Nonperturbative signatures of fractons in the twisted multiflavour Schwinger model

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ECT* workshop, Trento

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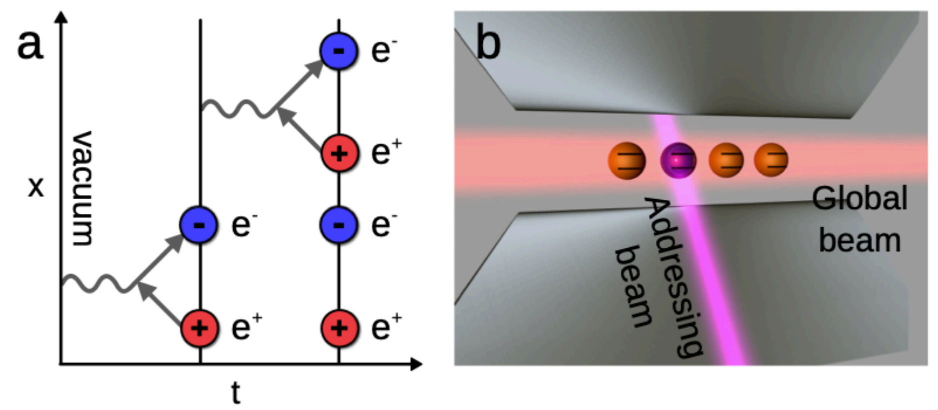
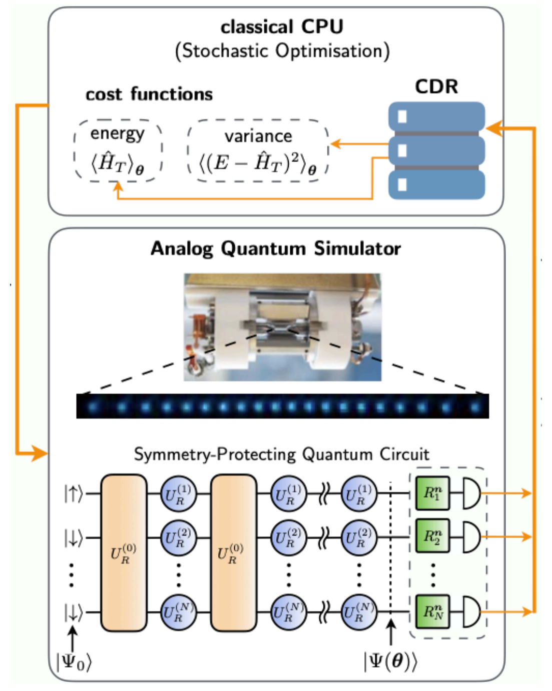
Quantum simulation of gauge theories

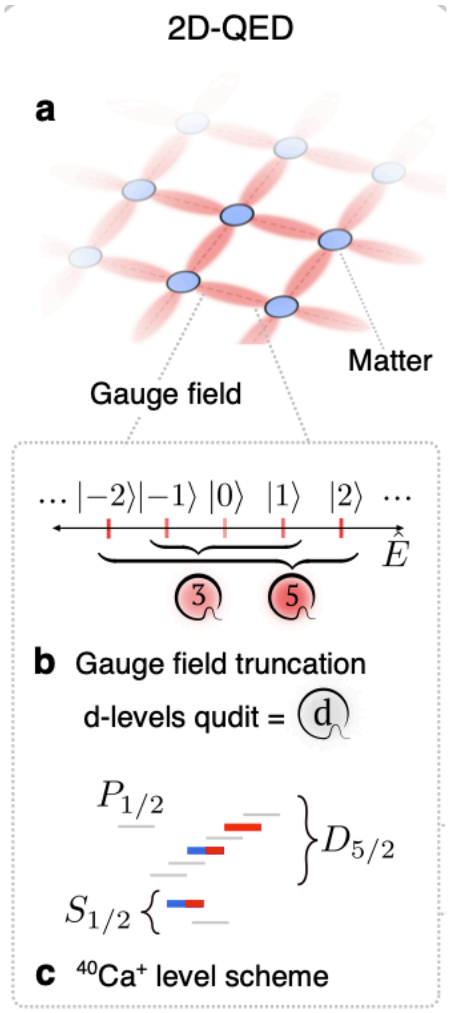
- A variety of quantum simulators devices available
- Lattice Hamiltonian approach, various encodings
- What kind of physics can we look at?



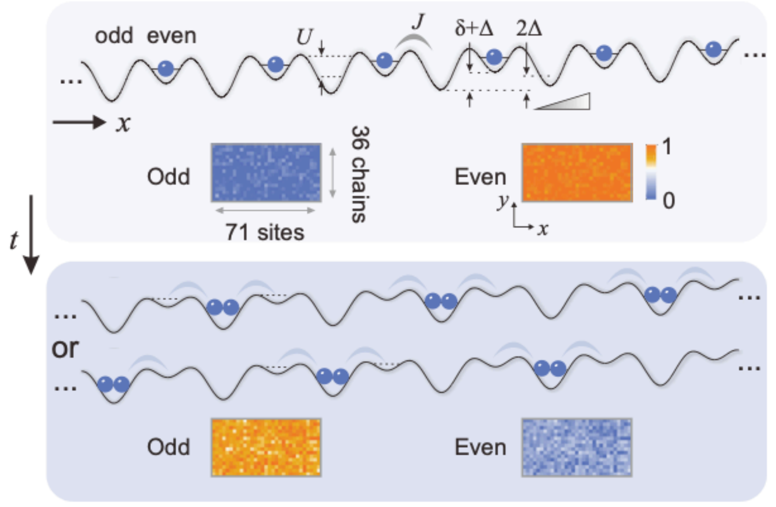
Martinez et al., Nature 534, 516-519 (2016)



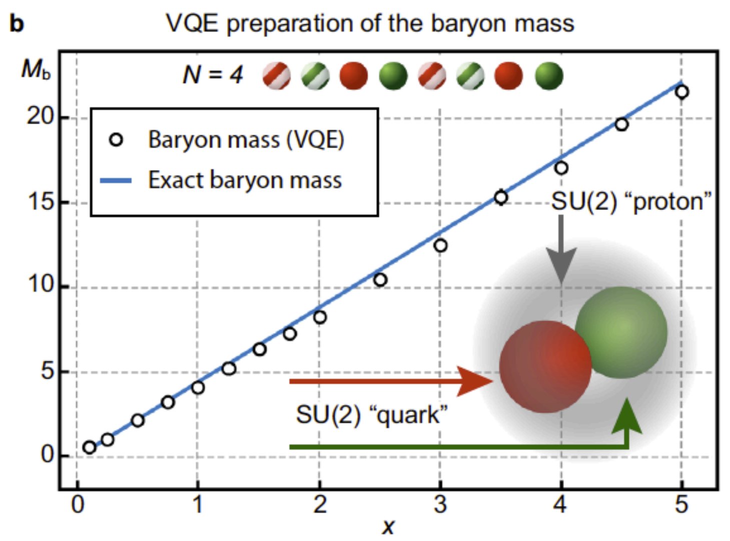
Kokail et al., Nature 569, 355-360 (2019)



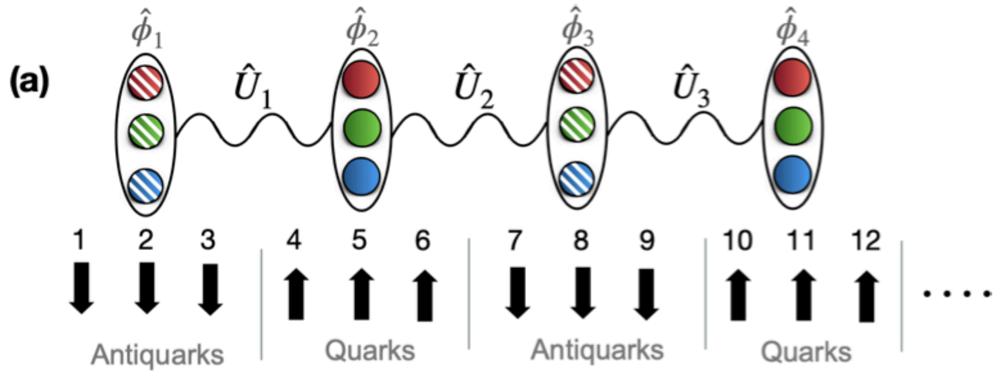
Meth et al.,Nat.Phys. 21, 570-576 (2025)



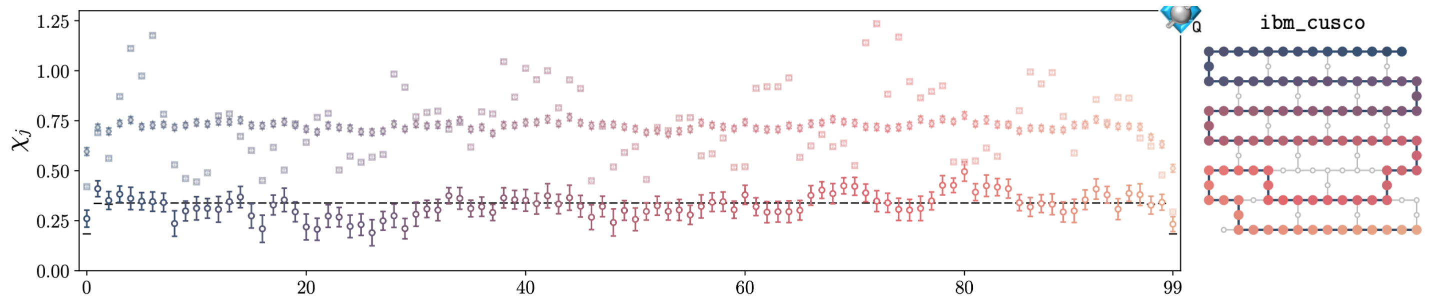
Yang et al., Nature 587, 392-396 (2020)



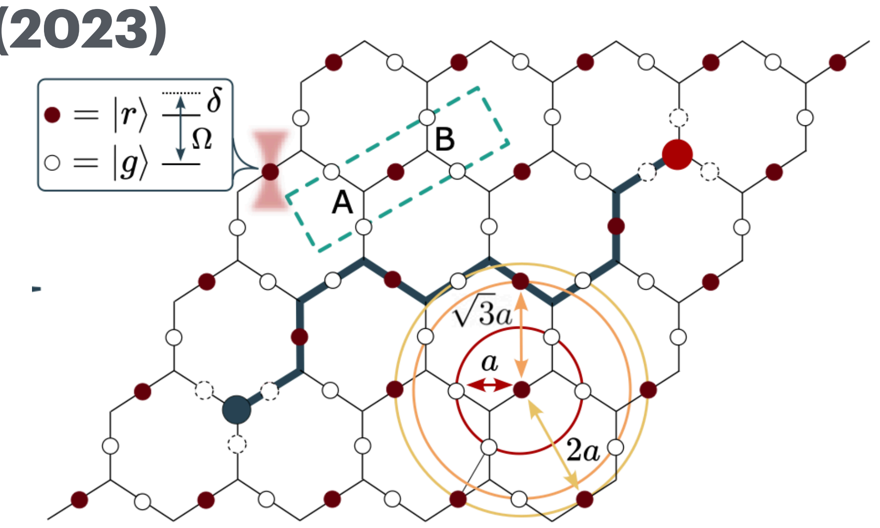
Atas et al., Nat. Comm. 12, 6499 (2021)



Atas et al., PRR 5, 033184 (2023)



Farrell et al., PRX Quantum 5, 020315 (2024)

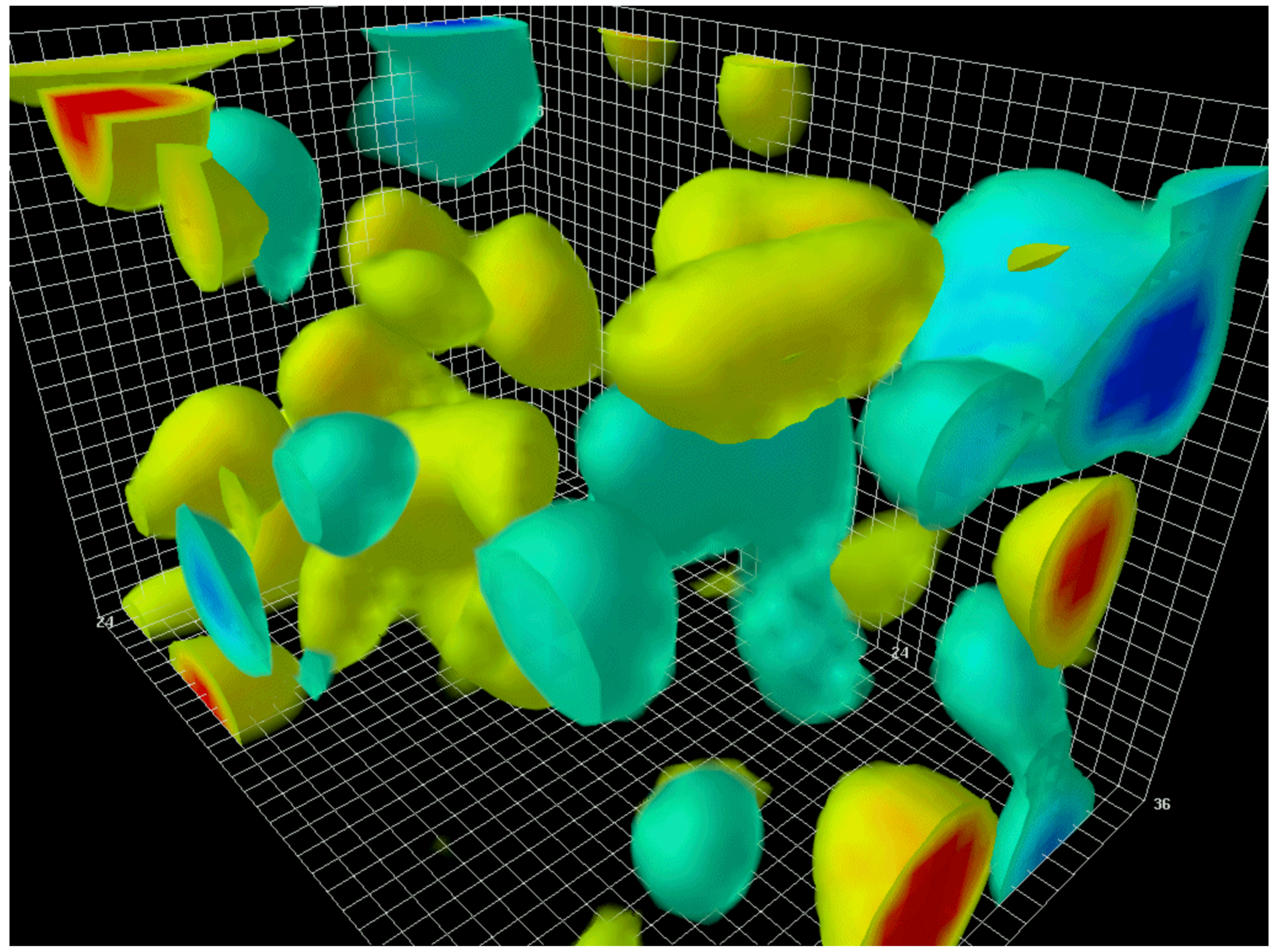


González-Cuadra et al., Nature 642, 321-326 (2025)

...and others!

Topology of vacuum in gauge theories

- How does the **physical vacuum of QCD** look like?
- What **topological properties** does it have?
- What properties does **matter** have in the physical vacuum? (**chiral symmetry breaking**, **confinement**, etc.)



Source: Visual QCD Archive

Instantons in Yang-Mills

- Yang-Mills action:

$$S_{YM} = \frac{1}{2g^2}\int d^4x \text{Tr}(G_{\mu\nu}G^{\mu\nu}), \text{ where } G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu].$$

- Rewrite as $S_{YM} = \frac{1}{4g^2}\int d^4x \text{Tr}(G_{\mu\nu} \mp \tilde{G}^{\mu\nu})^2 \pm 2\text{Tr}(G_{\mu\nu}\tilde{G}^{\mu\nu}) \geq \frac{1}{2g^2}\int d^4x \text{Tr}(G_{\mu\nu}\tilde{G}^{\mu\nu})$,

  where $\tilde{G}^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$.

Instantons in Yang-Mills

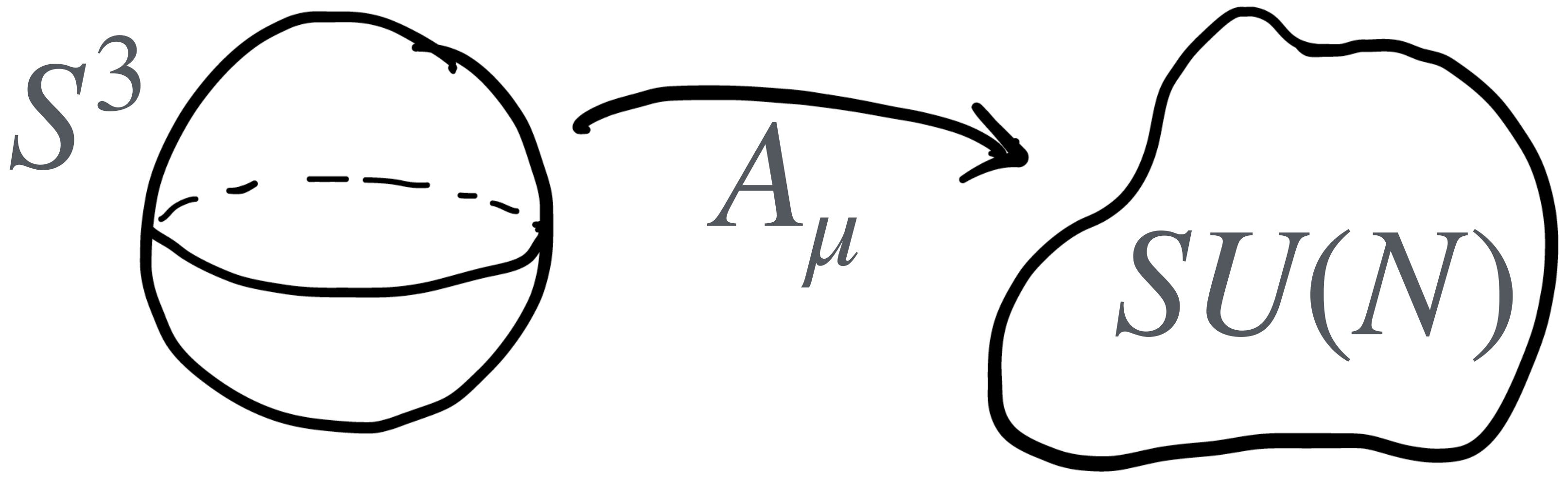
- Inequality saturated in case $G_{\mu\nu} = \pm \tilde{G}_{\mu\nu} \rightarrow D_\mu G^{\mu\nu} = D_\mu \tilde{G}^{\mu\nu} = 0$, thus instanton solutions of the YM equations.

- Minimal value of $S_{YM} = \frac{1}{2g^2}\int d^4x \text{Tr}(G_{\mu\nu}\tilde{G}^{\mu\nu}) = \frac{8\pi^2}{g^2}|Q|$, where

$Q = \frac{1}{16\pi^2}\int d^4x \text{Tr}(G_{\mu\nu}\tilde{G}^{\mu\nu})$ is the Pontryagin (or topological) charge.

Instantons and topological vacua

- Distinct vacua with different topological number exist due to $\pi_3(SU(N)) = \mathbb{Z}$ (index theorem)



- Instantons connect different topological vacua $|\nu\rangle$ and $|\nu \pm Q\rangle$.

- Therefore, physical vacuum is a superposition of topological vacua $|\theta\rangle = \sum_\nu e^{i\theta\nu} |\nu\rangle$.

Fractional instantons (*fractons*)

- Instantons provide **microscopic explanation of chiral symmetry breaking**; however, no confinement :(
- On different geometries, index theorem doesn't restrict topological charge —> **fractional charges possible**
- Fractons can **carry center flux**, thus **might** be important for **confinement**

Let us look at a simple model

Multi-flavour Schwinger model as a prototype model for topological gauge theory phenomena

Shifman, Smilga, PRD **50**, 7659 (1994)

In SUSY Yang-Mills theories, gluino condensate $\langle \bar{\lambda}\lambda \rangle \neq 0$
($\lambda \sim$ Majorana field, superpartner of gluon)

Topological sectors; usually ν integer

However, from path integral (at small mass) $\langle \bar{\lambda}\lambda \rangle = -\,\partial_m \ln Z\Big|_{m=0}$ with $Z = \sum_\nu Z_\nu$, where $Z_\nu = z_\nu m^{\nu N_c}$.

which implies $\langle \bar{\lambda}\lambda \rangle = 0$

How to reconcile? → Admit presence of **fractional topological sectors**

$$\langle \bar{\lambda}\lambda \rangle = -\lim_{m\to 0} \frac{z_{1/N_c}}{z_0}$$

Can we probe fractons in a simpler theory (and on a quantum simulator)?

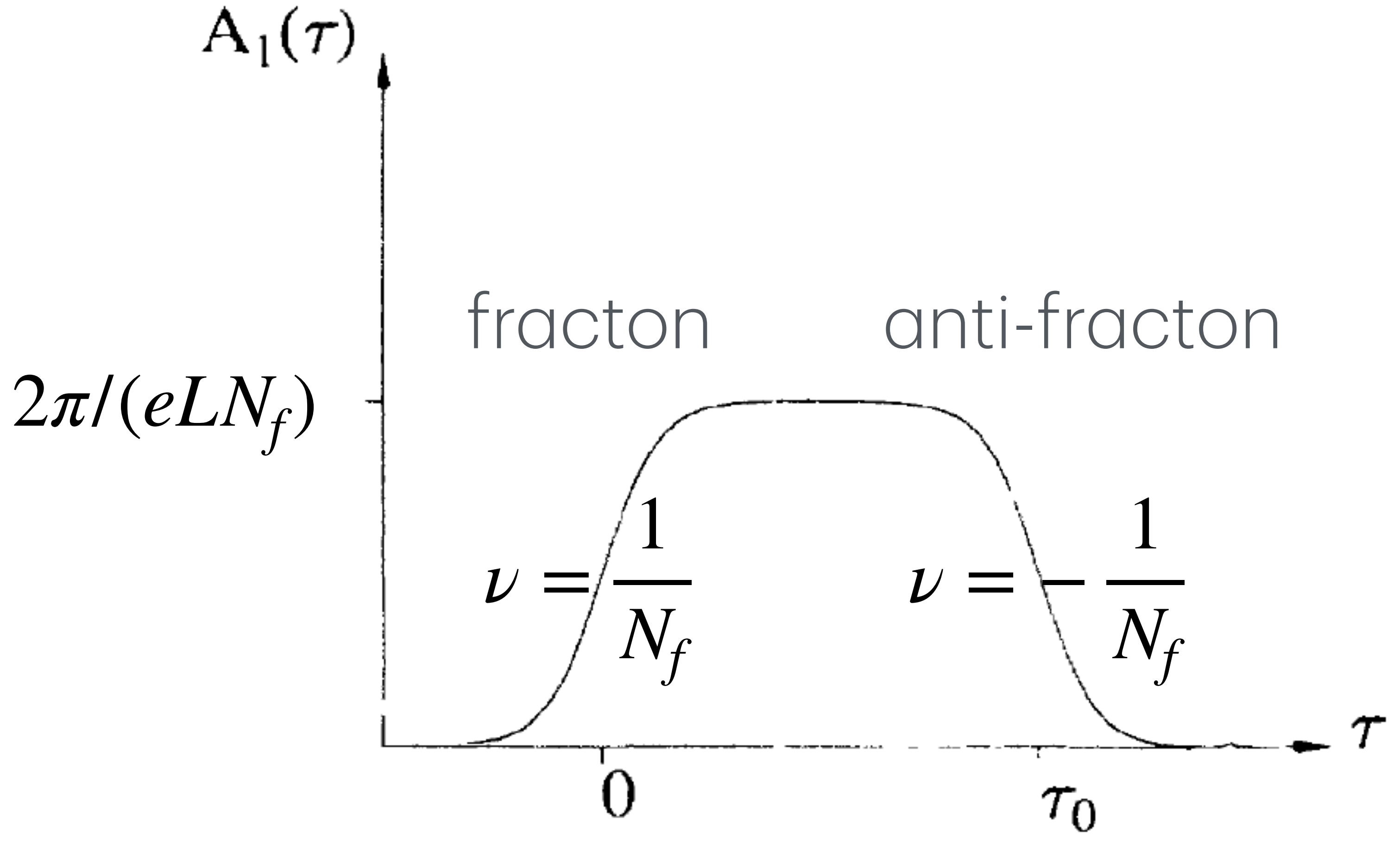
Fractons in continuum Schwinger Model

Shifman, Smilga, PRD **50**, 7659 (1994)

Because of $e^{ie\int dxA_1} = const$, A_1 defined up to different windings

# of windings of A_1 define topological invariant (Pontryagin charge) $\nu = \frac{e}{4\pi}\int_V dxd\tau\epsilon_{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)$

Fractional windings are usually confined



Topological charge: $\nu = \frac{eL}{2\pi}\int_{-\infty}^{\infty}\dot{A}(\tau)d\tau = A(\infty) - A(-\infty) = 0.$

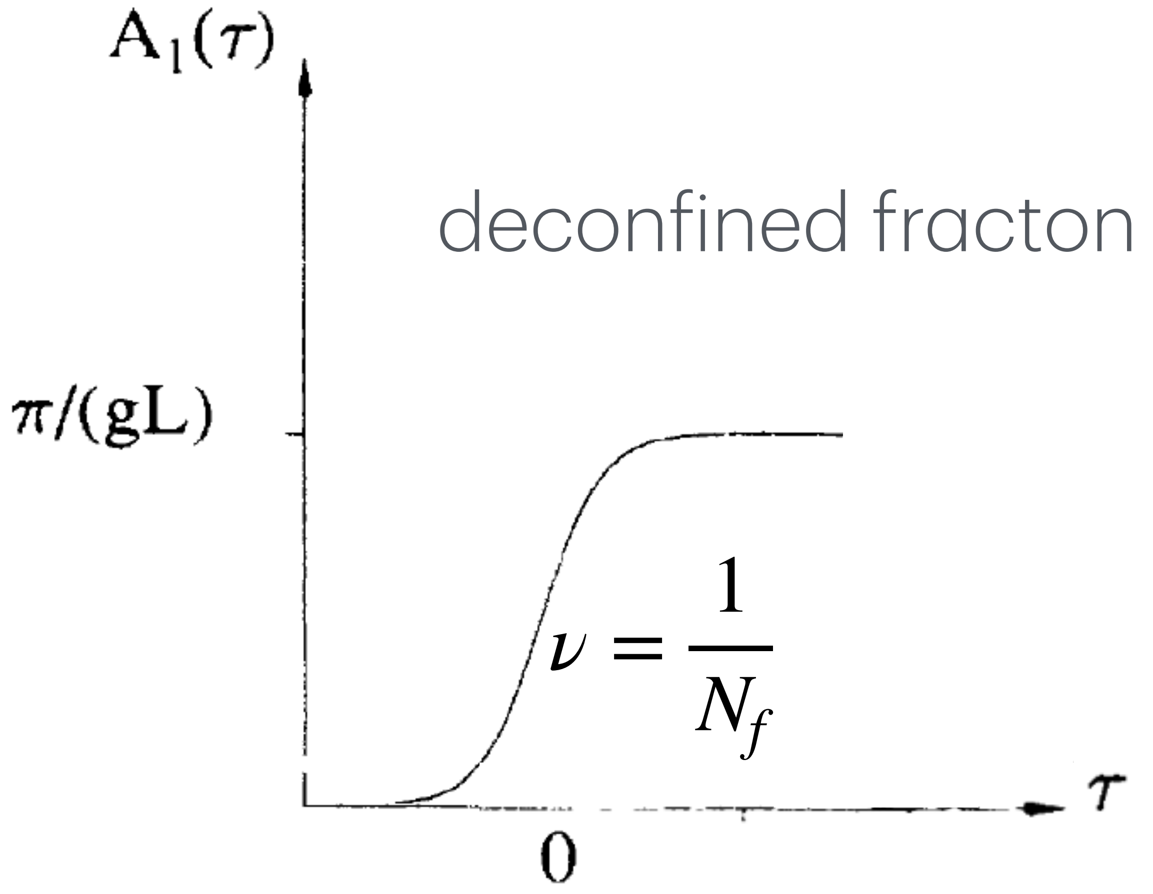
How to reveal them?

Fractons and chiral condensate

Path integral solution for chiral condensate $\langle\bar{\psi}\psi\rangle = -\partial_m \ln Z\Big|_{m=0}$ with $Z = \sum_\nu Z_\nu$, where $Z_\nu = z_\nu m^{\nu N_f}$.

If ν **integer** (no deconfined fractons): $\langle\bar{\psi}\psi\rangle = \lim_{m\to 0} \frac{z_1 m}{z_0} = 0.$

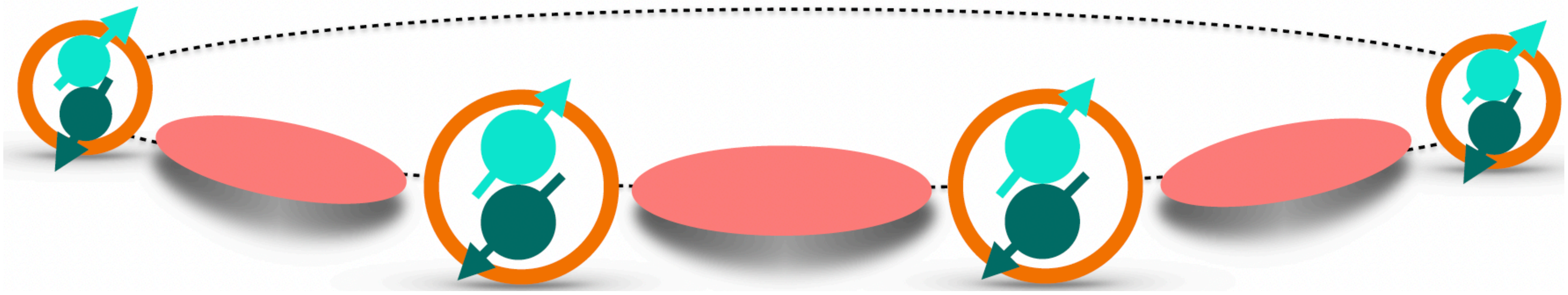
Conversely $\langle\bar{\psi}\psi\rangle \neq 0$ implies the existence of $\nu = \frac{1}{N_f}$ $\left(\langle\bar{\psi}\psi\rangle = \lim_{m\to 0} \frac{z_{1/N_f}}{z_0} \neq 0.\right)$



Topological charge: $\nu = \frac{eL}{2\pi}\int_{-\infty}^{\infty} \dot{A}(\tau)d\tau = A(\infty) - A(-\infty) = \frac{1}{N_f}.$

Do these *fractons* become relevant?

Multi-flavour Schwinger Model on a lattice



Lattice Hamiltonian: $H = \sum_{\sigma=1}^{N_f} J \sum_x (\hat{\psi}^\dagger_{\sigma,x} \hat{U}_{x,x+1} \hat{\psi}_{\sigma,x+1} + \text{h.c.}) + m_\sigma \sum_x \hat{\psi}^\dagger_{\sigma,x} \hat{\psi}_{\sigma,x} + \frac{1}{2} \sum_x \left(\hat{E}_{x,x+1}\right)^2$

Discretized Gauss's law: $E_{x,x+1} - E_{x-1,x} - e \sum_{\sigma=1}^{N_f} \hat{\psi}^\dagger_{\sigma,x} \hat{\psi}_{\sigma,x} + 1 = 0.$

How much of the physics does the lattice model retain?
(cutoff S, lattice spacing a,...)

Quantum link vs truncated Schwinger model

Original EM field

E-field: E

Link operator: U

Algebra:

$$[E, U^{(\dagger)}] = \pm U^{(\dagger)}$$

$$[U, U^{\dagger}] = 0$$

$$\langle m' | S_n^+ | m \rangle = \delta_{m',m+1} \sqrt{1 - m(m+1)/[S(S+1)]}$$

Quantum link model

E-field: $E = eS^z$

Link operator: $U = S^+ / \sqrt{S(S+1)}$

Algebra:

$$[E, U^{(\dagger)}] = \pm U^{(\dagger)}$$

$$[U, U^{\dagger}] = \pm \frac{2}{\sqrt{S(S+1)}} E$$

Nucl. Phys. B492, 455 (1997)

$$\langle m' | \tilde{S}^+ | m \rangle \equiv \delta_{m',m+1}$$

Truncated Schwinger model

E-field: $E = eS^z$

Link operator: $U = \tilde{S}^+$

Algebra:

$$[E, U^{(\dagger)}] = \pm U^{(\dagger)}$$

$$[U, U^{\dagger}] = (|S\rangle\langle S| - |-S\rangle\langle -S|)$$

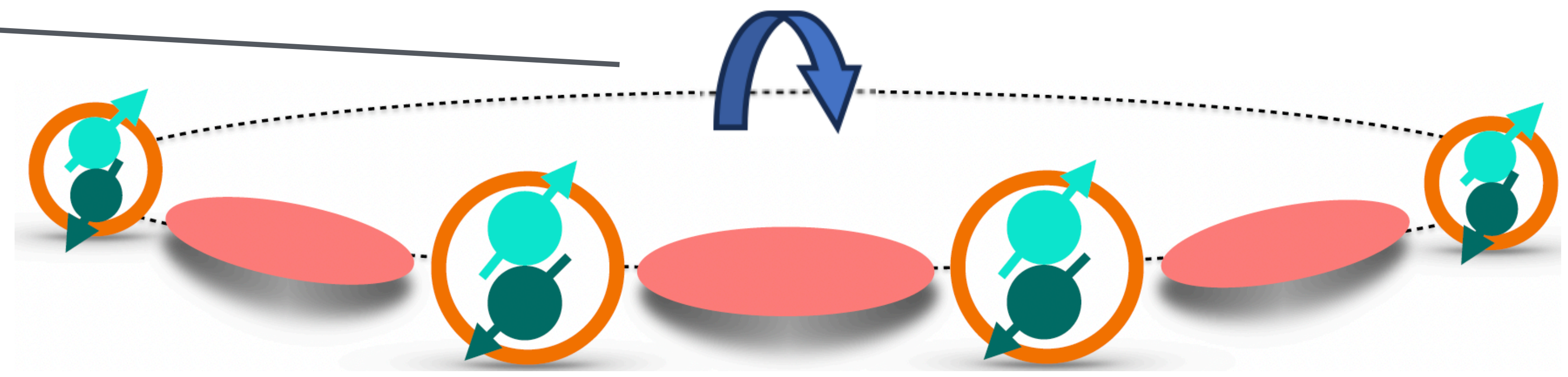
PRL109, 125302 (2012)
PRB 107, 205112 (2023)

Boundary conditions with a twist

Flavour-twisted boundary conditions

"torons" $\psi_p(x = L) = e^{2\pi i p/N_f}\psi_p(x = 0)$

break chiral symmetry



Normally, allowed large gauge transformation is

$$A_1(x) \longrightarrow \mathcal{S}[A_1(x)] = A_1(x) + \frac{2\pi}{eL} \text{ and } \psi_p(x) \longrightarrow \mathcal{S}[\psi_p(x)] = e^{-i\frac{2\pi x}{L}}\psi_p(x).$$

Boundary conditions with a twist

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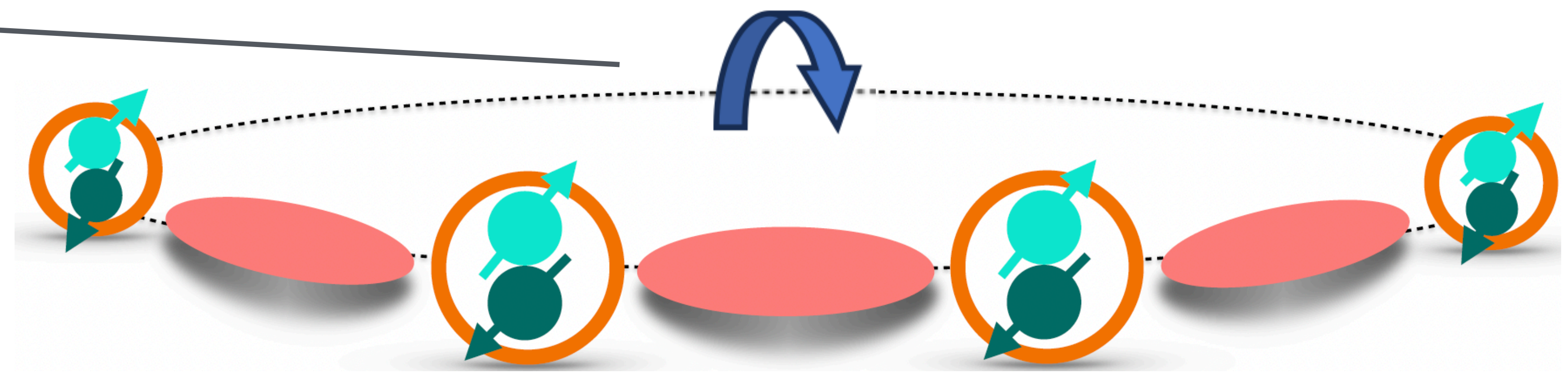


Under torons, however, one can additionally transform

$A_1(x) \longrightarrow \tilde{\mathcal{S}}[A_1(x)] = A_1(x) + \frac{2\pi}{N_f e L}$ and $\psi_p(x) \longrightarrow \tilde{\mathcal{S}}[\psi_p(x)] = e^{-i\frac{2\pi x}{N_f L}} \psi_{\tilde{p}}(x)$, where

$\tilde{p} = p + 1 \mod N_f.$

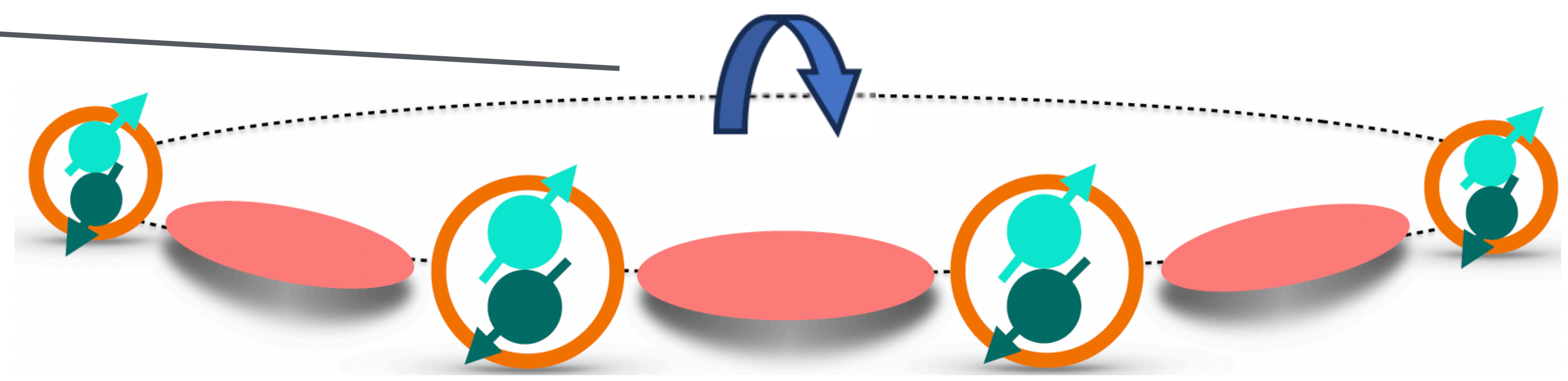
Testing the chiral condensate for $N_f = 2$

Flavour-twisted boundary conditions

"torons"

$$\psi_1(x = L) = \psi_1(x = 0)$$
$$\psi_2(x = L) = -\psi_2(x = 0)$$

break chiral symmetry

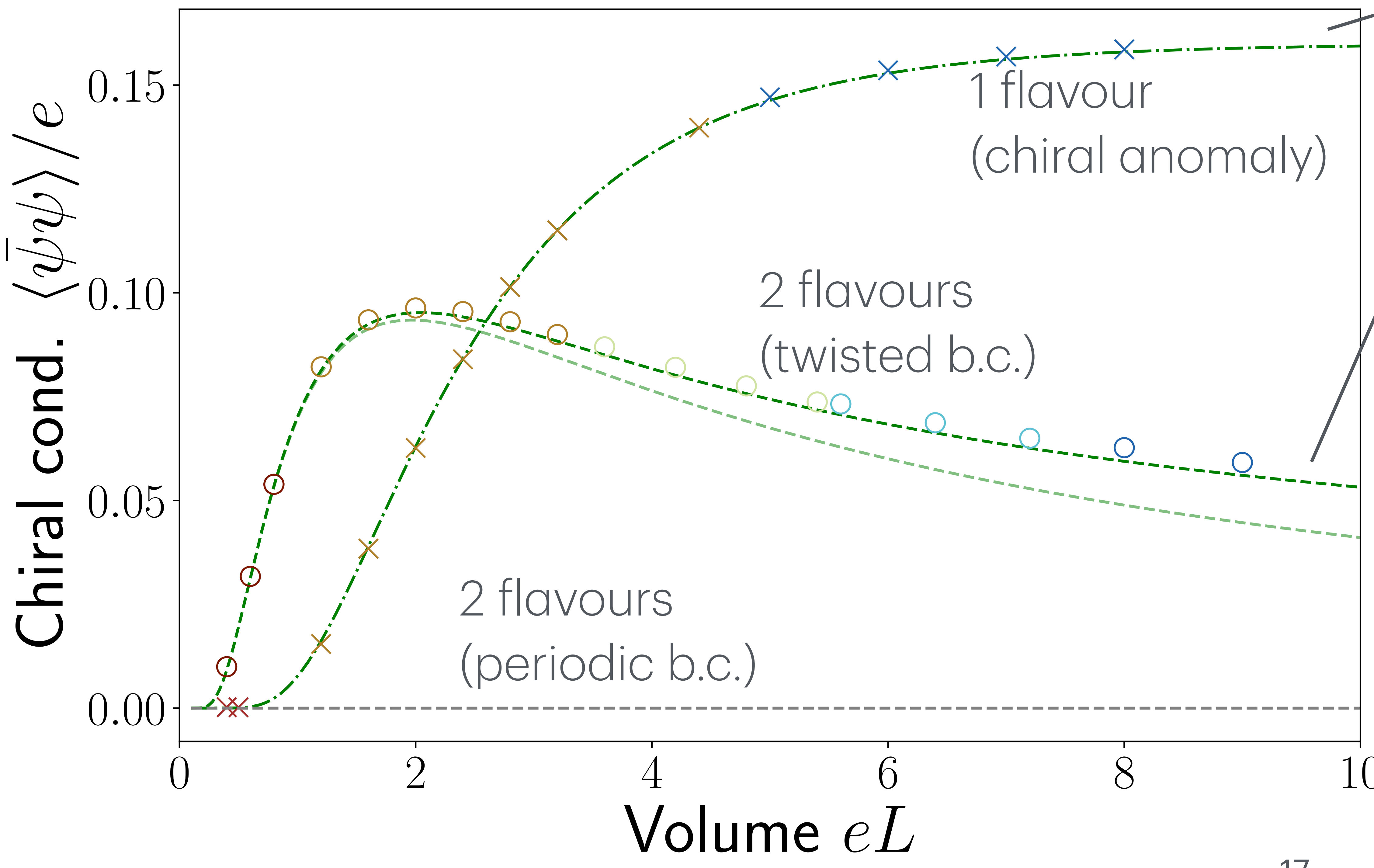


continuum

Shifman, Smilga, PRD, 1994

$$\langle \bar{\psi}\psi \rangle \propto \frac{1}{N_f L} \exp\left(-\frac{\pi}{N_f \mu L} \right), \quad \mu^2 = N_f e^2/\pi$$

(at small L)



Note: In QED, finite volume effect

Formulated positively: finite volume reveals fractons!

Drastically different behaviour for one-flavour and two-flavour models!

Can we probe it on a quantum device?

numerics

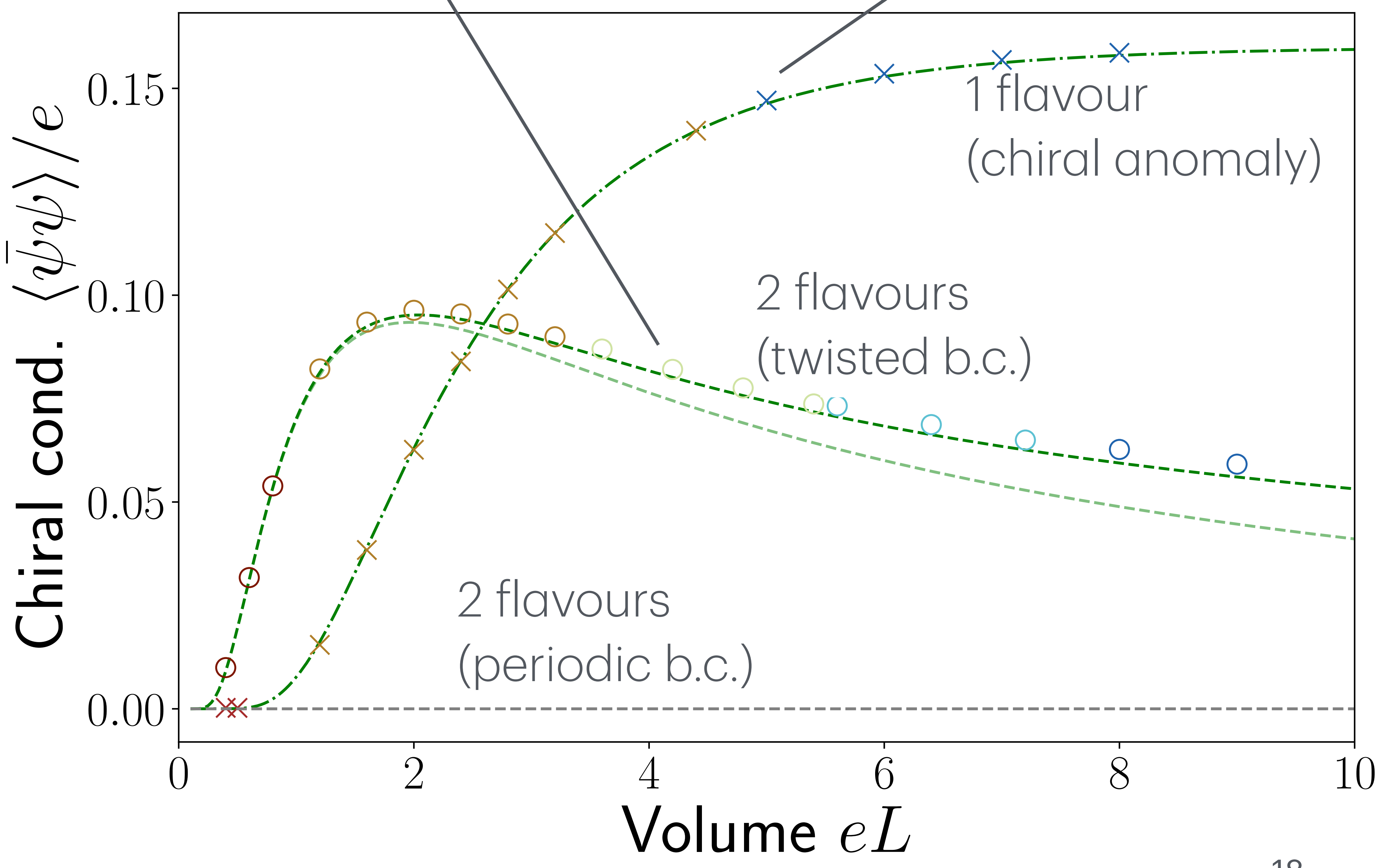
$ea \in [0.1, 0.5]; m/e = 0$

S-cutoff $= \pm 3$

numerics

$ea \in [0.05, 0.5]; m/e = -1/4$

S-cutoff $= \pm 3$



Shifman, Smilga, PRD, 1994

$$\langle\bar{\psi}\psi\rangle \propto \frac{1}{N_f L} \exp\left(-\frac{\pi}{N_f \mu L}\right), \quad \mu^2 = N_f e^2/\pi$$

(at small L)

Visible in very coarse-grained and highly truncated systems.

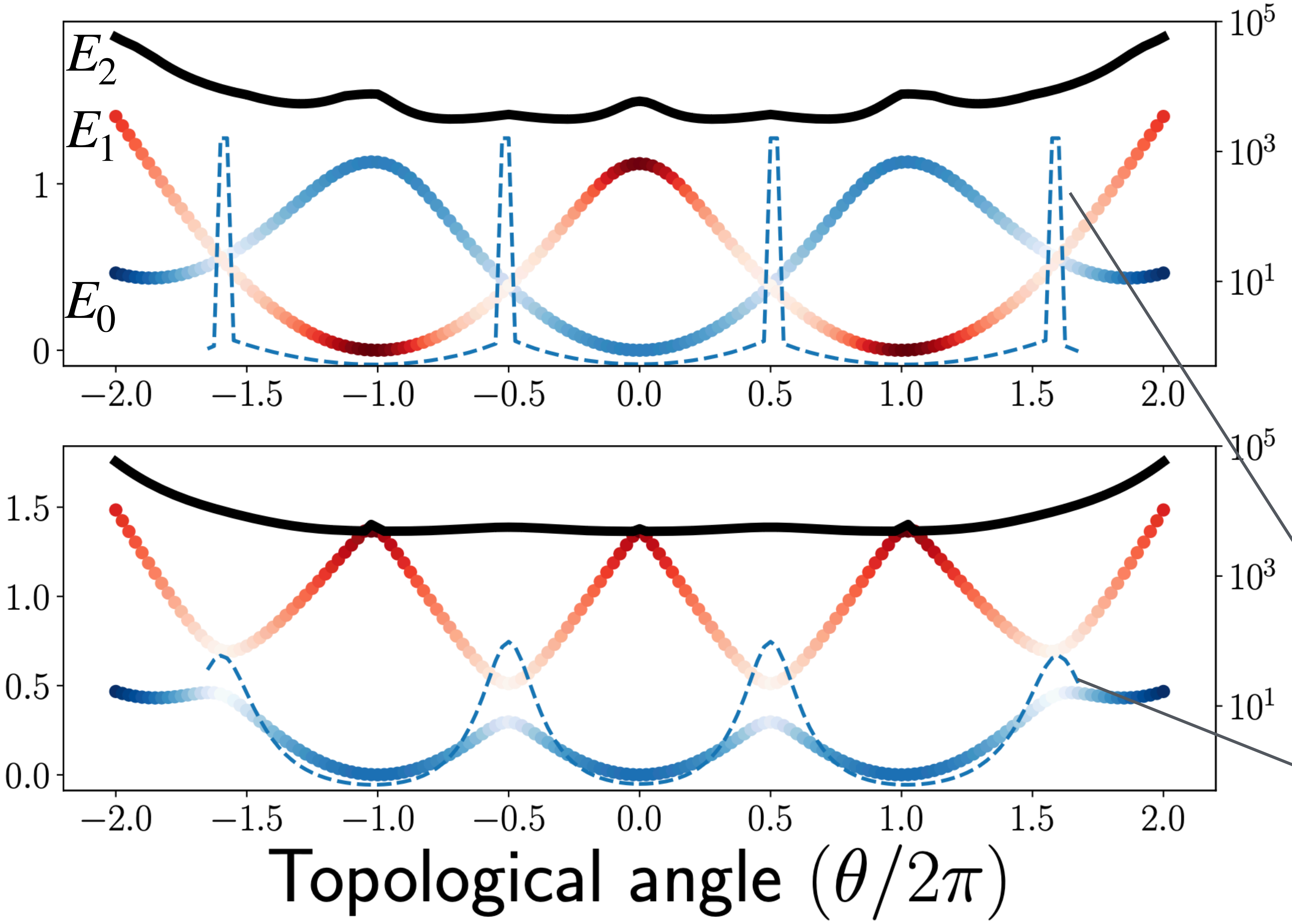
→ Perfect playground for quantum simulators!

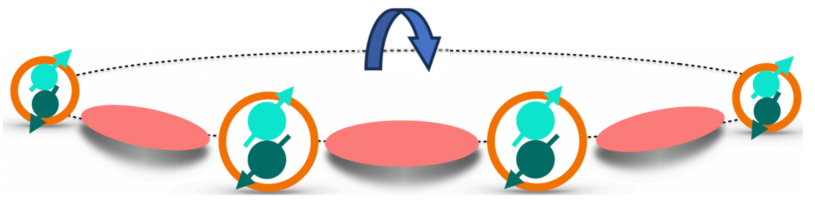
In the non-perturbative regime $m/e \sim 1$

$$Z = \sum_{\nu} e^{i\nu\theta} Z_{\nu}$$

Periodic under

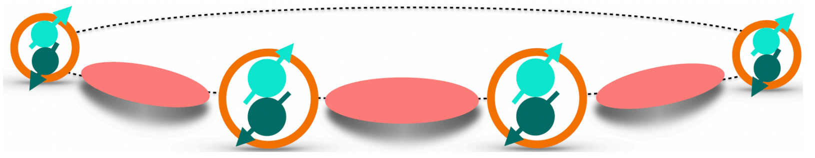
$$\theta \to \theta + 2\pi/\nu$$





Twisted bc

$\nu = 1/2$



Periodic bc

$\nu = 1$

Fidelity suscepitibily

Lattice result

$L = (4$ sites $+ 4$ links$)$

S-cutoff $= \pm 2$

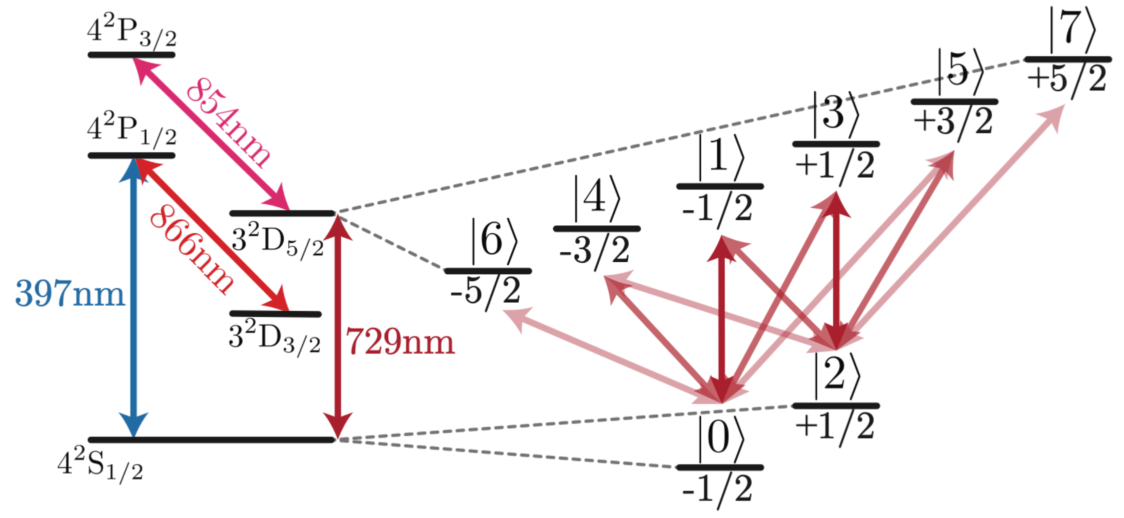
$ea = 1; m/e = 0.4$

Consistent with small m continuum predictions (perturbative)

$$E_k(\theta) = -2m \exp\left(-\frac{\pi}{N\mu eL}\right)\cos\left(\frac{\theta + 2\pi k}{N}\right).$$

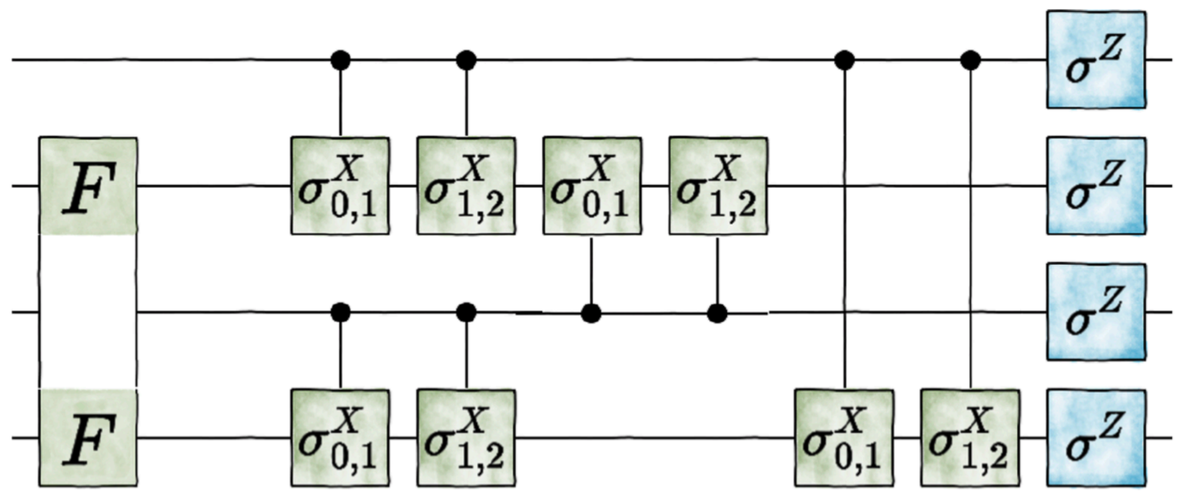
Quantum simulation with qudits

- **Integrate-out fermions** and **encode gauge fields** into **qudits** (local qudit Hamiltonian)



Ringbauer et al., Nat. Phys. 18, 1053-1057 (2021)

- Formulate **variational qudit circuit** and use **favourite minimisation procedure** (VQE, VarQITE, SC-ADAPT-VQE, etc.)



Quantum simulation with qudits

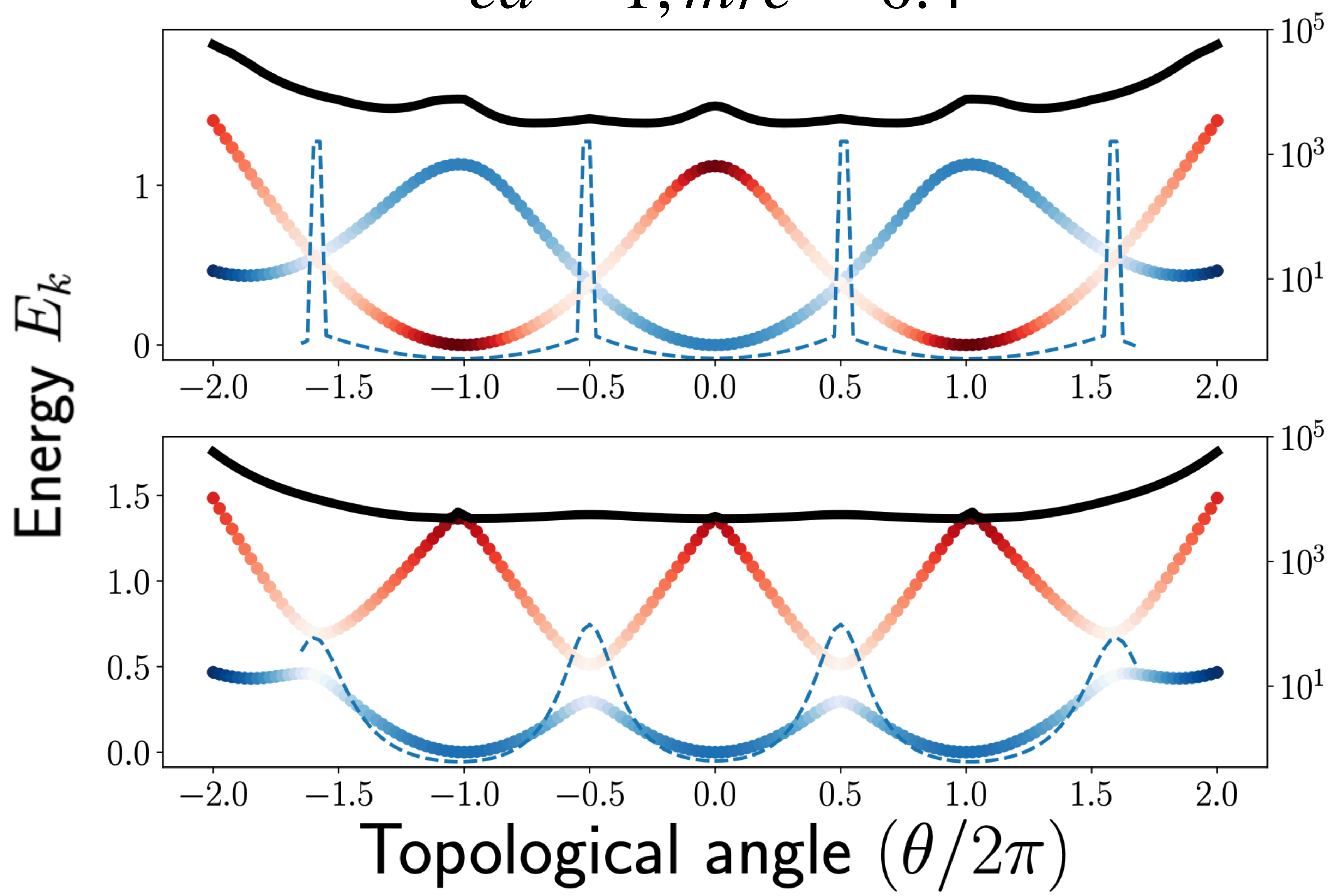
- Energy oscillations for periodic vs flavour-twisted bc (after a $\pi-$shifting in θ)

TN result

$L = (4 \text{ sites} + 4 \text{ links})$

$S\text{-cutoff } = \pm 2$

$ea = 1; m/e = 0.4$

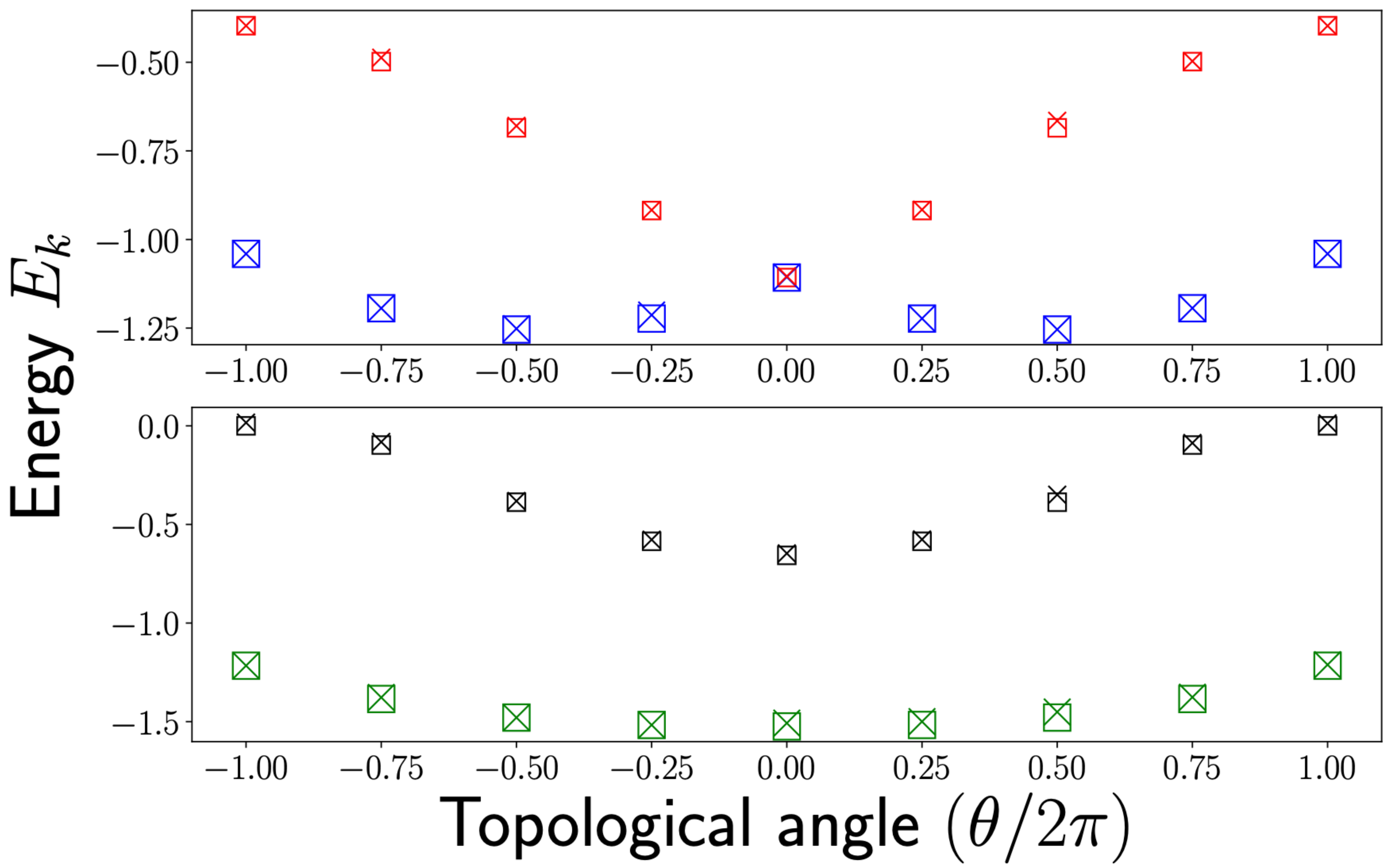


VarQITE (numerics)

$L = (2 \text{ sites} + 2 \text{ links})$

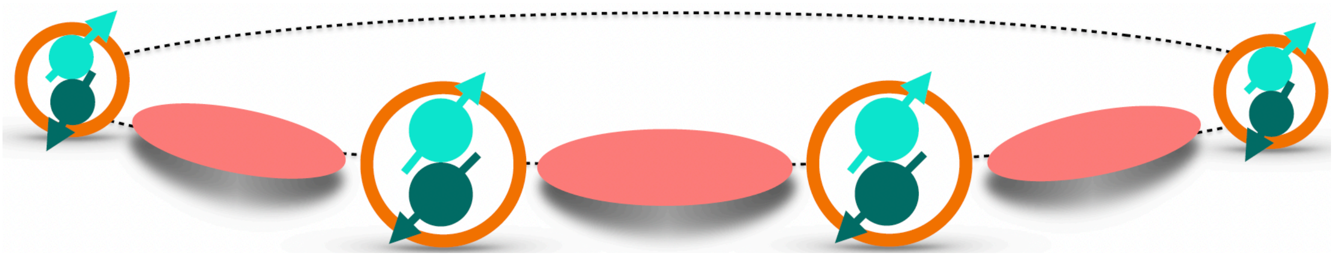
$S\text{-cutoff } = \pm 1$

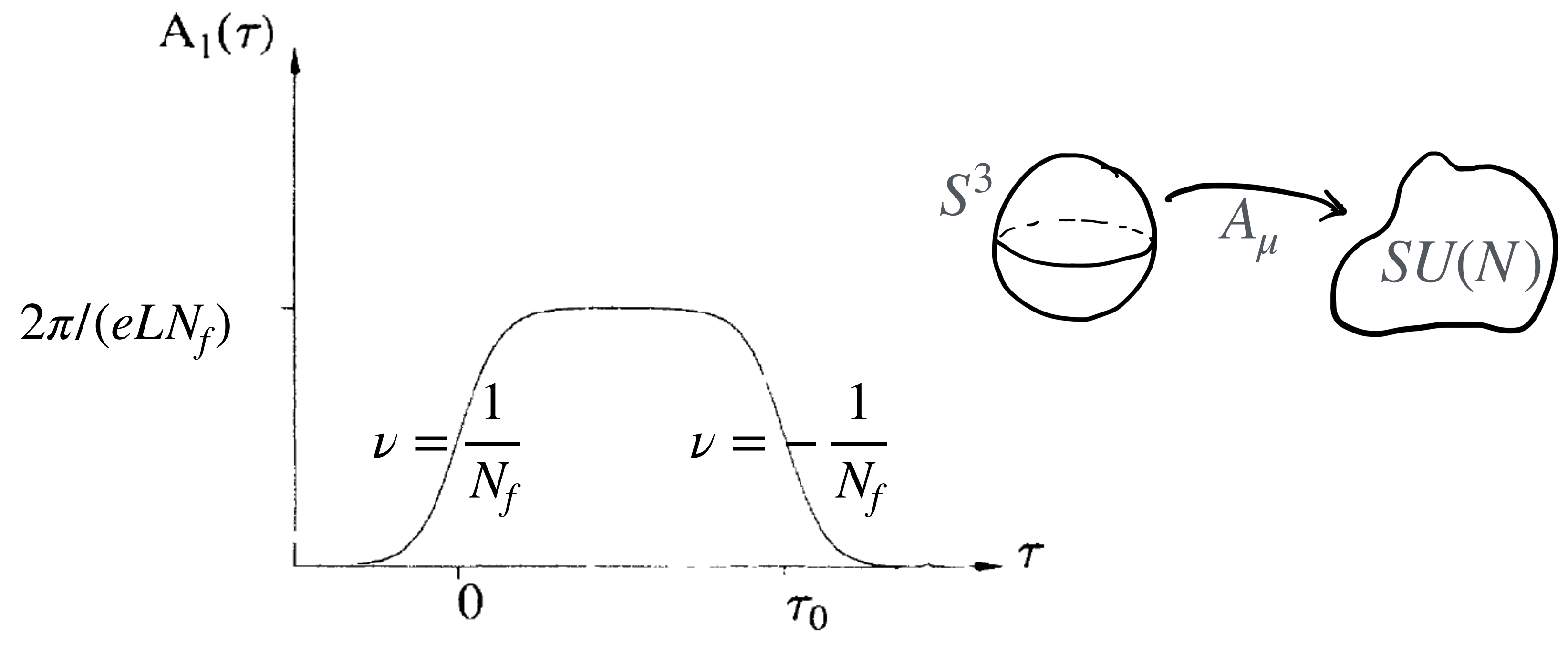
$ea = 1; m/e = 1$



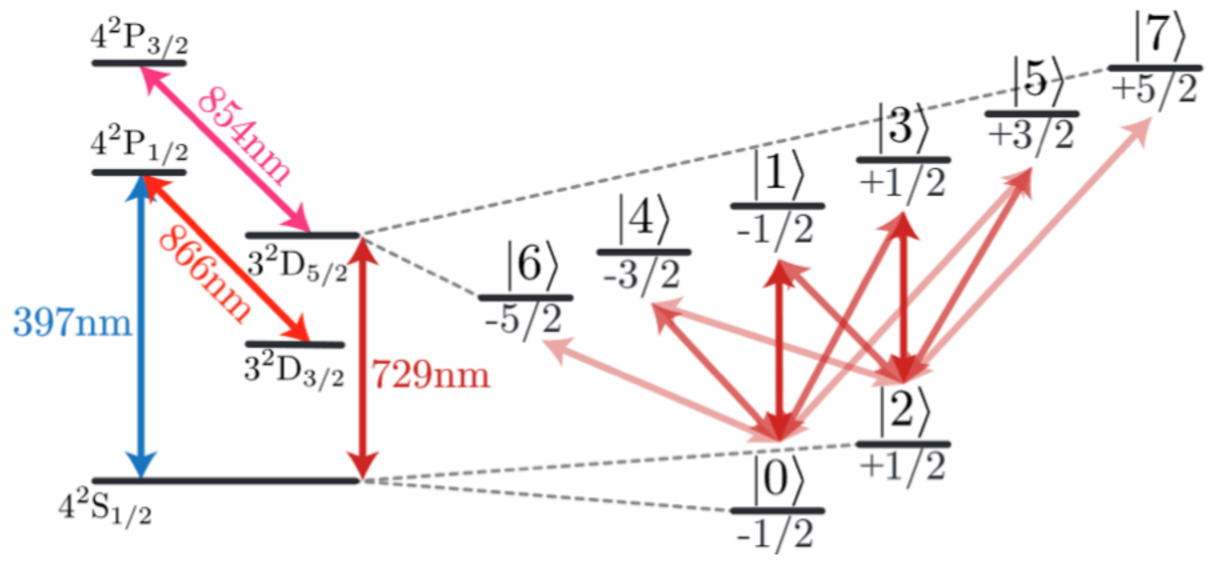
Conclusion and outlook

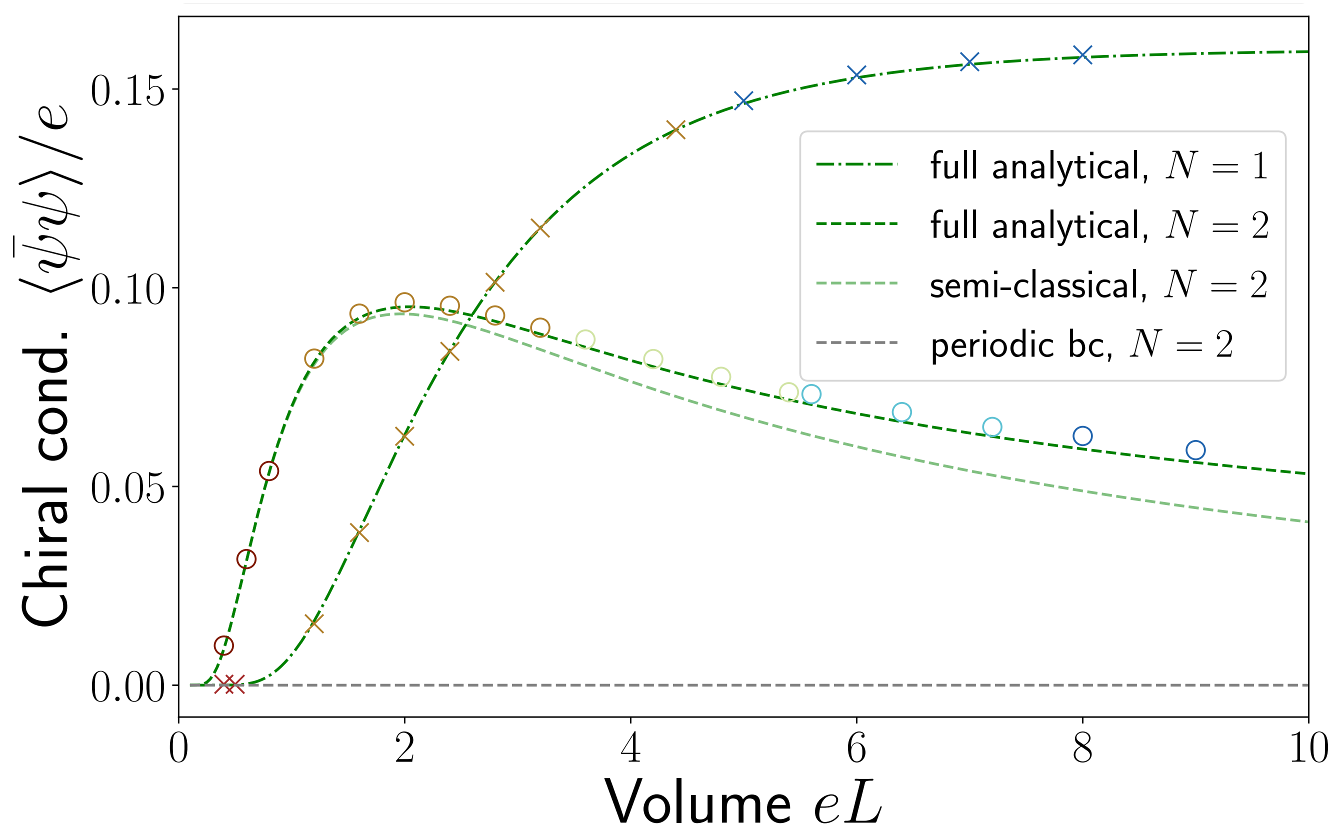
- Detection of non-trivial topology in gauge theories
- Fascinating target within reach: fracton excitations
- Challenging non-perturbative regimes and continuum physics in quantum simulators





→ Next steps: Implementation on a real quantum device...





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THANK YOU!