Static Quark Potential in a $\mathrm{SU}(2)$ Quantum Link Model

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Introduction to the $\mathrm{SU}(2)$ QLM

Wilson SU(2) LGT

Described by *link variables* $u_{x,i}$ and *canonic momentum operators* $\vec{L}_{x,i}$ and $\vec{R}_{x,i}$

$$u_{x,i} = u^0_{x,i}\mathbb{1} + iu^a_{x,i}\sigma_a \qquad u^0_{x,i}u^0_{x,i} + u^a_{x,i}u^a_{x,i} = 1 \quad \wedge \quad u^\rho_{x,i} \in \mathbb{R}$$

$$\left[L^a_{x,i}, u_{y,j}\right] = -\delta_{xy}\delta_{ij}\frac{\sigma^a}{2}u_{y,j} \qquad \left[R^a_{x,i}, u_{y,j}\right] = \delta_{xy}\delta_{ij}u_{y,j}\frac{\sigma^a}{2}$$

$$\left[L^a_{x,i}, L^b_{y,j}\right] = i\delta_{xy}\delta_{ij}\epsilon_{abc}L^c_{x,i} \qquad \left[R^a_{x,i}, R^b_{y,j}\right] = i\delta_{xy}\delta_{ij}\epsilon_{abc}R^c_{x,i} \qquad \left[L^a_{x,i}, R^b_{y,j}\right] = 0$$

Wilson $\mathrm{SU}(2)$ LGT

Hamiltonian derived by Kogut and Susskind

$$\mathcal{H} = \underbrace{g^2 \sum_{x,i} \left(\vec{L}^2_{x,i} + \vec{R}^2_{x,i} \right)}_{\mathcal{H}_E} + \underbrace{\frac{1}{g^2} \sum_P \mathrm{Tr} \left(u_P + u_P^\dagger \right)}_{\mathcal{H}_M}$$

with the Gauss law condition

$$G^a_x \left|\Psi\right\rangle = \sum_i \left(L^a_{x,i} + R^a_{x-\hat{i},i} \right) \left|\Psi\right\rangle = 0$$

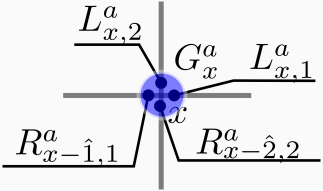

Figure: Olesen, `wiese.itp.unibe.ch/theses/olesen_phd.pdf`

SU(2) QLM

Promote link variables $u_{x,i}$ to link operators $U_{x,i}$

$$U_{x,i} = U^0_{x,i}\mathbb{1} + iU^a_{x,i}\sigma^a \qquad U^\dagger_{x,i} = U^0_{x,i}\mathbb{1} - iU^a_{x,i}\sigma^a$$

$$[L^a_{x,i}, U_{y,j}] = -\delta_{xy}\delta_{ij}\frac{\sigma^a}{2}U_{y,j} \qquad [R^a_{x,i}, U_{y,j}] = \delta_{xy}\delta_{ij}U_{y,j}\frac{\sigma^a}{2}$$

$$\left[L^a_{x,i}, L^b_{y,j}\right] = i\delta_{xy}\delta_{ij}\epsilon_{abc}L^c_{x,i} \qquad \left[R^a_{x,i}, R^b_{y,j}\right] = i\delta_{xy}\delta_{ij}\epsilon_{abc}R^c_{x,i} \qquad \left[L^a_{x,i}, R^b_{y,j}\right] = 0$$

$$[U^0_{x,i}, U^0_{y,j}] = 0 \qquad [U^0_{x,i}, U^a_{y,j}] = i\delta_{xy}\delta_{ij}\left(R^a_{x,i} - L^a_{x,i}\right) \qquad \left[U^a_{x,i}, U^b_{y,j}\right] = i\delta_{xy}\delta_{ij}\epsilon_{abc}\left(R^c_{x,i} + L^c_{x,i}\right)$$

$U^\rho_{x,i}$, $L^a_{x,i}$ and $R^a_{x,i}$ generate $\mathfrak{so}(5)$ link algebra $\rightarrow$ chose finite representation

$\mathrm{SU}(2)$ QLM

Hamiltonian derived by Kogut and Susskind

$$\mathcal{H} = \underbrace{g^2 \sum_{x,i} \left(\vec{L}^2_{x,i} + \vec{R}^2_{x,i} \right)}_{\mathcal{H}_E} + \underbrace{\frac{1}{g^2} \sum_{P} \mathrm{Tr} \left(U_P + U_P^\dagger \right)}_{\mathcal{H}_M}$$

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Comparison

Wilson LGT

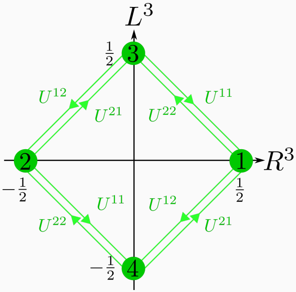
- ∞-dimensional local Hilbertspace
- needs to be discretized to simulate thus breaking gauge invariance
- related to physical theory by continuum limit

QLM

- finite dimensional local Hilbertspace
- retains exact gauge invariance
- related to physical theory by dimensional reduction

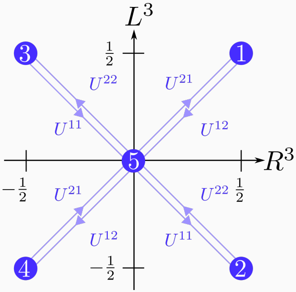
Representations of $\mathrm{SO}(5)$

Comparison



$\{4\}$-**representation**

- decomposes into $\{1,2\}+\{2,1\}$
- no vacuum state
- delocalized flux
- electric Hamiltonian is constant

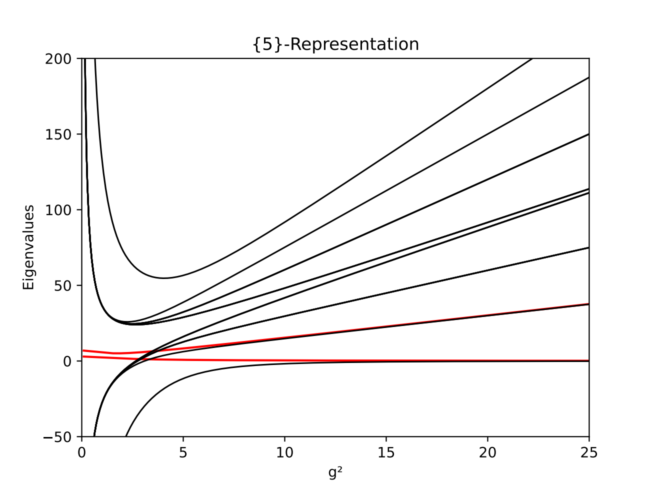


$\{5\}$-**representation**

- decomposes into $\{2,2\}+\{1,1\}$
- vacuum state exists
- localized flux
- similar to low order character expansion

Figures: Banerjee et al. Phys. Rev. B 97 (2018) 205108

Exact diagonalization of the plaquette



$\{5\}$ representation (black)

Wilson $SU(2)$ ground and 1st excited state (red)

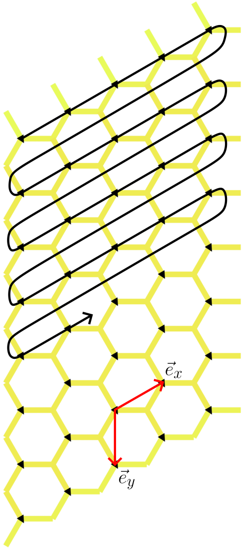
Analytic values for Wilson $SU(2)$ plaquette: Huang et al. Nuclear Physics B 309 (1988) 733

Numerics

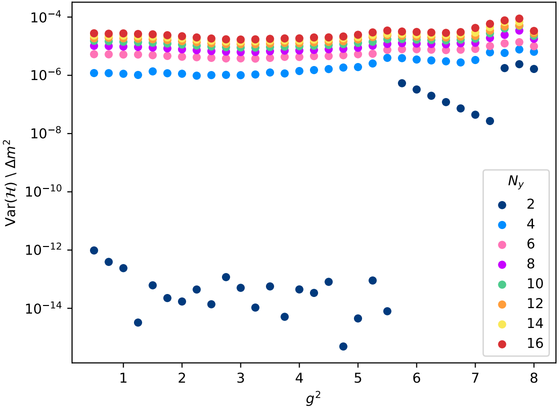
Simulation

Simulate $5 \times N_y$ systems using MPS and DMRG

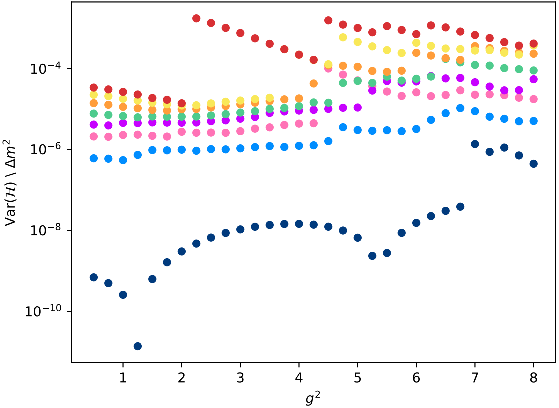
- $N_y \in \{2, 4, \ldots, 14, 16\}$
- $g^2 \in \{0.5, 0.75, \ldots, 7.75, 8\}$
- periodic boundaries in x-direction
- closed boundaries in y-direction
- with and without external $\frac{1}{2}$-charges to induce string in y-direction



Degree of convergence

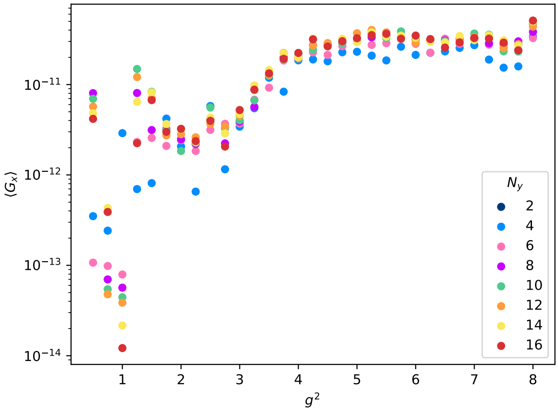


without external charges

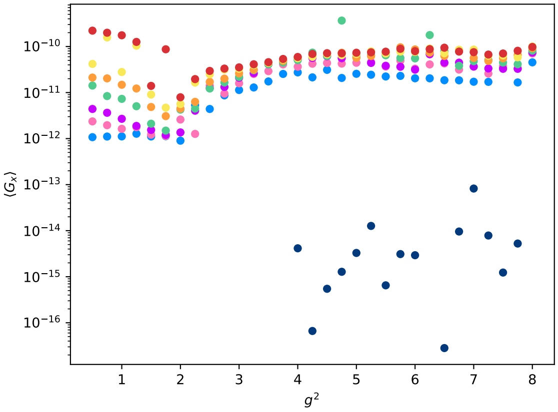


with external charges

Gauss's Law

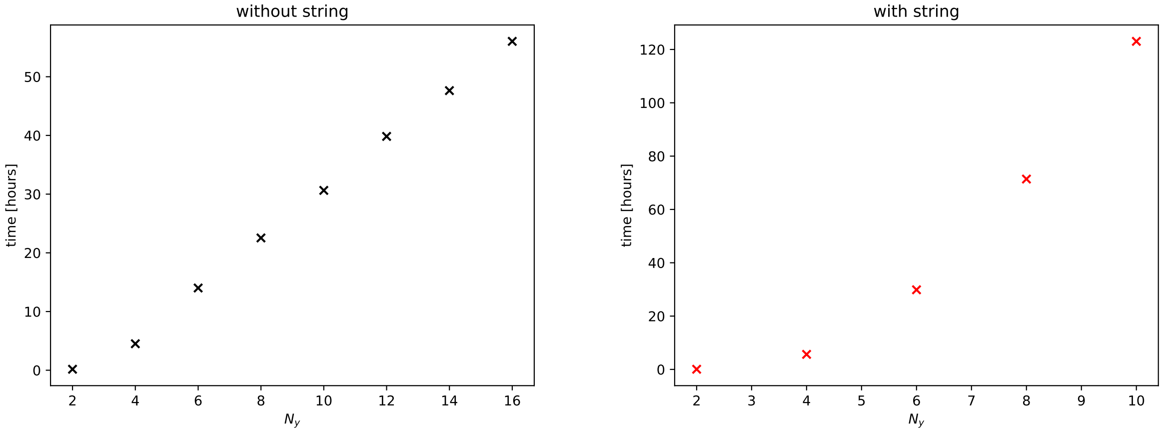


without external charges



with external charges

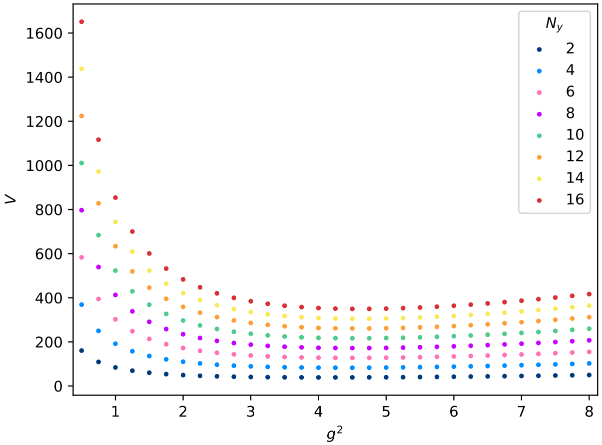
Runtime


without string
time [hours]
0
10
20
30
40
50
2
4
6
8
10
12
14
16
N_y
with string
time [hours]
0
20
40
60
80
100
120
2
3
4
5
6
7
8
9
10
N_y

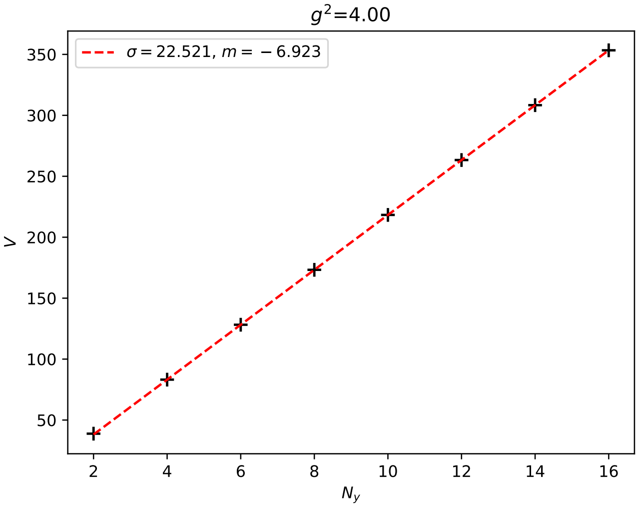
Setting the scale

Static Quark Potential

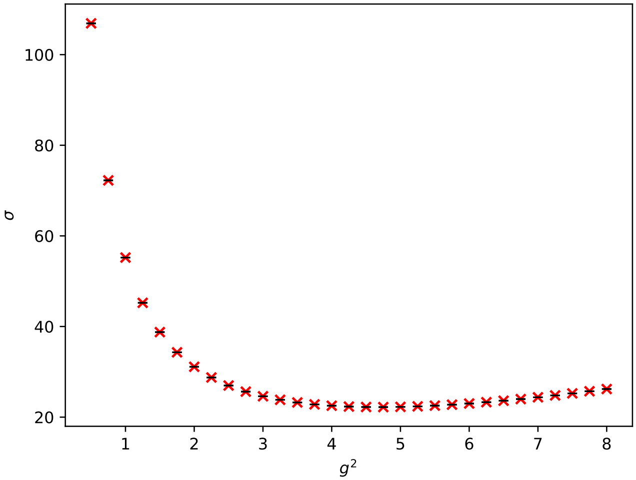
Subtract energies of empty systems from systems with strings to obtain the Static Quark Potential.



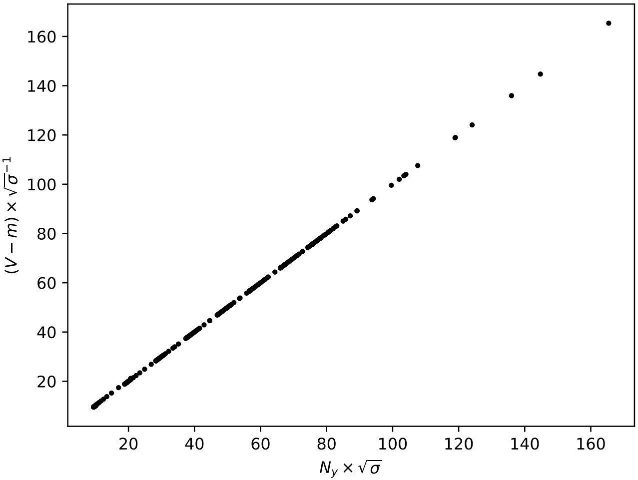
Static Quark Potential



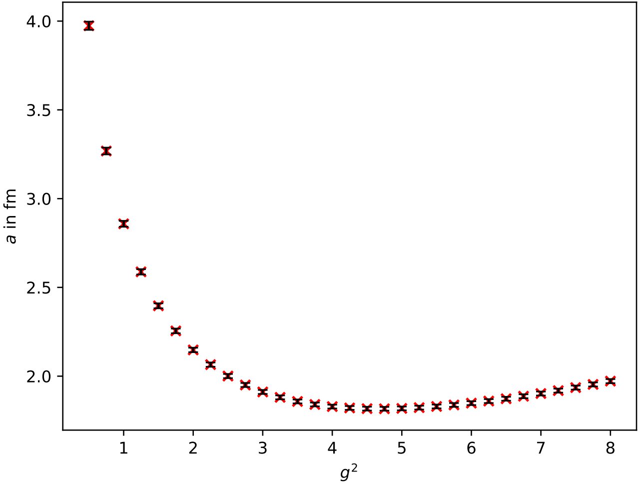
The String Tension



Universal Curve

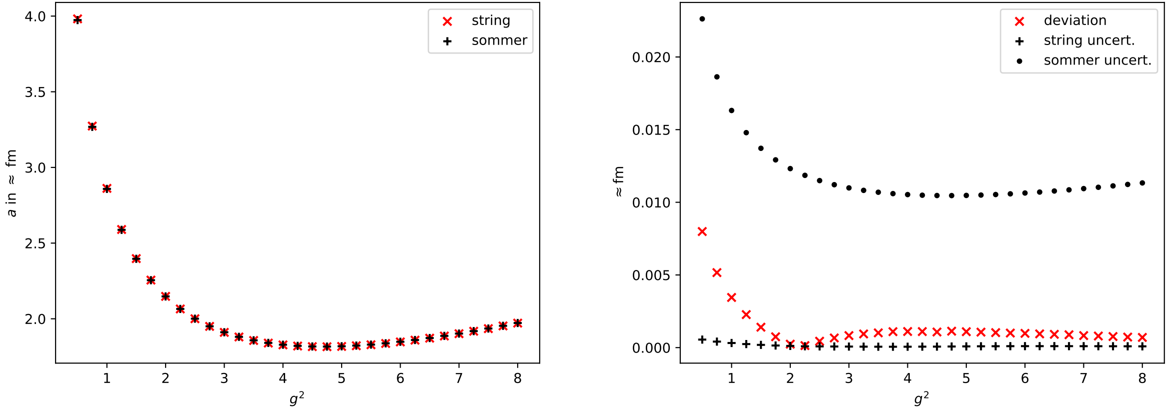


Sommer Parameter

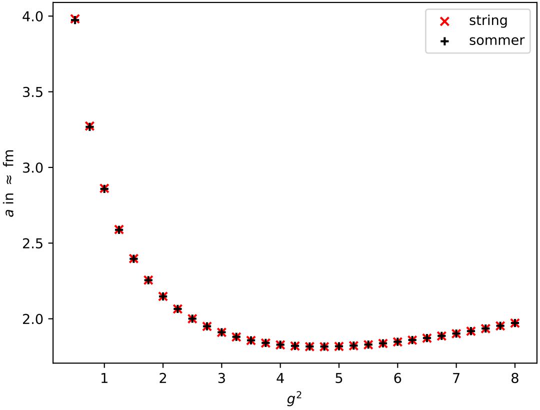


$$\forall g^2 : \left(N_y^2 \frac{dV}{dN_y} \right)_{N_y = 0.49\,\text{fm}} = 1.65$$

Comparison


string
sommer
a in ≈ fm
4.0
3.5
3.0
2.5
2.0
1
2
3
4
5
6
7
8
g^2
deviation
string uncert.
sommer uncert.
≈ fm
0.020
0.015
0.010
0.005
0.000

Comparison



- no continuum limit
- divergent lattice spacing in either limit
- $g^2_{\text{min}} = (4.687 \pm 0.002)$
  $a_{\text{min}} = (1.815 \pm 0.002)$ fm

Outlook

Outlook

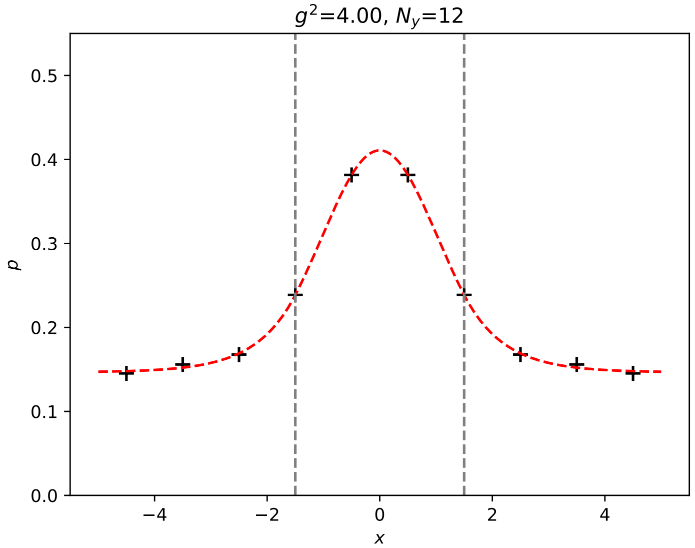
Possible next steps

- add fermions
- larger representations
- VQE
- real time evolution
- non-abelian conserved quantities in DMRG
- augmented TTN / isoTNS

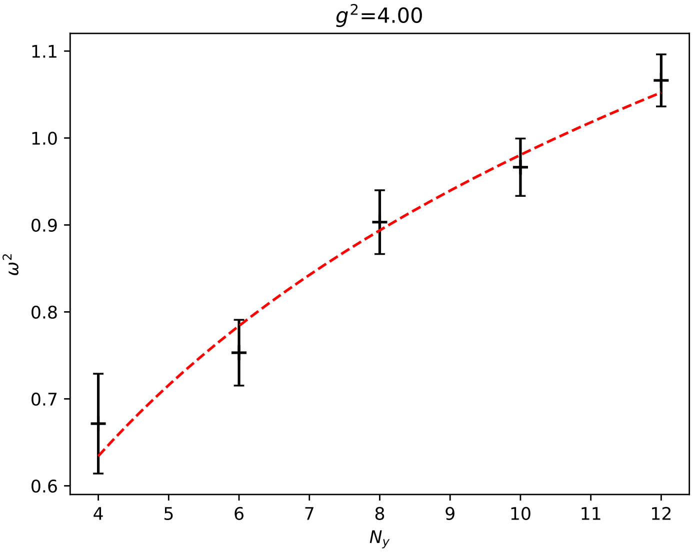
The End

Thanks for listening

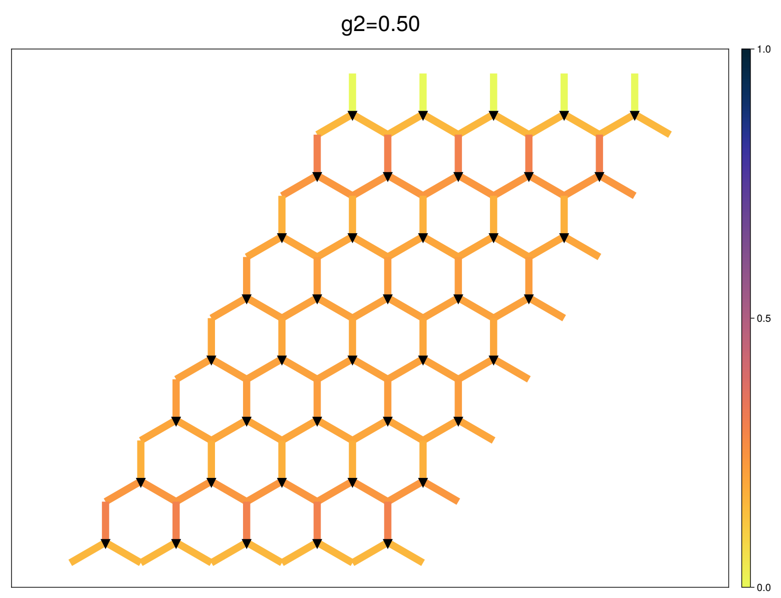
Bonus



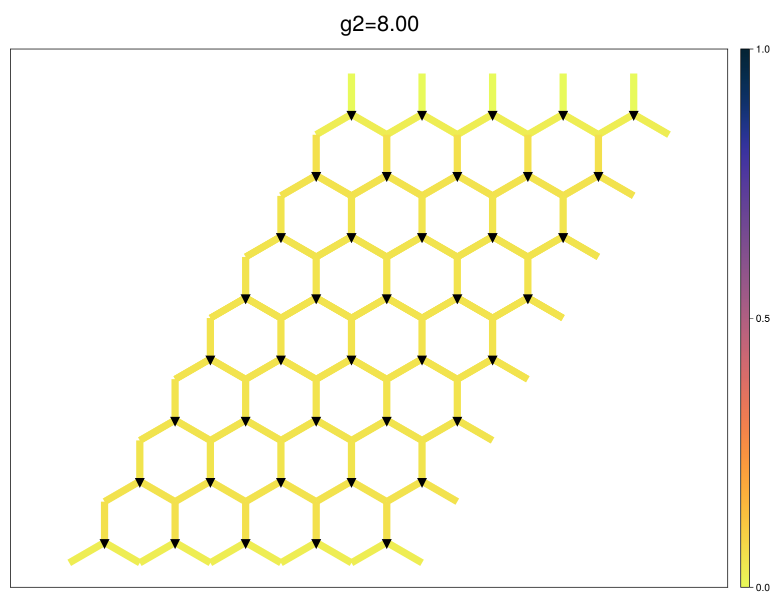
Bonus



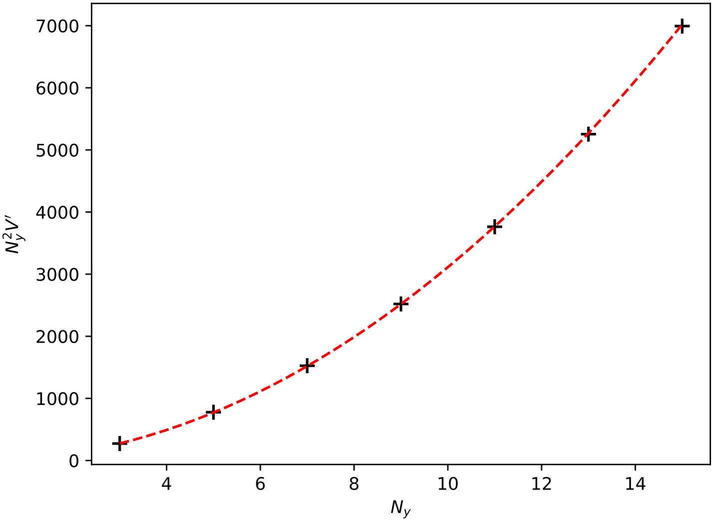
Bonus



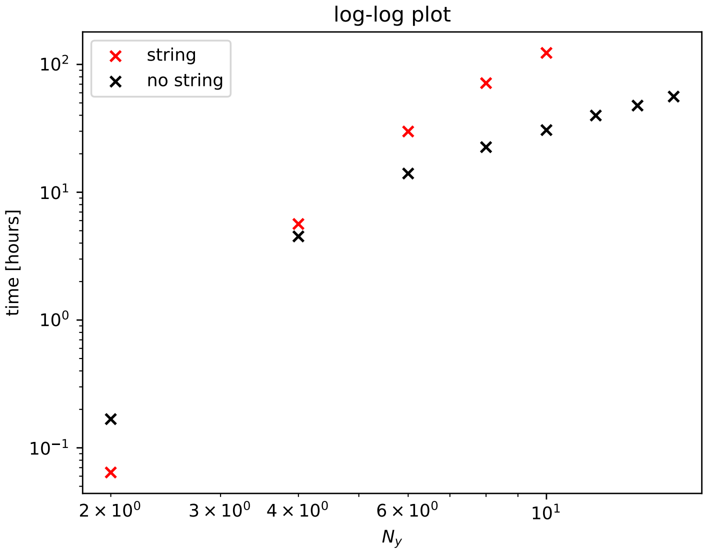
Bonus



Bonus

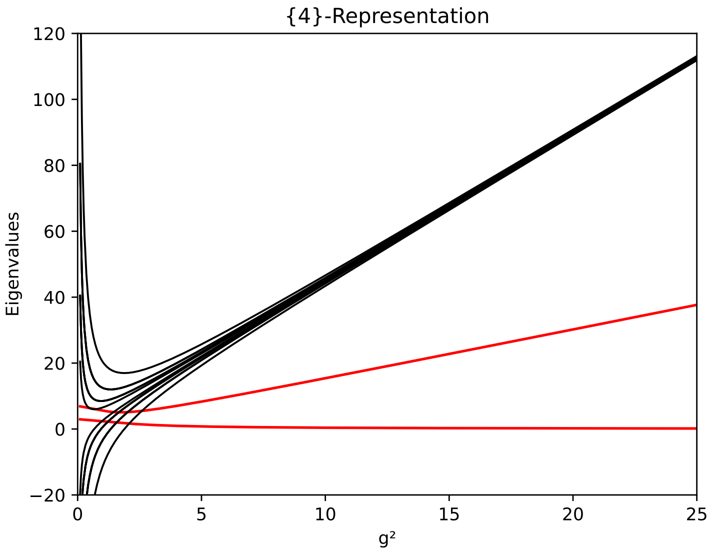


Bonus

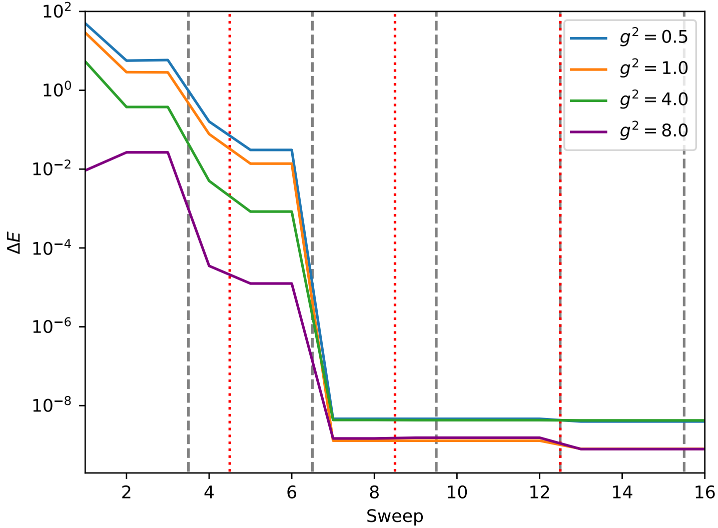


log-log plot
string
no string
time [hours]
10^2
10^1
10^0
10^{-1}
2×10^0
3×10^0
4×10^0
6×10^0
10^1
N_y

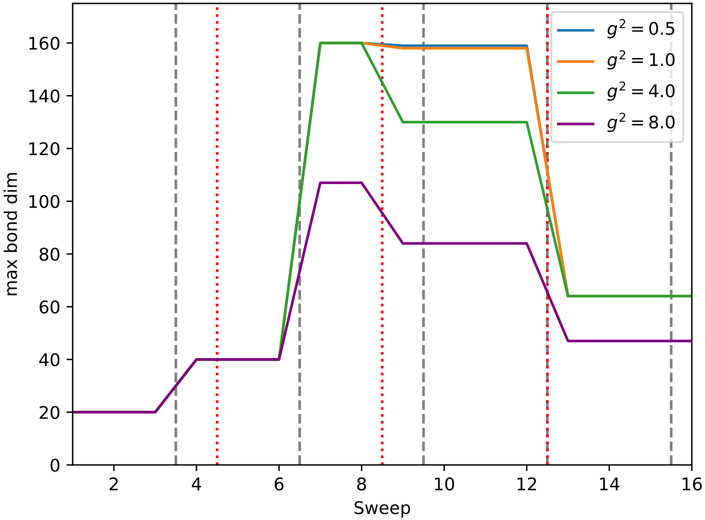
Bonus


{4}-Representation
Eigenvalues
120
100
80
60
40
20
0
−20
0
5
10
15
20
25
g²

Bonus

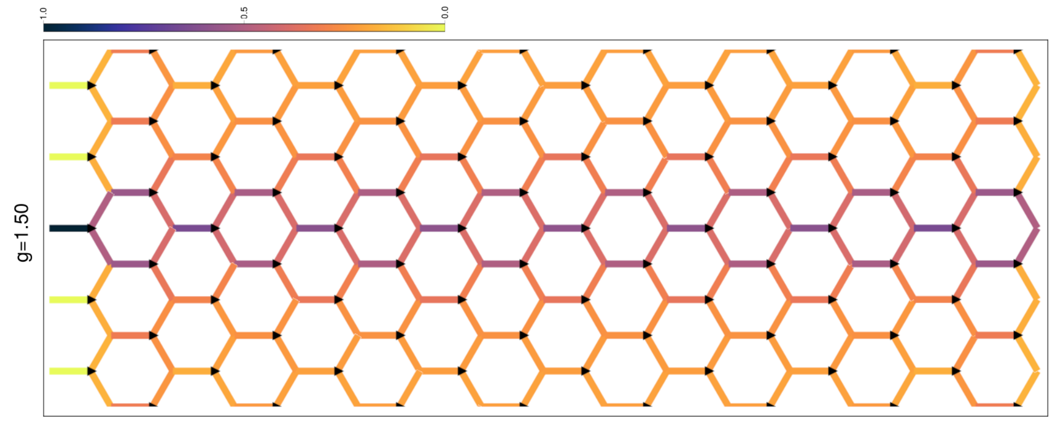

10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
ΔE
2
4
6
8
10
12
14
16
Sweep
$g^2 = 0.5$
$g^2 = 1.0$
$g^2 = 4.0$
$g^2 = 8.0$

Bonus

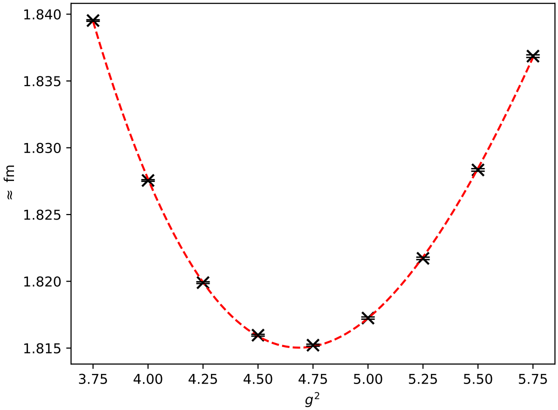


$g^2 = 0.5$
$g^2 = 1.0$
$g^2 = 4.0$
$g^2 = 8.0$
max bond dim
Sweep
0
20
40
60
80
100
120
140
160
2
4
6
8
10
12
14
16

Bonus



Bonus



Fit cubic polynomial without linear term:

- $\chi^2_{red} = 2.12$
- $g^2_{\text{min}} = (4.687 \pm 0.002)$
- $a_{\text{min}} = (1.8154 \pm 0.0020)$ fm