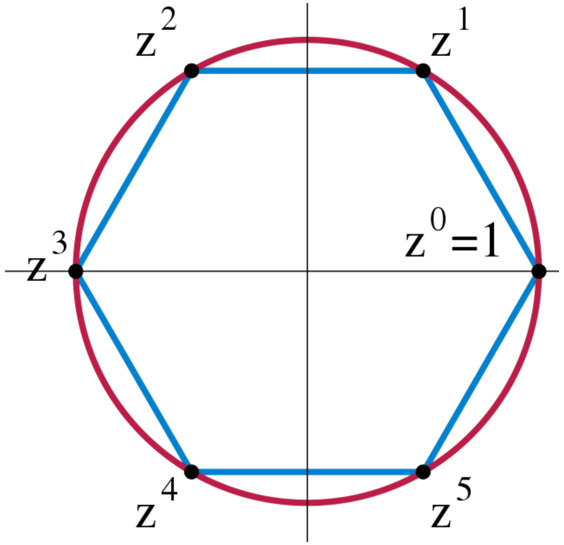
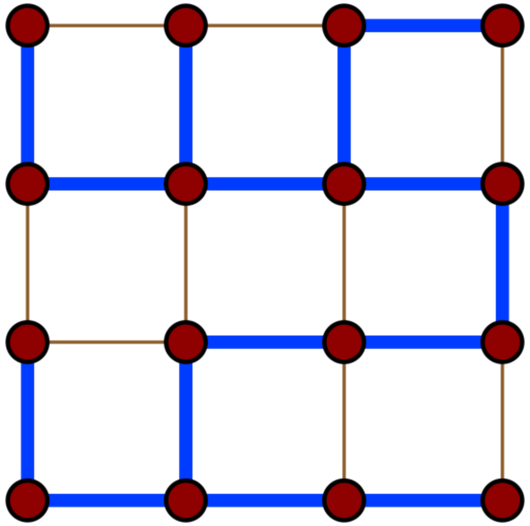
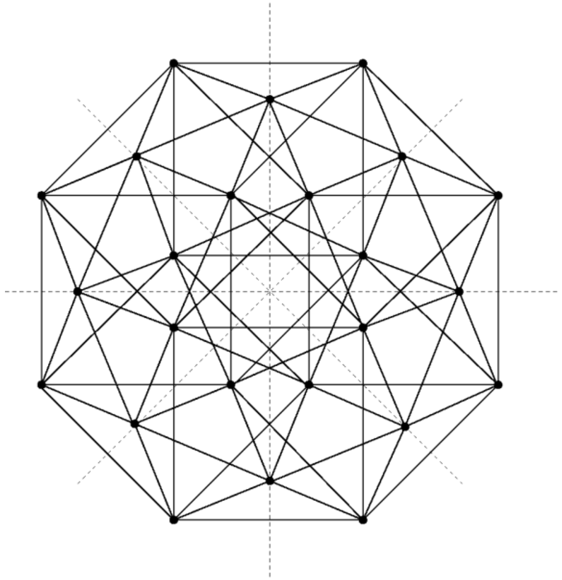
How many states are gauge-invariant?





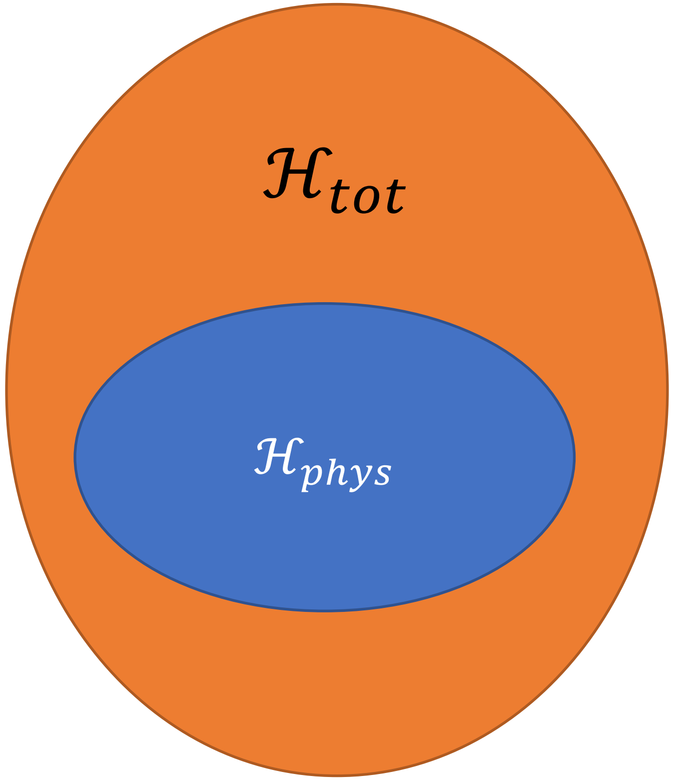


Alessandro Mariani
University of Turin, Italy

Gauge-invariant states

$$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$$

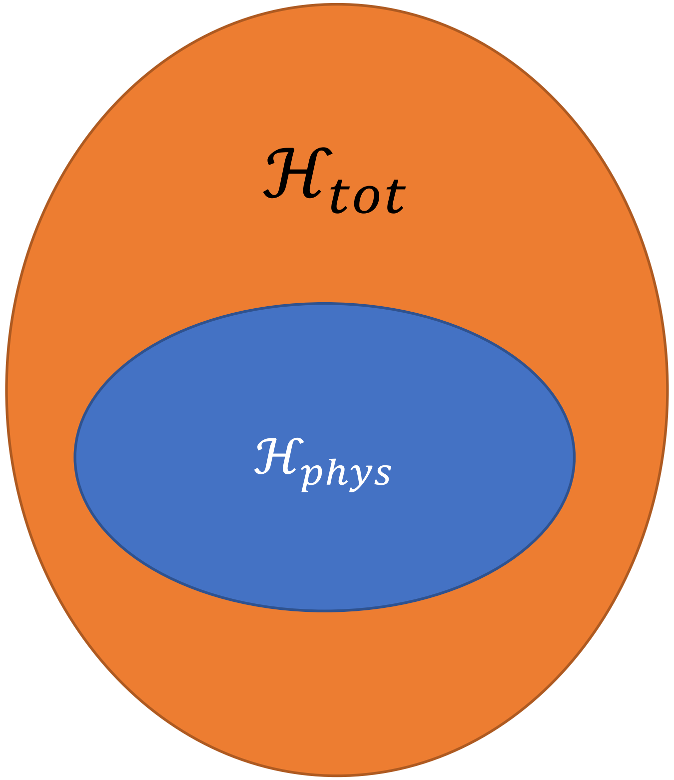
Only states which satisfy the **Gauss law** are **physical**.



Gauge-invariant states

$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$

Only states which satisfy the **Gauss law** are **physical**.



When working in gauge-invariant formulations:

It is useful to know **how many states are gauge-invariant:**
1) Resource estimation
2) Crosscheck

Many schemes for truncating the Hilbert space

Bosonic QFTs have infinite-dimensional
Hilbert space

Many ways have been designed to make
the Hilbert space finite-dimensional:

Many schemes for truncating the Hilbert space

Bosonic QFTs have infinite-dimensional
Hilbert space

Many ways have been designed to make
the Hilbert space finite-dimensional:

Quantum Link Models

Truncation in electric field basis

Finite subgroups

Orbifold

q-deformation

Mixed basis

Fuzzy

Finite subsets

Many more…

Many schemes for truncating the Hilbert space

Bosonic QFTs have infinite-dimensional
Hilbert space

Many ways have been designed to make
the Hilbert space finite-dimensional:

Quantum Link Models

Truncation in electric field basis

Finite subgroups

Orbifold

q-deformation

Mixed basis

Fuzzy

Finite subsets

Simple formula for #
gauge-invariant states

Many more…

Many schemes for truncating the Hilbert space

Bosonic QFTs have infinite-dimensional
Hilbert space

Many ways have been designed to make
the Hilbert space finite-dimensional:

Quantum Link Models

Truncation in electric field basis

Finite subgroups

Orbifold

q-deformation

Mixed basis

Fuzzy

Finite subsets

Simple formula for #
gauge-invariant states

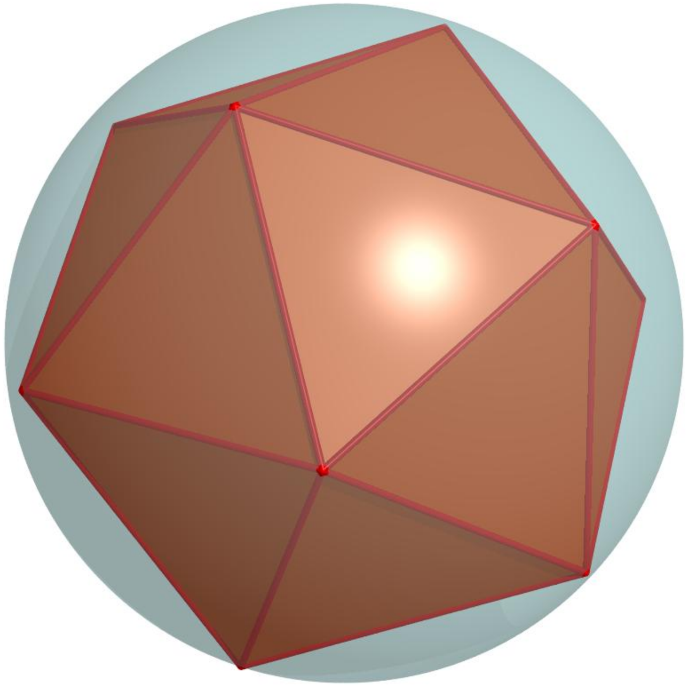
Many more…

Harder

Finite gauge groups

Idea: replace the gauge group (a Lie group) with a **finite subgroup** G.

e.g. $\mathbb{Z}_N \leq U(1)$,
$Q_8 \leq SU(2)$,
$S(1080) \leq SU(3)$



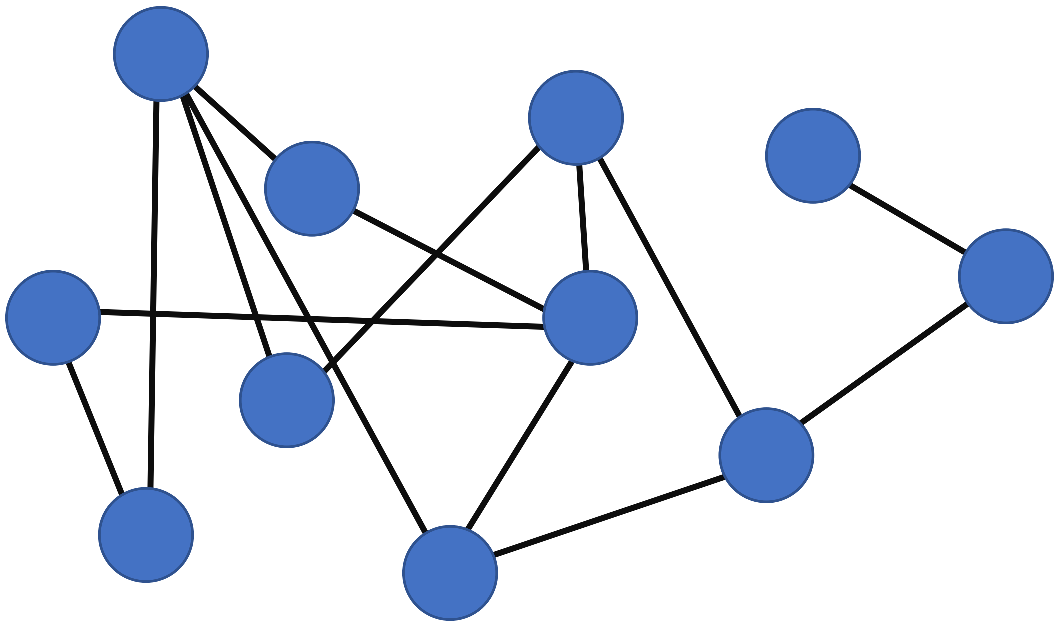
The link variable $U \in G$ can take only finitely-many values.
⟶ **Hilbert space is finite-dimensional**.

- Continuum limit via improved actions [Alexandru et al ’19, ‘22]
- Can construct Hamiltonian [Orland ‘91, Harlow & Ooguri ’18, Mariani, Pradhan, Ercolessi ’23]

[Hasenfratz & Niedermayer ’01]

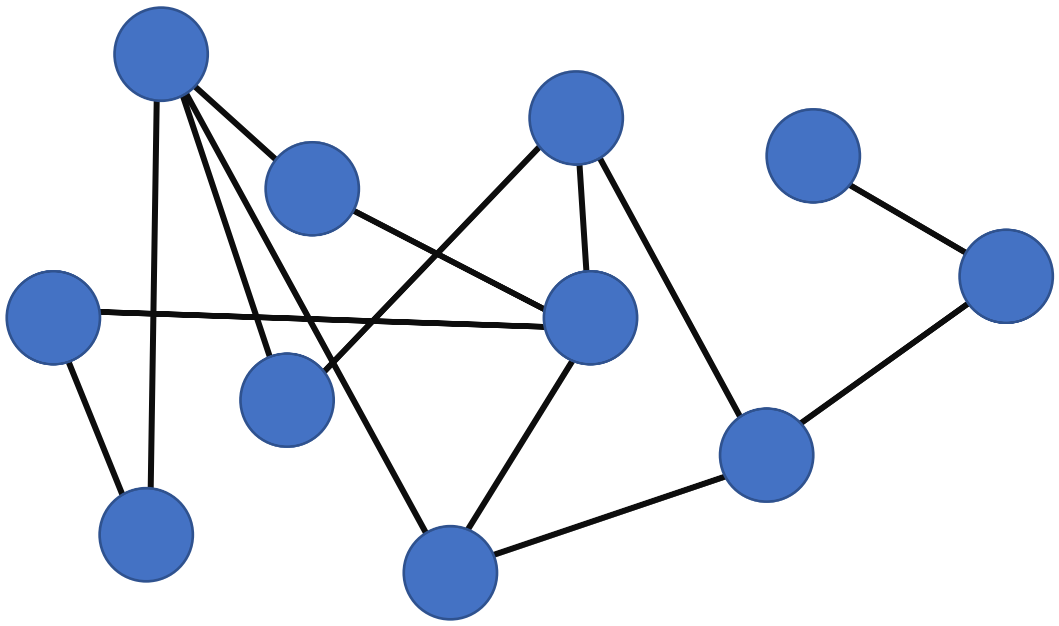
Gauge invariant states

Setting: discretize space as a **graph**:



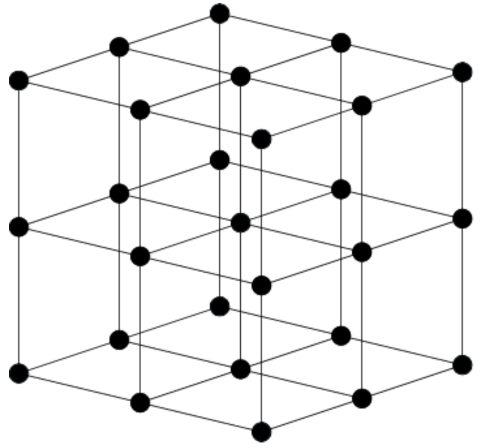
Gauge invariant states

Setting: discretize space as a **graph**:

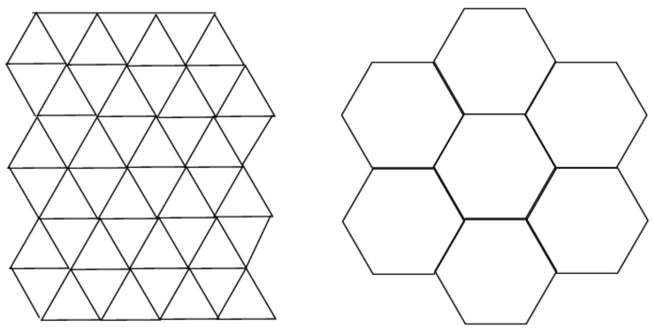


Many interesting special cases:

hypercubic lattices in d dimensions, with open, periodic or mixed boundary conditions

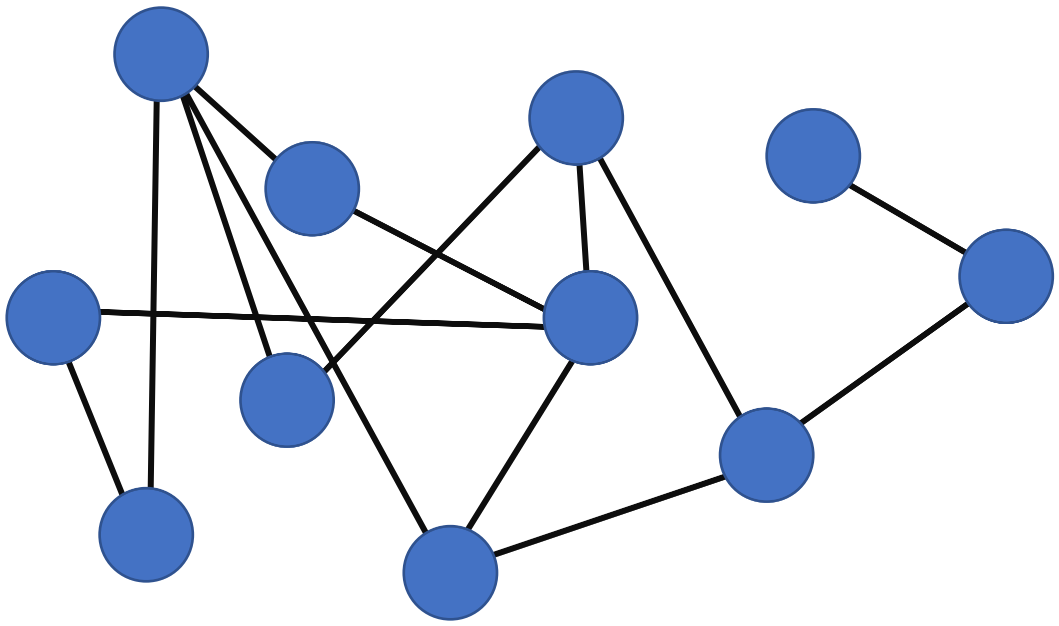


triangular, honeycomb lattices, etc.



Gauge invariant states

Setting: discretize space as a **graph**:



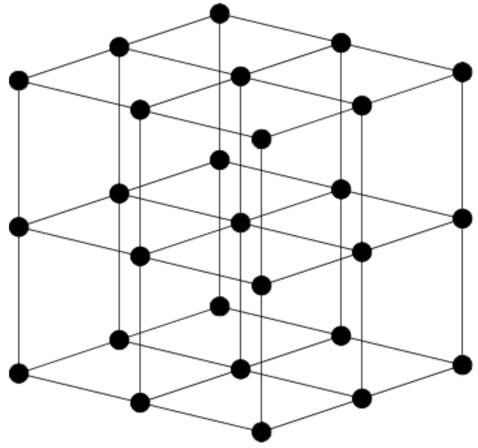
Put one group element $g_l \in G$ per link l

Orthonormal basis of Hilbert space $|g_l\rangle$

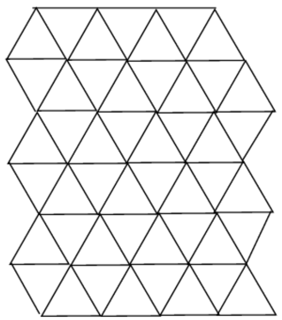
$$\mathcal{H}_{tot} = \bigotimes_{links} \mathbb{C}[G]$$

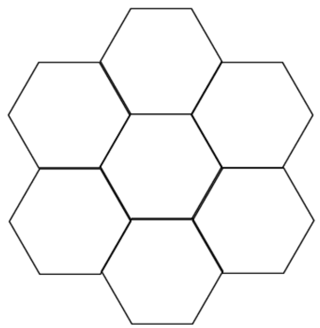
Many interesting special cases:

hypercubic lattices in d dimensions, with open, periodic or mixed boundary conditions



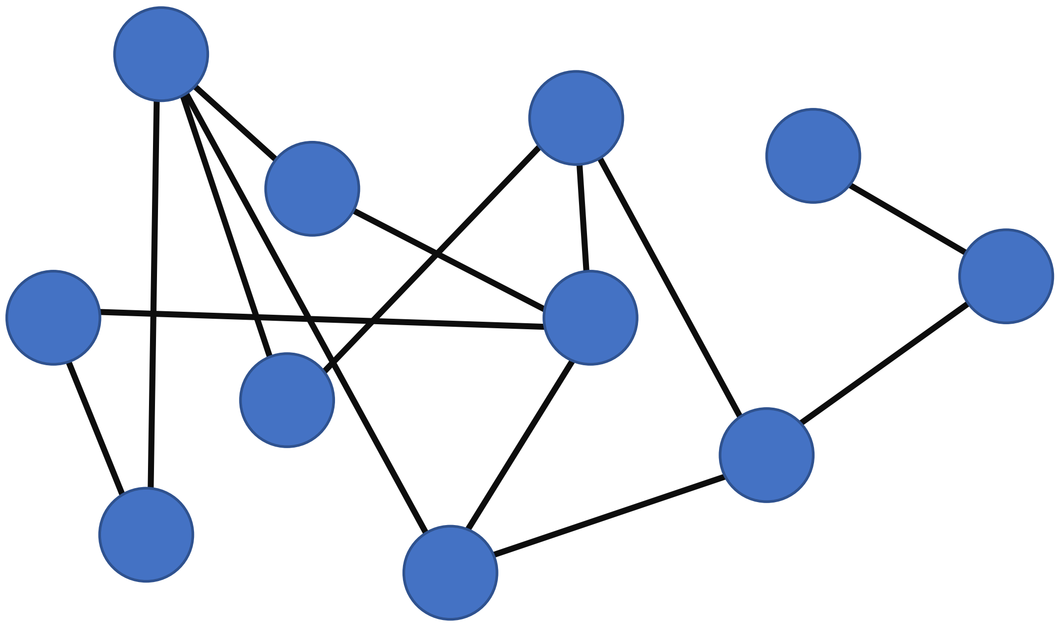
triangular, honeycomb lattices, etc.





Gauge invariant states

Setting: discretize space as a **graph**:



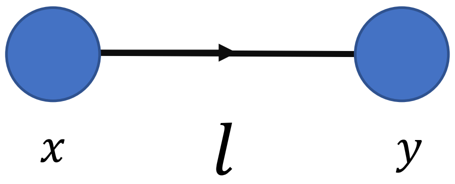
Put one group element $g_l \in G$ per link l

Orthonormal basis of Hilbert space $|g_l\rangle$

$$\mathcal{H}_{tot} = \bigotimes_{links} \mathbb{C}[G]$$

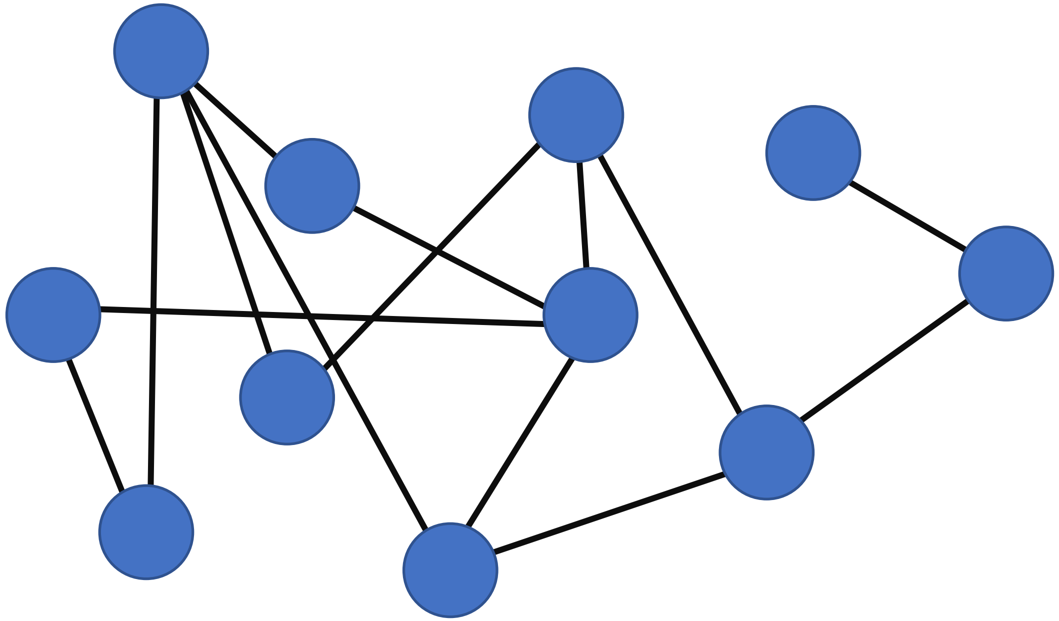
Gauge transformations act as:

$$\mathcal{G}|g_l\rangle = |g_x g_l g_y^{-1}\rangle$$



Gauge invariant states

Setting: discretize space as a **graph**:



Put one group element $g_l \in G$ per link l

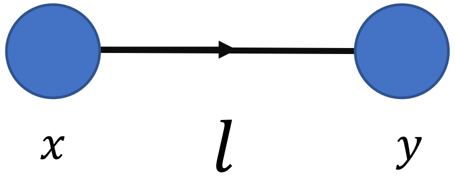
Orthonormal basis of Hilbert space $|g_l\rangle$

$$\mathcal{H}_{tot} = \bigotimes_{links} \mathbb{C}[G]$$

Gauge transformations act as:

$$\mathcal{G}|g_l\rangle = |g_x g_l g_y^{-1}\rangle$$

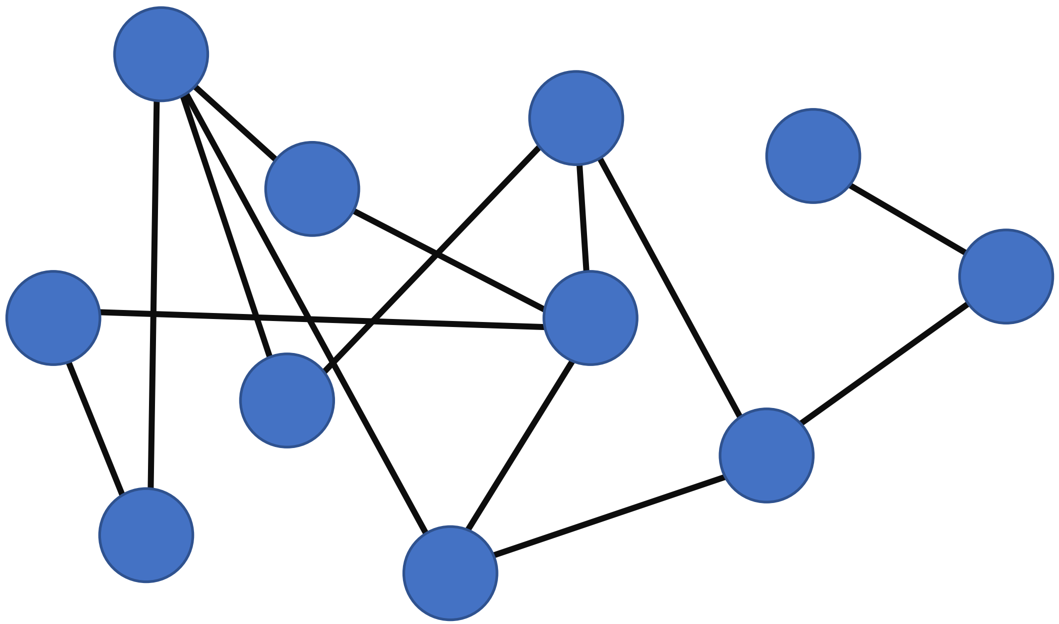
Same action
On every link



Gauge invariant states

Same action
On every link

Setting: discretize space as a **graph**:



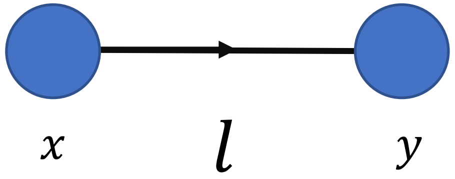
Put one group element $g_l \in G$ per link l

Orthonormal basis of Hilbert space $|g_l\rangle$

$$\mathcal{H}_{tot} = \bigotimes_{links} \mathbb{C}[G]$$

Gauge transformations act as:

$$\mathcal{G}|g_l\rangle = |g_x g_l g_y^{-1}\rangle$$



Gauge-invariant states satisfy:

$$\mathcal{G}|\psi\rangle = |\psi\rangle$$

They form the physical Hilbert space

$$\mathcal{H}_{phys}$$

Counting gauge-invariant states

Write down explicit projector $P: \mathcal{H}_{tot} \to \mathcal{H}_{phys}$

$$P = \frac{1}{|G|^V} \sum_{\mathcal{G} \in G^V} \mathcal{G}$$

$$P^2 = P$$

[Mariani, Pradhan, Ercolessi ‘23]

[Mariani ‘24, ‘25]

Counting gauge-invariant states

Write down explicit projector $P: \mathcal{H}_{tot} \to \mathcal{H}_{phys}$

$$P = \frac{1}{|G|^V} \sum_{\mathcal{G} \in G^V} \mathcal{G}$$

$$\dim \mathcal{H}_{phys} = \operatorname{tr} P = \frac{1}{|G|^V} \sum_{\mathcal{G} \in G^V} \operatorname{tr} \mathcal{G}$$

$P^2 = P$

eigenvalue 1 ⟶ $\in \mathcal{H}_{phys}$

eigenvalue 0 ⟶ not in $\mathcal{H}_{phys}$

[Mariani, Pradhan, Ercolessi ‘23]

[Mariani ‘24, ‘25]

The dimension of the physical subspace

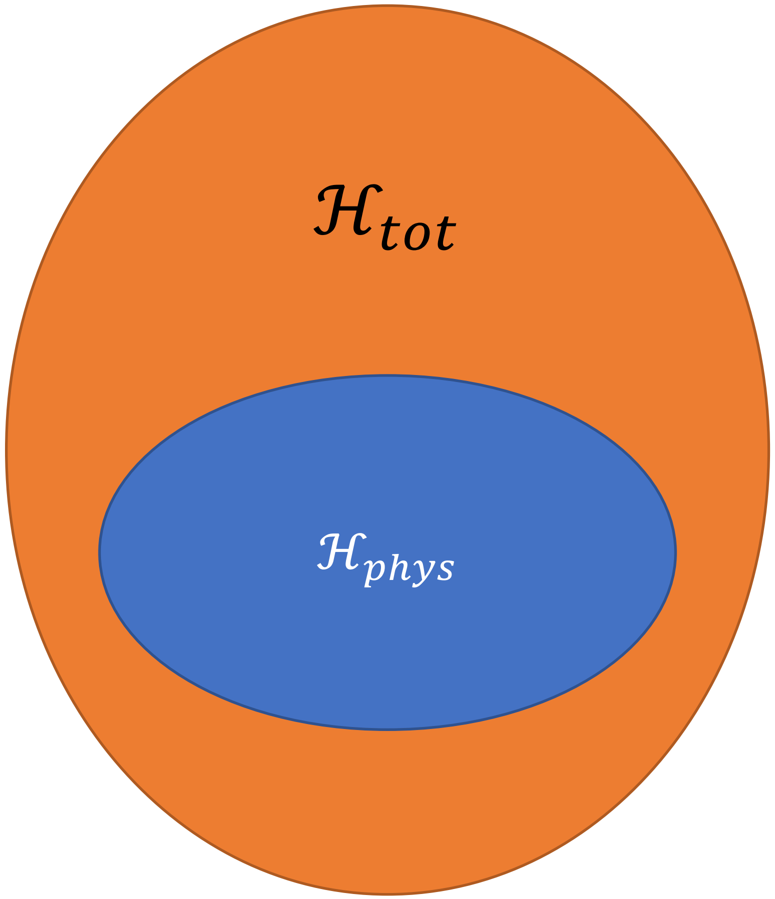
For a **pure gauge theory** with arbitrary finite group G on an arbitrary lattice with V sites and E links:

$$\dim \mathcal{H}_{tot} = |G|^E$$

$$\dim \mathcal{H}_{phys} = \sum_C \left(\frac{|G|}{|C|}\right)^{E-V}$$

C are the **conjugacy classes** of G, i.e. g_1 and g_2 are in the same C iff $g_2 = g\, g_1\, g^{-1}$.

Remember assumption: Gauss law the same everywhere.

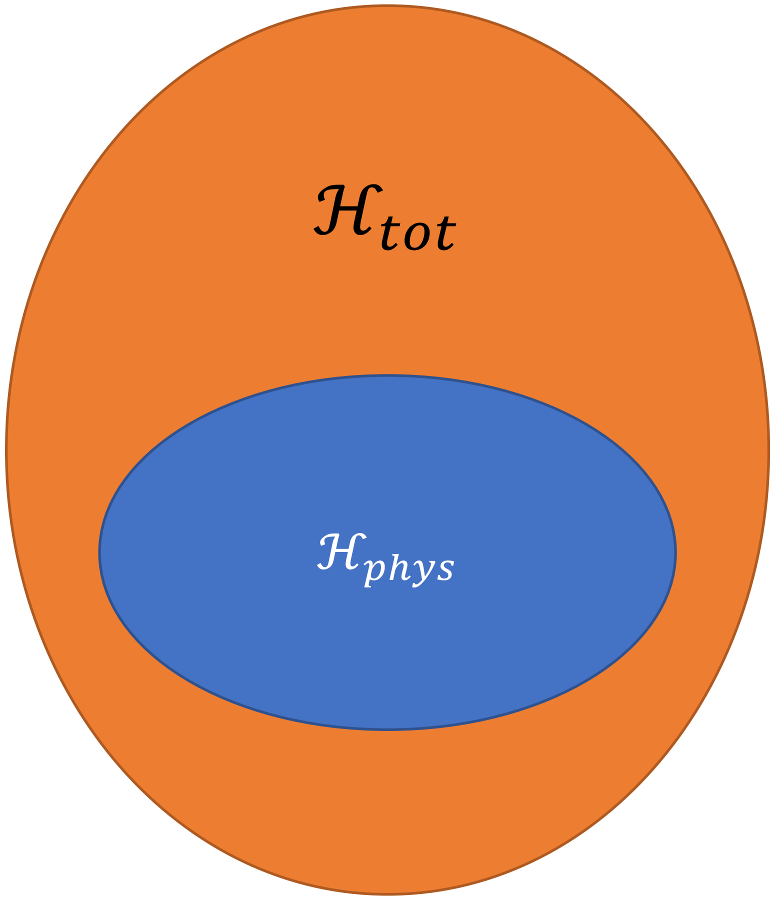


[Mariani, Pradhan, Ercolessi ‘23]

[Mariani ‘24]

Matter fields and twisted boundaries

As of this week, formulas also for scalar and fermionic matter, as well as twisted boundary conditions:



Matter fields and twisted boundaries

As of this week, formulas also for scalar and fermionic matter, as well as twisted boundary conditions:

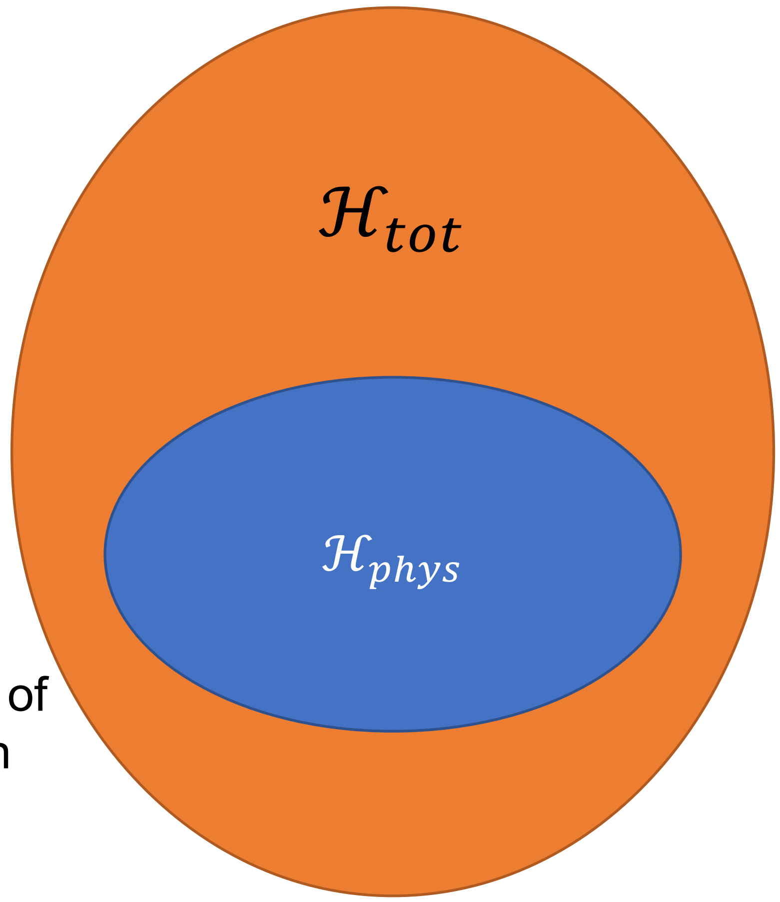
Scalar fields:

$$\dim \mathcal{H}_{phys} = \sum_{C} \left(\frac{|G|}{|C|} \right)^{E-V} |\mathrm{Fix}(C)|^{V}$$

fixed points of group action

Scalar field valued in an arbitrary finite set S, its local Hilbert space is $\mathbb{C}[S]$.

G acts on S via group action.



arXiv:2509.02173

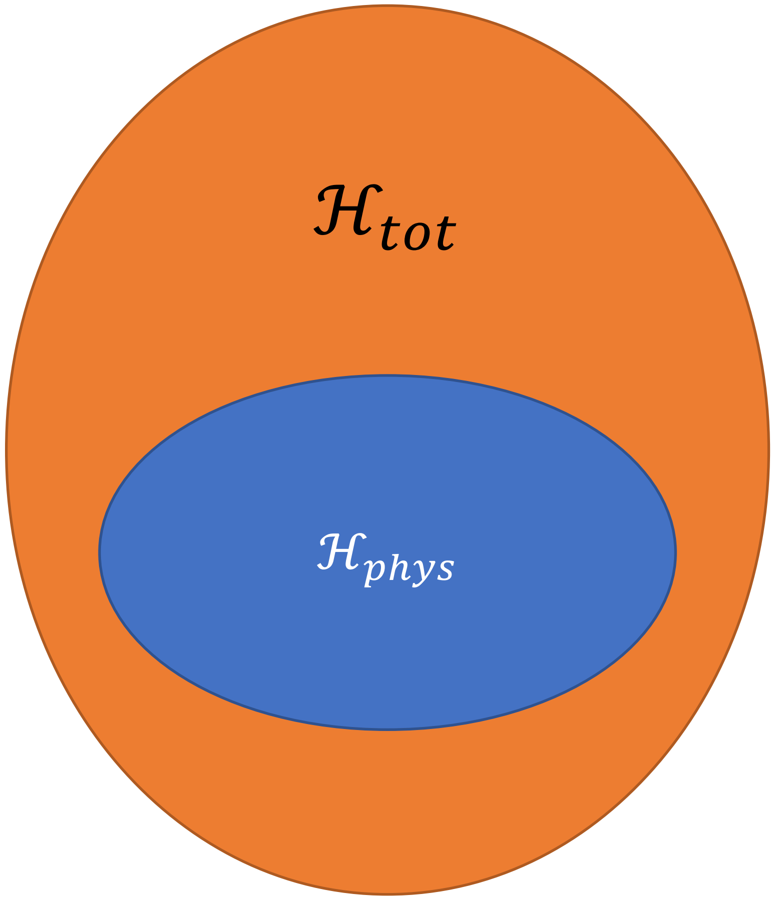
Matter fields and twisted boundaries

As of this week, formulas also for scalar and fermionic matter, as well as twisted boundary conditions:

Fermion fields:

$$\dim \mathcal{H}_{phys} = \sum_{C} \left(\frac{|G|}{|C|}\right)^{E-V} \det\big(1+\rho(C)\big)^{VN_s}$$

Fermions live in ρ representation of G.



arXiv:2509.02173

Matter fields and twisted boundaries

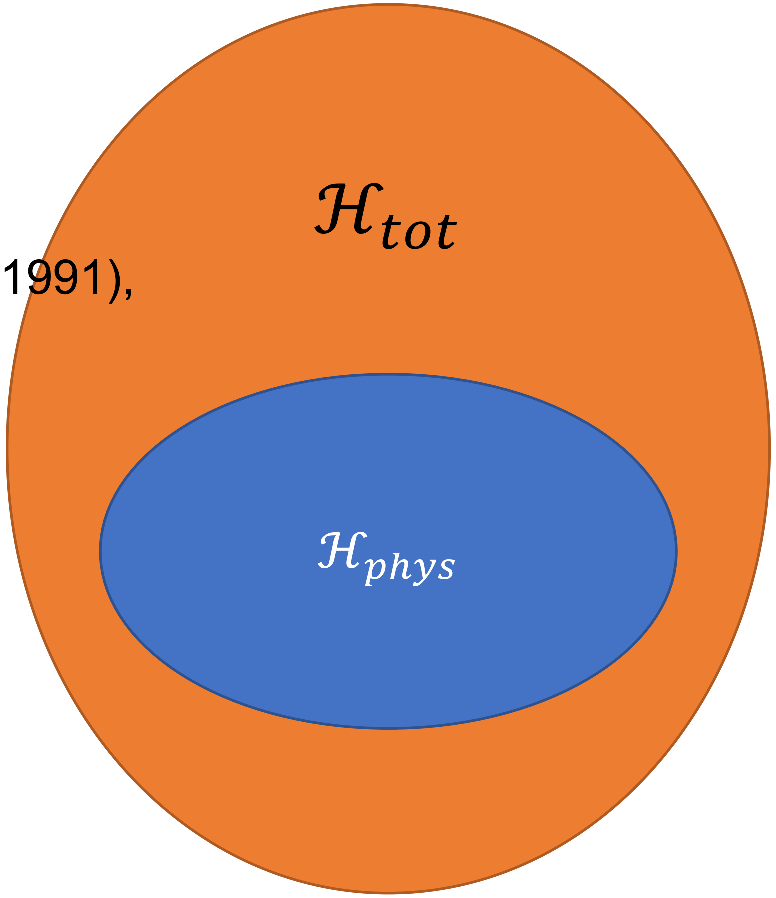
As of this week, formulas also for scalar and fermionic matter, as well as twisted boundary conditions:

e.g. C-periodic boundary conditions: [Kronfeld & Wiese (1991), Wiese (1992)]

$$\dim \mathcal{H}_{phys} = \sum_{C,\, C=C^{-1}} \left(\frac{|G|}{|C|}\right)^{E-V}$$

i.e. sum over only those conjugacy classes C which are self-inverse (they contain all their inverses).

Gauss law acts differently on boundary links.



arXiv:2509.02173

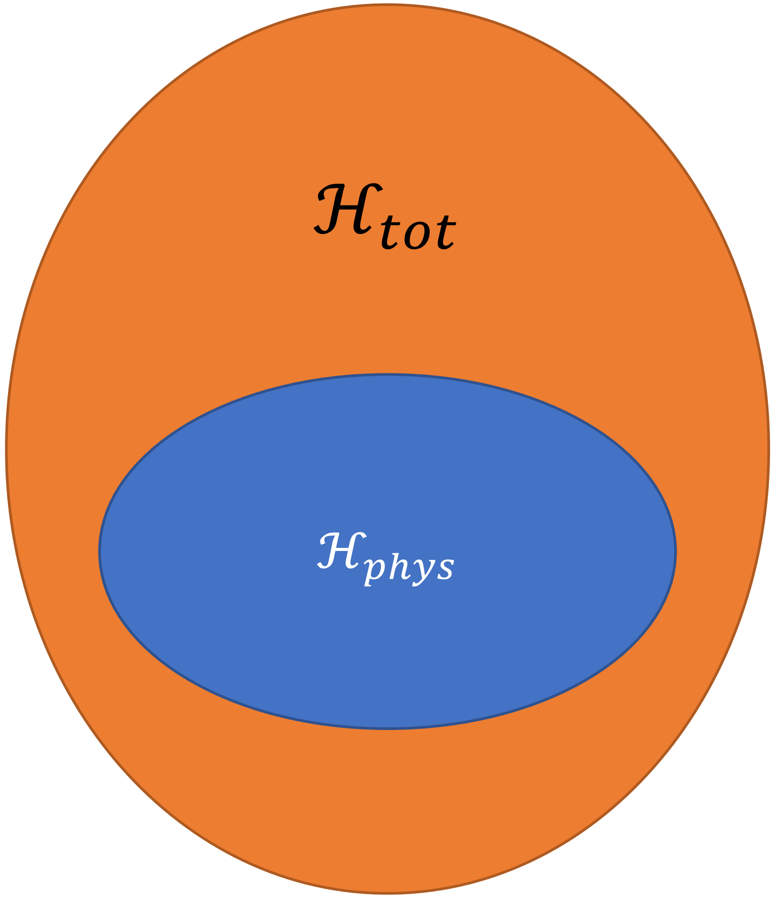
How to describe the physical subspace?

[Durhuus '80, Sengupta '94, Lévy '04]

$$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$$

(traced) Wilson loops **do not** necessarily span $\mathcal{H}_{phys}$

For $SU(N)$ Wilson loops in the fundamental span $\mathcal{H}_{phys}$.



See [Mariani '24] for a summary.

How to describe the physical subspace?

[Durhuus ’80, Sengupta ’94, Lévy ‘04]

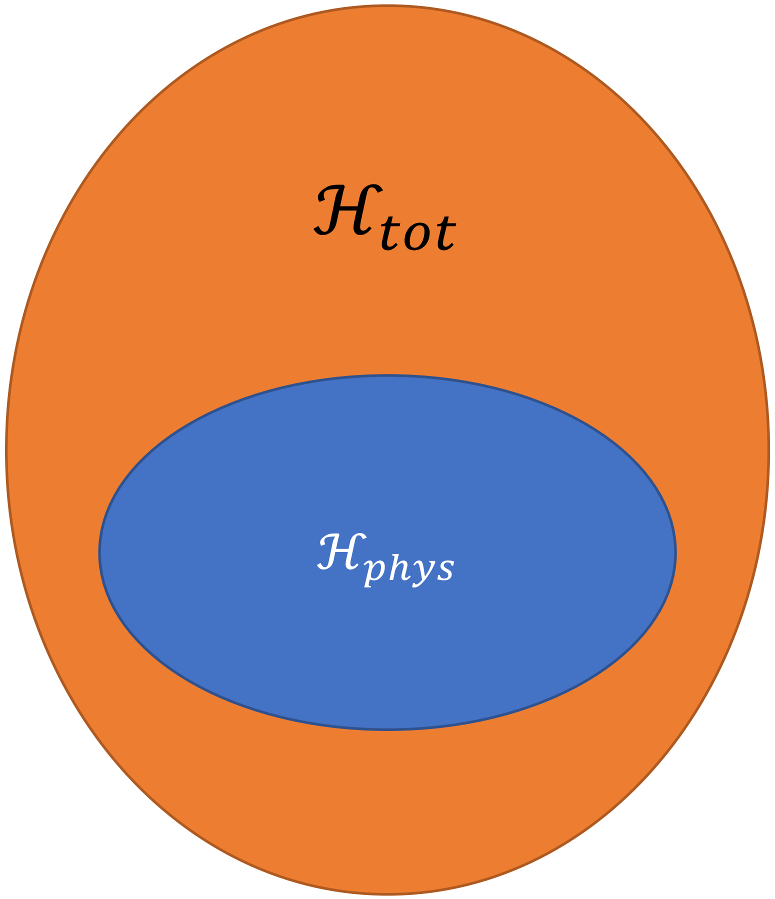
$$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$$

(traced) Wilson loops **do not** necessarily span $\mathcal{H}_{phys}$

For $SU(N)$ Wilson loops in the fundamental span $\mathcal{H}_{phys}$.

For direct products of $SU(N)$, $U(N)$, $SO(N), O(N)$ and Abelian groups, Wilson loops span $\mathcal{H}_{phys}$, but **all irreps may be needed** (e.g. $SO(2N)$).

For other groups such as G_2 it is **not known**.

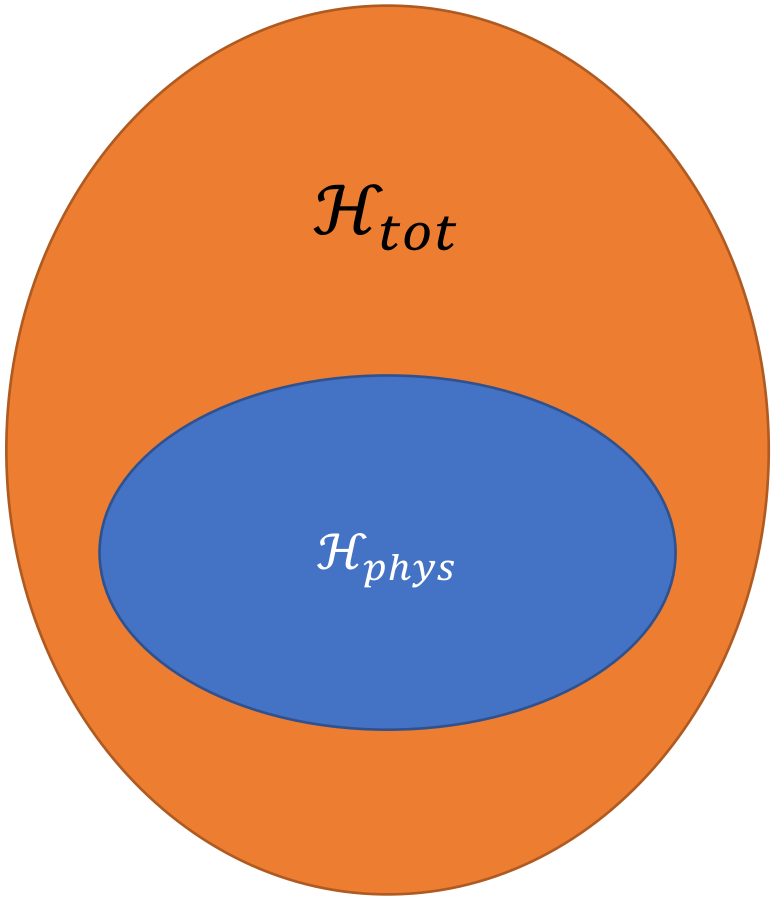


See [Mariani ‘24] for a summary.

How to describe the physical subspace?

[Durhuus ’80, Sengupta ’94, Lévy ‘04]

$$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$$



(traced) Wilson loops **do not** necessarily span $\mathcal{H}_{phys}$

For $SU(N)$ Wilson loops in the fundamental span $\mathcal{H}_{phys}$.

For direct products of $SU(N)$, $U(N)$, $SO(N), O(N)$ and Abelian groups, Wilson loops span $\mathcal{H}_{phys}$, but **all irreps may be needed** (e.g. $SO(2N)$).

For other groups such as G_2 it is **not known**.

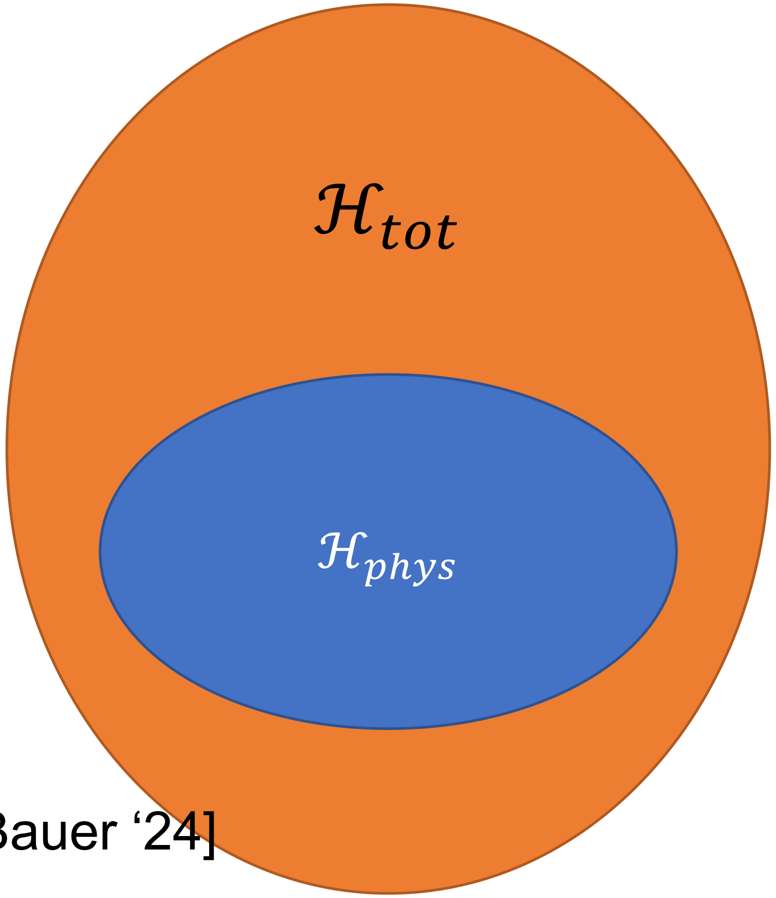
↓

Cannot use Wilson loops for general description.
(various other implications: entanglement entropy, etc)

See [Mariani ‘24] for a summary.

How to describe the physical subspace?

$$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$$



Not everything which works for $SU(N)$ also works for truncations.

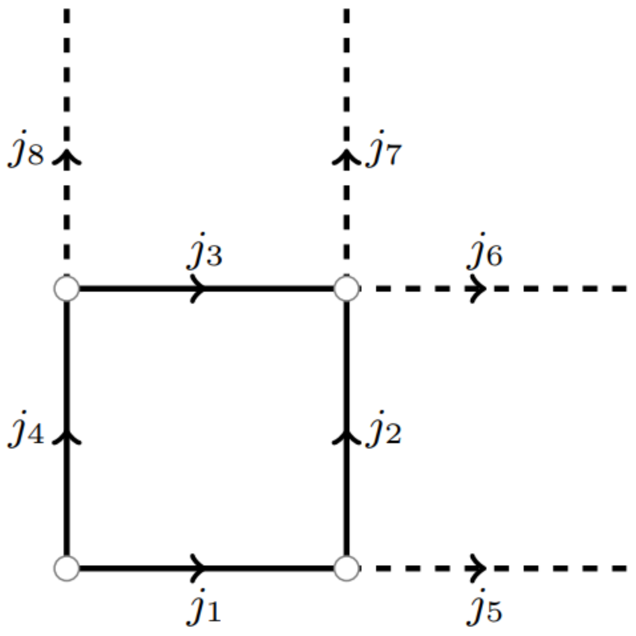
Alternatives:
- Representation basis / spin networks [Baez '94]
- untraced Wilson loops (maximal tree gauge fixing)
  Finite groups: [Mariani '24]
  Other contexts: [Grabowska, Kane, Bauer '24], [Burbano, Bauer '24]

What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

On each link the electric field takes a value

$$j_l = -s, -s+1, \dots, s-1, s$$



What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

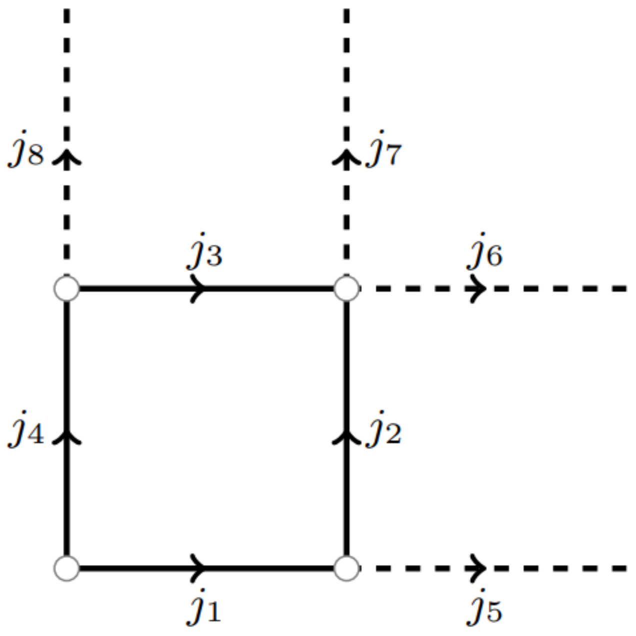
On each link the electric field takes a value

$$j_l = -s, -s+1, \ldots, s-1, s$$

Gauge-invariant states (for example on a square lattice) satisfy

$$j_1 + j_2 - j_3 - j_4 = 0$$

For the four links attached to the site.



What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

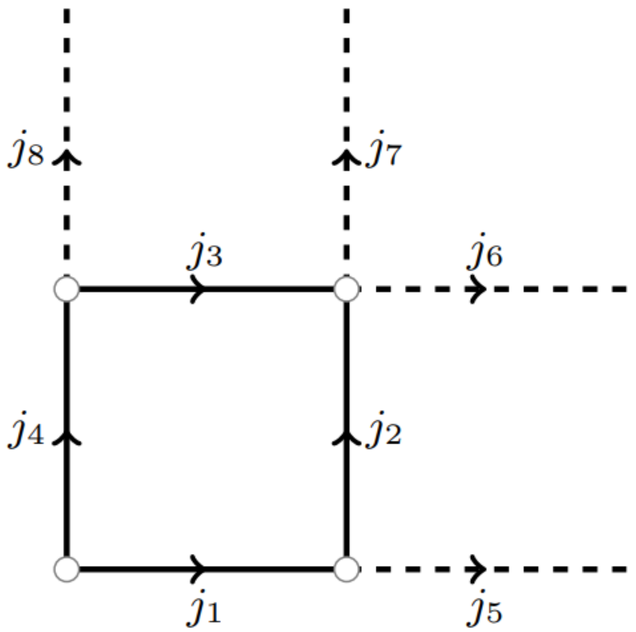
On each link the electric field takes a value

$$j_l = -s, -s+1, \dots, s-1, s$$

Gauge-invariant states (for example on a square lattice) satisfy

$$j_1 + j_2 - j_3 - j_4 = 0$$

For the four links attached to the site.



Already known to mathematicians [Beck, Zlaslavsky '03]: open problem in graph theory.

Mathematicians have shown that $\dim \mathcal{H}_{phys}^{QLM}$ = polynomial in s. [Kochol '02]

What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

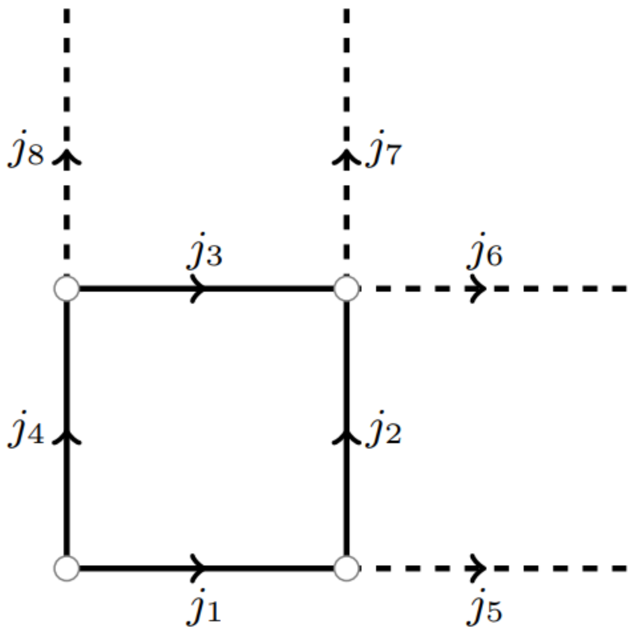
On each link the electric field takes a value

$$j_l = -s, -s+1, \dots, s-1, s$$

Gauge-invariant states (for example on a square lattice) satisfy

$$j_1 + j_2 - j_3 - j_4 = 0$$

For the four links attached to the site.



$\dim \mathcal{H}_{phys}^{QLM}$ does not depend simply on E and V (more geometric info needed)

On an arbitrary graph, with arbitrary spin on each link, and arbitrary charges the problem is #P-hard. [e.g. Baldoni-Silva et al ’03]

What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

On each link the electric field takes a value

$$j_l = -s, -s+1, \dots, s-1, s$$

Gauge-invariant states (for example on a square lattice) satisfy

$$j_1 + j_2 - j_3 - j_4 = 0$$

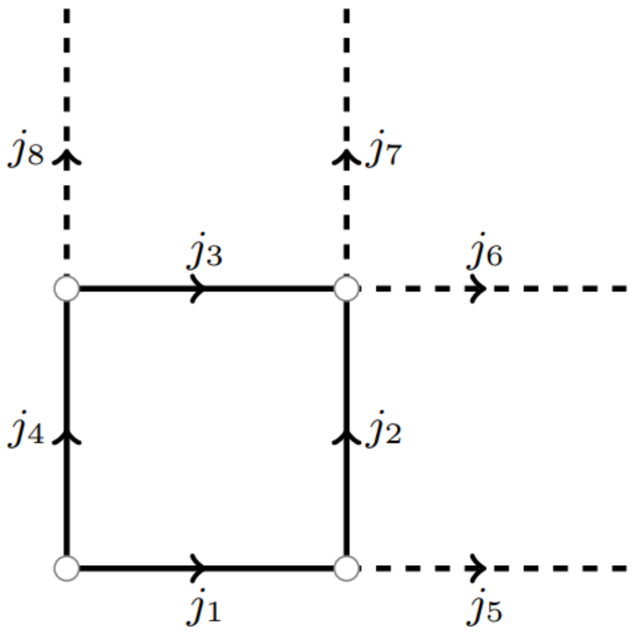
For the four links attached to the site.

Compare with $\mathbb{Z}_N$, $N = 2s + 1$ where the condition is

$$j_1 + j_2 - j_3 - j_4 = 0 \ (\text{mod } N)$$

where $j = 0, 1, \dots, N-1$. Then here the answer is

$$N^{E-V+1}$$



What about electric field truncations?

Choose an eigenbasis of the electric field $|jmn\rangle$, where j indexes the irreps of $SU(N)$. Truncate to $j \leq j_{\text{max}}$.

What about electric field truncations?

Choose an eigenbasis of the electric field $|jmn\rangle$, where j indexes the irreps of $SU(N)$. Truncate to $j \leq j_{\text{max}}$.

Can write again:

$$\dim \mathcal{H}_{phys} = \text{tr}\, P = \frac{1}{|G|^V} \sum_{\mathcal{G} \in G^V} \text{tr}\, \mathcal{G}$$

What about electric field truncations?

Choose an eigenbasis of the electric field $|jmn\rangle$, where j indexes the irreps of $SU(N)$. Truncate to $j \leq j_{\max}$.

Can write again:

$$\dim \mathcal{H}_{phys} = \operatorname{tr} P = \frac{1}{|G|^V} \sum_{\mathcal{G} \in G^V} \operatorname{tr} \mathcal{G}$$

To compute the trace, need character identity:

$$\sum_{j \in \mathrm{Irrep}} \chi_j(g)^* \chi_j(h) = \begin{cases} \frac{|G|}{|C|} & \text{if } g, h \in C \text{ (same conjugacy class)} \\ 0 & \text{otherwise} \end{cases}$$

But if we keep only some irreps (i.e. $j \leq j_{\max}$) the formula no longer simplifies.

Conclusions

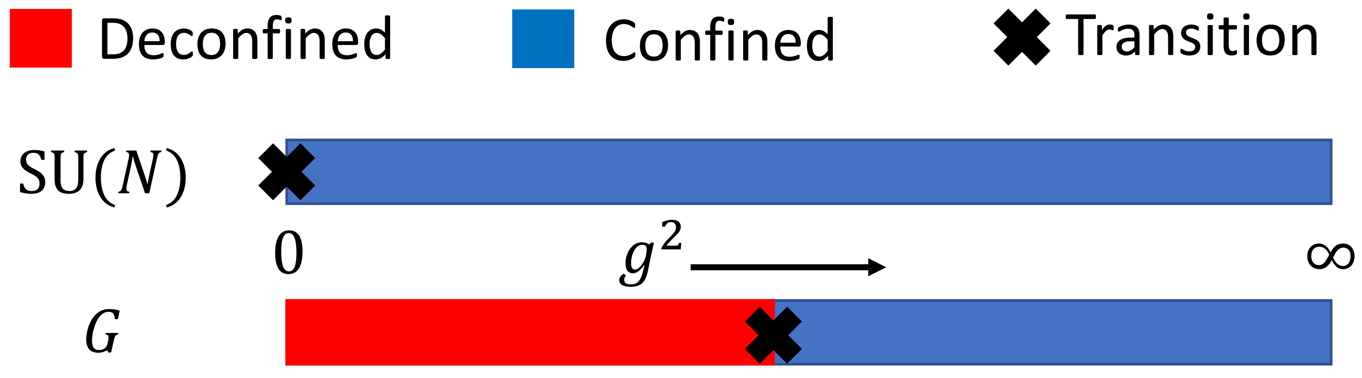
Can compute $\dim \mathcal{H}_{phys}$ for finite groups in various settings.

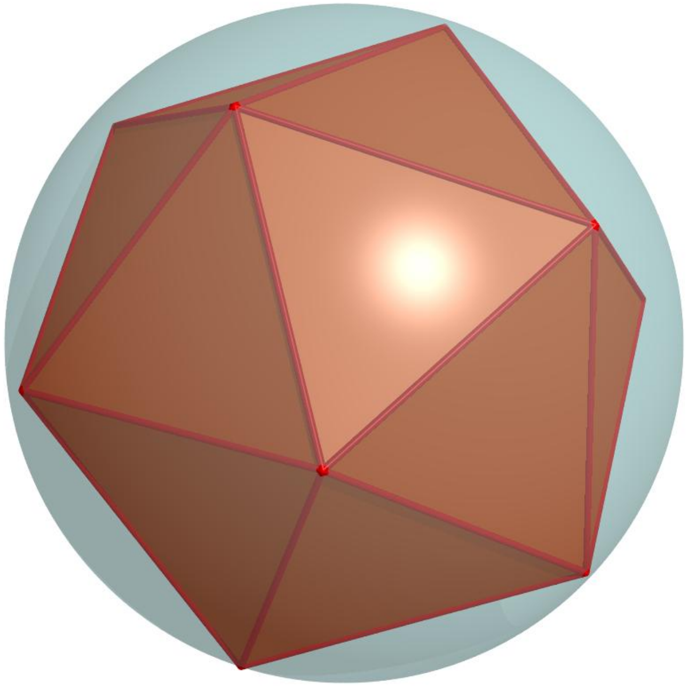
For other truncation methods, the problem is not so easy.

Not everything which works for $SU(N)$ also works for truncations thereof.

Finite subgroups

For finite subgroup $G \leq \mathrm{SU}(N)$ in 4D (zero temperature, pure gauge)





For G finite, the transition is only **first order**

→ effective theory

→ requires **improvement**

[Hasenfratz & Niedermayer ’01]