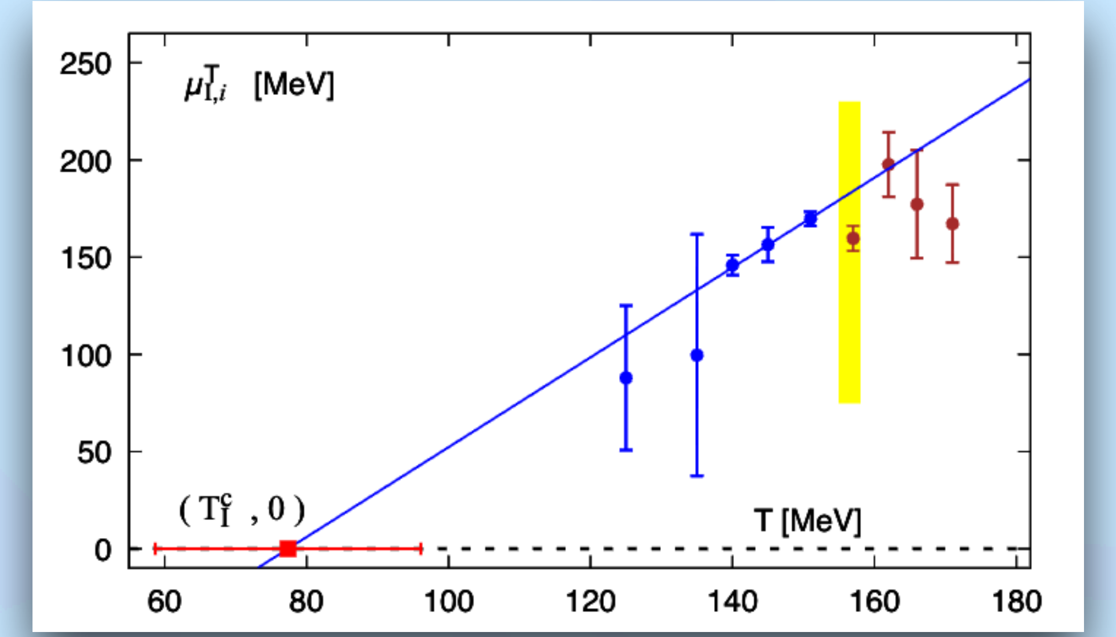
2. Find : μ_I^{ρ} at $T = T_I^c$: $\Rightarrow \mu_I^c$ $\{$ since, at $T = T_I^c$, $\mu_I^{\rho} = \mathrm{Re}\left(\mu_I^0\right)$ $\}$

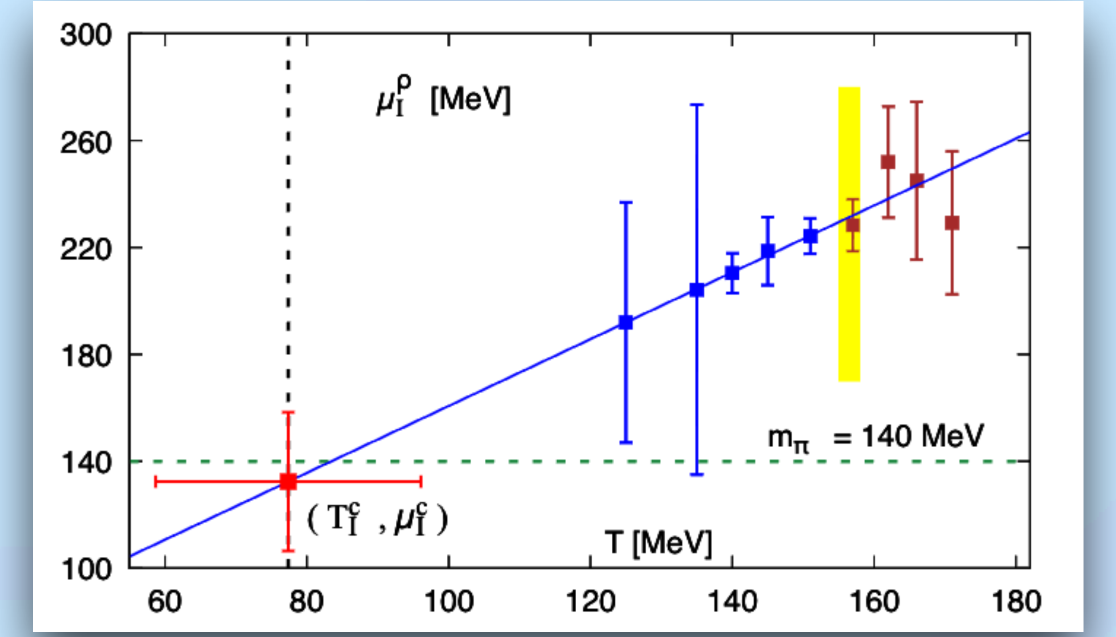
↓

Thus, obtain an estimate of a critical point $\left(\mu_I^c\,,T_I^c\right)$

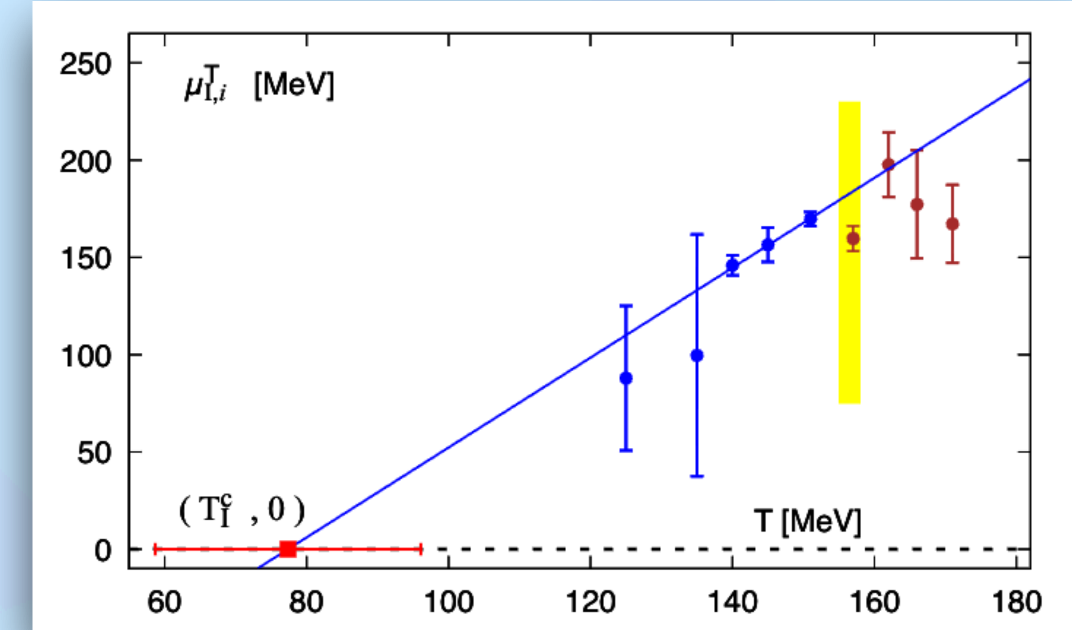


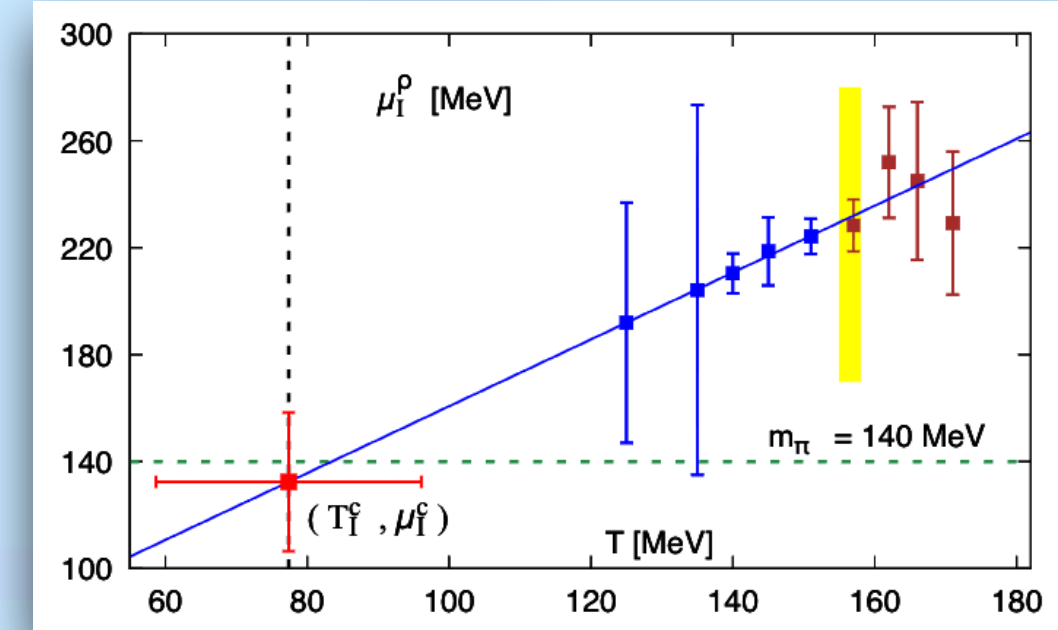
$\mu^T_{I,i}$ [MeV]
$(T^c_I, 0)$
T [MeV]
μ^ρ_I [MeV]
m_π = 140 MeV
(T^c_I, μ^c_I)
T [MeV]



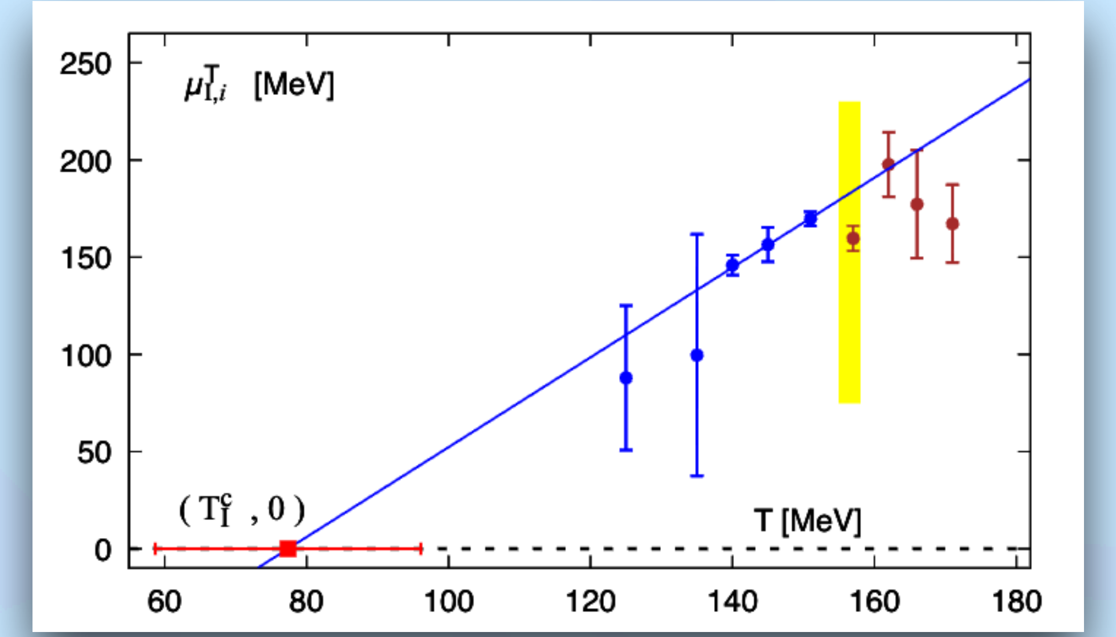


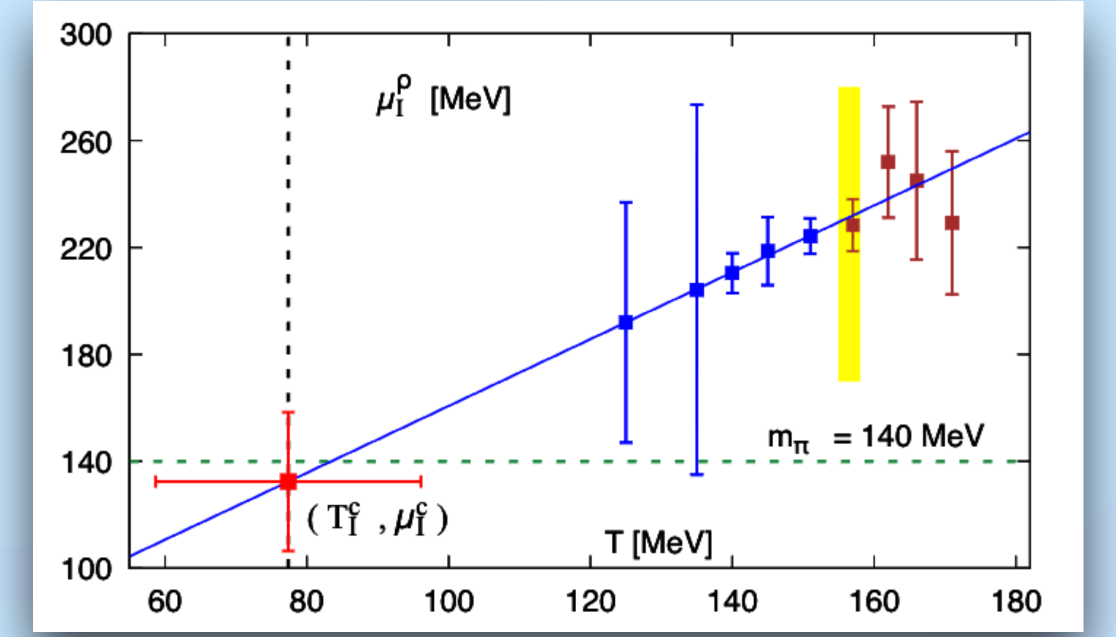
Linear extrapolation of $\mu^T_{I,i}$ and μ^ρ_I





Linear extrapolation of $\mu^T_{I,i}$ and μ^ρ_I + only in $T \in [125 : 151]$ MeV





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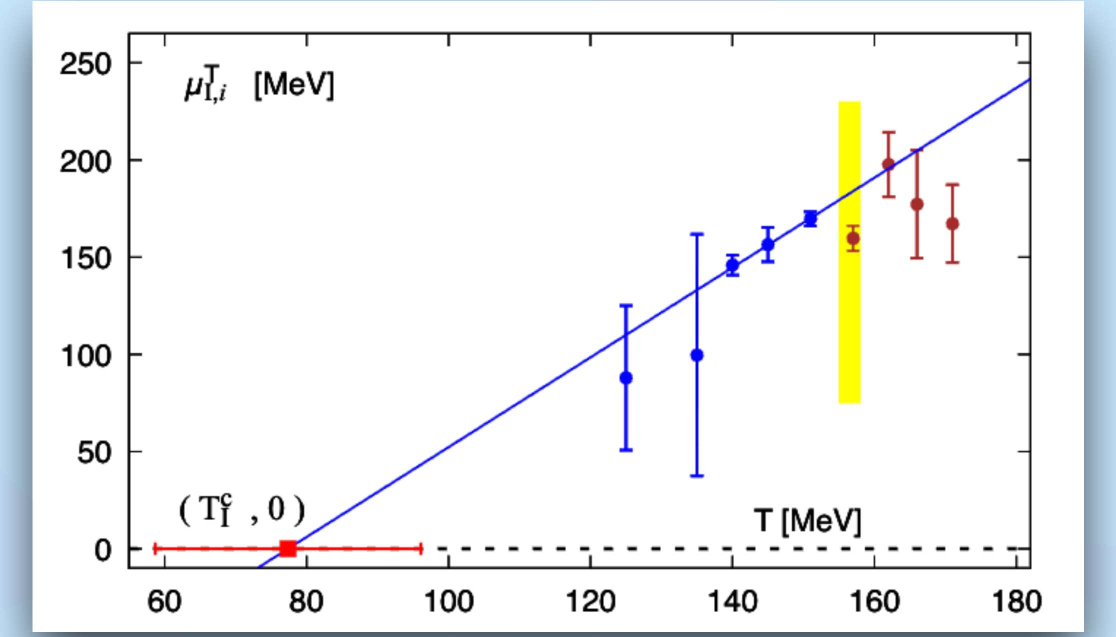
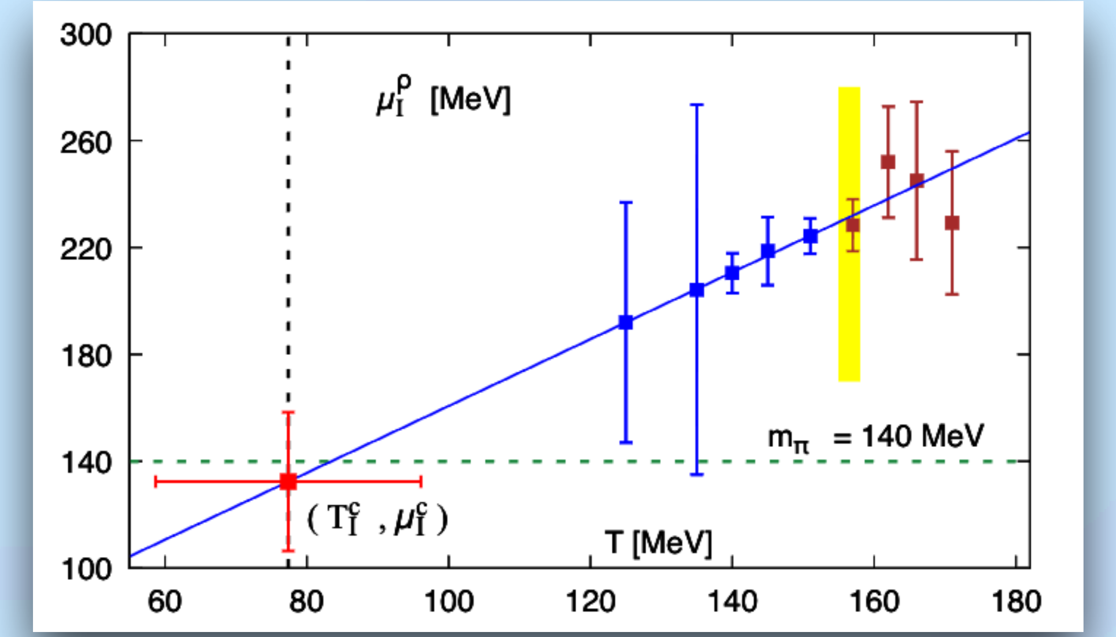
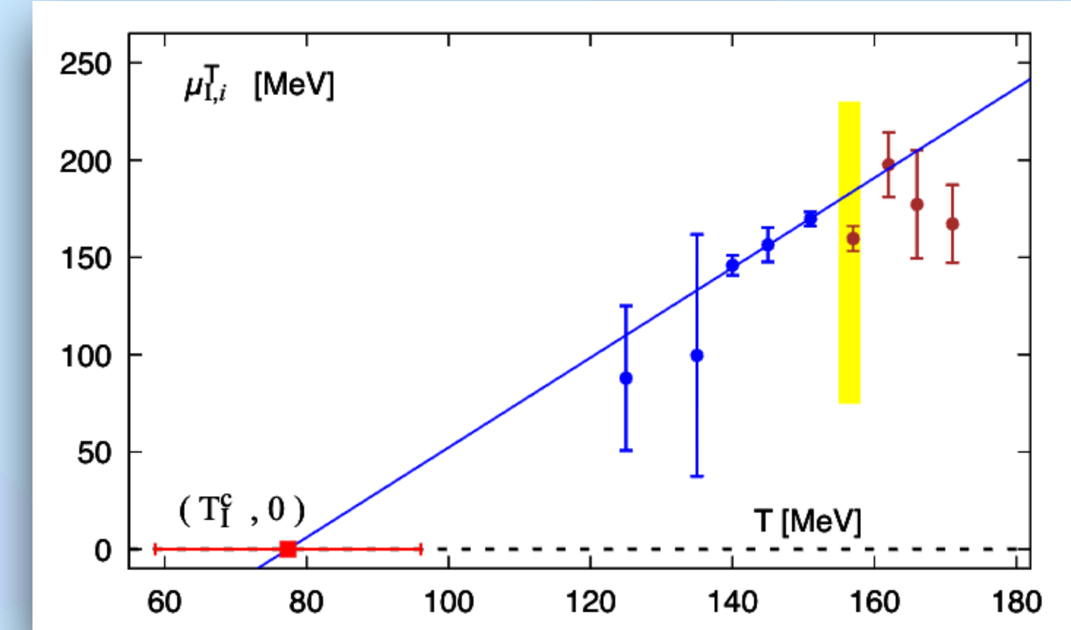
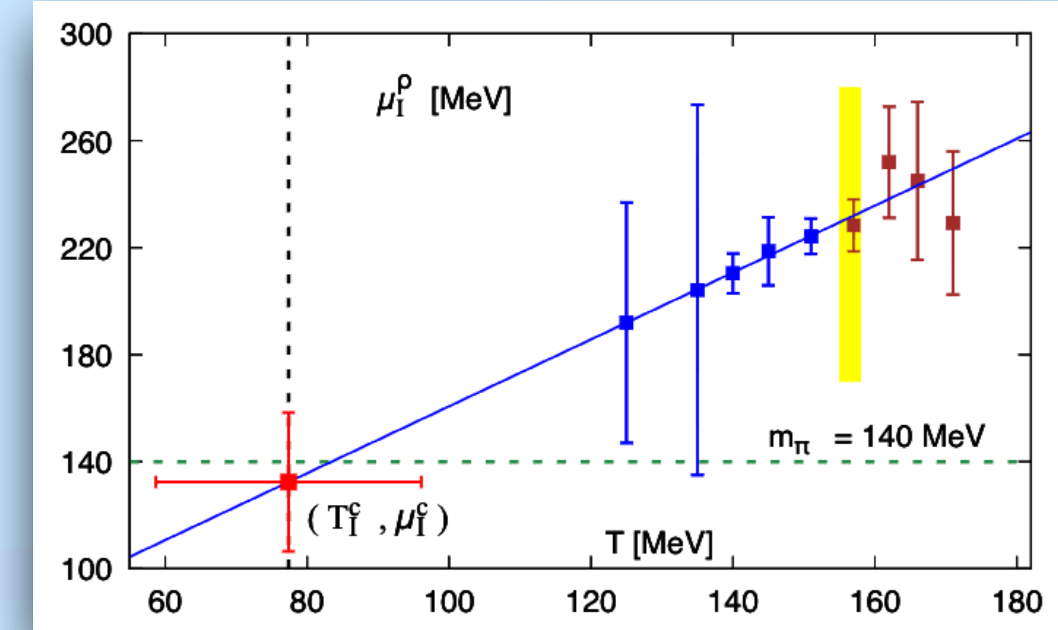




Linear extrapolation of $\mu^T_{I,i}$ and μ^ρ_I + only in $T \in [125 : 151]$ MeV

$$\left(\mu_I^c\,,\,T_I^c\right) :\, = \left(132.40 \pm 25.91\ ,\ 77.38 \pm 18.71\right)\ \text{MeV}$$

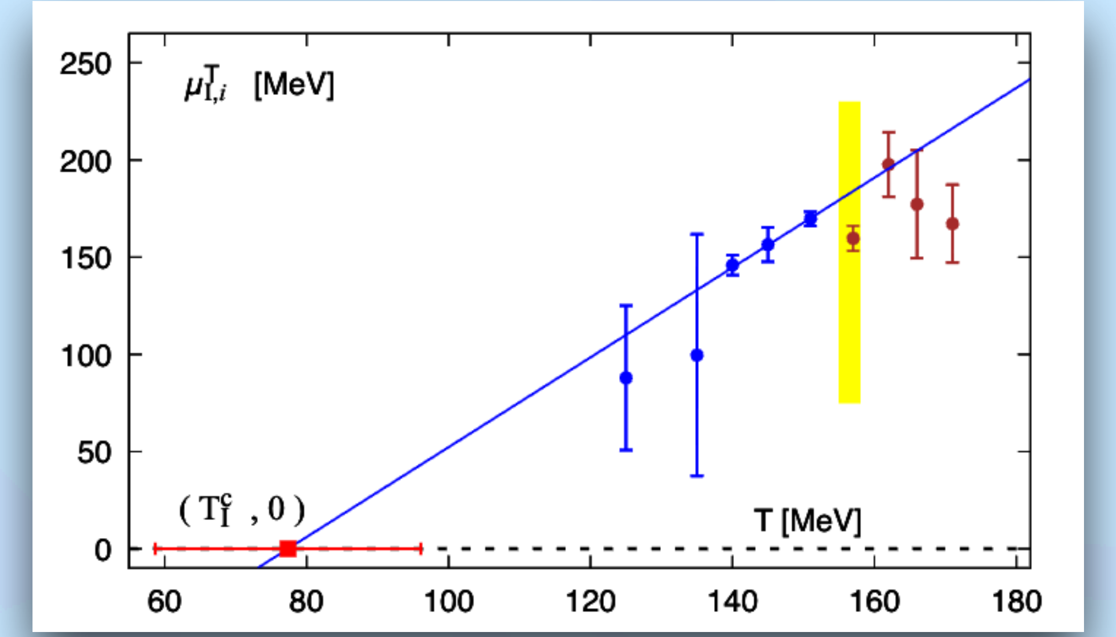


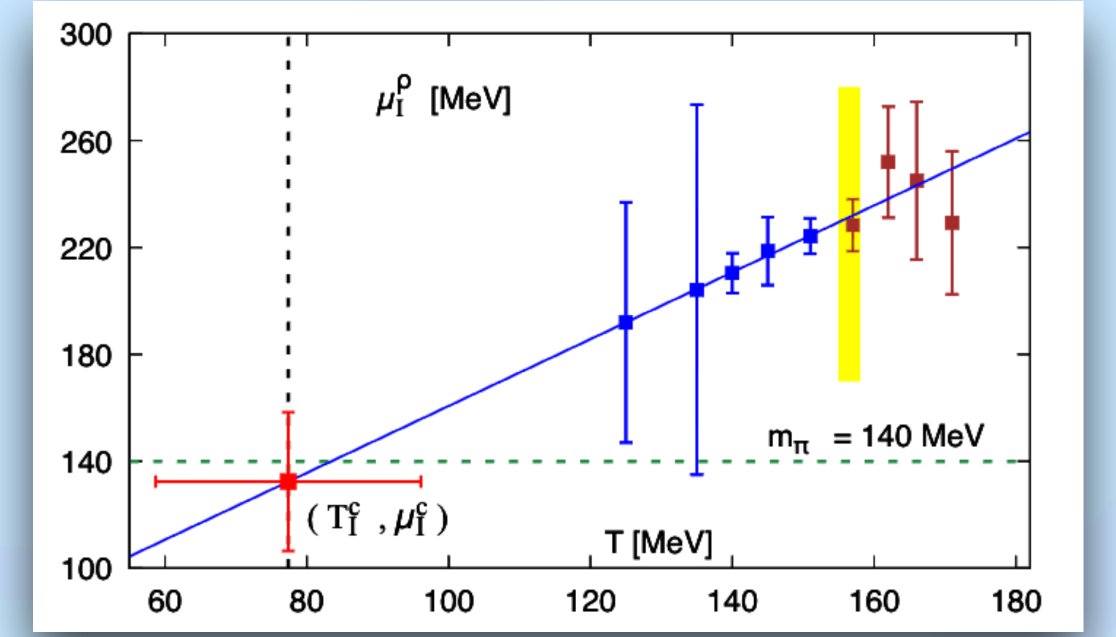


Linear extrapolation of $\mu^T_{I,i}$ and μ^ρ_I + only in $T \in [125 : 151]$ MeV

$$\left(\mu^c_I , T^c_I\right) : = \left(132.40 \pm 25.91 \ , 77.38 \pm 18.71\right) \text{ MeV}$$

This lies on the present state-of-the-art pion condensate line in the QCD isospin phase diagram. $\Longleftrightarrow$ Nomenclature with $\mu^c_I = m_\pi$ **(One of the achievements).**





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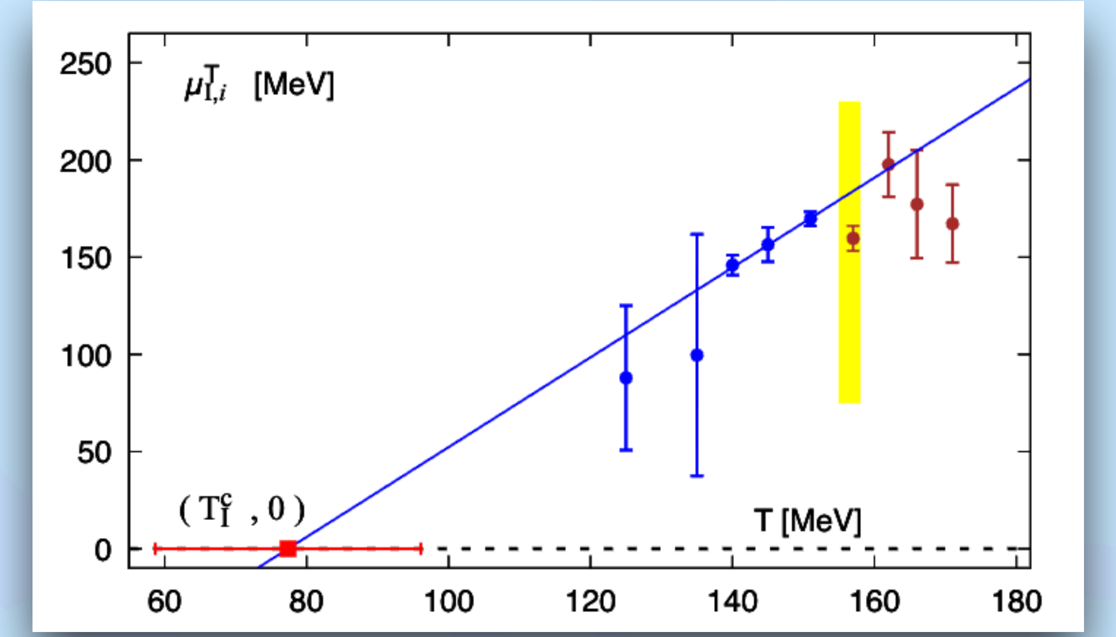
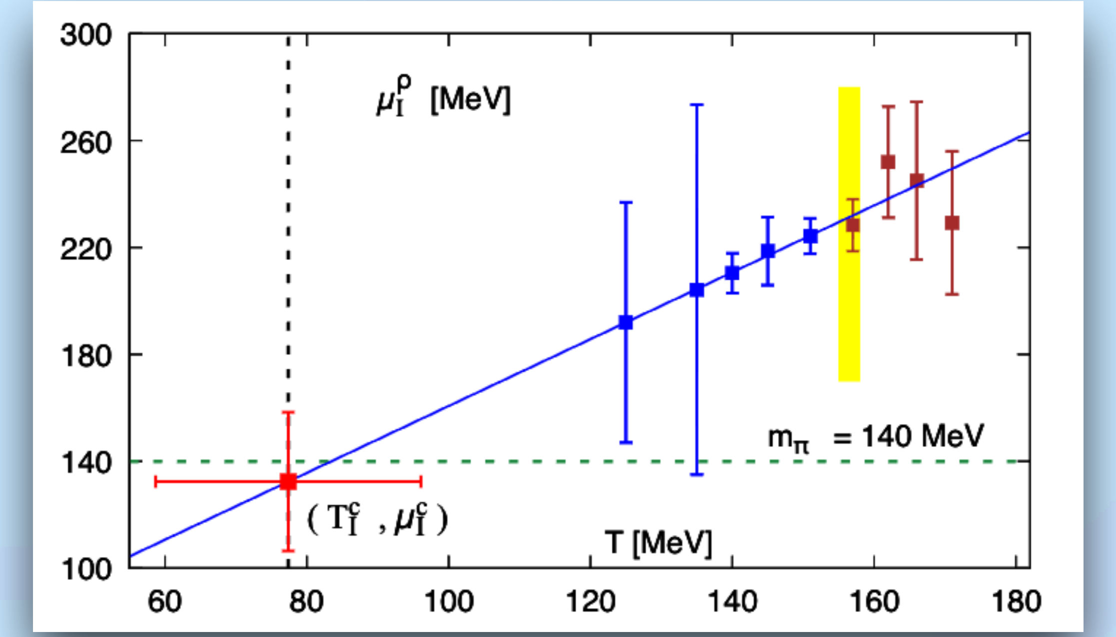
within error bars





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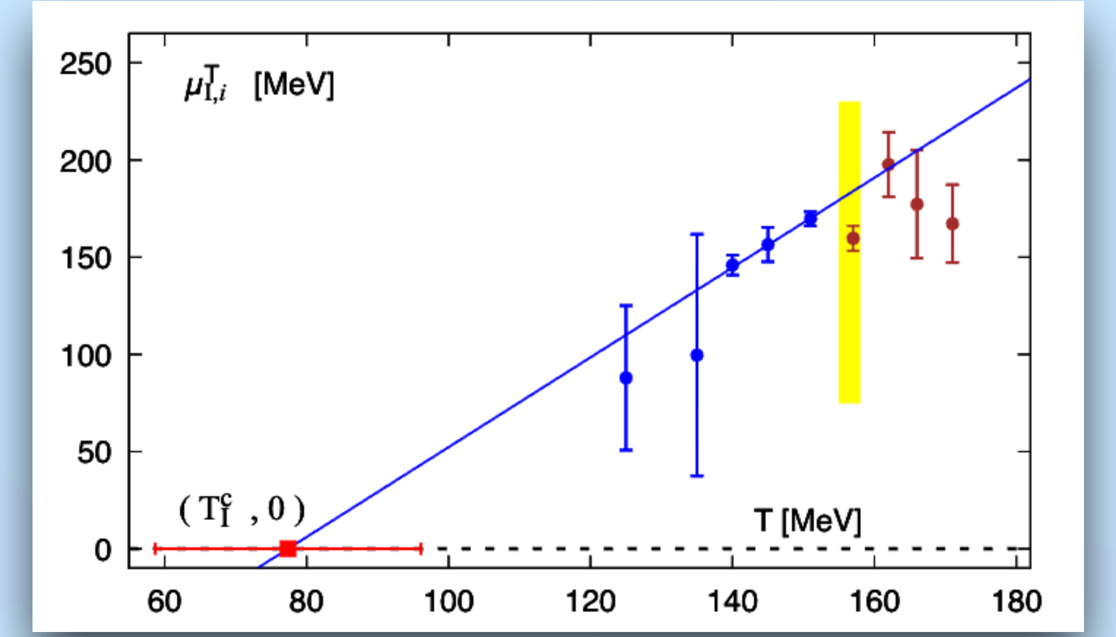

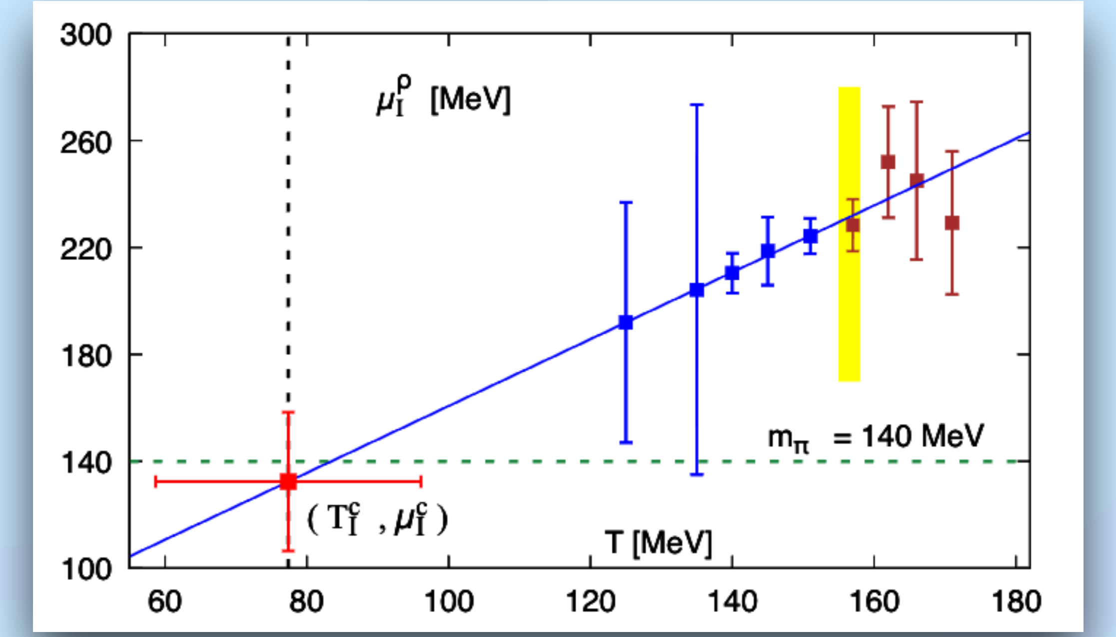

Linear extrapolation of $\mu^T_{I,i}$ and μ^ρ_I + only in $T \in [125 : 151]$ MeV

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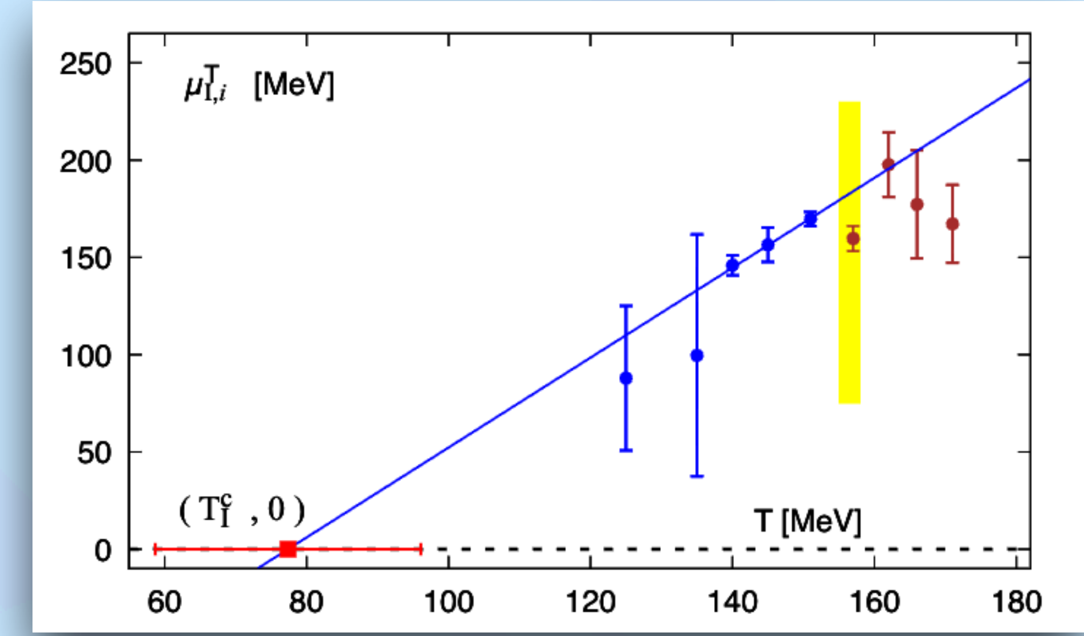
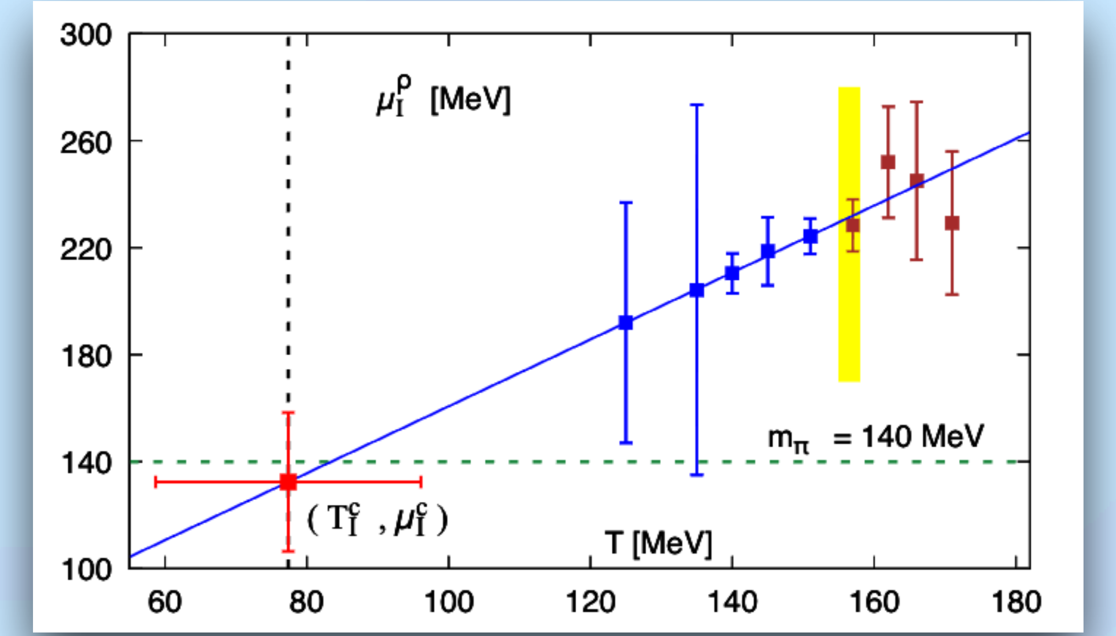
within error bars

Son, Stephanov : *PRL*

Andersen et. al. (2024) : *PRD*

Andersen et. al. (2025) : *PRD*





Linear extrapolation of $\mu^T_{I,i}$ and μ^ρ_I + only in $T \in [125 : 151]$ MeV

$$\left(\mu_I^c\,,\,T_I^c\right) := \left(132.40 \pm 25.91\ ,\ 77.38 \pm 18.71\right)\ \text{MeV}$$

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within error bars

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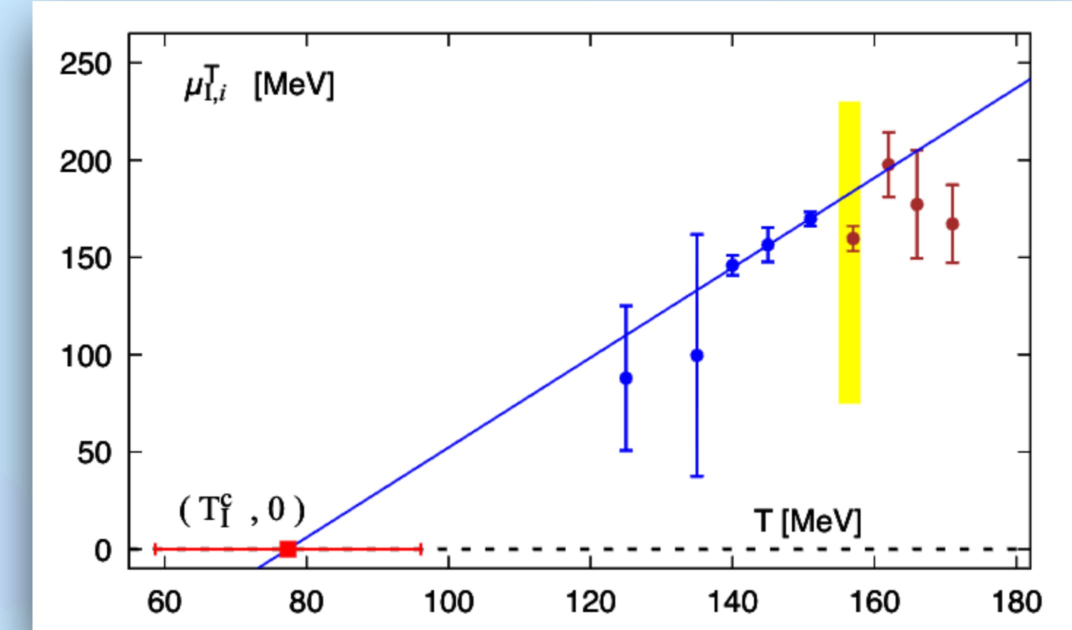
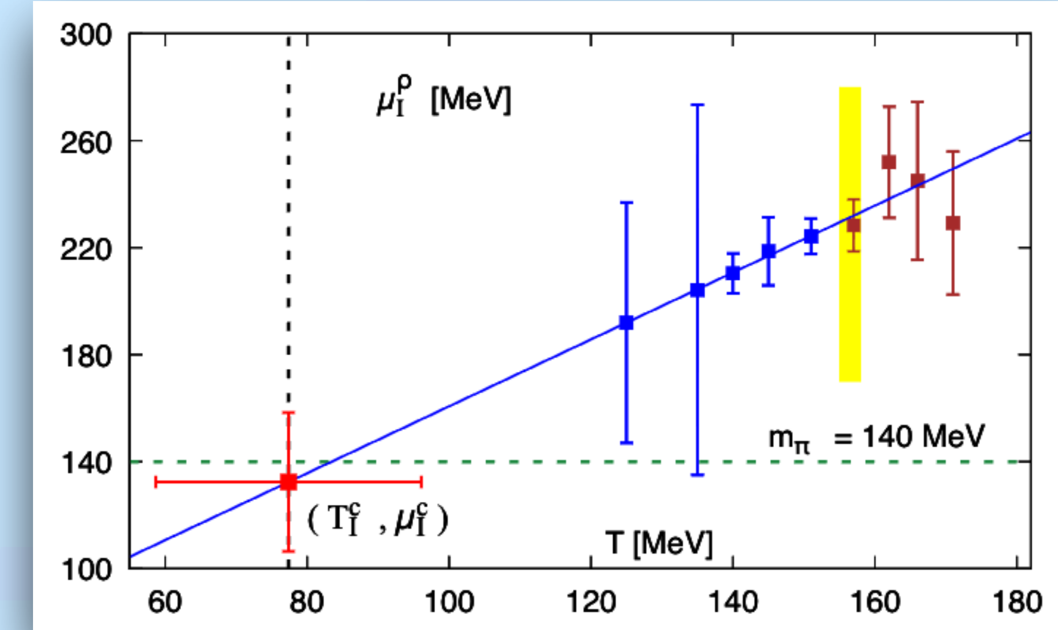
$\mu_I = \mu_u - \mu_d$





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within error bars

Son, Stephanov : *PRL*

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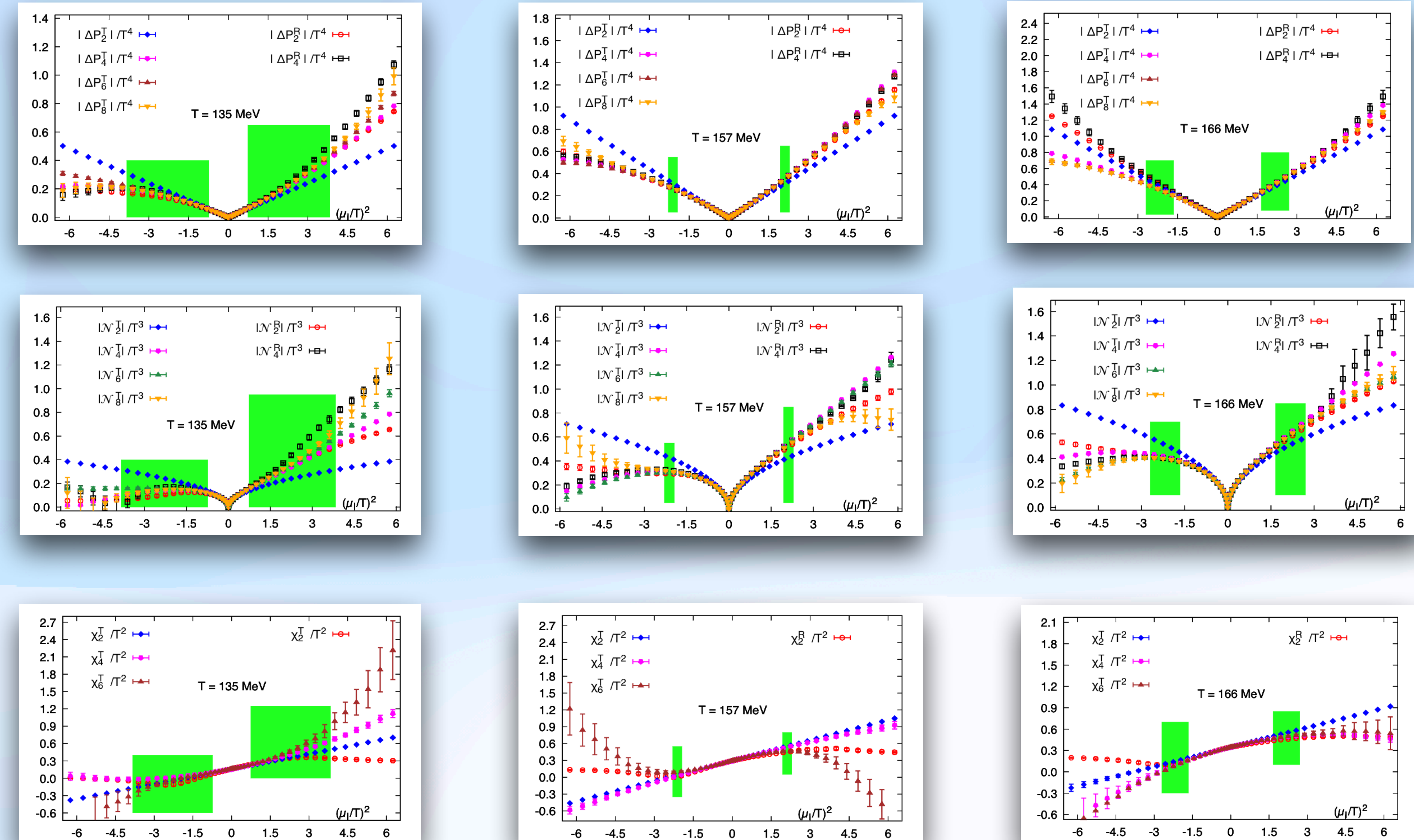
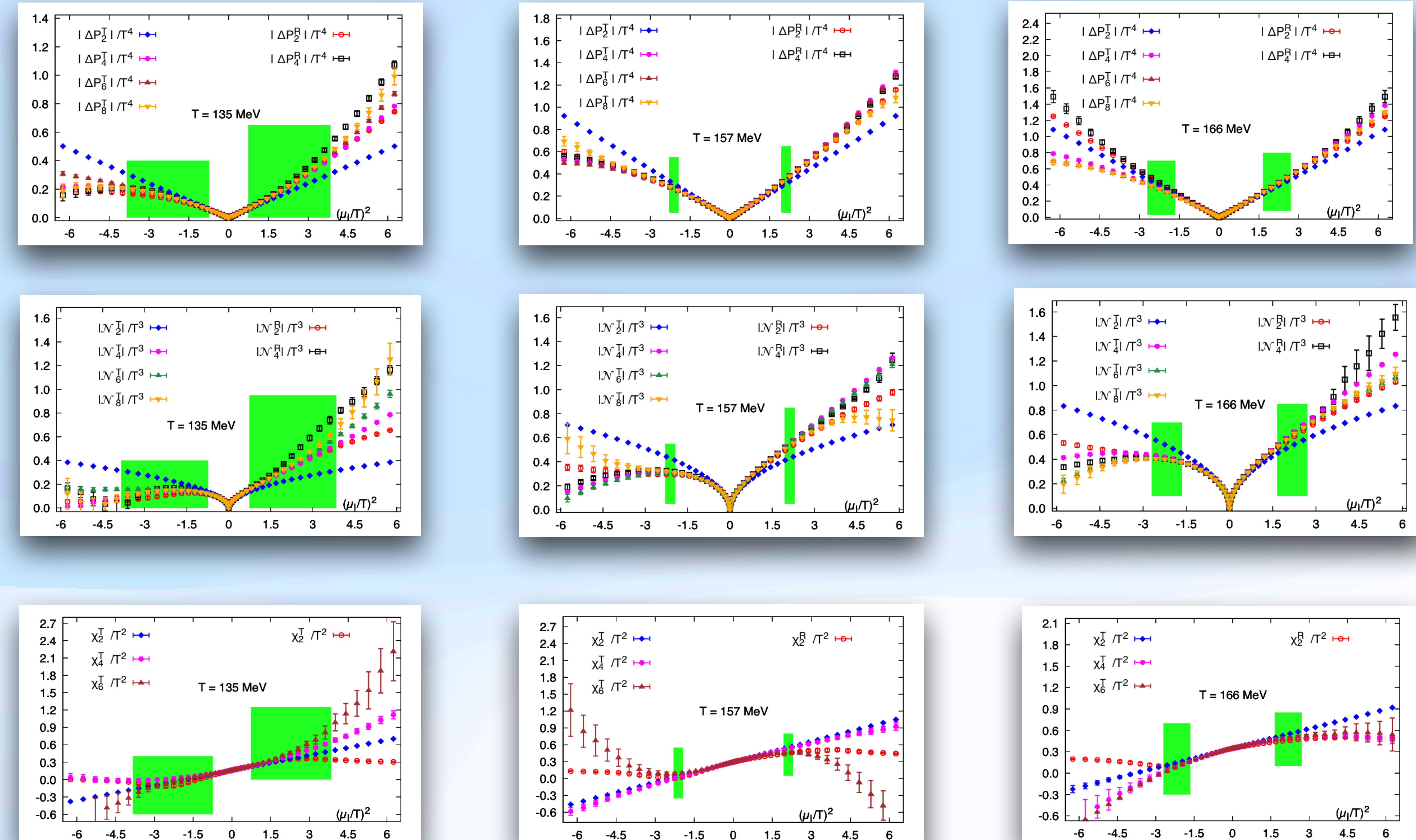
$\mu_I = \mu_u - \mu_d$

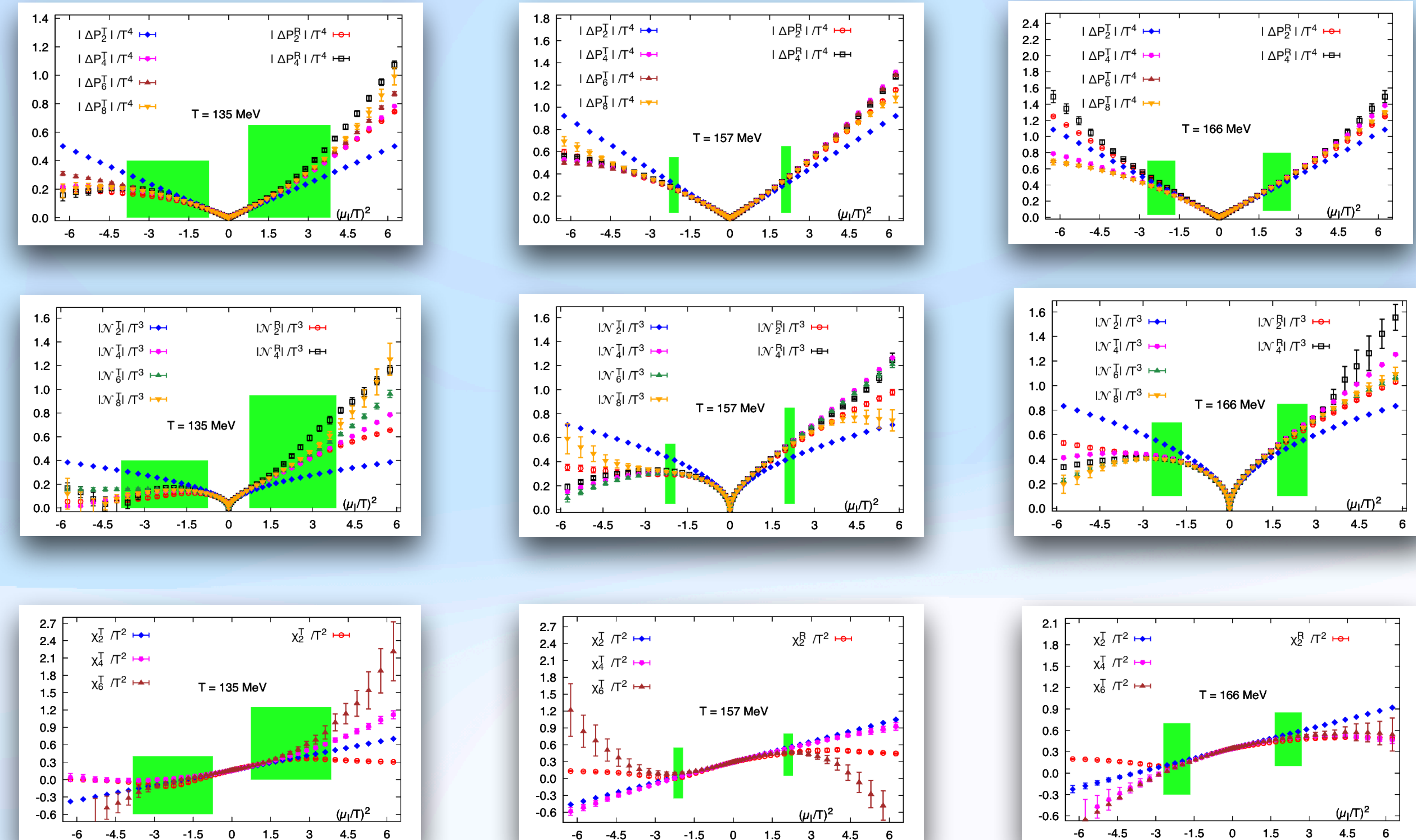
Some observations related to this RoC ...



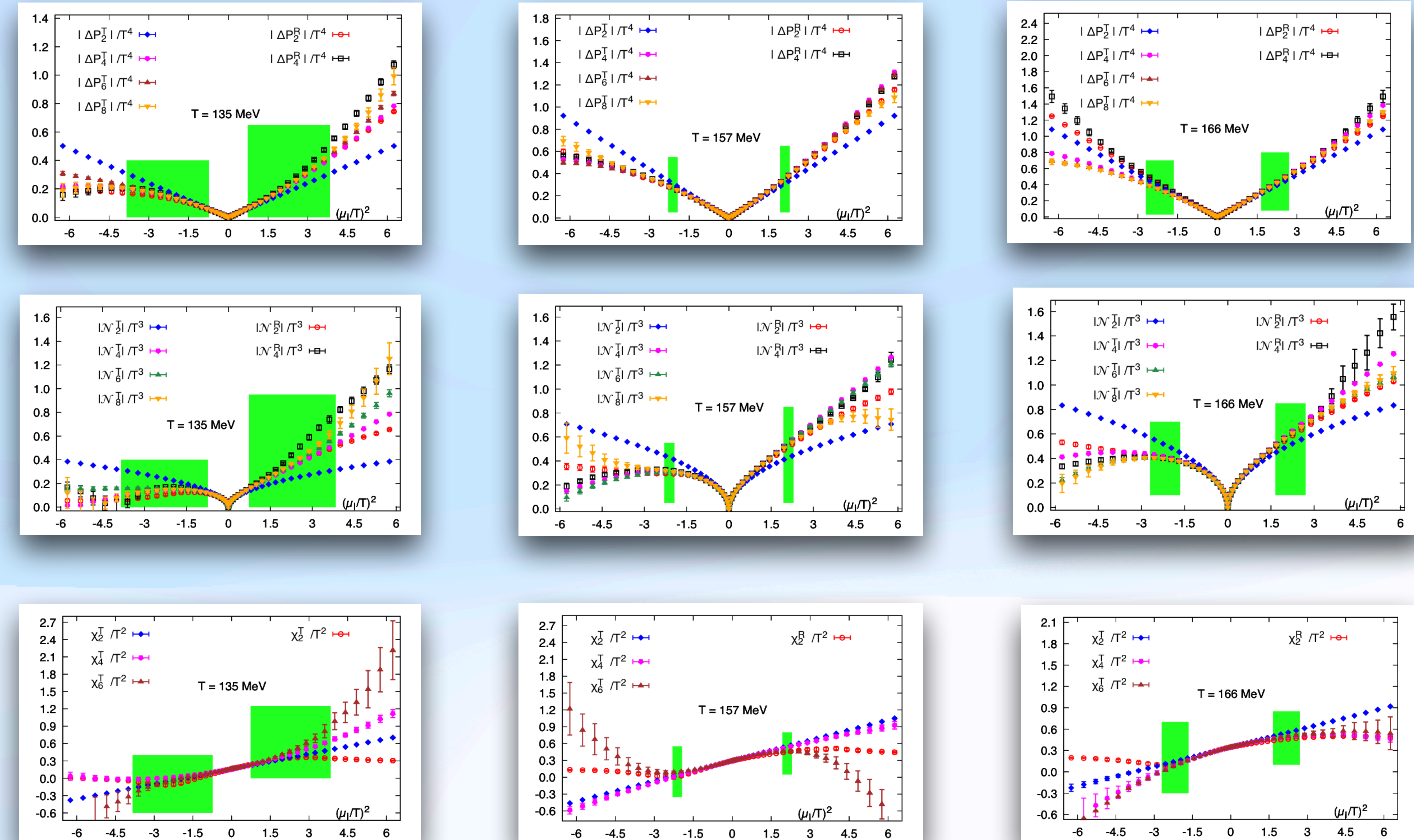
T = 135 MeV
T = 157 MeV
T = 166 MeV
$|\Delta P_2^I|/T^4$
$|\Delta P_4^I|/T^4$
$|\Delta P_6^I|/T^4$
$|\Delta P_8^I|/T^4$
$|\Delta P_2^R|/T^4$
$|\Delta P_4^R|/T^4$
$|\mathcal{N}_2^I|/T^3$
$|\mathcal{N}_4^I|/T^3$
$|\mathcal{N}_6^I|/T^3$
$|\mathcal{N}_8^I|/T^3$
$|\mathcal{N}_2^R|/T^3$
$|\mathcal{N}_4^R|/T^3$
χ_2^I/T^2
χ_4^I/T^2
χ_6^I/T^2
χ_2^R/T^2
$(\mu_I/T)^2$



- Order-by-order deviations **beyond** μ_I^ρ



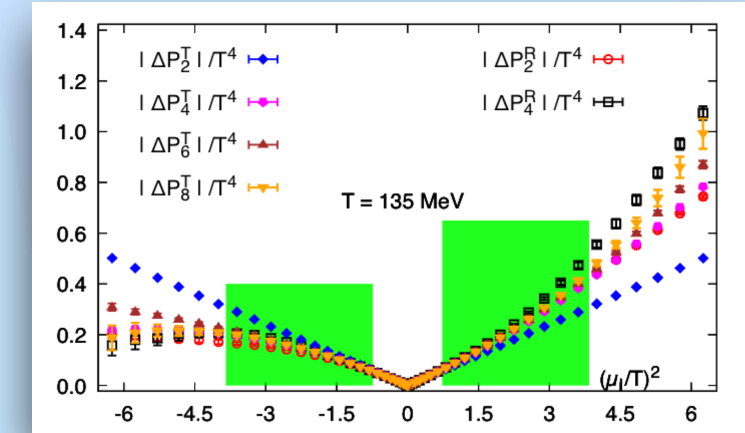
- Order-by-order deviations **beyond** μ_I^ρ → for **different** T

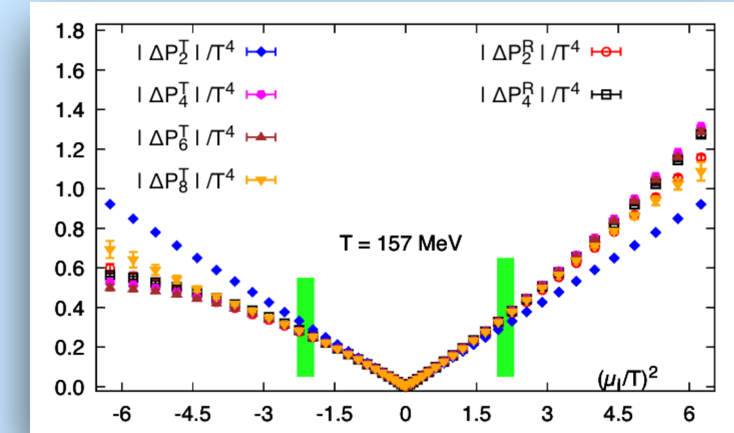


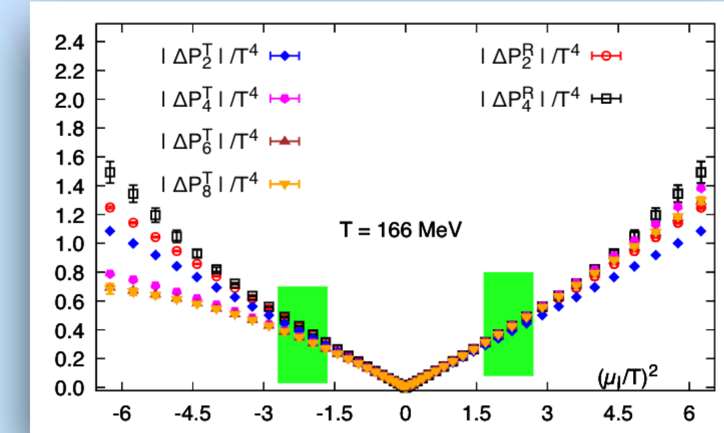
- Order-by-order deviations **beyond** μ_I^ρ

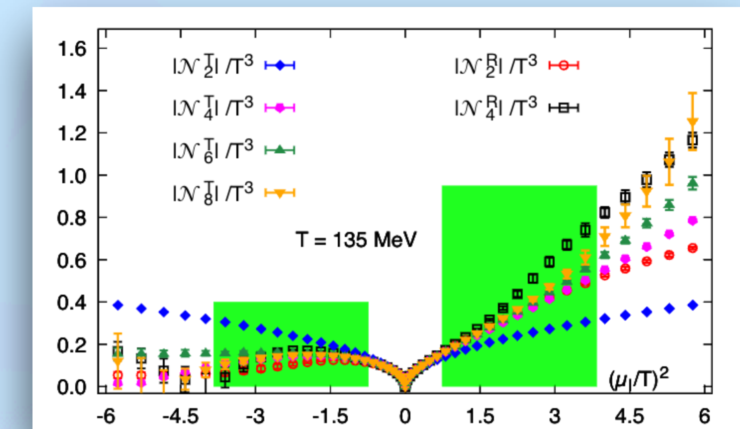
for **different** T

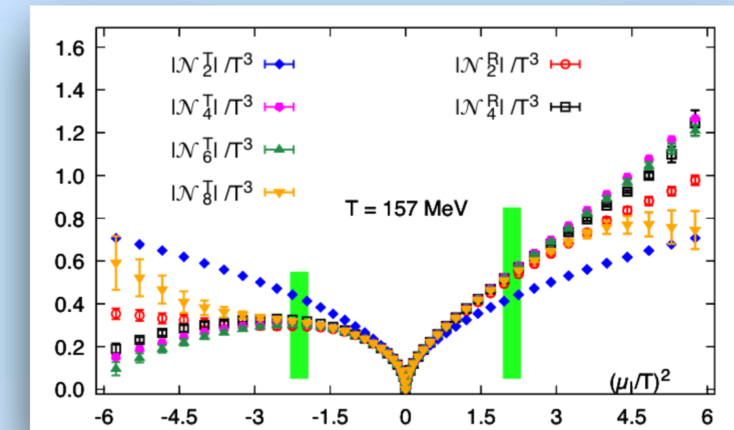
for **both Re** (μ_I) **and Im** (μ_I)

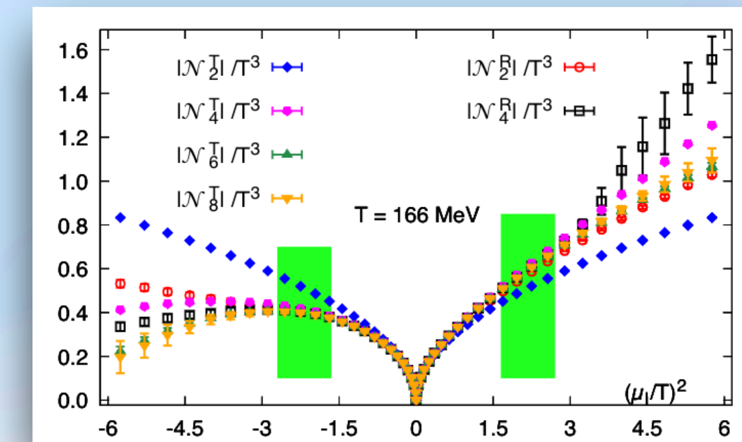


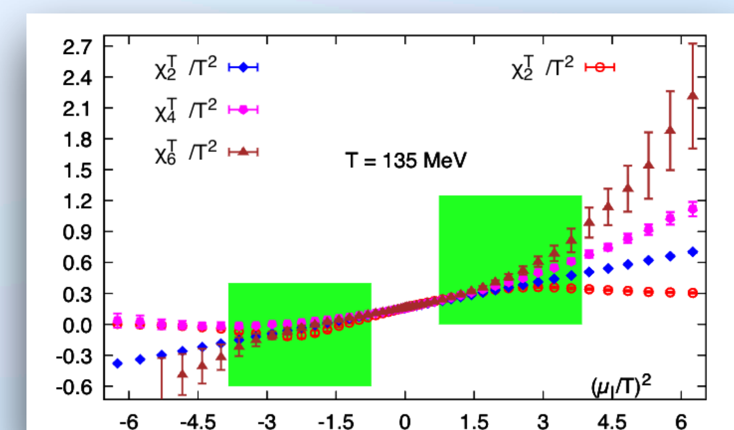


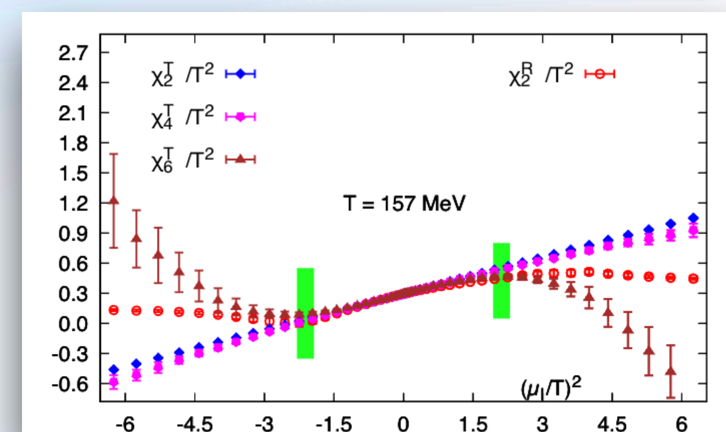


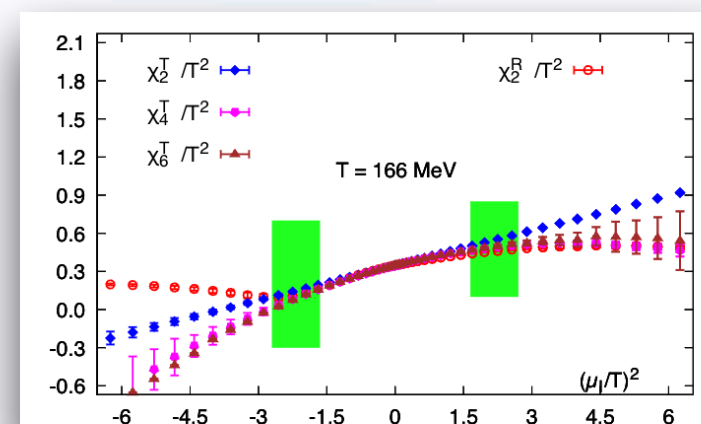




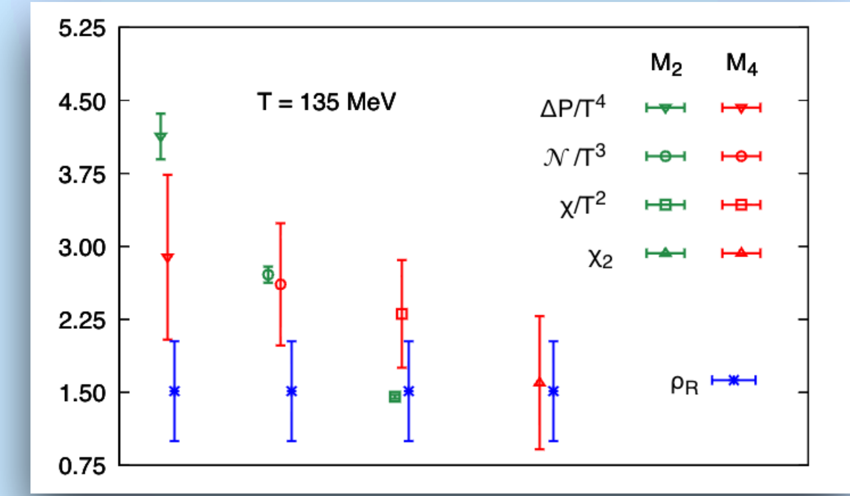
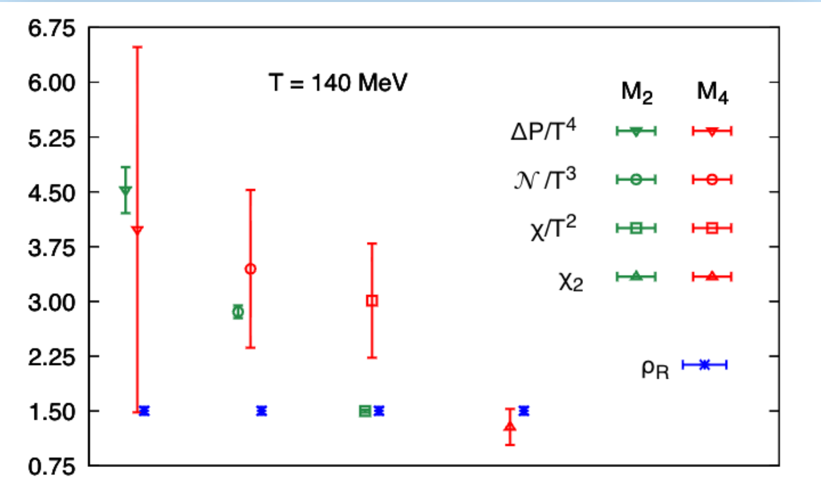
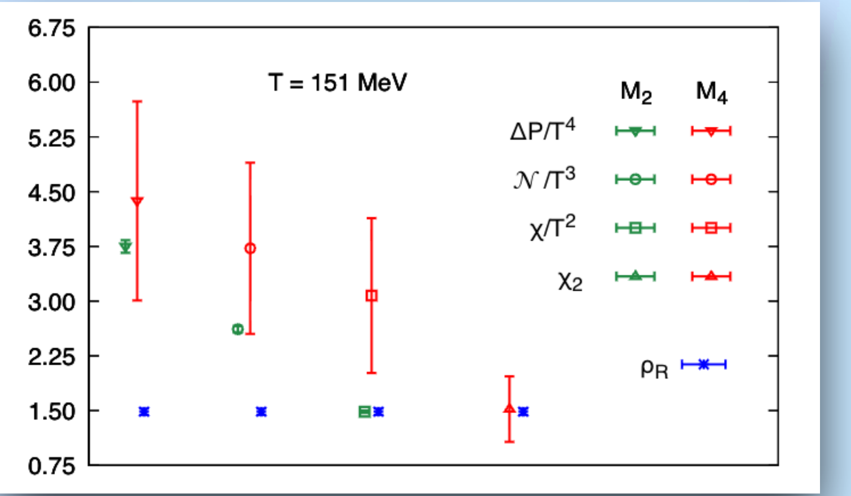


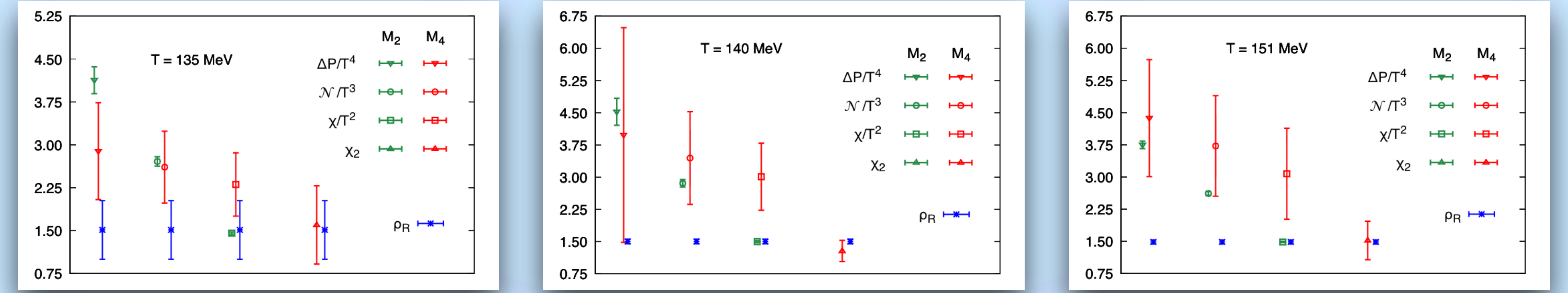






- Order-by-order deviations **beyond** μ_I^ρ → for **different** T ; for **both Re** (μ_I) **and Im** (μ_I)
- Qualitatively, deviations → $\chi > \mathcal{N} > \Delta P$


T = 135 MeV
M2
M4
ΔP/T4
𝒩/T3
χ/T2
χ2
ρR

T = 140 MeV
M2
M4
ΔP/T4
𝒩/T3
χ/T2
χ2
ρR

T = 151 MeV
M2
M4
ΔP/T4
𝒩/T3
χ/T2
χ2
ρR



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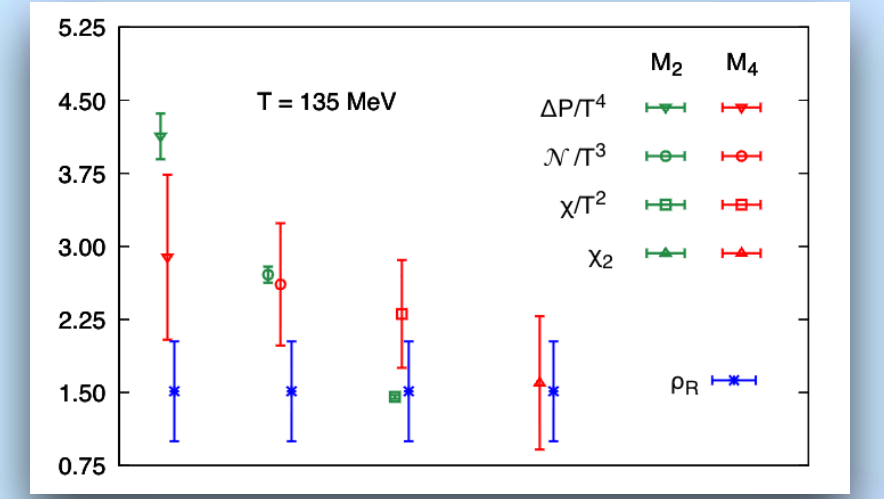

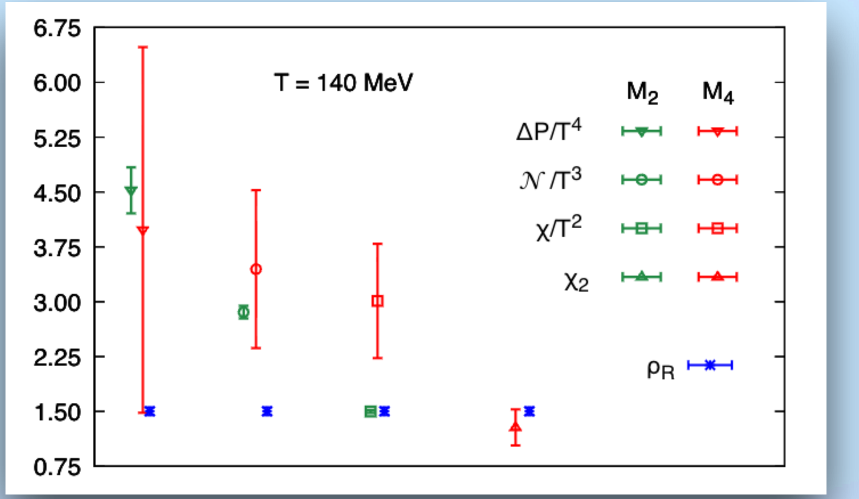

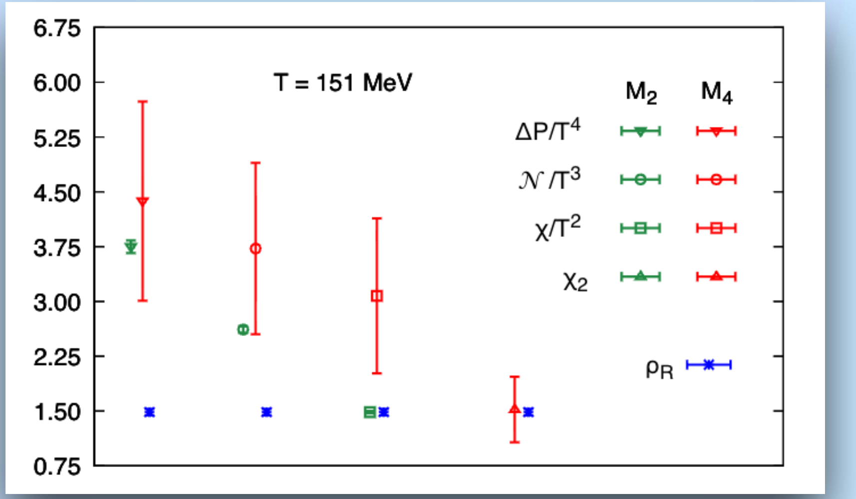

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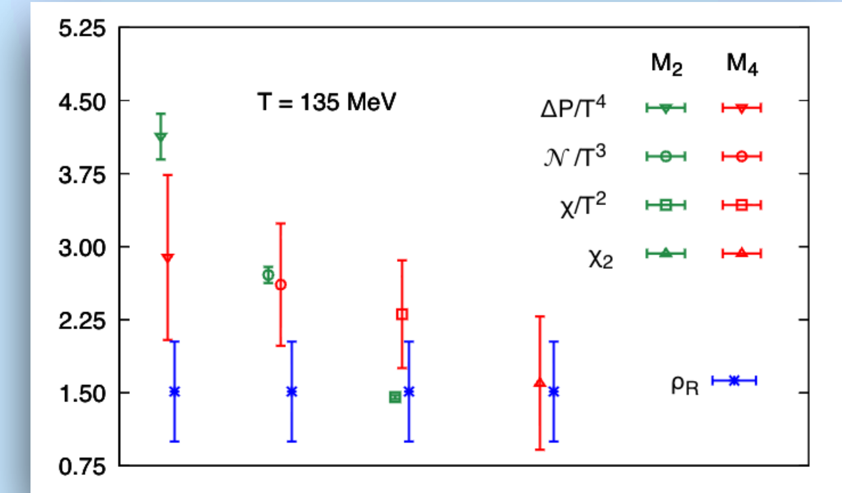

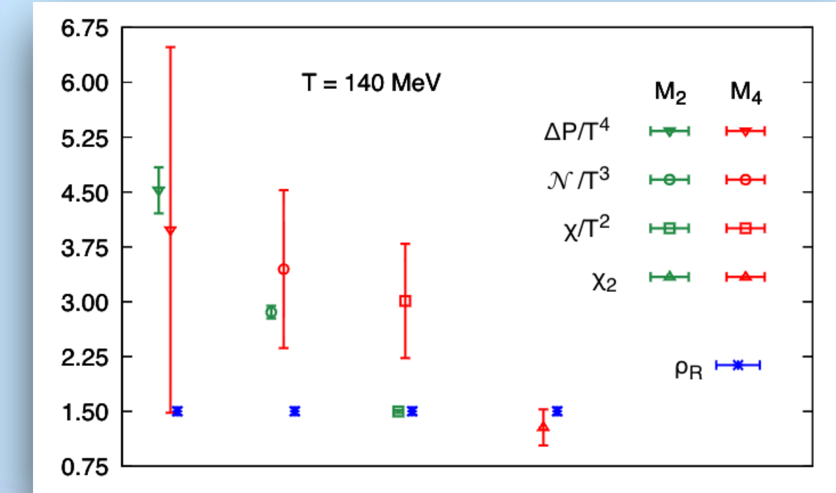

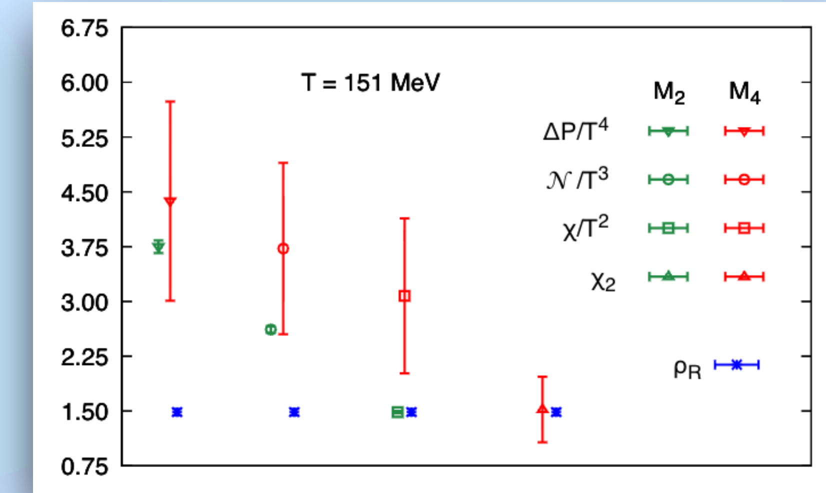

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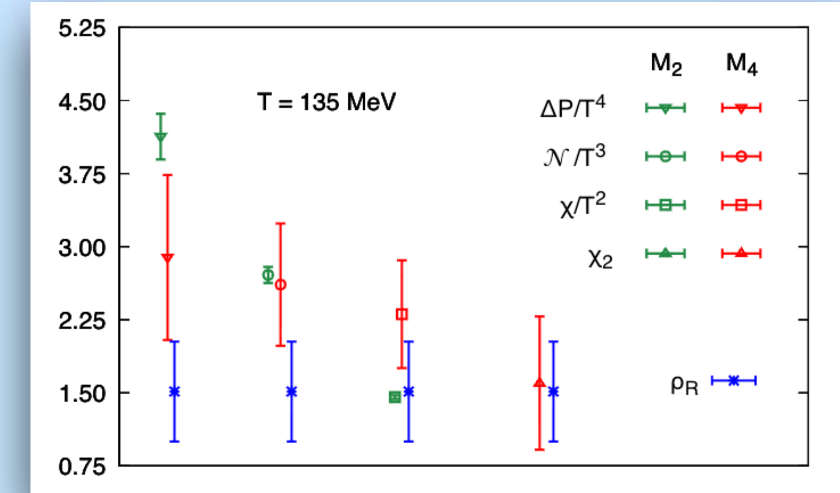

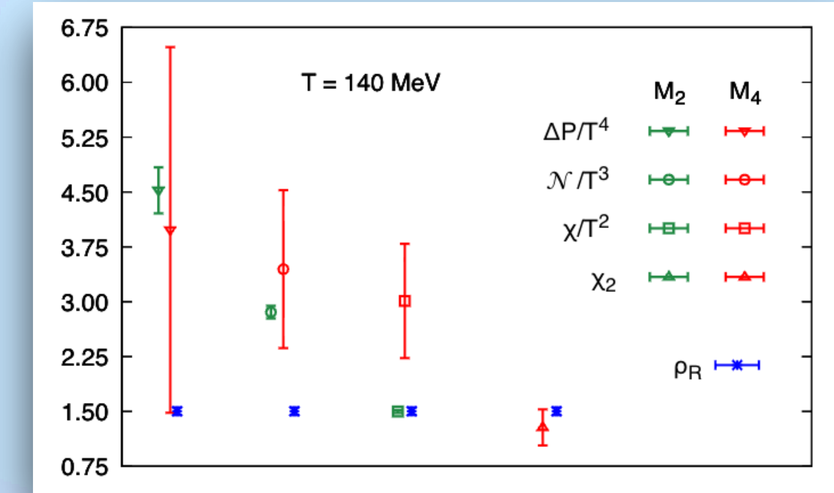

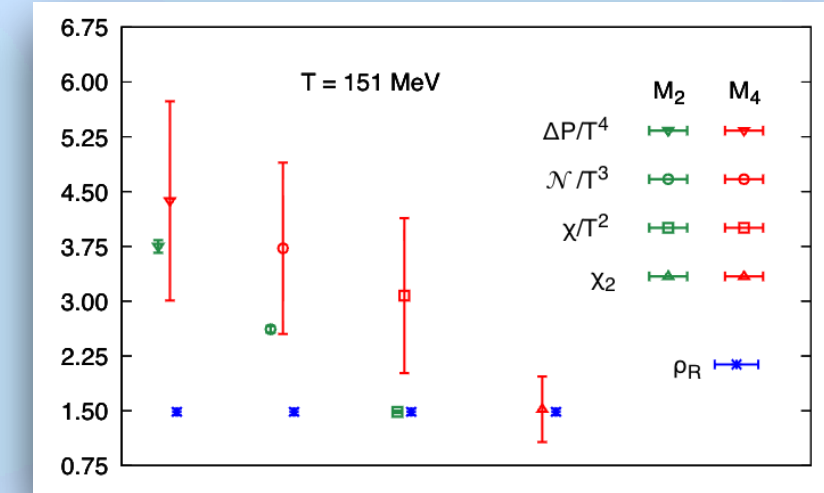

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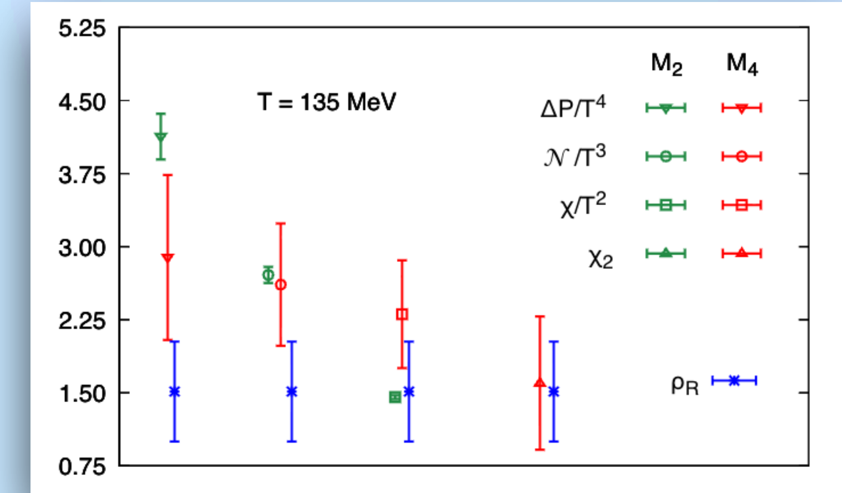

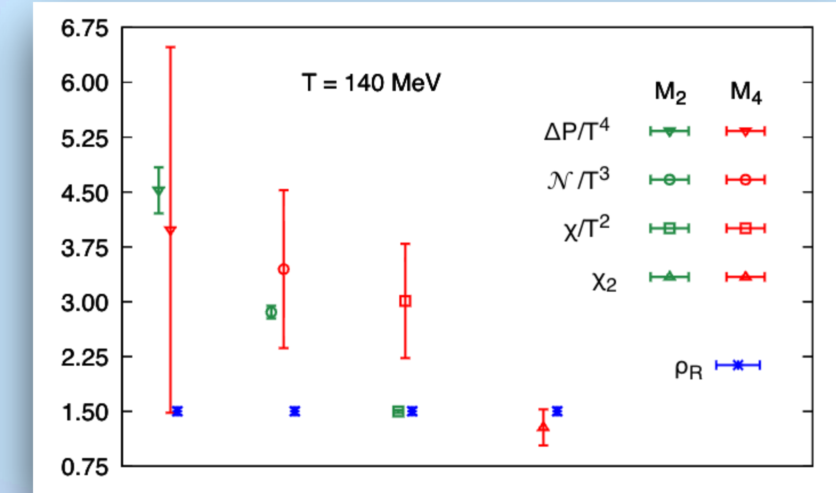

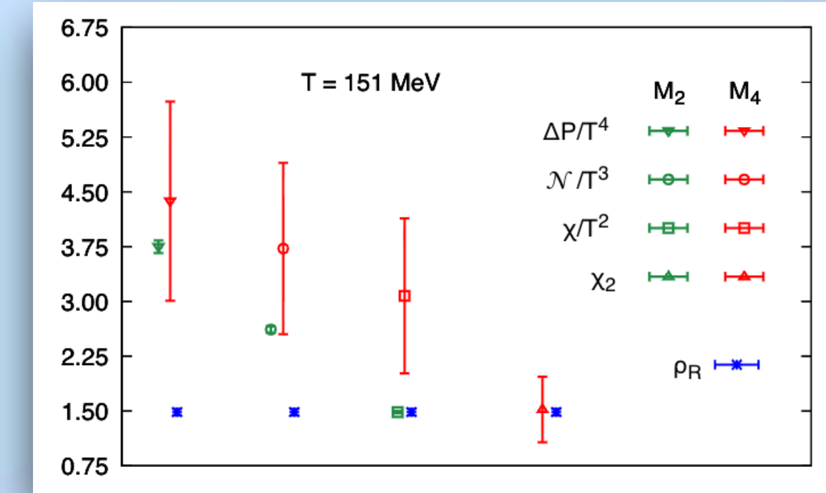

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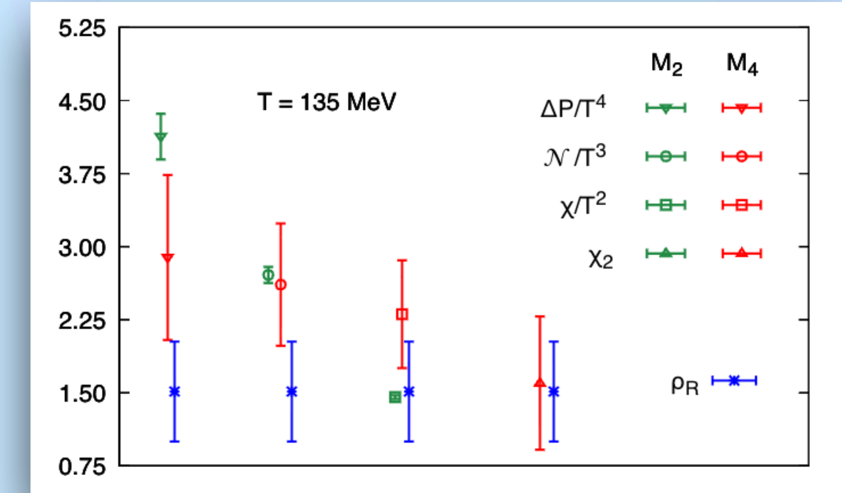

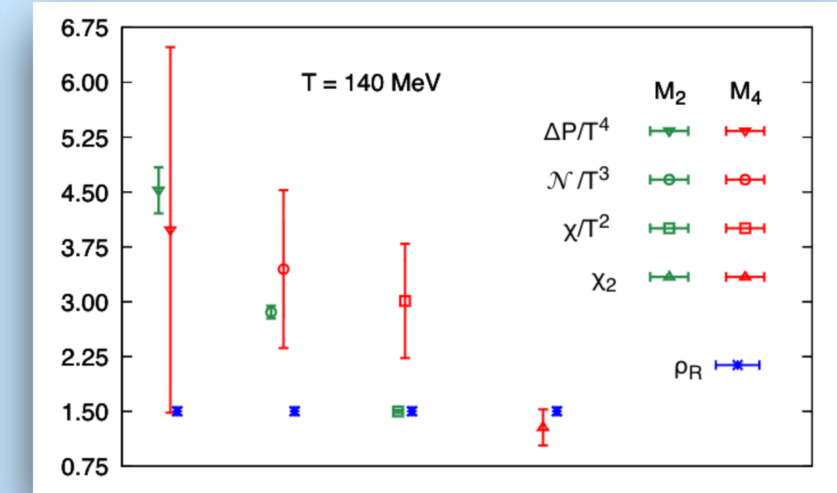

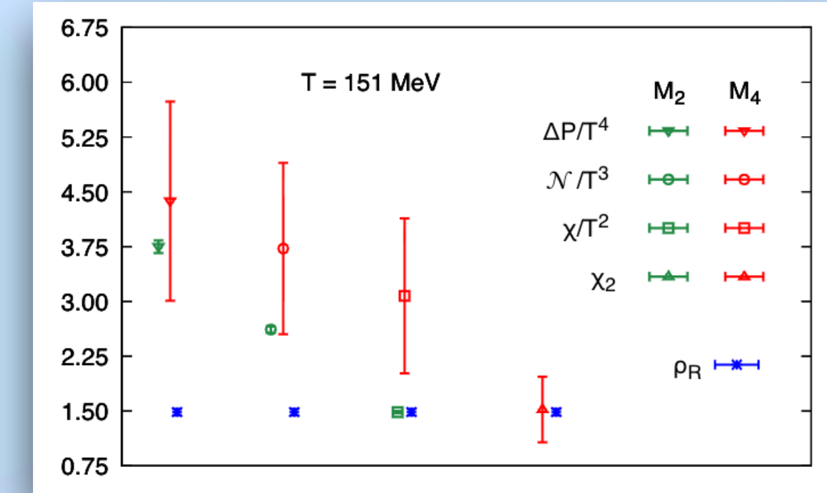

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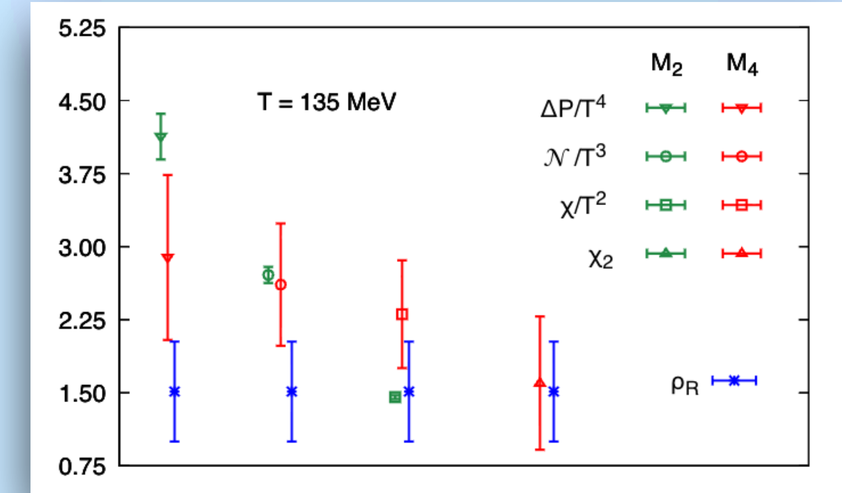

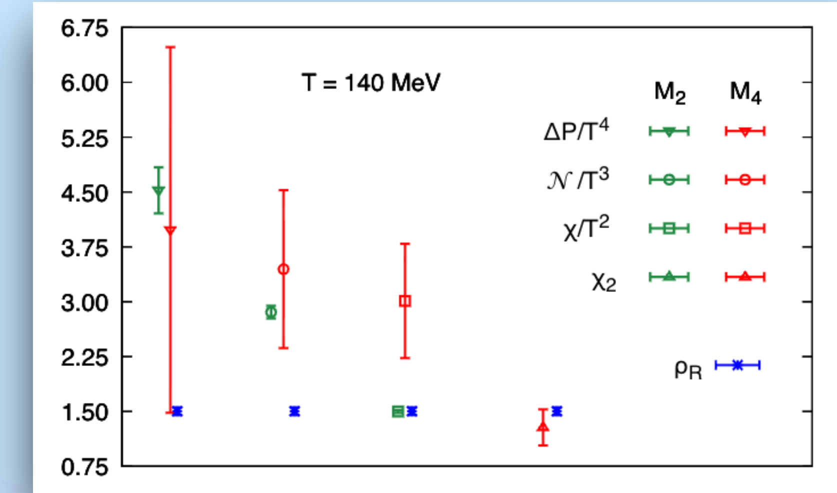

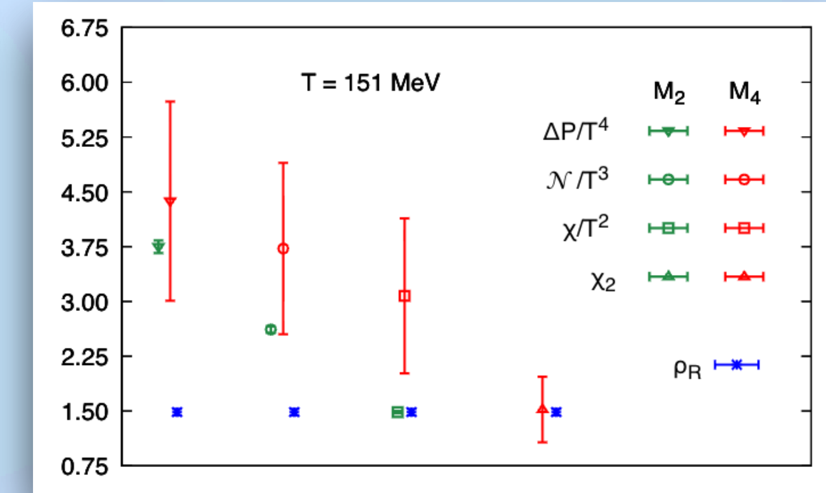

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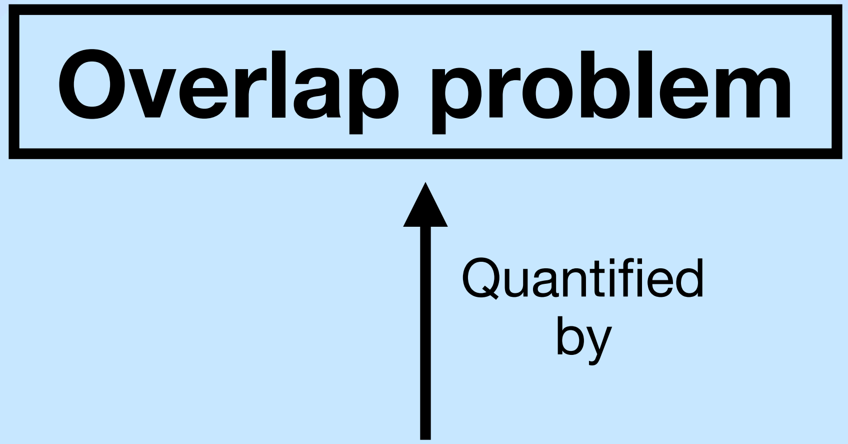
For higher order μ_I derivatives of ΔP

Overlap problem


Overlap problem
Quantified
by

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Results???

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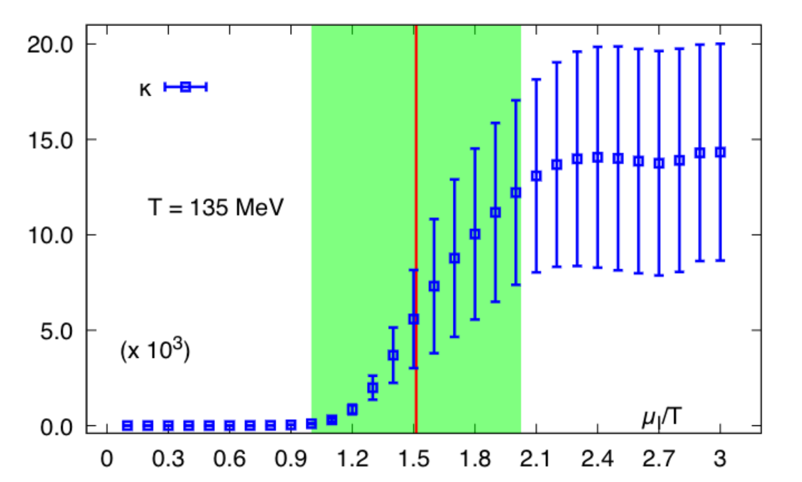

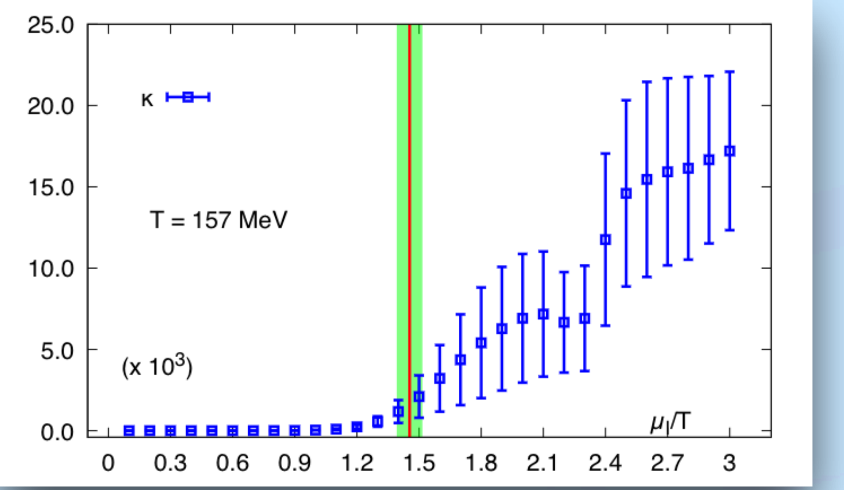

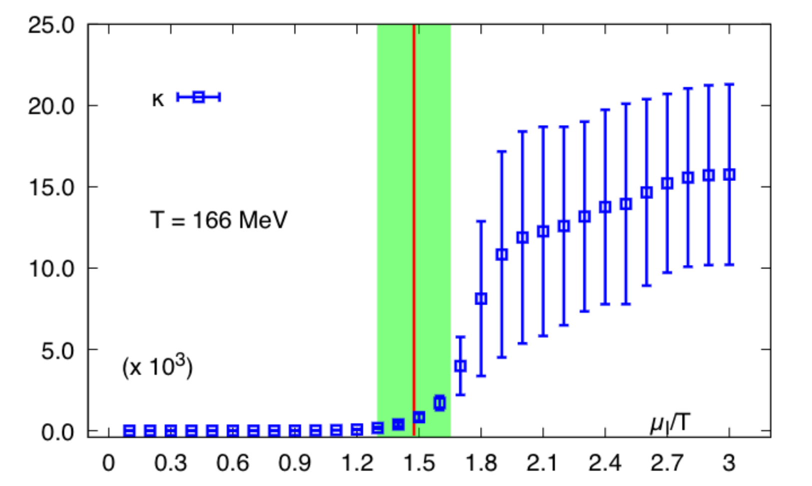

- **Controllable until** Radius of convergence (**drastic after that**) → for all the three T .
- Indicates the **efficacy of $\mu_I = 0$ extrapolations** to **determine finite μ_I observables** .
- This truly **breaks down** and is **unreliable** beyond the Radius of convergence μ_I^ρ

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With lot of future things to do...

Future works and Outlook

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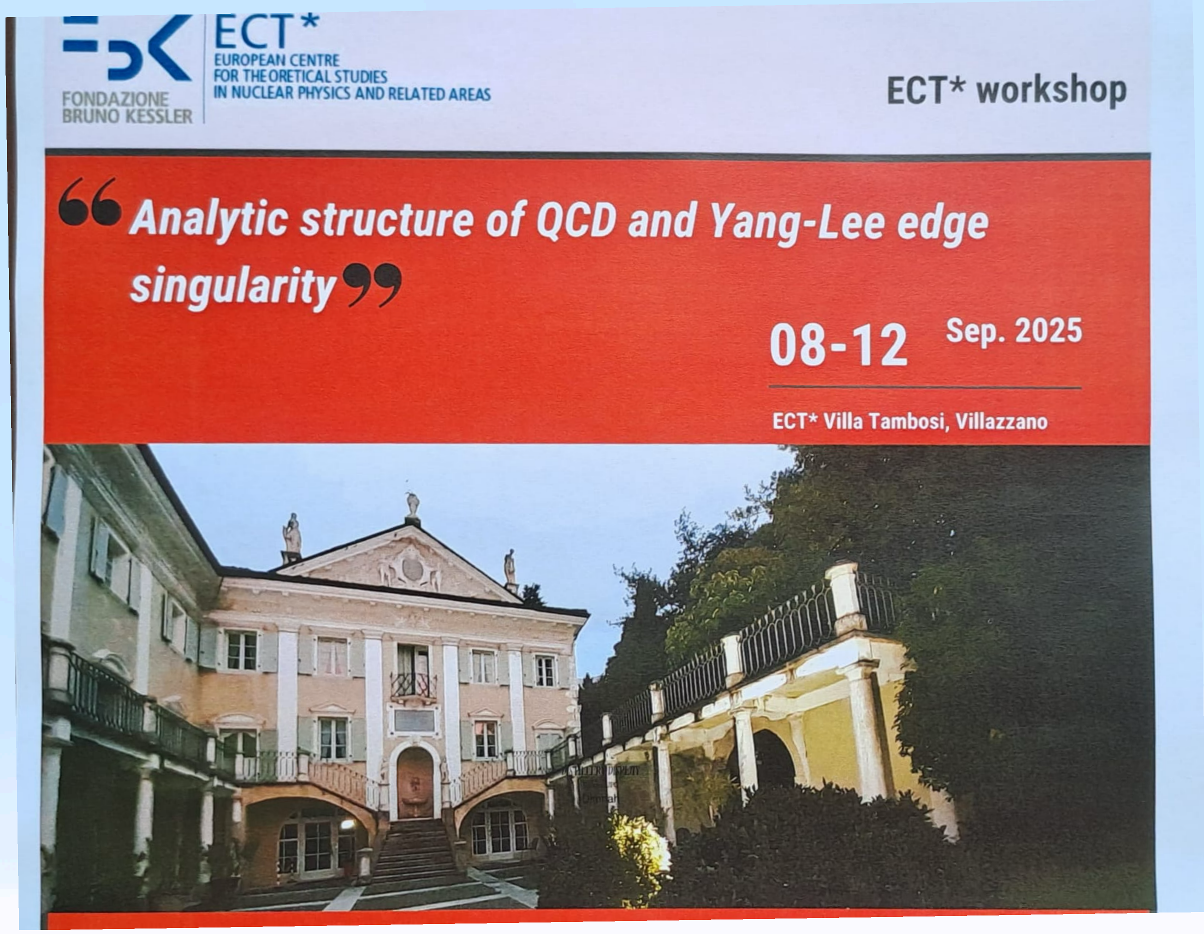
From a LY zero perspective



THANK YOU FOR YOUR ATTENTION

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Backup slides

The nomenclature of isospin

$$B = \frac{1}{3}\left(N_u + N_d + N_s\right) \ , \ S = -N_s \, , \ I = \frac{1}{2}\left(N_u - N_d\right)$$

$$\mu_B = \frac{3}{2}\left(\mu_u + \mu_d\right) \qquad \mu_I = \left(\mu_u - \mu_d\right)$$

$B_u = B_d = B_s = 1/3$ (Baryons are $3-$quark systems)

$|I_p| = |I_n| = 1/2$, since $2I + 1 = 2$ (proton and neutron)

Thus, $I_u = -I_d$ and $|I_u| = |I_d| = 1/2$