Quantum entanglement from local operators

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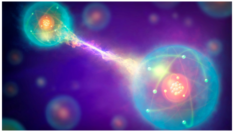
ECT* Trento, Hamiltonian Lattice Gauge Theories: Status, Novel Developments and Applications

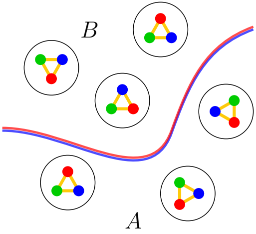
Based on a WIP with Marco Panero, Paolo Stornati and Luca Tagliacozzo

Entanglement in lattice field theories I

Entanglement in many-body systems $\longrightarrow$ applications in different areas of physics:



- Quantum phases of matter and quantum phase transitions [Vidal et al.; PRL **90** (2003)]
- High-energy physics
  - Confinement in gauge theories [Klebanov et al.; Nucl. Phys. B **796** (2008)]
- AdS/CFT and quantum gravity [Ryu, Takayanagi; PRL **96** (2006)]
- Complexity of quantum simulations, thermalization and its violation, scattering processes ...

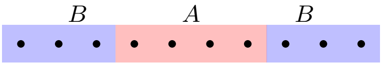


Dirac Medallists 2024

Horacio Casini, Marina Huerta, Shinsei Ryu, Tadashi Takayanagi

For "*pioneering contributions to the understanding of quantum entropy in gravity and quantum field theory*"

Bipartite entanglement



$$\rho_A = \mathrm{Tr}_B\, \rho$$

$$S = -\,\mathrm{Tr}_A(\rho_A \log \rho_A)$$

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}\, \rho_A^n \qquad S_2 \longleftrightarrow \mathrm{Tr}\, \rho_A^2$$

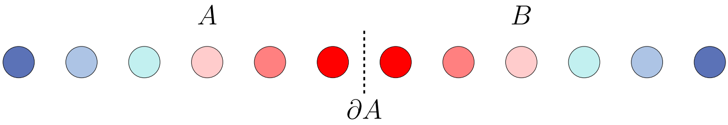
A hard quantity to handle

$$\rho_A = \sum_{P_{i_1},\ldots,P_{i_N}} c_{i_1 \ldots i_N} P_{i_1} \otimes \cdots \otimes P_{i_N}$$

$$P \in \{1, X, Y, Z\}$$

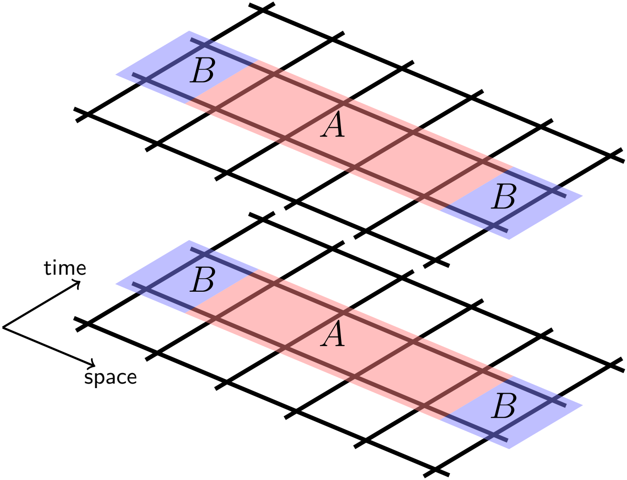
- Computing $c_{i_1 \ldots i_N}$ is exponentially hard
- Other methods: SWAP operators, random unitaries + classical shadows ...
- In general: progressively harder for larger systems

Area law: ground states of local Hamiltonian have few entanglement



$$\mathrm{Tr}\,\rho_A^2 \simeq \langle \mathcal{O}(\partial \mathrm{A}) \rangle \ ?$$

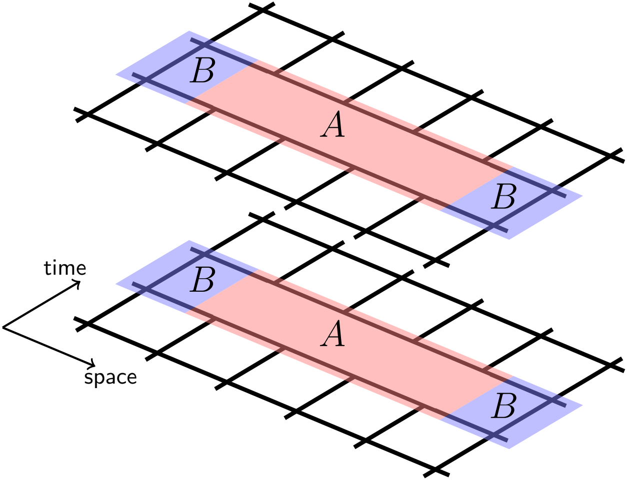
Intuition $2/3$: replica trick I



$Z^2 \rightarrow$ Decoupled replicas

[Calabrese, Cardy; J.Stat.Mech. 0406 (2004)]

Intuition $2/3$: replica trick I



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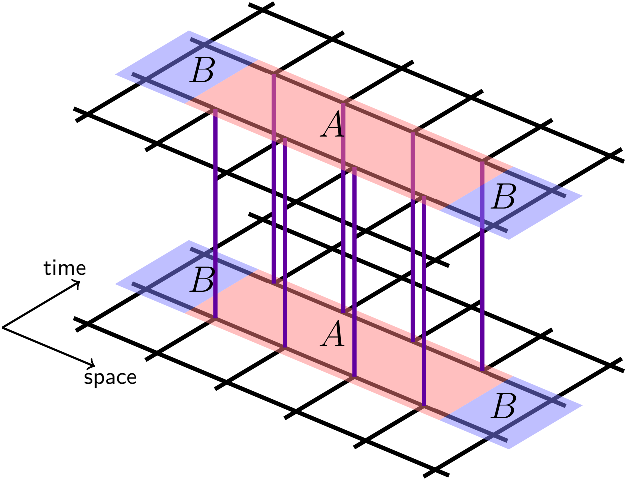
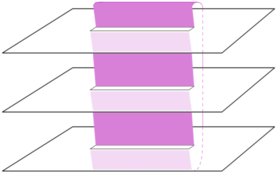


$Z^2 \to$ Decoupled replicas $\qquad Z_2 \to$ Coupled replicas $\qquad S_2 = -\log \frac{Z_2}{Z^2}$

[Calabrese, Cardy; J.Stat.Mech. 0406 (2004)]

Replicated manifolds

- Widely used in **CFT calculations**
  [Calabrese, Cardy; J.Stat.Mech. 0406 (2004)]



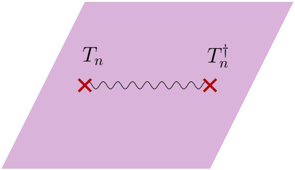
From [Cardy et al.; J.Stat.Phys **130** (2008)]

- Exploited in Lagrangian simulations
  [Buividovich, Polikarpov; Nucl.Phys.B **802** (2008)]
  [Itou et al.; PTEP (2016)]
  [Rabenstein et al.; PRD **100** (2019)]
  [AB, Panero; JHEP **06** (2024)]

Twist fields

- The replica trick implies that Rényi entropies can be computed as correlators of local operators

$$\mathrm{Tr}\, \rho_A^n = \langle T_n T_n^\dagger \rangle$$



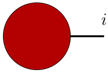
- A lattice characterization is missing...

Goals

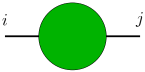
Lattice characterization of twist operators

Systematic way to measure entanglement with local operators

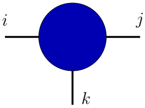
Tensor diagrams



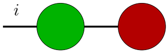
v_i vector



M_{ij} matrix



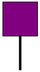
A_{ijk} rank-3 tensor



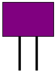
$M_{ij}v_j$ matrix-vector multiplication

States as tensors

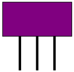
$|\psi\rangle = \sum_i c_i \, |i\rangle$



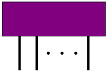
$|\psi\rangle = \sum_{ij} c_{ij} \, |ij\rangle$



$|\psi\rangle = \sum_{ijk} c_{ijk} \, |ijk\rangle$



$|\psi\rangle = \sum_{i_1 i_2 \dots i_N} c_{i_1 i_2 \dots i_N} \, |i_1 i_2 \dots i_N\rangle$

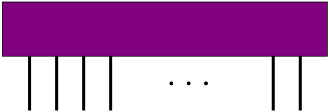


Quick dictionary: Tensor Networks II

Exponential size

- Full representation of a quantum state: exponential in system size

$$|\psi\rangle = \sum_{i_1 i_2 \dots i_N} c_{i_1 i_2 \dots i_N} |i_1 i_2 \dots i_N\rangle$$

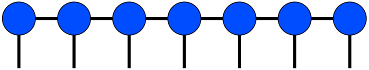


$$\sim \exp(N)$$

MPS

- Matrix Product States: **compressed** representation for a $1d$ system

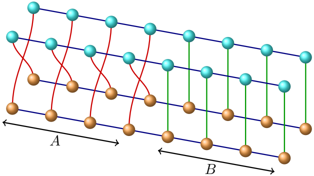
$$c_{i_1 i_2 \dots i_N} \to [A^1]_{i_1}^{\alpha_1} [A^2]_{i_2}^{\alpha_1 \alpha_2} \dots [A^N]_{i_N}^{\alpha_N}$$

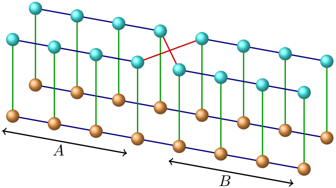


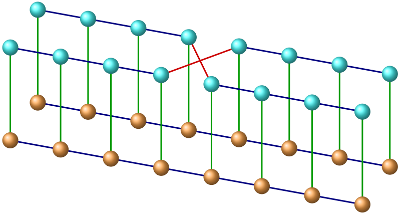
$$\sim \mathrm{poly}(N)$$

Intuition 3/3: MPS

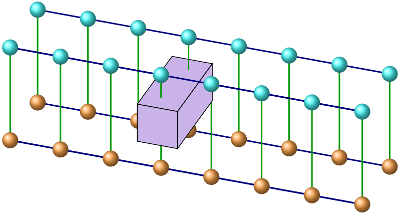
$\mathrm{Tr}\,\rho_A^2 =$



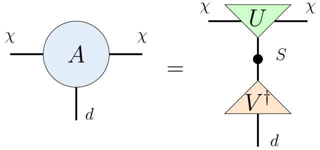


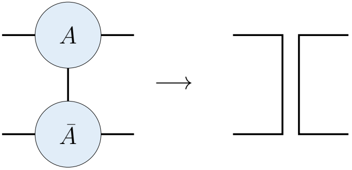


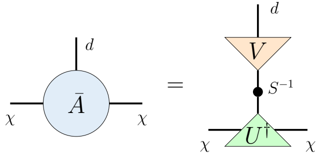
$\overset{?}{\simeq}$

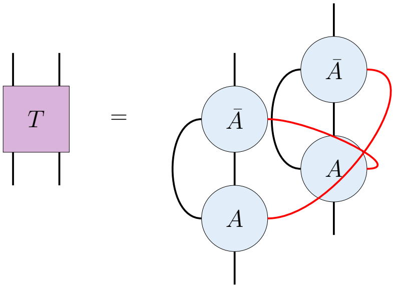


Inverting the MPS









An *exact* contraction

Invert the MPS

- If one inverts tensor by tensor we get

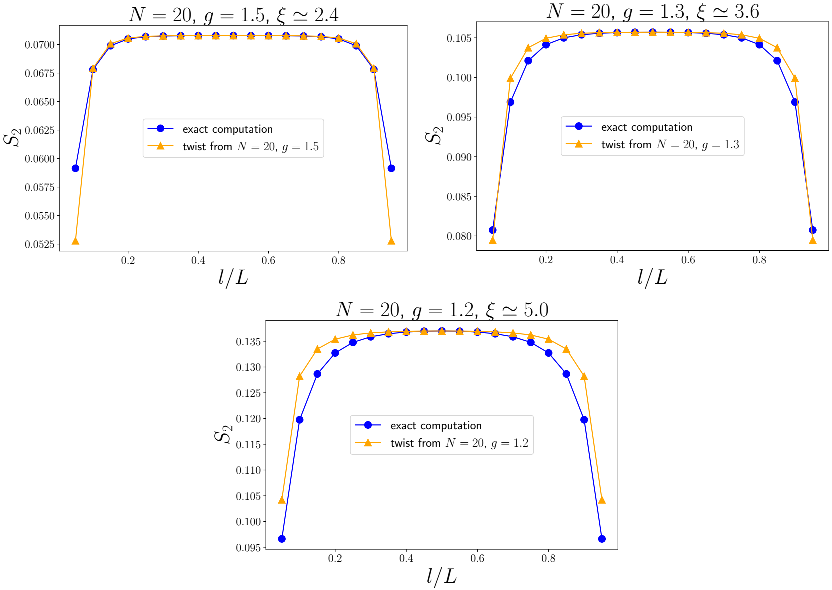
$$\mathrm{Tr}\,\rho_A^2 = \langle T \rangle$$

- However: in general T depends on the site
- Unclear how to derive it, but likely it implies the **knowledge of S_2**

What can we learn?

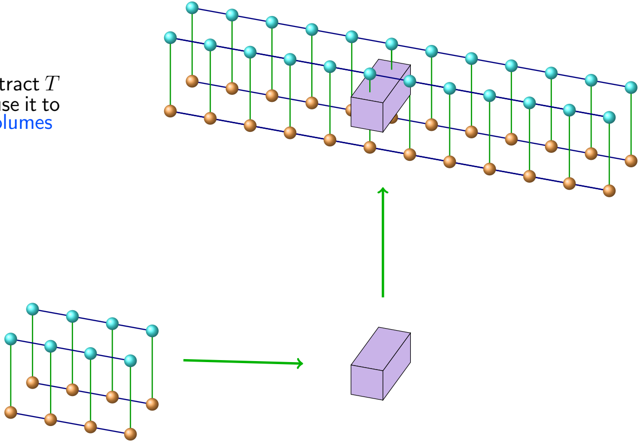
- An **exact** scheme might be too costly
- Can we design an approximate yet systematically improvable scheme?
- A first idea: use the **same operator** to compute S_2 in the whole system
- Benchmark: transverse field Ising model with OBC

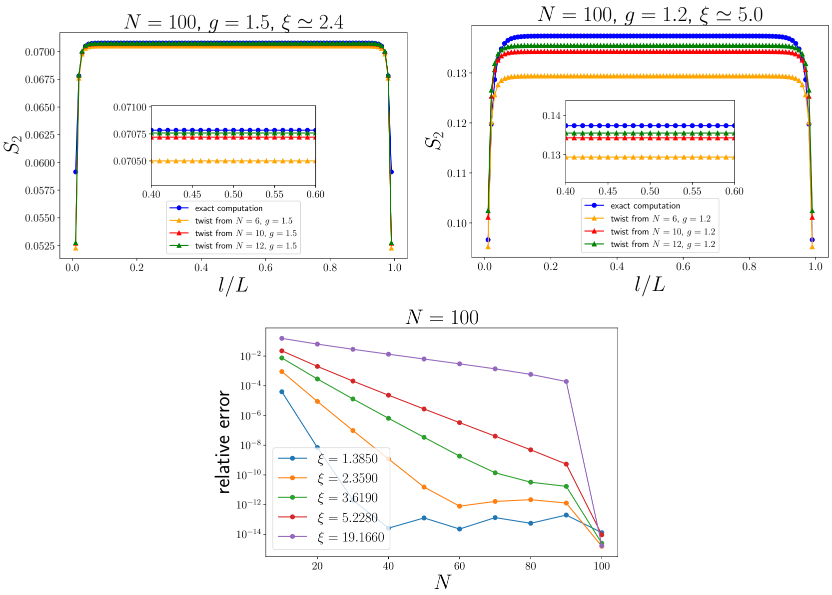
Strategies to compute S_2 I



Strategies to compute S_2 II

Another strategy: extract T for small volumes and use it to compute S_2 in large volumes

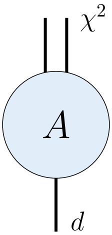




N = 100, g = 1.5, ξ ≃ 2.4
S2
l/L
exact computation
twist from N = 6, g = 1.5
twist from N = 10, g = 1.5
twist from N = 12, g = 1.5
N = 100, g = 1.2, ξ ≃ 5.0
exact computation
twist from N = 6, g = 1.2
twist from N = 10, g = 1.2
twist from N = 12, g = 1.2
N = 100
relative error
N
ξ = 1.3850
ξ = 2.3590
ξ = 3.6190
ξ = 5.2280
ξ = 19.1660

Matrix inversion

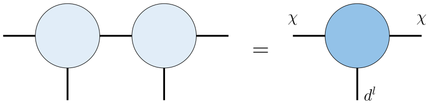
- Larger ξ requires larger χ
- The "inversion" progressively **fails** to give access to the virtual legs



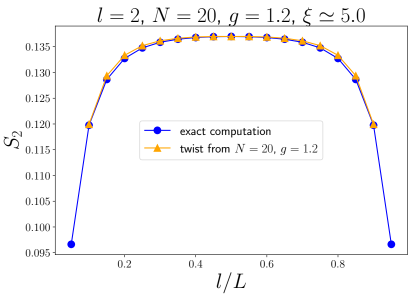
$$\chi^2 > d$$

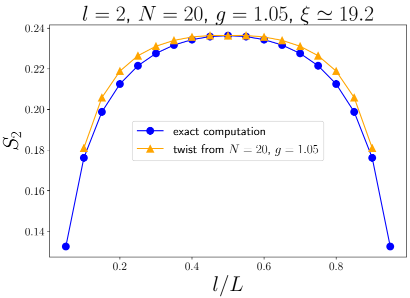
Larger tensors

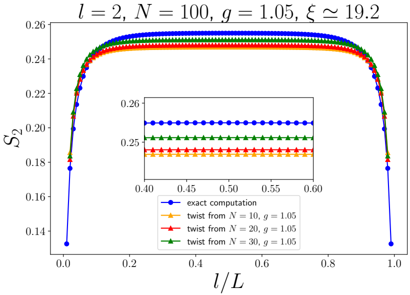
- Contract together l tensors
- **Trade-off**: larger matrices provide a better inversion but are more difficult to measure

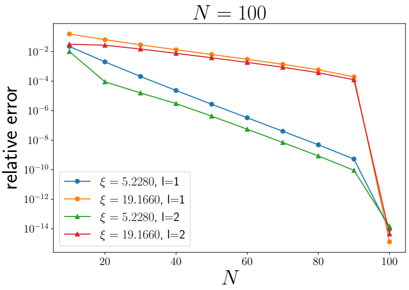


Larger twists II









Conclusions and outlook

Conclusions

- **Local operators** inspired to **twist fields** can provide a measure of entanglement
- Approximate and systematically improvable scheme at least in gapped systems
- Larger twist provide a progressively better approximation

Outlook

- Tomography of such operators?
- Precise lattice definition of twist fields from CFT?
- How to deal with large ξ?
- MPO version of large twists?